Mapping

By Olivier Croissant

1 Introduction

Let assume we want to keep the variance and a nth derivative (0<=n<=infini)

so $Var[C_t] = Var[C_1 + C_2]$

and because

$$\left(\frac{\partial}{\partial r_t}\right)^n NPV[C_t] = t^n NPV[C_t] = t^n Z_t C_t$$

if we assume that the zero coupons at time e t_1 and t_2 are normal with a covariance matrix equal to:

$$Cov = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

So the risk on a zero coupon paying C_1 at time t_1 is equal to

$$Var[C_1] = \left(\sigma_{B_{z_1}} \frac{\partial}{\partial B_{z_1}} NPV[C_1]\right)^2 = \sigma_1^2 C_1^2 \qquad \text{(EQ 1)}$$

this imply that the right side of the equation (1) is equal to

$$Var[C_1 + C_2] = \sigma_1^2 Z_1^2 C_1^2 + 2\rho\sigma_1\sigma_2 Z_1 C_1 Z_2 C_2 + \sigma_2^2 Z_2^2 C_2^2 \quad \text{(EQ 2)}$$

So by comparison the left fide of equation 1 should be:

$$Var[C_t] = \sigma_t^2 Z_t^2 C_t^2 \quad \text{(EQ 3)}$$

but we don't know σ_t

let assume that we have an interpolation method that doesn't depends on the yield curve:

$$\sigma_t = \sigma_t(\sigma_1, t_1, \sigma_2, t_2, t)$$

then we can easily show that:

$$\sigma_1^2 Z_1^2 C_1^2 + 2\rho\sigma_1\sigma_2 Z_1 C_1 Z_2 C_2 + \sigma_2^2 Z_2^2 C_2^2 = \sigma_t^2 Z_t^2 C_t^2$$

$$t^n Z_t C_t = t_1^n Z_1 C_1 + t_2^n Z_2 C_2$$

let $x = \dfrac{Z_1 C_1}{Z_t C_t}$, this imply that

$$x^2[\sigma_1^2 t_2^{2n} + (-2\rho\sigma_1\sigma_2(t_1 t_2)^n + \sigma_2^2 t_1^{2n})] + 2x[\rho\sigma_1\sigma_2(t_2 t)^n - \sigma_2^2(t_1 t)^n] + (\sigma_2^2 t^{2n} - \sigma_t^2 t_2^{2n}) = 0$$

then

$$x = \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2}$$

$$\Delta = \sigma_t^2 t_2^{2n}(\sigma_1^2 t_2^{2n} - 2\rho\sigma_1\sigma_2(t_1 t_2)^n + \sigma_2^2 t_1^{2n}) - (1-\rho^2)\sigma_1^2\sigma_2^2(t_2 t)^{2n}$$

So the solution of the mapping problem is:

$$\boxed{\begin{aligned} C_1 &= \left(\frac{Z_t C_t}{Z_1}\right)\frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2} \\ C_2 &= \frac{Z_t C_t}{t_2^n Z_2}\left(t^n - t_1^n \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2}\right) \\ \Delta &= \sigma_t^2 t_2^{2n}(\sigma_1^2 t_2^{2n} - 2\rho\sigma_1\sigma_1(t_1 t_2)^n + \sigma_2^2 t_1^{2n}) - (1-\rho^2)\sigma_1^2\sigma_2^2(t_2 t)^{2n} \end{aligned}}$$

those formula make sense when x greater than 0 and less than 1

So the natural sensitivities transform like:

$$dr_t = \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2} dr_{t_1} + \frac{1}{t_2^n}\left(t^n - t_1^n \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2}\right) dr_{t_2}$$

$$0 \le \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2} \le 1$$

and due to the non dependence of σ_t on the yield curve,

$$\overline{d^2 r_t} = 0$$

the preceding differential form is integrable and allow us to deduce the interpolation function which is “implicit ” in the choice made to map the cash flows:

$$f(t) = \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2} r_1 + \frac{1}{t_2^n}\left(t^n - t_1^n \frac{(t_1 t)^n \sigma_2^2 - (t_2 t)^n \rho \sigma_1 \sigma_2 \pm \sqrt{\Delta}}{t_2^{2n}\sigma_1^2 - 2(t_1 t_2)^n \rho \sigma_1 \sigma_2 + t_1^{2n}\sigma_2^2} \right) r_2$$

here the parameter n can be used to adjust the smoothness of this interpolation

the left and right derivatives are dependant on the function $\sigma_t(t)$

more precisely:

$$\left(\left. \frac{df}{dt} \right|_{t = t_i} \right)^+ = \left(\left. \frac{df}{dt} \right|_{t = t_i} \right)^-$$

the shape of the interpolation of the standard deviation of the zero coupon $\sigma_t(t)$ can also be used to perform this smoothness:

The standard deviation of the zero coupon is linked to the standard deviation of the zero coupon continuous yield by

: $$\sigma_{Z_1} = \sigma_{y,1} \left| \frac{\partial B_1}{\partial y_1} \right| = \sigma_{y,1} t_1$$

we assume most of the time that the most intrinsic quantity is the yield SD, It is why the interpolation is done via σ_{Z_t} / t but this approximate relationship induce non-smoothness at the level of bond SD, so it is sensible to want to modify it.

a parameter could be defined in the interpolation:

$$\boxed{\frac{\sigma_t}{t^p} = \frac{\sigma_1}{t_1^p} + \frac{t - t_1}{t_2 - t_1}\left(\frac{\sigma_2}{t_2^p} - \frac{\sigma_1}{t_1^p} \right)}$$

p and n will be used to adjust the smoothness of the yield curve and the volatility curve.

n =0 zero means keeping the present value

n=1 means keeping the general sensitivity

p=0 means interpolating linearly the bond standard deviation.

p=1 means interpolating linearly the yield standard deviation

So The mapping of the volatility is

$$d\sigma_t = \left(\frac{t_2 - t}{t_1^p (t_2 - t_1)}\right) d\sigma_1 + \left(\frac{t - t_1}{t_2^p (t_2 - t_1)}\right) d\sigma_2$$

with also

$$\overline{d^2\sigma_t} = 0$$

2 Second Order Mapping

If we write the mapping relationship as

$$dr_t = \sum_i \alpha_i dr_i$$

then by applying the chain rule of the second order we get :

$$d^2 r_t = \sum_{i,j} \frac{\partial \alpha_i}{\partial r_j} (dr_i \otimes dr_j) + \sum_i \alpha_i d^2 r_i$$

in the case where α_i doesn't depends on the r_j the preceding read :

$$d^2 r_t = \sum_i \alpha_i d^2 r_i$$

which give the preceding

$$\overline{d^2 r_t} = 0$$

if f is a function of the rate r_t then the mapping could be done as :

$$df = \frac{\partial f}{\partial r_t} dr_t = \sum_i \left(\frac{\partial f}{\partial r_t} \alpha_i \right) dr_i$$

and for the second order :

$$\overline{d^2 f} = \frac{\partial^2 f}{(\partial r_t)^2} (dr_t \otimes dr_t) + \frac{\partial f}{\partial r_t} \overline{d^2 r_t} = \sum_{i,j} \left(\frac{\partial^2 f}{(\partial r_t)^2} \alpha_i \alpha_j + \frac{\partial f}{\partial r_t} \frac{\partial \alpha_i}{\partial r_j} \right) (dr_i \otimes dr_j)$$

which for the case where α_i doesn't depends on the r_j :

$$\overline{d^2 f} = \sum_{i,j} \left(\frac{\partial^2 f}{(\partial r_t)^2} \alpha_i \alpha_j \right) (dr_i \otimes dr_j)$$

We can then generalize the preceding calculation to the case where we have crossed gammas :

$$f = f(r_t, v_t)$$

if we have the mappings :

$$dr_t = \sum_i \alpha_i dr_i \qquad dv_t = \sum_i \beta_i dv_i$$

the mappings are : :

$$df = \sum_i \left(\frac{\partial f}{\partial r_t}\alpha_i\right) dr_i + \sum_i \left(\frac{\partial f}{\partial v_t}\beta_i\right) dv_i$$

$$\overline{d^2} f = \sum_{i,j} \frac{\partial^2 f}{(\partial r_t)^2}\alpha_i\alpha_j(dr_i \otimes dr_j) + \sum_{i,j} \frac{\partial^2 f}{\partial r_t \partial v_t}\alpha_i\beta_j(dr_i \otimes dv_j) + \sum_{i,j} \frac{\partial^2 f}{(\partial v_t)^2}\beta_i\beta_j(dv_i \otimes dv_j)$$

3 Mapping of averaged instruments sensitivities

3.1 Theorie

the sensivities associated with the averaged instruments are defined by kernels :

$$df = \int_{-\infty}^{\infty} ds\left(\frac{\delta f}{\delta r_s(s)} dr_s\right)$$

and

$$d^2 f = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dq \left(\frac{\delta^2 f}{\delta r_s(s)\delta r_q(q)}(dr_s \otimes dr_q)\right) + \int_{-\infty}^{\infty} ds\left(\frac{\delta f}{\delta r_s(s)} d^2 r_s\right)$$

So by appling the mapping equations :

$$df = \int_{-\infty}^{\infty} ds\left(\frac{\delta f}{\delta r_s(s)}\sum_i \alpha_i(s)dr_i\right) = \sum_i\left(\left(\int_{-\infty}^{\infty} ds\left(\frac{\delta f}{\delta r_s(s)}\alpha_i(s)\right)\right)dr_i\right)$$

and is the case where where α_i doesn't depends on the r_j :

$$\overline{d^2} f = \sum_{i,j}\left(\int_{-\infty}^{\infty} ds\int_{-\infty}^{\infty} dq\frac{\delta^2 f}{\delta r_s(s)\delta r_q(q)}(\alpha_i(s)\alpha_j(q))\right)(dr_i \otimes dr_j)$$

3.2 Application to the case of averaged forward rates

3.2.1 First Order

In this case the sensitivities are

$$\frac{\delta f_{d_1, d_2, T}}{\delta r_s(s)} = \Theta(s-d_2)\Theta(d_2+T-s)\frac{s}{T(d_2-d_1)}e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)}$$
$$-\Theta(s-d_1)\Theta(d_1+T-s)\frac{s}{T(d_2-d_1)}e^{r_{d_2-d_1+s}(d_2-d_1+s)-r_s s}$$

and

so the integral reads : $df_{d_1, d_2, T} = \sum_i \Pi_i dr_i$ where

$$\lambda_{1,i}(s) = \frac{(t_i s)^n \sigma_i^2 - (t_{i+1} s)^n \rho_{i,i+1} \sigma_i \sigma_{i+1} \pm \sqrt{\Delta_{i,i+1}(s)}}{t_{i+1}^{2n} \sigma_i^2 - 2(t_i t_{i+1})^n \rho_{i,i+1} \sigma_i \sigma_{i+1} + t_i^{2n} \sigma_{i+1}^2}$$

$$\lambda_{2,i}(s) = \left(\frac{1}{t_i^n}\left(s^n - t_{i-1}^n \frac{(t_{i-1} s)^n \sigma_i^2 - (t_i s)^n \rho_{i-1,i} \sigma_{i-1} \sigma_i \pm \sqrt{\Delta_{i-1,i}(s)}}{t_i^{2n} \sigma_{i-1}^2 - 2(t_{i-1} t_i)^n \rho \sigma_{i-1} \sigma_i + t_{i-1}^{2n} \sigma_i^2}\right)\right)$$

$$\alpha_i(s) = \Theta(s - t_i)\Theta(t_{i+1} - s)\lambda_{1,i}(s) + \Theta(s - t_{i-1})\Theta(t_i - s)\lambda_{2,i}(s)$$

$$\Delta_{i,i+1}(s) = (\sigma_{i,i+1}(s))^2 t_{i+1}^{2n} (\sigma_i^2 t_{i+1}^{2n} - 2\rho \sigma_i \sigma_i (t_i t_{i+1})^n + \sigma_i^2 t_i^{2n}) - (1 - \rho_{i,i+1}^2) \sigma_i^2 \sigma_{i+1}^2 (t_{i+1} s)^{2n}$$

$$\sigma_{i,i+1}(s) = \sigma_i \left(\frac{s}{t_i}\right)^p + \frac{s - t_i}{t_{i+1} - t_i}\left(\sigma_{i+1}\left(\frac{s}{t_{i+1}}\right)^p - \sigma_i \left(\frac{s}{t_i}\right)^p\right)$$

$$\Pi_i = \Pi_{1,i} + \Pi_{2,i} + \Pi_{3,i} + \Pi_{4,i}$$

and

$$\Pi_{1,i} = \Theta(Min\{d_2 + T, t_{i+1}\} - Max\{d_2, t_i\})$$

$$\times \int_{Max\{d_2, t_i\}}^{Min\{d_2 + T, t_{i+1}\}} ds \left(\frac{se^{r_s s - r_{d_1 - d_2} + s(d_1 - d_2 + s)}}{T(d_2 - d_1)}\right) \lambda_{1,i}(s)$$

$$\Pi_{2,\,i} = -\Theta(Min\{d_1 + T, t_{i+1}\} - Max\{d_1, t_i\})$$
$$\times \int_{Max\{d_1,\, t_i\}}^{Min\{d_1 + T,\, t_{i+1}\}} ds \left(\frac{se^{r_{d_2 - d_1 + s}(d_2 - d_1 + s) - r_s s}}{T(d_2 - d_1)} \right) \lambda_{1,\, i}(s)$$

$$\Pi_{3,\,i} = \Theta(Min\{d_2 + T, t_i\} - Max\{d_2, t_{i-1}\})$$
$$\times \int_{Max\{d_2,\, t_{i-1}\}}^{Min\{d_2 + T,\, t_i\}} ds \left(\frac{se^{r_s s - r_{d_1 - d_2 + s}(d_1 - d_2 + s)}}{T(d_2 - d_1)} \right) \lambda_{2,\, i}(s)$$

$$\Pi_{4,\,i} = -\Theta(Min\{d_1 + T, t_i\} - Max\{d_1, t_{i-1}\})$$
$$\times \int_{Max\{d_1,\, t_{i-1}\}}^{Min\{d_1 + T,\, t_i\}} ds \left(\frac{se^{r_{d_2 - d_1 + s}(d_2 - d_1 + s) - r_s s}}{T(d_2 - d_1)t_i^n} \right) \lambda_{2,\, i}(s)$$

The sign being chosen such that the payment are of the same sign than the cashflow.

3.2.2 Second order

The expression of the kernel associated with the second order is :

So the expression of the second order sensitivities will be :

where

$$\frac{\delta^2 f_{d_1, d_2, T}}{\delta r_s(s)\delta r_q(q)} =$$

$$\frac{s(s-1)}{T(d_2-d_1)} e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)} \delta(q-s)\Theta(s-d_2)\Theta(d_2+T-s)$$

$$-\frac{2(d_1-d_2+s)s}{T(d_2-d_1)} e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)} \delta(q-(d_1-d_2+s))\Theta(s-d_2)\Theta(d_2+T-s)$$

$$+\frac{s(s+1)}{T(d_2-d_1)} e^{r_{d_2-d_1+s}(d_2-d_1+s) - r_s s} \delta(q-s)\Theta(s-d_1)\Theta(d_1+T-s)$$

$$\overline{d^2 f_{d_1, d_2, T}} = \sum_{i,j} (\chi_{1,ij} + \chi_{2,ij} + \chi_{3,ij} + \chi_{4,ij} + \chi_{5,ij} + \chi_{6,ij} + \chi_{7,ij} + \chi_{8,ij} + \chi_{9,ij})(dr_i \otimes dr_j)$$

$$\chi_{1,ij} = \Theta(Min\{d_2+T, t_{i+1}, t_{j+1}\} - Max\{d_2, t_i, t_j\})$$

$$\times \int_{Max\{d_2, t_i, t_j\}}^{Min\{d_2+T, t_{i+1}, t_{j+1}\}} ds \frac{s^2}{T(d_2-d_1)} e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)} \lambda_{1,i}(s)\lambda_{1,j}(s)$$

$$\chi_{2,ij} = \Theta(Min\{d_2+T, t_{i+1}, t_j\} - Max\{d_2, t_i, t_{j-1}\})$$
$$\times 2\int_{Max\{d_2, t_i, t_{j-1}\}}^{Min\{d_2+T, t_{i+1}, t_j\}} ds \frac{s^2}{T(d_2-d_1)} e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)} \lambda_{1,i}(s)\lambda_{2,j}(s)$$

$$\chi_{3,ij} = \Theta(Min\{d_2+T, t_i, t_j\} - Max\{d_2, t_{i-1}, t_{j-1}\})$$
$$\times \int_{Max\{d_2, t_{i-1}, t_{j-1}\}}^{Min\{d_2+T, t_i, t_j\}} ds \frac{s^2}{T(d_2-d_1)} e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)} \lambda_{2,i}(s)\lambda_{2,j}(s)$$

$$\chi_{4,ij} = -\Theta(Min\{d_2+T, t_{i+1}, t_{j+1}+d_1-d_2\} - Max\{d_2, t_i, t_j+d_1-d_2\})$$
$$\times 2\int_{Max\{d_2, t_i, t_j+d_1-d_2\}}^{Min\{d_2+T, t_{i+1}, t_{j+1}+d_1-d_2\}} ds \frac{(d_1-d_2+s)s}{T(d_2-d_1)} e^{r_s s - r_{d_1-d_2+s}(d_1-d_2+s)} \lambda_{1,i}(s)\lambda_{1,j}(d_1-d_2+s)$$

$$\chi_{5,ij} = -\Theta(Min\{d_2 + T, t_{i+1}, t_j + d_1 - d_2\} - Max\{d_2, t_i, t_{j-1} + d_1 - d_2\})$$

$$\times 4\int_{Max\{d_2, t_i, t_{j-1} + d_1 - d_2\}}^{Min\{d_2 + T, t_{i+1}, t_j + d_1 - d_2\}} ds \frac{(d_1 - d_2 + s)s}{T(d_2 - d_1)} e^{r_s s - r_{d_1 - d_2 + s}(d_1 - d_2 + s)} \lambda_{1,i}(s)\lambda_{2,j}(d_1 - d_2 + s)$$

$$\chi_{6,ij} = -\Theta(Min\{d_2 + T, t_i, t_j + d_1 - d_2\} - Max\{d_2, t_{i-1}, t_{j-1} + d_1 - d_2\})$$

$$\times 2\int_{Max\{d_2, t_{i-1}, t_{j-1} + d_1 - d_2\}}^{Min\{d_2 + T, t_i, t_j + d_1 - d_2\}} ds \frac{(d_1 - d_2 + s)s}{T(d_2 - d_1)} e^{r_s s - r_{d_1 - d_2 + s}(d_1 - d_2 + s)} \lambda_{2,i}(s)\lambda_{2,j}(d_1 - d_2 + s)$$

$$\chi_{7,ij} = \Theta(Min\{d_1 + T, t_{i+1}, t_{j+1}\} - Max\{d_1, t_i, t_j\})$$

$$\times \int_{Max\{d_1, t_i, t_j\}}^{Min\{d_1 + T, t_{i+1}, t_{j+1}\}} ds \frac{s^2}{T(d_2 - d_1)} e^{r_{d_2 - d_1 + s}(d_2 - d_1 + s) - r_s s} \lambda_{1,i}(s)\lambda_{1,j}(s)$$

$$\chi_{8,ij} = \Theta(Min\{d_1+T, t_{i+1}, t_j\} - Max\{d_1, t_i, t_{j-1}\})$$
$$\times 2\int_{Max\{d_1, t_i, t_{j-1}\}}^{Min\{d_1+T, t_{i+1}, t_j\}} ds \frac{s^2}{T(d_2-d_1)} e^{r_{d_2-d_1+s}(d_2-d_1+s) - r_s s} \lambda_{1,i}(s)\lambda_{2,j}(s)$$

$$\chi_{9,ij} = \Theta(Min\{d_1+T, t_i, t_j\} - Max\{d_1, t_{i-1}, t_{j-1}\})$$
$$\times \int_{Max\{d_1, t_{i-1}, t_{j-1}\}}^{Min\{d_1+T, t_i, t_j\}} ds \frac{s^2}{T(d_2-d_1)} e^{r_{d_2-d_1+s}(d_2-d_1+s) - r_s s} \lambda_{2,i}(s)\lambda_{2,j}(s)$$

3.3 Application to the averaged exchange rates

3.3.1 first order

$$\begin{cases} \dfrac{\delta F_x(t,T)}{\delta r_{p,s}(s)} = \dfrac{1}{T}\displaystyle\int_t^{t+T} X \times s e^{s(r_{p,s} - r_{2,s})} ds \\ \dfrac{\delta F_x(t,T)}{\delta r_{2,s}(s)} = -\dfrac{1}{T}\displaystyle\int_t^{t+T} X \times s e^{s(r_{p,s} - r_{2,s})} ds \end{cases}$$

so

$$dF_x(t,T) = \sum_i \Pi_{p,i} dr_i + \sum_i \Pi_{2,i} dr_i + \left\{ \frac{1}{T} \int_t^{t+T} s e^{s(r_{p,s} - r_{2,s})} ds \right\} dX$$

Where

$$\begin{cases} \Pi_{p,i} = \dfrac{\Theta(Min\{t+T, t_{i+1}\} - Max\{t, t_i\})}{T} \displaystyle\int_{Max\{t,t_i\}}^{Min\{t+T, t_{i+1}\}} X \times s e^{s(r_{p,s} - r_{2,s})} \lambda_{1,i}(s) ds \\ \Pi_{2,i} = \dfrac{-\Theta(Min\{t+T, t_{i+1}\} - Max\{t, t_i\})}{T} \displaystyle\int_{Max\{t,t_i\}}^{Min\{t+T, t_{i+1}\}} X \times s e^{s(r_{p,s} - r_{2,s})} \lambda_{2,i}(s) ds \end{cases}$$

3.3.2 Second order

$$\begin{cases} \dfrac{\delta^2 F_x(t,T)}{\delta r_{p,s}(s) \delta r_{p,q}(q)} = \dfrac{1}{T} \displaystyle\int_0^T X \times (t+s)^2 e^{(t+s)(r_{p,t+s} - r_{2,t+s})} ds \\ \dfrac{\delta^2 F_x(t,T)}{\delta r_{p,s}(s) \delta r_{2,q}(q)} = -\dfrac{1}{T} \displaystyle\int_0^T X \times (t+s)^2 e^{(t+s)(r_{p,t+s} - r_{2,t+s})} ds \\ \dfrac{\delta^2 F_x(t,T)}{\delta r_{2,s}(s) \delta r_{2,q}(q)} = \dfrac{1}{T} \displaystyle\int_0^T X \times (t+s)^2 e^{(t+s)(r_{p,t+s} - r_{2,t+s})} ds \end{cases}$$

So

where

$$\overline{d^2F_x(t,T)} = \sum_{i,j} \chi_{pp,ij}(dr_{p,i} \otimes dr_{p,j}) + \chi_{p2,ij}(dr_{p,i} \otimes dr_{2,j}) + \chi_{22,ij}(dr_{2,i} \otimes dr_{2,j})$$

$$+ \sum_i \Pi_{p,i}(dr_i \otimes dX) + \sum_i \Pi_{2,i}(dr_i \otimes dX)$$

$$\chi_{pp,ij} = \frac{\Theta(Min\{t+T, t_{i+1}, t_{j+1}\} - Max\{t, t_i, t_j\})}{T} \int_{Max\{t, t_i, t_j\}}^{Min\{t+T, t_{i+1}, t_{j+1}\}} X \times s^2 e^{s(r_{p,s} - r_{2,s})} \lambda_{1,i}(s) \lambda_{1,j}(s) ds$$

$$\chi_{p2,ij} = \frac{-2\Theta(Min\{t+T, t_{i+1}, t_j\} - Max\{t, t_i, t_{j-1}\})}{T} \int_{Max\{t, t_i, t_{j-1}\}}^{Min\{t+T, t_{i+1}, t_j\}} X \times s^2 e^{s(r_{p,s} - r_{2,s})} \lambda_{1,i}(s) \lambda_{2,j}(s) ds$$

$$\chi_{22,ij} = \frac{\Theta(Min\{t+T, t_i, t_j\} - Max\{t, t_{i-1}, t_{j-1}\})}{T} \int_{Max\{t, t_{i-1}, t_{j-1}\}}^{Min\{t+T, t_i, t_j\}} X \times s^2 e^{s(r_{p,s} - r_{2,s})} \lambda_{2,i}(s) \lambda_{2,j}(s) ds$$

4 Generalisations

4.1 From Cov(discrete rates) to Cov(continuous rates)

We will have a finite number of factors that describes the state of every interest rate curve, so we need a way to transform the above sensitivities into standardized ones than use only a set of perfectly defined ones imposed by external constraints (available statistical data, computation time,...).

SO to relate the natural sensitivities to the standardized one (determined by external constraints.). We will start from the natural relationship that should exist between the interest rates and the standardized maturities.Such a relationship will be called an interpolation method.This interpolation method could includes more parameters that will fit the curve to the market curve.

$$r_t = f_t(r_1, \ldots.., r_n, p)$$

p stand for the set of other parameters that makes the curve matches exactly the known points of the markets.

It is necessary to have a function f differentiable with respect to the standardized rates.and this requirement is not really a problem.This has nothing to do with the differentiability of the rate with the maturity that could not be the case.For example a linear interpolation of the zero coupons rate is not differentiable with respect to the maturity but is differentiable with respect to the rates in the proportional shift model.

It is important to understand the difference between the fitting of the curve to the market price for selfdiscounting instruments and the determination of the dynamic behavior of this curve.

The dynamic behavior of the curve should be arbitrage free for pricing derivative. But for very short term, it is not required.The reason is that the risk management framework that we are developing is a local one. allowing us to look at instantaneous risk.And Instantaneously, there is no arbitrage relationship

Let’s assume that we know the covariance matrix of the standardized maturities which will be the case in this probabilistic risk management framework, the stochastic behavior of the whole curve is determined.Particularly, the volatility curve is perfectly determined. there is an implicit volatility curve associated with the interpolation technique we have chosen for the zero coupons curve.It is given by the Ito lemma: the volatility associated with the rate r(t) should be:

$$v_t = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial f_t}{\partial r_i}\right)\left(\frac{\partial f_t}{\partial r_j}\right) C_{ij}$$

More generally the covariance between two rate r_{t_1} and r_{t_2} is:

$$Cov\left(r_{t_1}, r_{t_2}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial f_{t_1}}{\partial r_i}\right)\left(\frac{\partial f_{t_2}}{\partial r_j}\right) C_{ij}$$

this volatility curve doesn't necessarily reflect the one implicit in the market.

It's why we must, in order to insure consistency, determine the interpolation of the zero rate and the interpolation of the volatility in the same time.Another note will be dedicated to this problem.Let assume that for the moment we have those two interpolation available.

4.2 From sensivility(continuous rate) to sensibility(discrete rare)

If we come back to our main target which is to determine the standardized sensitivities we will use to compute the risk, we have use a guiding principle.

The only guiding principle we have is to act such that the risk computed with the complete set of sensitivity (the natural ones) should be equal to the risk computed with the standardized one.The risk is a function of the sensitivities and the covariance of the market so the equation from which we will draw the standardized sensitivities is:

$$Risk[NatSensitivities, Cov_C] = Risk[StdSensivitie(s, Cov_D)]$$

where Cov_C is the covariance matrix associated with the continuous rate (the one computed in the preceding section) and Cov_D is the covariance matrix associated with the standardized discrete maturities.

this equation could be very difficult to solve,

Lets take a simple case:

Lets assume that the risk is expressed by the covariance operators and that the natural sensitivities are represented by distribution valued kernels (that include the discrete natural sensitivities as we will see).

Let assume than the P&L of the portfolio is represented by W, the equation is:

$$Trace\left[(dW_C \otimes dW_C)Cov_C + \frac{1}{2}(d^2W_C\ \ Cov_C)^2\right]$$

$$=$$

$$Trace\left[(dW_D \otimes dW_D)Cov_D + \frac{1}{2}(d^2W_D\ \ Cov_D)^2\right]$$

Where dW_C and d^2W_C are the kernels representing the natural sensitivities

and dW_D and d^2W_D represent the vector of standardized sensitivities.

All the operations included the preceding equation are the one naturally associated with the natural linear structure implied.

In order to simplify again let's neglect the second order first, this will allow to determine the first order sensitivities, then by reinjecting the second order terms into the equation we will determine the second order sensitivities.This result in a decoupling of the preceding equation into:

$$Trace[(dW_C \otimes dW_C)Cov_C] = Trace[(dW_D \otimes dW_D)Cov_D]$$
$$Trace[(d^2W_C \; Cov_C)^2] = Trace[(d^2W_D \; Cov_D)^2]$$

but we know the relationship between Cov_C and Cov_D that we can represent by the following equation:

$$Cov_C = (\partial F)^* Cov_D \, \partial F$$

So a possible solution is then:

$$dW_D = (dW_C)^* \, \partial F$$
$$d^2W_D = \partial F \; d^2W_D \; (\partial F)^*$$

which means that:

$$\frac{\partial W}{\partial r_i} = \int_{-\infty}^{\infty} ds \left[\left(\frac{\partial F_s}{\partial r_i} \right) \frac{\delta W}{\delta r_s(s)} \right]$$

and

$$\frac{\partial^2 W}{\partial r_i \partial r_j} = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dq \left[\left(\frac{\partial F_s}{\partial r_i} \right) \left(\frac{\partial F_q}{\partial r_j} \right) \frac{\delta^2 W}{\delta r_s(s) \delta r_q(q)} \right]$$

4.3 Linear mapping

If we assume that the rate rare linearly interpollated between points given by the markets then the function F(t) is defined by :

$$F_t(\Delta r_1, ..., \Delta r_n) = F_{0,t} + \Delta r_j + \left(\frac{t - t_j}{t_{j+1} - t_j}\right)(\Delta r_{j+1} - \Delta r_j) \qquad \text{with j such } (t \in [t_j, t_{j+1}[\ )$$

where the points t_j are the maturities chosen for the risk management.

then ;

$$\text{For each t let chose j such that } (t \in [t_j, t_{j+1}[\ ) \qquad \begin{cases} \dfrac{\partial F_t}{\partial r_j} = \left(\dfrac{t_{j+1} - t}{t_{j+1} - t_j}\right) \\ \dfrac{\partial F_t}{\partial r_{j+1}} = \left(\dfrac{t - t_j}{t_{j+1} - t_j}\right) \end{cases}$$

So the integration rules

$$\int_{-\infty}^{\infty} \delta(s - s_0) f(s) ds = f(s_0)$$

allow us to compute easely the standardized derivatives in case of distribution-type natural sensitivities like the one for averaging instruments.