General Hedge Theory

By Olivier Croissant

1 First order hedging

1.1 Computation of the mono dimensional projectors

Let be C the covariance matrix of the problem and let state

$$X^{\dagger} = {}^{t}X \cdot C$$

The hedging problem is understood as the minimisation of the first order risk that means

$$Min\{Risk[hedgedPosition]\} \equiv Min\left[{}^{t}\left(x + \sum_i \alpha_i H_i\right) C \left(x + \sum_i \alpha_i H_i\right)\right]$$

So we want to solve :

$$\frac{\partial}{\partial \alpha_k}\left\{{}^{t}\left(x + \sum_i \alpha_i H_i\right) C \left(x + \sum_i \alpha_i H_i\right)\right\} = 0$$

which develops in :

$$\frac{\partial}{\partial\alpha_k}\left\{{}^t xCx + \sum_i ({}^t xCH_i + {}^t H_i Cx) + \sum_{i,j} ({}^t H_j CH_i + {}^t H_i CH_j)\right\} = 0$$

Because C is symetric this equation can be rewritten as s

$$H_k{}^\dagger x + \sum_i \alpha_i H_i{}^\dagger H_k = 0$$

If k belong to the set indexing the hedge factors, and a belong to the set indexing the dimension A of the problem :

let ‘s define :

$$M_{i,j} = H_i{}^\dagger H_j$$

Then the solution of the preceding system can be written as :

$$\alpha_i(X) = -\sum_j M^{-1}{}_{i,j} H_j{}^\dagger x$$

and the projection of x on the subspace defined by $H = Vect[H_1, ..., H_n]$ is :

$$P[x, H] = \sum_i \alpha_i(X) H_i = -\sum_{i,j} M^{-1}{}_{i,j} H_j{}^\dagger x H_i$$

so introducing the explicit dimensions

$$P[H]_{a,b} = -\sum_{i,j} (H^\dagger H)^{-1}{}_{i,j} H_{i,a} H_j{}^\dagger{}_b$$

So we can write the projection acting on vector x as :

$$\boxed{P[x, H] = \sum_{i,j} H_i \left\{ (H^\dagger H)^{-1}{}_{i,j} H_j{}^\dagger x \right\}}$$

1.2 Computation of the bi dimensional projector

and then the action on symetric matrices is ;

$$P[S, H]_{a,b} = \sum_{c,d} {}^t P[H]_{a,c} S_{c,d} P[H]_{d,b} = \sum_{c,d} P[H]_{c,a} S_{c,d} P[H]_{d,b}$$

which give the following result :

$$P[S, H]_{a,b} = \sum_{c,d,i,j,k,l} (H^\dagger H)^{-1}{}_{k,l} H_{k,c} H_l{}^\dagger{}_a S_{c,d} (H^\dagger H)^{-1}{}_{i,j} H_{i,d} H_j{}^\dagger{}_b$$

then by regrouping the terms :

$$P[S, H]_{a,b} = \sum_{c,d,i,j,k,l} (H^\dagger H)^{-1}{}_{k,l} H_l{}^\dagger{}_a H_{k,c} S_{c,d} H_{i,d} (H^\dagger H)^{-1}{}_{i,j} H_j{}^\dagger{}_b$$

if we create

$$\langle H | S | H \rangle_{k,i} = \sum_{c,d} H_{k,c} S_{c,d} H_{i,d}$$

then we have :

$$P[S,H]_{a,b} = \sum_{i,j,k,l} \langle H|S|H\rangle_{k,i}\left(\left\{(H^{\dagger}H)^{-1}{}_{k,l}H_l{}^{\dagger}\right\}\otimes\left\{(H^{\dagger}H)^{-1}{}_{i,j}H_j{}^{\dagger}\right\}\right)_{a,b}$$

and so

$$\boxed{P[S,H] = \sum_{i,j,k,l} \langle H|S|H\rangle_{k,i}\left\{(H^{\dagger}H)^{-1}{}_{k,l}H_l{}^{\dagger}\right\}\otimes\left\{(H^{\dagger}H)^{-1}{}_{i,j}H_j{}^{\dagger}\right\}}$$

1.3 Computation of the hedging effect:

We mean the reduction of risk due to hedging is equal to :

$$\Delta V = {}^{t}(x - P[x,H])C(x - P[x,H]) - {}^{t}xCx$$

If we introduce the formula for the projector it give us:

$$\boxed{\Delta V = -\sum_{i,j}(H^{\dagger}H)^{-1}{}_{i,j}(H^{\dagger}{}_i x)(H^{\dagger}{}_j x)}$$

2 Second Order Hedging

2.1 Computation of the mono dimensional projector

let assume we have a position described by its first derivative vector : x and a second derivative matrix : S , then we have a set of hedging instrument decribed by their first derivative H_i and their second derivative matrix : T_i

We want to minimize the second order variance :

$$^t\left(x + \sum_i \alpha_i H_i\right) C \left(x + \sum_i \alpha_i H_i\right) + \frac{1}{2} Tr\left[\left(S + \sum_i \alpha_i T_i\right) C \left(S + \sum_i \alpha_i T_i\right) C\right]$$

Then, the derivative with respect to α_k is :

$$H_k^\dagger x + \sum_i \alpha_i H_i^\dagger H_k + \frac{1}{2} Tr[SCT_k C] + \frac{1}{2} \sum_i \alpha_i Tr[T_i C T_k C]$$

So the hedge ratios α_k are solution of the system :

$$\sum_i \alpha_i \left(H_i^\dagger H_k + \frac{1}{2} Tr[T_i C T_k C]\right) = -H_k^\dagger x - \frac{1}{2} Tr[SCT_k C]$$

Let s adopt a notation for the matrix that solve the system above :

$$\left\langle \begin{bmatrix} H \\ T \end{bmatrix}, \begin{bmatrix} H \\ T \end{bmatrix} \right\rangle_{i,j} = H_i^\dagger H_j + \frac{1}{2} Tr[T_i C T_j C]$$

then we can write :

$$\alpha_i = -\sum_j \left\langle \begin{bmatrix} H \\ T \end{bmatrix}, \begin{bmatrix} H \\ T \end{bmatrix} \right\rangle^{-1}_{i,j} \left(H_j^\dagger x + \frac{1}{2} Tr[SCT_j C]\right)$$

The projector has in fact two component : the vector part of it and the matrix part of it . It is why we are going to define a more general space as :

$$M_2 = R_n \oplus S[R_n \otimes R_n]$$

Which is the direct sum of the space of vectors and the space of symetric matrices.This space is associated with a scalar product :

$$\left\langle \begin{bmatrix} x \\ X \end{bmatrix}, \begin{bmatrix} y \\ Y \end{bmatrix} \right\rangle = {}^t x C y + \frac{1}{2} Tr[XCYC]$$

In this new space, the projector is very similar to the projector in the first order case.:

$$\boxed{P\left[\begin{bmatrix} x \\ S \end{bmatrix}, \begin{bmatrix} H \\ T \end{bmatrix}\right] = \begin{bmatrix} \sum_{i,j} \left\langle \begin{bmatrix} H \\ T \end{bmatrix}, \begin{bmatrix} H \\ T \end{bmatrix} \right\rangle^{-1}_{i,j} \left(H_j^{\dagger} x + \frac{1}{2} Tr[SCT_j C]\right) H_i \\ \sum_{i,j} \left\langle \begin{bmatrix} H \\ T \end{bmatrix}, \begin{bmatrix} H \\ T \end{bmatrix} \right\rangle^{-1}_{i,j} \left(H_j^{\dagger} x + \frac{1}{2} Tr[SCT_j C]\right) T_i \end{bmatrix}}$$

In Fact we can easily show that if we consider this space M_2 as a vector space in which everything happens, then we can build a matrix C_2 associated with the scalar product such as the formula look identical for the second order and the first order.

Let assume that the second order matrix of the hedge instrument are zero, then we find the effect of the projectors on the second order martix immediatly:

$$P\left[\begin{bmatrix} x \\ S \end{bmatrix}, \begin{bmatrix} H \\ 0 \end{bmatrix}\right] = \begin{bmatrix} \sum_{i,j} \left\langle \begin{bmatrix} H \\ 0 \end{bmatrix}, \begin{bmatrix} H \\ 0 \end{bmatrix} \right\rangle^{-1}_{i,j} (H_j^{\dagger} x) H_i \\ 0 \end{bmatrix}$$

the raison for this 0 is that because the space on which we project is defined by the hedging instrument, and if these one have a zero matrix component, then the result wil have a zero matrix component.This result is totally different from the result of the effect of the first order projector on the second order matrix.

2.2 Computation of the hedging effect:

By using the homology with the first order case (That can be made rigourous as , we have seen) :we have:

$$\boxed{\Delta V = -\sum_{i,j} \left\langle \begin{bmatrix} H \\ T \end{bmatrix}, \begin{bmatrix} H \\ T \end{bmatrix} \right\rangle^{-1}_{i,j} \left\langle \begin{bmatrix} H_i \\ T_i \end{bmatrix}, \begin{bmatrix} x \\ S \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} H_i \\ T_j \end{bmatrix}, \begin{bmatrix} x \\ S \end{bmatrix} \right\rangle}$$