Emergent Space-Time, Entanglement Entropy, and Error Correction in Black Holes

Olivier Croissant
Emerging Pricing Technologies, Paris, France

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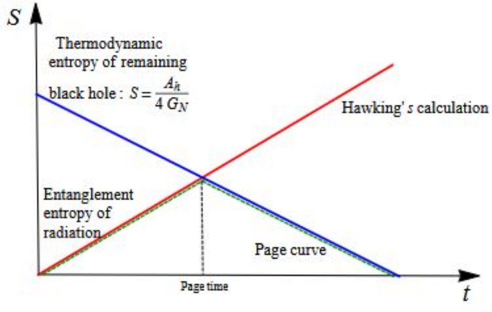
Motivation

- Understand how space-time can emerge from quantum information.
- Resolve the black hole information paradox.
- Explore quantum error correction as a language for fundamental physics.

Black Hole Information Paradox

- Hawking: Black holes radiate and may lose mass **(Hawking radiation)**.
- Leads to a loss of information — apparent contradiction with unitary evolution in quantum mechanics.
- Entropy seems to increase irreversibly, conflicting with quantum predictability.

Page Curve and Quantum Extremal Surfaces



- Entanglement entropy initially increases, then decreases (**Page curve**).
- Quantum extremal surface (QES) prescription matches predictions from AdS/CFT.

Emergence and Error Correction

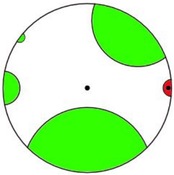
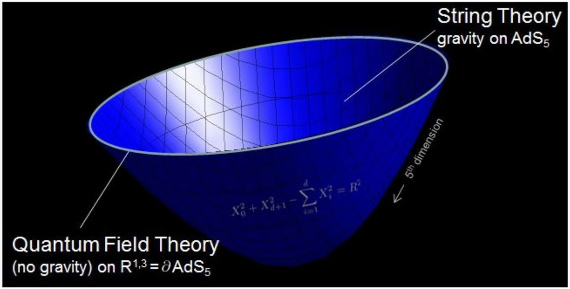


Figure 4. Correcting for erasures in AdS/CFT. Bulk quantum information at point in the center is protected in the CFT against the erasure of the boundary of any one of the green regions, but bulk information at the point near the boundary is completely lost by an erasure of the boundary of the red region.

- Quantum error correcting codes map effective degrees of freedom into fundamental ones.
- Protects bulk information against boundary erasures.
- Central bulk operators are more protected than near-boundary ones.

Holographic Principle and Complexity



- All degrees of freedom in a gravitational theory are encoded on a lower-dimensional boundary.
- **Complexity barrier:** space-time behaves classically unless you do something exponentially hard.
- **Locality***: locality holds unless exponential complexity is reached.

Towards de Sitter Holography: Pseudo-Entropy

- Recent proposals (e.g., by Takayanagi) explore holographic duals in de Sitter space.
- **Pseudo-entropy:** a generalization of entanglement entropy, potentially applicable in time-dependent or cosmological settings.
- This may open new avenues for understanding quantum gravity in expanding universes.

Reference: T. Akutagawa, T. Takayanagi, Z. Wei, *Pseudo Entropy in de Sitter Space*, arXiv:2303.12172

Main Formula:

$$S_A^{\text{pseudo}} = -\mathrm{Tr}_A[\rho_{AB}(\theta) \log \rho_{AB}(\theta)]$$

where $\rho_{AB}(\theta)$ is a pseudo-density matrix parameterized by an angle θ, used to interpolate between time slices in de Sitter.

How Pseudo-Entropy Helps in de Sitter Holography

- In AdS/CFT, entanglement entropy is computed using extremal surfaces anchored on boundaries.
- In de Sitter, the absence of a clear spatial boundary complicates this picture.
- **Pseudo-entropy** offers a way to define entropy across time-like separated regions, crucial in cosmological horizons.
- Provides an alternative Page curve consistent with unitary evolution in de Sitter.
- Helps extend the holographic dictionary to spacetimes with positive cosmological constant.

Quantum Fisher Information Metric

- The quantum Fisher information (QFI) is a Riemannian metric on the space of quantum states.
- It quantifies distinguishability between infinitesimally close quantum states.
- Derived from the fidelity $F(\rho, \rho + d\rho) = 1 - \frac{1}{8} g_{\mu\nu} d\lambda^\mu d\lambda^\nu + \cdots$
$$g_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\rho(L_\mu L_\nu + L_\nu L_\mu)\right]$$
- L_μ is the symmetric logarithmic derivative (SLD), defined via:
- $\partial_\mu \rho = \frac{1}{2}(L_\mu \rho + \rho L_\mu)$
- QFI leads to the geodesic distance used in quantum state space analysis.

Geodesics in Quantum Information Space

- In quantum geometry, a geodesic is the shortest path between two quantum states, respecting the underlying metric.
- One natural metric: the **quantum Fisher information metric**, derived from the second-order expansion of fidelity.
- Let $\rho(\lambda)$ be a family of quantum states parameterized by λ. The geodesic minimizes:

$$D(\rho_0, \rho_1) = \int_0^1 \sqrt{g_{\lambda\lambda}}\, d\lambda$$

- Where $g_{\lambda\lambda} = \mathrm{Tr}[\partial_\lambda \rho(\lambda)\, L_\lambda]$ and L_λ is the symmetric logarithmic derivative.
- Such geodesics form the basis for defining divergences and triangle inequalities in quantum state space.

Example: QFI for a Qubit

- Consider a qubit state parameterized by θ:
$$\rho(\theta) = \frac{1}{2}(\mathbb{I} + \theta\sigma_z)$$
- For this family, we compute the symmetric logarithmic derivative (SLD) L_θ:
$$L_\theta = \sigma_z \quad \text{(SLD derived directly from symmetry)}$$
- The quantum Fisher information becomes:
$$F_Q(\theta) = \mathrm{Tr}[\rho(\theta)L_\theta^2] = \mathrm{Tr}\left[\frac{1}{2}(\mathbb{I} + \theta\sigma_z) \cdot \sigma_z^2\right] = 1$$
- This simple result means the quantum Fisher metric is flat and constant in this parameterization.
- Can be used to compute geodesics and distances in quantum state space.

Example: QFI for Two Qubits

- Consider a two-qubit entangled state depending on parameter θ:

$$|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$$

- The density matrix is: $\rho(\theta) = |\psi(\theta)\rangle\langle\psi(\theta)|$
- The SLD can be computed from:

$$\partial_\theta \rho = \frac{1}{2}(L_\theta \rho + \rho L_\theta)$$

- Resulting in:

$$F_Q(\theta) = 4(\langle\partial_\theta\psi|\partial_\theta\psi\rangle - |\langle\psi|\partial_\theta\psi\rangle|^2) = 4$$

- This indicates strong sensitivity of the entangled state to changes in θ.
- Useful in quantum metrology and illustrates richer geometry in multi-qubit systems.

Information Geometry and Duality

- Divergences (like KL divergence or pseudo-entropy) define dual coordinate systems:
  - Primal coordinates θ: parametrizing exponential family distributions
  - Dual coordinates $\eta = \partial_\theta \psi(\theta)$: obtained via Legendre transform
- The Fisher information metric is defined as:

$$g_{ij} = \frac{\partial^2 \psi(\theta)}{\partial \theta_i \partial \theta_j}$$

- Dual connections:
  - ∇: flat in θ (exponential geodesics)
  - ∇^*: flat in η (mixture geodesics)
- This structure is called "dually flat" and enables Pythagorean identities for projections.

Quantum Information Geometry and Duality

- Quantum analogs of classical divergences (e.g., Umegaki entropy) define quantum statistical geometry.
- The quantum Fisher information metric plays a central role:

$$g_{ij} = \mathrm{Tr}[\partial_i \rho \cdot L_j] \text{ (for suitable SLD or RLD choice)}$$

- Quantum duality is more subtle due to non-commutativity:
  - No unique geodesics — multiple monotone metrics exist (Petz classification)
  - Still, *approximate dual coordinates* can be defined from convex functions
- Quantum geodesics and divergences (like relative entropy) still yield a structure for projections and learning.

Entanglement Entropy as Quantum Divergence

- Entanglement entropy can be interpreted as a type of quantum divergence.
- Inspired by information geometry, where divergence measures induce geometric structure.
- We propose a **Pythagorean-like identity** involving pseudo-entropies:

$$S(\rho\|\sigma) + S(\sigma\|\tau) = S(\rho\|\tau) + \text{correction terms}$$

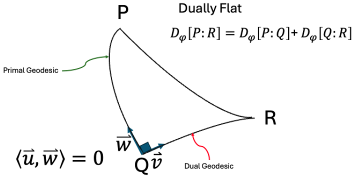
- One divergence (e.g., $S(\sigma\|\tau)$) must be computed along a geodesic in the dual quantum space.
- Suggests a geometric interpretation of entropy as distances between states in a quantum manifold.

Locality from Pythagorean Theorem in CFT and dS

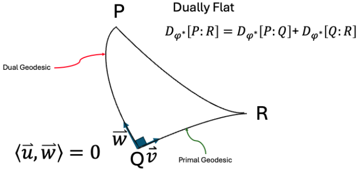
- Both AdS/CFT and proposed dS/CFT scenarios admit emergent locality.
- Locality emerges from the geometric interpretation of entanglement entropy.
- The Pythagorean theorem holds in dually flat information geometry:
$$D(p\|r) = D(p\|q) + D(q\|r)$$
- This structure encodes spatial separation via quantum divergences.
- In dS space, pseudo-entropy replaces von Neumann entropy, but preserves this geometric identity.
- In boundary CFTs, modular flow and entanglement wedges also obey this additive decomposition.

Commutative Diagram of Divergence Computation

- The structure of dual coordinate systems in information geometry leads naturally to a commutative diagram.
- Each path in the diagram represents a consistent computation of divergences:
  - Direct divergence $D_\varphi(P\|R)$
  - Indirect path via projection: $D_\varphi(P\|Q) + D_\varphi(Q\|R)$
- This identity expresses the Primal Generalized Pythagorean theorem.
- A dual version also holds:
  - $D_{\varphi^*}(P\|R) = D_{\varphi^*}(P\|Q) + D_{\varphi^*}(Q\|R)$
  - Represents the Dual Generalized Pythagorean theorem.



Primal Generalized Pythagore



Dual Generalized Pythagore

CFT/dS Divergence Diagram

- This diagram expresses a commutative relation between divergences computed in the boundary CFT and the bulk dS space.
- The upper row shows a conditional entropy identity in the CFT:
  $S(\rho_A \| P_B) + S(\rho_B \| P_C) = S(\rho_A \| \rho_C)$
- The lower row shows the corresponding identity with dualized entropies in dS:
  $S(\rho_A^* \| \rho_B^*) + S(\rho_B^* \| \rho_C^*) = S(\rho_A^* \| \rho_C^*)$
- Suggests a holographic duality where information-theoretic locality is preserved through dual entropic flow.
- Naturally, ρ_A^* is the projection in the bulk of the CFT density matrix ρ_A.
- These dual identities enable a consistent notion of locality in both dually flat classical and quantum information geometries.
- Such locality notions are extended to holographic scenarios like AdS/CFT and dS/CFT.

Conclusion

- Entanglement entropy, QES, and error correction help recover predictability in black holes.
- Space-time may be an emergent, redundant, and protected construct.
- Complexity provides a veil hiding radical non-locality.
- Pseudo-entropy and new de Sitter proposals extend holography beyond AdS.
- A divergence-based view of pseudo-entropy opens paths toward quantum geometry.