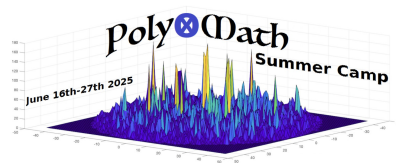# POLYMATH CAMP

Probabilistic Perspectives on the Longest Increasing Subsequence: From Ulam to Baik-Deift-Johansson

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1
INTRODUCTION: ULAM'S PROBLEM AND PATIENCE SORTING

In the 1960s, mathematician Stanislaw Ulam posed a seemingly simple question that would lead to deep connections between combinatorics, probability theory, and even quantum physics:

"Given a random shuffling of the numbers 1 through n, what is the expected length of the longest increasing subsequence?"

1.1 BASIC DEFINITIONS

Definition 1.1 (Increasing Subsequence)**.** *Let π be a permutation of $\{1, 2, \ldots, n\}$. An **increasing subsequence** of π is a sequence of indices $(i_1, i_2, \ldots, i_k)$ such that:*

$$\begin{cases} i_1 < i_2 < \cdots < i_k \\ \pi(i_1) < \pi(i_2) < \cdots < \pi(i_k) \end{cases} \tag{1}$$

Example 1.2. *Consider the permutation $\pi = (7, 2, 8, 1, 3, 4, 10, 6, 9, 5)$.*

- *The subsequence $(2, 3, 4, 6, 9)$ is increasing (length 5)*

- *The subsequence $(1, 3, 4, 6, 9)$ is also increasing (length 5)*

- *The subsequence $(7, 8, 10)$ is increasing (length 3)*

The longest increasing subsequence has length 5.

1.2 THE PATIENCE SORTING ALGORITHM

To make this problem more tangible, we introduce a card game called **patience sorting**. This game not only provides intuition but also gives us an algorithm to find the longest increasing subsequence.

Definition 1.3 (Patience Sorting Rules)**.** *Given a deck of cards numbered 1 to n:*

1. *Cards are turned over one at a time*

2. *Each card can either:*

   - *Be placed on top of a higher-numbered card in an existing pile*
   - *Start a new pile to the right of all existing piles*

3. *Goal: End with as few piles as possible*

Example 1.4. *To illustrate the game, consider a shuffled deck of 10 cards in the order: $(7, 2, 8, 1, 3, 4, 10, 6, 9, 5)$. We will follow the greedy strategy, which always places a card on the leftmost valid pile.*

We begin with the top card, 7, placed face up on the table. The next card, 2, can go on top of 7. The following card, 8, is larger than 2, so it starts a new pile to the right. Then comes 1: it could be placed on the 8 (second pile) or on the 2 (first pile), but the greedy rule requires placing it on the leftmost option, so on top of the 2. Continuing in this way, the piles evolve as follows:

1	**1**	**1**	**1**	**1**	**1**	**1**	5
2	2 **3**	2 **3**	2 **3**	2 **3** 6	2 **3** 6	2 **3** 6	
7 **8**	7 8	7 8 **4**	7 8 **4 10**	7 8 **4** 10	7 8 **4** 10 **9**	7 8 **4** 10 **9**	

Figure 1: Illustration of the patience sorting game using the greedy algorithm for this permutation.

Theorem 1.5 (Patience Sorting Characterization). *With deck π, patience sorting played with the greedy strategy (always place on the leftmost possible pile) ends with exactly $\ell(\pi)$ piles, where $\ell(\pi)$ is the length of the longest increasing subsequence. Furthermore, any legal strategy ends with at least $\ell(\pi)$ piles.*

Proof. We prove this equality by establishing both directions.

- **Number of piles $\geq \ell(\pi)$.**

  Let $(a_1, \ldots, a_k)$ be any increasing subsequence of π with $k = \ell(\pi)$. We claim that under any legal strategy, these cards must occupy distinct piles.

  Indeed, when card a_i is played, card a_{i-1} is already placed on some pile with top card $c \leq a_{i-1}$. Since $a_{i-1} < a_i$, we have $c < a_i$, so a_i cannot be placed on this pile. By induction, all k cards require distinct piles, giving at least $\ell(\pi)$ piles total.

- **Greedy strategy achieves $k \leq \ell(\pi)$ piles.**

  Under the greedy strategy, when placing card x on pile $j > 1$, we can create a back-pointer from x to the top card y of pile $j - 1$ (since $x > y$).

  If the greedy strategy uses k piles, let a_k be the top card of pile k. Following back-pointers gives a sequence $a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_k$ where each a_i sits on pile i. This sequence is increasing (by pointer construction) and appears in order in π (since each pointer was created when the later card was played). Thus we have an increasing subsequence of length k, so $\ell(\pi) \geq k$.

  Therefore, we get $k = \ell(\pi)$. $\square$

Example 1.6. *If we consider again the permutation $\pi = (7, 2, 8, 1, 3, 4, 10, 6, 9, 5)$, then the pointers will look like this:*
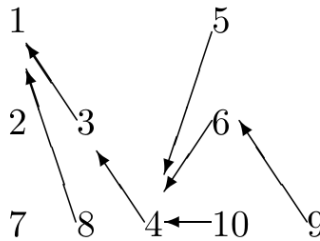

Figure 2: Illustration of the pointers for this permutation.

And we recover the LIS $(1, 3, 4, 6, 9)$ (although it is not the unique LIS of this permutation).

2
THE ERDŐS-SZEKERES THEOREM

Before diving into Ulam's problem, we need a fundamental result from 1935 that provides bounds on the longest monotone subsequences.

Theorem 2.1 (Erdős-Szekeres, 1935)**.** *Any sequence of at least $n^2 + 1$ distinct real numbers contains either:*

- *An increasing subsequence of length $n + 1$, or*

- *A decreasing subsequence of length $n + 1$*

Proof. One possible proof consists of proving this theorem by contradiction, by considering a sequence $a_1, a_2, \ldots, a_{n^2+1}$ of distinct numbers, and assuming that all $x_i \leq n$ and all $y_i \leq n$.

- **Step 1: Labeling.** For each position $i \in \{1, 2, \ldots, n^2 + 1\}$, assign a pair (x_i, y_i) where:

$$x_i = \text{length of longest increasing subsequence ending at position } i \qquad (2)$$
$$y_i = \text{length of longest decreasing subsequence ending at position } i \qquad (3)$$

- **Step 2: Distinctness of labels.** We claim that all pairs (x_i, y_i) are distinct. Indeed, consider two positions $i < j$:

  1. **Case 1:** If $a_i < a_j$, then we can extend any increasing subsequence ending at i by appending a_j. Thus:
  $$x_j \geq x_i + 1 > x_i$$

  2. **Case 2:** If $a_i > a_j$, then we can extend any decreasing subsequence ending at i by appending a_j. Thus:
  $$y_j \geq y_i + 1 > y_i$$

  Since all numbers are distinct, exactly one of these cases holds, so $(x_i, y_i) \neq (x_j, y_j)$.

- **Step 3: Pigeonhole principle.** We have $n^2 + 1$ distinct pairs (x_i, y_i).

  If all $x_i \leq n$ and all $y_i \leq n$, then there are at most $n \times n = n^2$ possible pairs. By the pigeonhole principle, this is impossible, since we have $n^2 + 1$ distinct elements, so $n^2 + 1$ distinct pairs.

Therefore, either some $x_i > n$ or some $y_i > n$, giving us either an increasing subsequence of length $> n$ or a decreasing subsequence of length $> n$. $\qquad \square$

3
THE SCHENSTED CORRESPONDENCE AND YOUNG TABLEAUX

3.1 Young Tableaux

Definition 3.1 (Partition and Young Diagram)**.** *A **partition** $\lambda = (\lambda_1, \ldots, \lambda_j)$ of an integer $n \geq 1$ is a sequence with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq 1$ and $\sum_i \lambda_i = n$. We identify a partition with its **Young diagram** (or Ferrers diagram) of n cells with λ_i cells in row i.*

Example 3.2. *$\lambda = (5, 4, 3, 3, 1)$ is a valid partition for $n = 16$. We can identify this partition with its corresponding Young diagram (16 cells in total, including 5 on row 1, 4 on row 2, 3 on rows 3 & 4, ..., as shown below:*
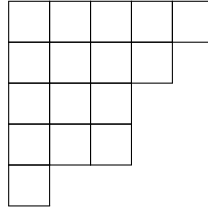

Figure 3: Young diagram for the partition $\lambda = (5, 4, 3, 3, 1)$.

Proposition 3.3 (Hook formula)**.** *The number of Young tableaux of a given shape λ is denoted by d_λ and is given by the **hook formula** of Frame-Robinson:*

$$d_\lambda = \frac{n!}{\prod_c h_c}$$

With h_c the hook length of cell c, equal to the number of cells to the right of c in the same row, plus the number of cells below c in the same column, plus the cell c itself.

Proof. Cf. Iyad and Anita's presentation. $\qquad\square$

Definition 3.4 (Standard Young Tableau)**.** *A **standard Young tableau** of shape λ is a filling of the Young diagram with numbers $1, 2, \ldots, n$ such that:*

- *Numbers increase along each row (left to right)*
- *Numbers increase down each column (top to bottom)*

Example 3.5. *Considering the same partition $\lambda = (5, 4, 3, 3, 1)$ for $n = 16$, we can derive an example of a representation of one of its Young Tableaux:*

Figure 4: Example of Young tableau for the partition $\lambda = (5, 4, 3, 3, 1)$.

3.2 THE SCHENSTED CORRESPONDENCE

The Schensted correspondence is a bijection between permutations and pairs of standard Young tableaux.

Theorem 3.6 (Schensted Correspondence)**.** *There exists a bijection between:*

- *Permutations of $\{1, 2, \ldots, n\}$*
- *Pairs (P, Q) of standard Young tableaux of the same shape λ a partition of n*

Moreover, the length of the first row of P equals the length of the longest increasing subsequence of the permutation.

Remark 3.7 (Construction of the Schensted Correspondence)**.** *The construction uses a recursive "bumping" procedure similar to patience sorting. When inserting a number into a tableau, it either extends a row or "bumps" a larger number to the next row.*

- *The P-tableau (insertion tableau) is constructed by successive **Row Insertion** ($T \leftarrow x$) of the input elements.*

  1. ***Start:** Let T be the current P-tableau (initially empty). Insert the next element, x, into the **first row** of T.*
  2. ***Insertion Rule:** In the current row R:*
     - *Find the **smallest entry** y in R such that $y > x$.*
     - ***Replace** y with x (this element y is "bumped").*
     - *Recursively insert the bumped element y into the next row, $R + 1$.*
  3. ***Termination:** If no such entry y exists (i.e., x is greater than or equal to all elements in R), **append** x to the right end of R.*

- *The Q-tableau (recording tableau) records the history of the shape growth.*

  1. *The Q-tableau must always maintain the **same shape** as the P-tableau.*
  2. *After the m-th element is inserted into P (creating a new cell), the **newly added cell** in Q is filled with the integer m (the index of the element being inserted).*

*The resulting P-tableau has rows that are strictly increasing and columns that are strictly increasing (a **Semi-Standard Young Tableau** or **Standard Young Tableau** if input elements are distinct). The resulting Q-tableau is always a **Standard Young Tableau** (strictly increasing rows and columns) whose entries record the order of insertion.*

Example 3.8. *If we consider again the permutation $\pi = (7, 2, 8, 1, 3, 4, 10, 6, 9, 5)$, the construction of the Schensted correspondence will give something like this:*

	2	2 8	1 8	1 3	1 3 4	1 3 4 10	1 3 4 6	1 3 4 6 9	1 3 4 5 9
P	7	7	2	2 8	2 8	2 8	2 8 10	2 8 10	2 6 10
			7	7	7	7	7	7	7 8

	1	1 3	1 3	1 3	1 3 6	1 3 6 7	1 3 6 7	1 3 6 7 9	1 3 6 7 9
Q	2	2	2	2 5	2 5	2 5	2 5 8	2 5 8	2 5 8
			4	4	4	4	4	4	4 10

Figure 5: Construction of the Schensted correspondence for the permutation $\pi = (7, 2, 8, 1, 3, 4, 10, 6, 9, 5)$.

Remark 3.9. *We notice that the first row of the P-tableau is identical to the row of top cards in the patience sorting of π. Thus, the length of the first row, λ_1, is equal to the length $l(\pi)$ of the LIS of the permutation π. This bijection allows us to derive formulae such as the probability $P(L_n = l)$, where L_n is the length of the LIS for a uniform random permutation of $\{1, 2, \ldots, n\}$.*

Corollary 3.10 (Distribution Formula)**.** *For a uniform random permutation π_n, letting $L_n = \ell(\pi_n)$:*

$$\mathbb{P}(L_n = \ell) = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \lambda_1 = \ell}} d_\lambda^2$$

where $\lambda \vdash n$ means that λ is a partition of n, and d_λ is the number of standard Young tableaux of shape λ.

4
CONVERGENCE TO A CONSTANT: THE SUBADDITIVE APPROACH

4.1 Hammersley's Representation

Hammersley gave a representation of L_n using a Poisson point process that reveals a powerful superadditive structure. This shifts the discrete combinatorial problem into a continuous probabilistic framework.

Definition 4.1 (Poisson Point Process)**.** *A Poisson point process of intensity λ in a region $R \subseteq \mathbb{R}^d$ is a random collection of points such that:*

1. *For any bounded region $A \subseteq R$, the number of points $N(A)$ in A follows a Poisson distribution with parameter $\lambda \cdot |A|$.*

2. *For disjoint regions, the numbers of points are independent random variables.*

Definition 4.2 (Hammersley's Process)**.** *Consider a Poisson point process of intensity 1 in the quadrant $[0, \infty)^2$. Define:*

$$\hat{L}(x, t) = maximum \ number \ of \ points \ on \ any \ increasing \ path \ from \ (0, 0) \ to \ (x, t)$$

where an increasing path is a sequence of points $(x_1, t_1), \dots, (x_k, t_k)$ with $x_1 < \cdots < x_k$ and $t_1 < \cdots < t_k$.

Remark 4.3. *The key property of this process is its superadditivity. In the following lemma, we have to understand that the terms on the right-hand side arise from independent parts of the process.*

Lemma 4.4 (Superadditivity of Expectation)**.** *Let $m(x, t) = \mathbb{E}[\hat{L}(x, t)]$. For $0 < y < x$ and $0 < s < t$:*

$$m(x, t) \geq m(y, s) + m(x - y, t - s)$$

Proof Sketch. An increasing path from $(0, 0)$ to (x, t) can be formed by concatenating a path within the box $[0, y] \times [0, s]$ with a path within the box $[y, x] \times [s, t]$. Because the points in these two disjoint boxes are independent, we can optimize the paths in each box separately and combine them. The expected length of the optimal path in the first box is $m(y, s)$, and by translation invariance, the expected length in the second is $m(x - y, t - s)$. Taking the expectation over all such concatenated paths gives the inequality. □

4.2 The Poissonization Strategy

The superadditive property is the key to proving convergence, but the original problem on a fixed set of n numbers doesn't have this structure. The key insight is to use a "Poissonization-Depoissonization" two-step trick. The full proof is highly technical, but we will outline the strategy, which is quite intuitive.

Theorem 4.5 (Convergence to a Constant)**.** *There exists a constant $c > 0$ such that, for the length L_n of the longest increasing subsequence of a random permutation of $\{1, \dots, n\}$:*

$$\frac{L_n}{\sqrt{n}} \to c \quad in \ probability \ as \ n \to \infty$$

Proof Sketch. As mentioned earlier, the proof is very complex, but a good strategy is to prove this through two main conceptual steps: "poissonization" & "depoissonization."

- **Step 1: Poissonization ($\rightarrow$ Moving to an Easier World)**

  Idea: Instead of analyzing a permutation of a fixed number n of elements, we analyze the LIS for a random number of points, where this random number $N(n)$ follows a Poisson distribution with mean n. We call the LIS length in this case $L_{N(n)}$.

  Intuition: The Poisson process has a wonderful independence property that the fixed-n problem lacks. The number of points in disjoint regions are independent. This independence is exactly what allows us to use powerful convergence results like Kingman's subadditive ergodic theorem on Hammersley's process. This theorem guarantees that for the Poissonized problem, the average length grows predictably:

  $$\frac{\mathbb{E}[L_{N(n)}]}{\sqrt{n}} \xrightarrow{n \to \infty} c$$

  Working in the "Poisson world" makes the problem tractable by unlocking these powerful analytic tools.

- **Step 2: Depoissonization ($\rightarrow$ Returning to the Real World)**

  Idea: This step bridges the result from the "easy" Poisson world back to the original fixed-n problem. It argues that what holds for $L_{N(n)}$ must also hold for L_n.

  Intuition: It relies on two properties:

  1. **Concentration:** A Poisson variable $N(n)$ with mean n is very likely to be very close to n. For example, the probability that $|N(n) - n|$ is larger than $n^{2/3}$ is exponentially small:

     $$\mathbb{P}(|N(n) - n| > n^{2/3}) \leq 2 \exp\left(-\frac{n^{1/3}}{3}\right) \to 0$$

  2. **Stability:** The length of the LIS is not very sensitive to small changes in the number of elements. Adding or removing k elements can change the LIS length by at most k. This gives us a "Lipschitz" property:
     $$|L_{N(n)} - L_n| \leq |N(n) - n|$$

  Since $N(n)$ is almost certainly very close to n, the stability property implies that $L_{N(n)}$ must be very close to L_n. Therefore, their asymptotic behavior, when scaled by $\sqrt{n}$, must be identical. If we have proven that $\frac{L_{N(n)}}{\sqrt{n}} \to c$, it forces the conclusion that $\frac{L_n}{\sqrt{n}} \to c$ as well.

  $\square$

5
THE VERSHIK-KEROV-LOGAN-SHEPP RESULT

After Hammersley's work established that $\mathbb{E}[L_n]/\sqrt{n}$ converges to a constant c, the problem shifted to finding its value. In 1977, the answer was found independently by two groups using a deep connection to the shapes of Young Tableaux.

Theorem 5.1 (Vershik-Kerov, Logan-Shepp, 1977). *The constant for the expected length of the LIS is exactly 2, i.e.:*

$$\lim_{n \to \infty} \frac{\mathbb{E}[L_n]}{\sqrt{n}} = 2$$

Proof Sketch. Once again, the proof is highly advanced, but its central idea is to rephrase the question about the longest increasing subsequence into a question about the *shape* of a "typical" Young Tableau.

- **The Schensted Connection:** Recall that the Schensted correspondence creates a bijection between a permutation π_n and a pair of Standard Young Tableaux (P, Q) of the same shape λ. Crucially, the length of the first row of the tableau, λ_1, is equal to the length of the LIS of the permutation, L_n. Therefore, finding the expected value of L_n is equivalent to finding the expected length of the first row of a Young tableau, averaged over all permutations.

$$\mathbb{E}[L_n] = \mathbb{E}[\lambda_1]$$

- **The "Typical" Shape:** For large n, Vershik-Kerov and Logan-Shepp showed that if we pick a permutation uniformly at random, the corresponding tableau shape λ is highly likely to be close to a specific, deterministic curve. To be precise, they scaled the Young diagram down by $\sqrt{n}$. As $n \to \infty$, the boundary of the shape converges to a curve.

- **The Limiting Curve:** This limiting curve is described by a function $\Omega(u)$. While its explicit formula is complex, Logan and Shepp discovered that the shape could also be described by its upper boundary, given by the function $y = f(x)$ (cf. Figure 6):

$$f(x) = \frac{2}{\pi} \left(x \sin^{-1}\left(\frac{x}{2}\right) + \sqrt{4 - x^2} \right)$$

- **Finding the Constant $c = 2$:** The connection is that the scaled length of the first row, $\lambda_1/\sqrt{n}$, converges to the **maximum x-coordinate** of this limiting curve. As we can see from the formula (where the term $\sqrt{4 - x^2}$ requires $x \le 2$), the curve is defined for $x \in [-2, 2]$. Its maximum extent along the x-axis is at $x = 2$.

Therefore, the expected scaled length of the first row converges to this maximum value:

$$\lim_{n \to \infty} \frac{\mathbb{E}[\lambda_1]}{\sqrt{n}} = \max(x) = 2$$

This beautiful argument connects a discrete combinatorial property (the length of the LIS) to the solution of a continuous variational problem in calculus (finding the limiting shape that maximizes a certain probability measure on tableaux).

$\square$

Remark 5.2. *This limiting shape is closely connected to the semicircle law from random matrix theory. The Young tableau limiting shape (right) can be understood as arising from the same mathematical framework that produces the semicircle distribution (left), both supported on the interval $[-2, 2]$.*
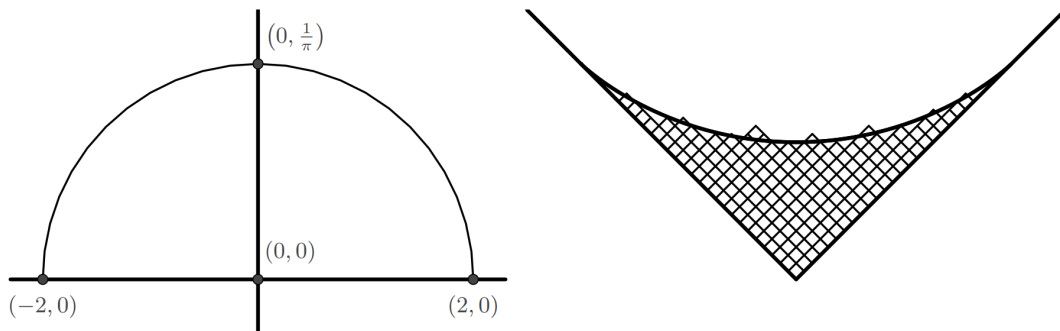

Figure 6: Left: Semicircle law $\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$. Right: Vershik-Kerov limiting shape for Young tableaux.

In the future, we can work more on the link between this theorem and the semi-circle law!

6
THE BAIK-DEIFT-JOHANSSON THEOREM

In 1999, a refinement was discovered concerning the fluctuations around the mean. Indeed, we just saw that $\lim_{n\to\infty} \mathbb{E}[L_n] = 2\sqrt{n}$. And by rescaling their difference by the correct factor, we get a probability distribution!

Definition 6.1 (The Tracy-Widom Distribution). *For $\beta \in \{1, 2, 4\}$, the Tracy–Widom distribution TW_β describes the fluctuations of the largest eigenvalue of Gaussian Orthogonal, Unitary, and Symplectic Ensembles (GOE, GUE, GSE).*
Let $q(x)$ be the Hastings–McLeod solution of the Painlevé II equation

$$q''(x) = xq(x) + 2q(x)^3, \qquad q(x) \sim \mathrm{Ai}(x) \ \ as \ x \to +\infty,$$

with Ai the Airy function. The cumulative distribution functions are:

$$\begin{cases} F_2(s) = \exp\left(-\int_s^\infty (x-s)\,q(x)^2\,dx\right), \\[2mm] F_1(s) = \exp\left(-\tfrac{1}{2}\int_s^\infty q(x)\,dx\right)\sqrt{F_2(s)}, \\[2mm] F_4\left(\dfrac{s}{\sqrt{2}}\right) = \cosh\left(\tfrac{1}{2}\int_s^\infty q(x)\,dx\right)\sqrt{F_2(s)}. \end{cases} \tag{4}$$

Proposition 6.2. *Some key properties of the Tracy-Widom Distribution:*

- *Support: $\mathbb{R}$, left-skewed, non-symmetric.*

- *Means: $\mathbb{E}[\mathrm{TW}_1] \approx -1.2065, \quad \mathbb{E}[\mathrm{TW}_2] \approx -1.7711, \quad \mathbb{E}[\mathrm{TW}_4] \approx -2.3069.$*

- *Variances: $\mathrm{Var}(\mathrm{TW}_1) \approx 1.6078, \quad \mathrm{Var}(\mathrm{TW}_2) \approx 0.8132, \quad \mathrm{Var}(\mathrm{TW}_4) \approx 0.5177.$*

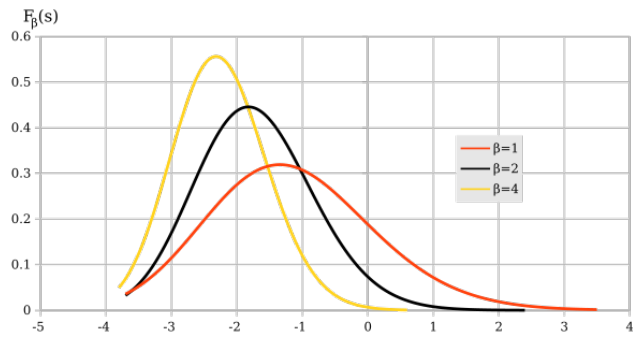- *Appears in many models: random matrices, longest increasing subsequences, growth processes, tilings.*

Figure 7: Densities of Tracy-Widom distributions for $\beta = 1, 2, 4$.

Theorem 6.3 (Baik-Deift-Johansson, 1999). *The rescaled fluctuations converge to the Tracy-Widom distribution:*

$$\mathbb{P}\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \le x\right) \to TW_2(x)$$

where TW_2 is the Tracy-Widom distribution.

Proof. This could be future work! $\qquad\qquad\square$

Remark 6.4. *This means that $\mathbb{E}[L_n] = 2\sqrt{n} + \mathbb{E}[\mathrm{TW}_2]\,n^{1/6} + o(n^{1/6})$, where $\mathbb{E}[\mathrm{TW}_2] \approx -1.7711$.*

7
APPLICATIONS AND CONNECTIONS

The longest increasing subsequence problem connects to numerous areas:

7.1 Random Matrix Theory

The same Tracy-Widom distribution appears as the distribution of the largest eigenvalue of random Hermitian matrices. This deep connection, while still somewhat mysterious, has led to profound insights in both fields.

7.2 Sequence Alignment

In bioinformatics, finding longest common subsequences is crucial for DNA sequence comparison. The LIS problem provides theoretical bounds for these algorithms.

7.3 Percolation Theory

The problem relates to first-passage percolation, where one studies shortest paths in random environments. The superadditive structure is analogous to the subadditive structure in percolation.

7.4 Queueing Theory

The patience sorting algorithm models certain priority queue systems where items must maintain order constraints.

7.5 Algebraic Combinatorics

The Schensted correspondence connects to:

- Representation theory of symmetric groups
- Symmetric functions
- The Robinson-Schensted-Knuth correspondence

8
CONCLUSION

What began as Ulam's simple question about shuffled cards has led us through:

- Elementary combinatorics (Erdős-Szekeres theorem)
- Probabilistic convergence theorems (subadditivity)

- Connections to Young tableaux (Schensted correspondence)

- Random matrix theory (Tracy-Widom distribution)

- Applications across mathematics and science

This journey illustrates how simple questions in mathematics can open doors to deep, unexpected connections across different fields. The longest increasing subsequence problem remains active in research, with applications ranging from theoretical computer science to statistical physics, demonstrating the unity and beauty of mathematics.

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