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Note

On Vector Partition Functions

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We present a structure theorem for vector partition functions. The proof rests on a formula due to Peter McMullen for counting lattice points in rational convex polytopes. © 1995 Academic Press, Inc.

INTRODUCTION

Let $A=(a_1, ..., a_n)$ be a $d\times n$-matrix of rank d with entries in $\mathbf{N}$, the set of non-negative integers. The corresponding *vector partition function* $\phi_A : \mathbf{N}^d \to \mathbf{N}$ is defined as follows: $\phi_A(u)$ is the number of non-negative integer vectors $\lambda=(\lambda_1, ..., \lambda_n)\in \mathbf{N}^n$ such that $A\cdot\lambda=\lambda_1 a_1+\cdots+\lambda_n a_n=u$. Equivalently, the function ϕ_A is defined by the formal power series:

$$\prod_{i=1}^{n} \frac{1}{(1-t_1^{a_{1i}}t_2^{a_{2i}}\cdots t_d^{a_{di}})} = \sum_{u\in\mathbf{N}^d} \phi_A(u_1, ..., u_d)\cdot t_1^{u_1}t_2^{u_2}\cdots t_d^{u_d}. \tag{1}$$

Vector partition functions appear in many areas of mathematics and its applications, including representation theory [9], commutative algebra [14], approximation theory [4] and statistics [5].

It was shown by Blakley [2] that there exists a finite decomposition of $\mathbf{N}^d$ such that ϕ_A is a polynomial of degree $n-d$ on each piece. Here we describe such a decomposition explicitly and we analyze how the polynomials differ from piece to piece. Our construction uses the geometric decomposition into chambers studied by Alekseevskaya, Gel'fand and Zelevinsky in [1]. Within each chamber we give a formula which refines

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the results by Dahmen and Micchelli in [4, §3]. The objective of this note is to provide polyhedral tools for the efficient computation of vector partition functions, with a view towards applications, such as the sampling algorithms in [6].

EXAMPLE ($n=6$, $d=3$). Consider the vector partition function

$$\phi_A : \mathbf{N}^3 \to \mathbf{N},\ (u, v, w) \mapsto \#\{\lambda \in \mathbf{N}^6 : A \cdot \lambda = (u, v, w)^t\}$$

associated with the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

In this instance the value of ϕ_A does not depend on the permutation of (u, v, w), so we may assume that $u \geqslant v \geqslant w$. Also, if $u+v+w \equiv 1 \bmod 2$ then $\phi_A(u, v, w) = 0$, so we shall assume that $u+v+w \equiv 0 \bmod 2$. Given these assumptions, we distinguish two cases:

Case 1. $u \geqslant v+w$. Then

$$\phi_A(u, v, w) = \frac{vw}{2} + \frac{vw^2}{8} - \frac{w^3}{24} + \begin{cases} 1 + v/2 + 2w/3 & \text{if } u \equiv 0 \; mod\, 2 \text{ and } v \equiv 0 \; mod\, 2, \\ 1/2 + v/2 + 5w/12 & \text{if } u \equiv 1 \; mod\, 2 \text{ and } v \equiv 1 \; mod\, 2, \\ 1/2 + 3v/8 + 13w/24 & \text{otherwise.} \end{cases}$$

Case 2. $u < v+w$. We set

$$\begin{aligned} \psi := {} & -u^2/8 + uv/4 + uw/4 - v^2/8 + vw/4 - w^2/8 \\ & + u^3/48 - u^2v/16 - u^2w/16 + uv^2/16 + uvw/8 + uw^2/16 \\ & - v^3/48 - v^2w/16 + vw^2/16 - w^3/16. \end{aligned}$$

Then

$$\phi_A(u, v, w) = \psi + \begin{cases} 1 + u/6 + v/3 + w/2 & \text{if } u \equiv 0 \; mod\, 2 \text{ and } v \equiv 0 \; mod\, 2, \\ 1/2 + u/6 + v/3 + w/4 & \text{if } u \equiv 1 \; mod\, 2 \text{ and } v \equiv 1 \; mod\, 2, \\ 1/2 + u/6 + 5v/24 + 3w/8 & \text{otherwise.} \end{cases}$$

Already this simple example illustrates the main feature of vector partition functions, which is the interplay of a structure of convex polyhedra (as seen in the distinction of cases 1 and 2) with a structure of finite abelian groups (as seen in the "mod"-subcases).

In order to deal with the general case, we introduce some notation. Let $pos(A) = \{\sum_{i=1}^{n} \lambda_i a_i \in \mathbf{R}^n : \lambda_1, ..., \lambda_n \geqslant 0\}$. For $\sigma \subset [n] := \{1, ..., n\}$ we consider the submatrix $A_\sigma := (a_i : i \in \sigma)$, the polyhedral cone $pos(A_\sigma)$, and the abelian group $\mathbf{Z}A_\sigma$ spanned by the the columns of A_σ. We may assume without loss of generality that A is surjective over $\mathbf{Z}$, that is, $\mathbf{Z}A = \mathbf{Z}^d$. This implies that the semigroup $\mathbf{N}A := pos(A) \cap \mathbf{Z}A$ is saturated.

The surjectivity assumption does not hold for the 3×6-matrix in our example. In order to apply the results below to such a case, one must choose a rational 3×3-matrix B which defines an isomorphism from $\mathbf{Z}A$ onto $\mathbf{Z}^3$ and then use the formula $\phi_A(u) = \phi_{BA}(Bu)$.

A subset σ of $[n]$ is a *basis* if $\#(\sigma) = rank(A_\sigma) = d$. The *chamber complex* is the polyhedral subdivision of the cone $pos(A)$ which is defined as the common refinement of the simplicial cones $pos(A_\sigma)$, where σ runs over all bases. Each *chamber* C (meaning: maximal cell in the chamber complex) is indexed by the set $\Delta(C) = \{\sigma \subset [n] : C \subseteq pos(A_\sigma)\}$. For each $\sigma \in \Delta(C)$, the group $\mathbf{Z}A_\sigma$ has finite index in $\mathbf{Z}^d$; write $G_\sigma := \mathbf{Z}^d/\mathbf{Z}A_\sigma$ for the group of residue classes. We say that σ is *non-trivial* if $G_\sigma \neq \{0\}$. For $u \in pos(A) \cap \mathbf{N}^d$, let $[u]_\sigma$ denote the image of u in G_σ.

In the small example above there are 12 chambers; they are grouped into two equivalence classes with respect to the S_3-symmetry. The following theorem is our main result.

THEOREM 1. *For each chamber C there exists a polynomial P of degree $n-d$ in $u = (u_1, ..., u_d)$, and for each non-trivial $\sigma \in \Delta(C)$ there exists a polynomial Q_σ of degree $\#(\sigma) - d$ in u and a function $\Omega_\sigma : G_\sigma \backslash \{0\} \to \mathbf{Q}$ such that, for all $u \in \mathbf{N}A \cap C$,*

$$\phi_A(u) = P(u) + \sum \{\Omega_\sigma([u]_\sigma) \cdot Q_\sigma(u) : \sigma \in \Delta(C) \text{ and } [u]_\sigma \neq 0\}.$$

Moreover, the "corrector polynomials" Q_σ satisfy the linear partial differential equations

$$\sum_{i=1}^{d} a_{ij} \frac{\partial Q_\sigma}{\partial u_i} \equiv 0 \qquad \text{for all } j \in \sigma \text{ such that } \sigma \backslash \{j\} \notin \Delta(C).$$

Remarks. (1) Therem 1 provides a generalization of the theory of *denumerants* (the $d = 1$ case) which can be found in Comtet's book [3, §2.6]. A nice MAPLE package for computing denumerants has been implemented by P. Lisoněk [10].

(2) Another important special case occurs when the matrix A is *unimodular*, which means that $G_\sigma = \{0\}$ for each basis σ. In this case ϕ_A is a polynomial function on each chamber [4, Corollary 3.1]. This happens, for instance, in the problem of counting non-negative integer matrices with prescribed row and column sums; see [5] for a general survey and see [13] for the computational state of the art.

(3) The main point of our formula is the "additive decoupling" of the correction term, which generalizes Theorem C in [3, §2.6]. The results of Dahmen and Micchelli in [4, §3] generalize a (somewhat weaker) classical theorem of Bell [3, §2.6, Thm. B]. The computational advantage of the additive decoupling is explained on page 114 in [3].

THE PROOF

We shall use notation which is standard in the theory of toric varieties; see e.g. [7]. Let N be a lattice of rank m, $M = Hom(N, \mathbf{Z})$ its dual lattice, and $N_\mathbf{Q}$ and $M_\mathbf{Q}$ the corresponding rational vector spaces. Suppose we are given a complete simplicial fan Σ in N having n rays, and non-zero lattice points $b_1, ..., b_n$ on these rays. (The b_i need not be primitive in N!) We identify the cones of Σ with subsets of $\{b_1, ..., b_n\}$. For each $1 \leqslant l \leqslant m$ and each cone $\tau = \{b_{\tau_1}, ..., b_{\tau_l}\}$, we let H_τ denote the group $\mathbf{Z}^\tau$ of integer valued functions on τ modulo the subgroup of those functions on τ which are restrictions from $M = Hom(N, \mathbf{Z})$. We say that τ is *non-trivial* if $H_\tau \neq \{0\}$.

Consider any convex polytope of the form

$$P_\gamma = \{x \in M_\mathbf{Q} : \langle x, b_i \rangle \leqslant \gamma_i \text{ for } i = 1, ..., n\}, \tag{2}$$

where $\gamma = (\gamma_1, ..., \gamma_n)$ ranges over the set $C(\Sigma)$ of all vectors in $\mathbf{Z}^n$ such that the normal fan of P_γ is coarser or equal to Σ. It is well known that there exists a poynomial function $F = F(\gamma)$ on $C(\Sigma)$ of degree m such that $\#(P_\gamma \cap M) = F(\gamma)$ provided that P_γ is integral, i.e., all vertices of P_γ lie in M. For a toric proof of this fact see e.g. [8, §5]. In general, however, the polytope P_γ is not integral, since the fan Σ is not assumed to be smooth. The following proposition characterizes the difference between $\#(P_\gamma \cap M)$ and the polynomial $F(\gamma)$ in the general case. If $\gamma \in \mathbf{Z}^n$, then we write $[\gamma]_\tau$ for the image of the function $\tau \to \mathbf{Z}$, $b_i \mapsto \gamma_i$ in the group H_τ.

PROPOSITION 2. *For every non-trivial cone $\tau \in \Sigma$ there exists a polynomial R_τ of degree $m - \#(\tau)$ in the variables $\gamma = (\gamma_1, ..., \gamma_n)$ and a function $\omega_\tau : H_\tau \backslash \{0\} \to \mathbf{Q}$ such that $\#(P_\gamma \cap M) - F(\gamma) = \sum \{\omega_\tau([\gamma]_\tau) \cdot R_\tau(\gamma) : \tau \in \Sigma$ and $[\gamma]_\tau \neq 0\}$ for all $\gamma \in C(\Sigma)$. Moreover, the polynomial R_τ depends only on those variables γ_i for which $\tau \cup \{b_i\} \in \Sigma$.*

We shall use the following theorem of McMullen. If F is a face of a polytope $P \subset M_{\mathbf{Q}}$ then $\nu(P, F)$ denotes the cone in N normal to F at P. Let $\mathscr{L}$ denote the set of all pairs (τ, L) where τ is a cone in N and L is an affine subspace of $M_{\mathbf{Q}}$ which is a translate of $\tau^{\perp}$.

THEOREM 3 (McMullen [11]). *There exists a function $\theta : \mathscr{L} \to \mathbf{Q}$ such that $\theta(\tau, L) = \theta(\tau, L + m)$ for all $m \in M$ and*

$$\#(P \cap M) = \sum_{F \text{ face of } P} \theta(\nu(P, F), \operatorname{aff}(F)) \cdot Vol(F) \textit{ for every polytope } P \textit{ in } M_{\mathbf{Q}}.$$

Here "Vol" denotes he standard volume form on the affine span aff(F) *of the face F.*

Proof. This is a special case of Theorem 3 in [11], provided one passes from simple valuations to general valuations using the technique in §3 of [11]. ∎

COROLLARY 4. *If P_γ is an integral polytope then the number of lattice points in P_γ equals*

$$F(\gamma) = \sum_{\tau \in \Sigma} \theta(\tau, \tau^{\perp}) \cdot Vol(P_\gamma^\tau), \tag{3}$$

where P_γ^r denotes the face of P_γ supported by τ.

Proof. If P_γ is integral then $\operatorname{aff}(P_\gamma^\tau)$ is a lattice translate of the linear subspace $\tau^{\perp}$. Therefore $\theta(\tau, \operatorname{aff}(P_\gamma^\tau)) = \theta(\tau, \tau^{\perp})$, and the claim follows directly from Theorem 3. ∎

We remark that formula (3) is a valid presentation for the polynomial function $F(\gamma)$ throughout the cone $C(\Sigma)$, not just for those special values of γ for which P_γ is integral.

Proof of Proposition 2. Let τ be a cone in Σ and let F_γ be the corresponding face of P_γ. As γ runs over $C(\Sigma)$, the volume of F_γ varies as a polynomial in γ of degree $dim(F_\gamma) = m - \#(\tau)$. We set $R_\tau(\gamma) := Vol(F_\gamma)$. This function is independent of a support parameter γ_i if the hyperplane $\langle x, b_i \rangle = \gamma_i$ does not intersect the face F_γ for general γ. The latter condition is equivalent to $\tau \cup \{b_i\}$ not being a cone of Σ. Hence R_τ has the property asserted in the second part of Proposition 2.

Consider any other vector $\gamma' \in C(\Sigma)$ and corresponding face $F_{\gamma'}$ of $P_{\gamma'}$. Note that $\operatorname{aff}(F_\gamma) = \{ y \in M : \forall b_i \in \tau : \langle y, b_i \rangle = \gamma_i \}$, and similarly for $F_{\gamma'}$. This implies

$$\begin{aligned}[\gamma]_\tau = [\gamma']_\tau &\Leftrightarrow \exists u \in M : \forall b_i \in \tau : \gamma_i = \gamma'_i + \langle u, b_i \rangle \\ &\Leftrightarrow \exists u \in M : \operatorname{aff}(F_\gamma) = \operatorname{aff}(F_{\gamma'}) + u.\end{aligned}$$

We can therefore define a function $\omega_\tau : H_\tau \backslash \{0\} \to \mathbf{Q}$ by setting

$$\omega_\tau([\gamma]_\tau) := \theta(\tau, \operatorname{aff}(F_\gamma)) - \theta(\tau, \tau^\perp).$$

Proposition 2 now follows immediately from Theorem 3 and Corollary 4. ∎

Proof of Theorem 1. We shall use representation techniques as in [12, §5]. Let $B = (b_1, ..., b_n)$ be an integer $(n-d) \times n$-matrix whose row space (over $\mathbf{Z}$) equals the kernel of A. In other words, we construct a short exact sequence of abelian groups

$$0 \longrightarrow \mathbf{Z}^{n-d} \xrightarrow{B^t} \mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^d \longrightarrow 0.$$

We set $m = n - d$ and $M = \mathbf{Z}^{n-d}$, and we consider the polytope P_γ in (2), for an arbitrary $\gamma \in \mathbf{Z}^n$. The map $x \mapsto \gamma - B^t \cdot x$ defines a bijection between the lattice points in P_γ and the set of elements $\lambda \in \mathbf{N}^n$ such that $A \cdot \lambda = A \cdot \gamma$. Therefore we have

$$\phi_A(A \cdot \gamma) = \#(P_\gamma \cap M). \tag{4}$$

We now fix a chamber C and we consider those vectors $\gamma \in \mathbf{N}^n$ such that $A \cdot \gamma$ lies in the interior of C. This determines the normal fan Σ of P_γ as follows:

$$\Sigma = \{ \{b_{\tau_1}, ..., b_{\tau_k}\} : [n] \backslash \{\tau_1, ..., \tau_k\} \in \Delta(C) \}.$$

Let us now fix $\sigma \in \Delta(C)$ and set $\tau := [n] \backslash \sigma$. In order to derive the first part of Theorem 1 from Proposition 2, it suffices to show that there exists a group isomorphism δ between H_τ and G_σ, which takes a class $[\gamma]_\tau$ in H_τ to the class of $[u]_\sigma$ in G_σ, where $u := A \cdot \gamma$. Indeed, in view of (4), we can then simply define $P(u) := F(\gamma)$, $\Omega_\sigma([u]_\sigma) := \omega_\tau([\gamma]_\tau)$, and $Q_\sigma(u) := R_\tau(\gamma)$ to get the desired formula for $\phi_A(u)$. (Note that $\#(\sigma) - d = m - \#(\tau)$.)

To define the group isomorphism δ, we consider the short exact sequence

$$0 \longrightarrow \mathbf{Z}^\sigma \xrightarrow{i} \mathbf{Z}^n \xrightarrow{\pi} \mathbf{Z}^\tau \longrightarrow 0,$$

where i and π are the obvious coordinate inclusion and projection respectively. We have

$$H_\tau = coker(\pi \circ B^t) \qquad \text{and} \qquad G_\sigma = coker(A \circ i).$$

Consider any element of G_σ, given by a representative $u \in \mathbf{Z}^{\mathrm{d}}$. We define $\delta(u)$ to be $\pi(\gamma)$, where γ is any preimage of u under A. This defines a unique element of H_τ because γ is well-defined up to $im(B^t) = ker(A)$. We have the equivalences

$$\begin{aligned} u \in \mathbf{Z}A_\sigma &\Leftrightarrow u \text{ has a preimage } \tilde{\gamma} \text{ under } A \text{ such that } \pi(\tilde{\gamma}) = 0 \\ &\Leftrightarrow \gamma - \tilde{\gamma} \in ker(A) = im(B^t) \\ &\qquad\qquad \text{for some } \tilde{\gamma} \in \mathbf{Z}^n \text{ such that } \pi(\tilde{\gamma}) = 0 \\ &\Leftrightarrow \pi(\gamma) = \pi(B^t \cdot \lambda) \text{ for some } \lambda \in \mathbf{Z}^{n-d}. \end{aligned}$$

This shows that u is zero in G_σ if and only if $\pi(\gamma) = \delta(u)$ is zero in H_τ. Therefore the group homomorphism δ is injective. But it is also surjective: if $v \in \mathbf{Z}^\tau$, then choose any $w \in \mathbf{Z}$, consider $v + w \in \mathbf{Z}^n$ and define $u = A(v+w)$. Then $\delta(u)$ and v represent the same element of H_τ. This completes the proof of the first part of Theorem 1.

To prove the second part we note that an element $j \in \sigma$ satisfies $\sigma \backslash \{j\} \notin \Delta(C)$ if and only if $\tau \cup \{b_j\} \notin \Sigma$. For such an index j, we apply the operator $\partial/\partial\gamma_j$ to the polynomial

$$R_\tau(\gamma) = Q_\sigma(A \cdot \gamma).$$

The result is zero, by Proposition 2, and consequently $\sum_{i=1}^d a_{ij} (\partial Q_\sigma / \partial u_i) \equiv 0$, as required. ∎

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