Combinatorial Relationship Between Finite Fields and Fixed Points of Functions Going Up and Down

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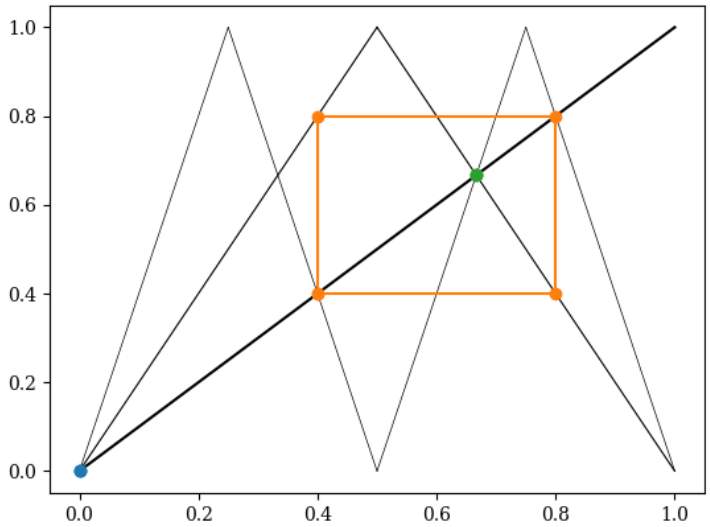
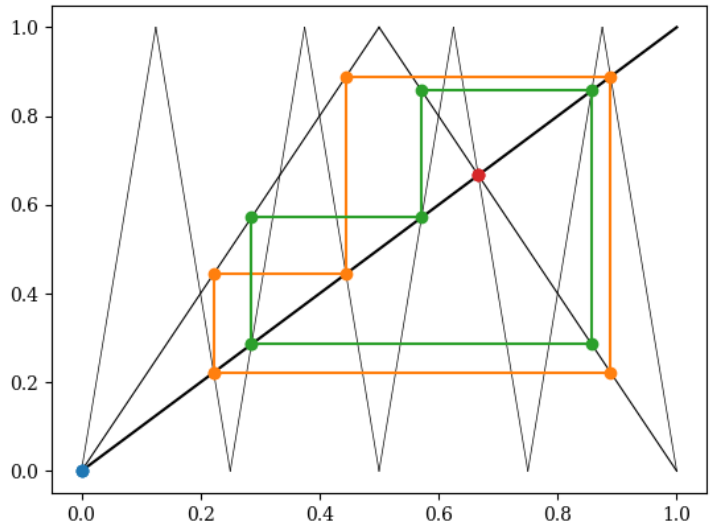
Abstract

Abstract We explore a combinatorial bijection between two seemingly unrelated topics: the roots of irreducible polynomials of degree m over a finite field $\mathcal{F}_p$ for a prime number p and the number of points that are periodic of order m for a continuous piece-wise linear function $g_p : [0, 1] \to [0, 1]$ that *goes up and down p times* with slope $\pm p$.

Functions going up and down

Let $I_k := \left[\frac{k}{p}, \frac{k+1}{p}\right]$ for $0 \leq k \leq p-1$ and $g_p : [0,1] \to [0,1]$ a continuous function linearly increasing from 0 to 1 on I_k for k even, and linearly decreasing from 1 to 0 on I_k for k odd.
Denote by g_p^n to the function obtained by taking g_p composed with itself n times. g_p^n has p^n fixed points.
A point x is called to be *periodic of order m* if $g_p^m(x) = x$, and $g_p^i(x) \neq x$ for $i = 1, 2, \ldots, m-1$.





Example: Functions g_2, g_2^2 and g_2^3 with its fixed points.

Main Observation

Theorem

The number $J_p(m)$ of points $x \in \mathrm{FP}(g_p^m)$ periodic of order m is equal to the number of roots of monic irreducible polynomials of degree m over the finite field $\mathcal{F}_p$.

Proof.

Both values can be computed by Möbius inversion formula. They satisfy the relation $\sum_{d|n} J_p(d) = \sum_{d|n} mI_p(d) = p^n$, in both cases (where $I_p(m)$ denotes the number of irreducible polynomials of degree m over $\mathcal{F}_p$ and the sum represent all elements of $\mathcal{F}_{p^n}$.) Then we find that $J_p(m) = mI_p(m) = \sum_{d|n} \mu(n/d)p^d$. □

Corollary

There is a bijection $B : \mathrm{FP}(g_p^n) \to \mathcal{F}_{p^n}$, such that $B(x)^p = B(g_p(x))$ for every $x \in \mathrm{FP}(g_p^n)$.

We provide here one such bijection, connecting some of the structure of both worlds.

Bijection via permutation π_{p^n}

We want to get a bijective proof. For this we construct a permutation map $\pi_{p^n} : \{0, 1, \ldots, p^n - 1\} \to \{0, 1, \ldots, p^n - 1\}$ that help us to create our bijection B, as follows:

Definition

Take π_{p^1} the identity map from 0 to $p - 1$. Then if $ap^{n-1} \leq k < (a+1)p^{n-1}$,

$$\pi_{p^n}(k) = \begin{cases} \pi_{p^{n-1}}(k - ap^{n-1}) + ap^{n-1} \text{ for } a \text{ even} \\ \pi_{p^{n-1}}(p^{n-1} - (k - ap^{n-1}) - 1) + ap^{n-1} \text{ for } a \text{ odd.} \end{cases}$$

Some examples for $p = 2$ and $p = 3$ of how π_{p^n} permute the numbers $0, 1, \ldots, p^n - 1$:

$\pi_{2^1} : 0, 1$

$\pi_{2^2} : 0, 1, 3, 2$

$\pi_{2^3} : 0, 1, 3, 2, 6, 7, 5, 4$

$\pi_{2^4} : 0, 1, 3, 2, 6, 7, 5, 4, 12, 13, 15, 14, 10, 11, 9, 8$

Main Theorem

Definition (Bijection B_α)

Let α be a primitive root (i. e. a generator of the multiplicative group) of $\mathcal{F}_{p^n}$ and $0 = x_0 < x_1 < \cdots < x_{p^n-1}$ be the fixed points of g_p^n. We define the bijection $B_\alpha : \mathrm{FP}(g_p^n) \to \mathcal{F}_{p^n}$ by $B_\alpha(x_0) = 0 \in \mathcal{F}_{p^n}$ or $B_\alpha(x_k) = \alpha^{\pi_{p^n}(k)}$ for all other $k > 0$.

Theorem

The function B_α is a bijection such that $(B_\alpha(x_k))^p = B_\alpha(g_p(x_k))$ for any fixed point $x_k \in \mathrm{FP}(g_p^n)$.

Proposition

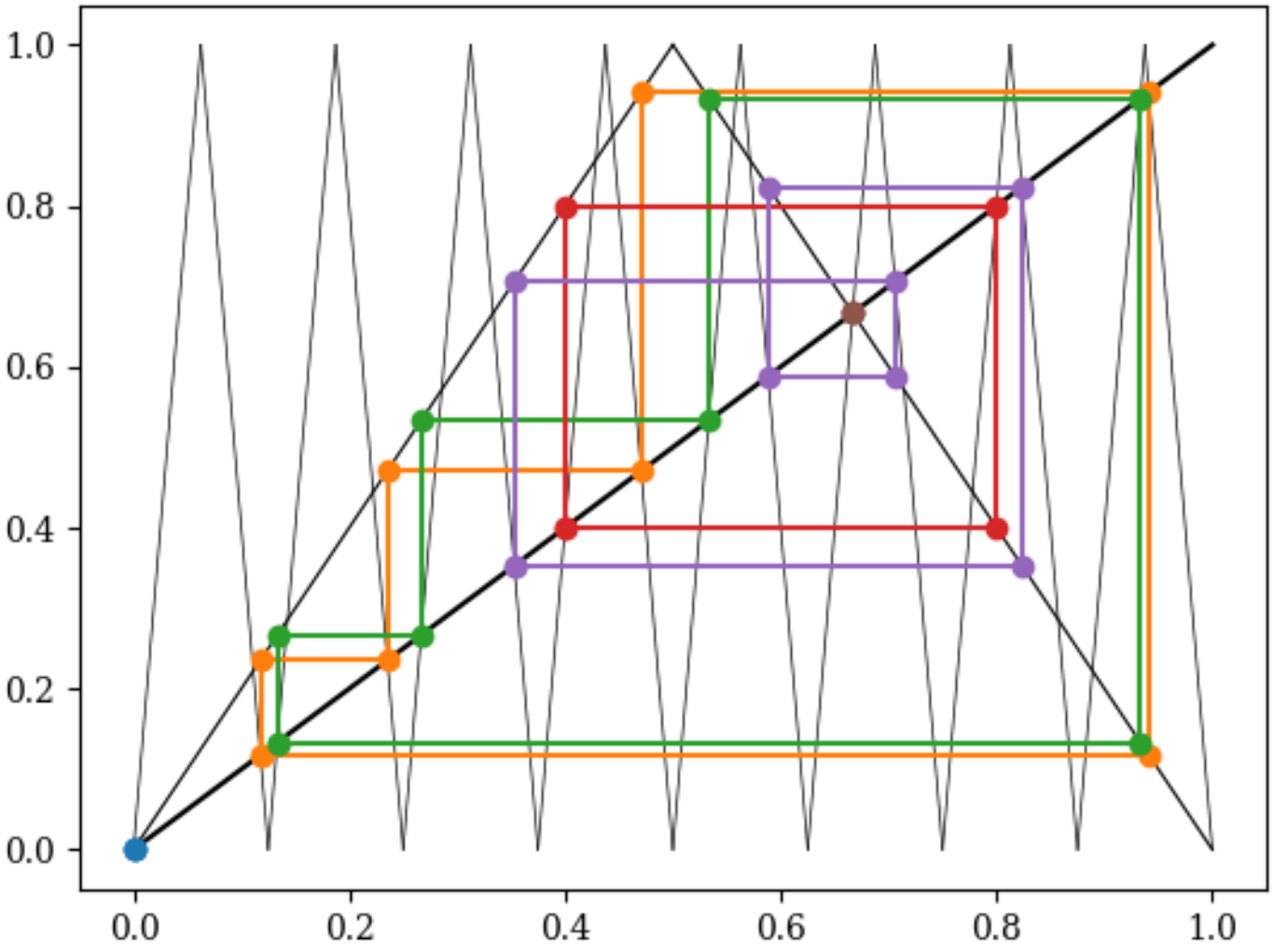
The bijection $B_\alpha : \mathrm{FP}(g_p^n) \to \mathcal{F}_{p^n}$ satisfy that $B_\alpha(x_i)B_\alpha(x_j) = B_\alpha(x_r)$ with

$$r = \pi_{p^n}^{-1}((\pi_{p^n}(i) + \pi_{p^n}(j)) \mod p^n - 1),$$

for any $0 < i, j < p^n$, where the class representative modulo $p^n - 1$ must be taken from 1 to $p^n - 1$.

Bijection B_α for $p = 2$, $n = 4$, and $\alpha^4 + \alpha + 1 = 0$ in $\mathcal{F}_{2^4}$

i	x_i	$g_p(x_i)$	$\pi_{p^n}(i)$	$B_\alpha(x_i)$
0	0.0	0.0	0	0
1	0.1176470588	0.2352941176	1	α
2	0.1333333333	0.2666666666	3	α^3
3	0.2352941176	0.4705882352	2	α^2
4	0.2666666666	0.5333333333	6	$\alpha^3 + \alpha^2$
5	0.3529411764	0.7058823529	7	$\alpha^3 + \alpha + 1$
6	0.4	0.8	5	$\alpha^2 + \alpha$
7	0.4705882352	0.9411764705	4	$\alpha + 1$
8	0.5333333333	0.9333333333	12	$\alpha^3 + \alpha^2 + \alpha + 1$
9	0.5882352941	0.8235294117	13	$\alpha^3 + \alpha^2 + 1$
10	0.6666666666	0.6666666666	15	1
11	0.7058823529	0.5882352941	14	$\alpha^3 + 1$
12	0.8	0.3999999999	10	$\alpha^2 + \alpha + 1$
13	0.8235294117	0.3529411764	11	$\alpha^3 + \alpha^2 + \alpha$
14	0.9333333333	0.1333333333	9	$\alpha^3 + \alpha$
15	0.9411764705	0.1176470588	8	$\alpha^2 + 1$



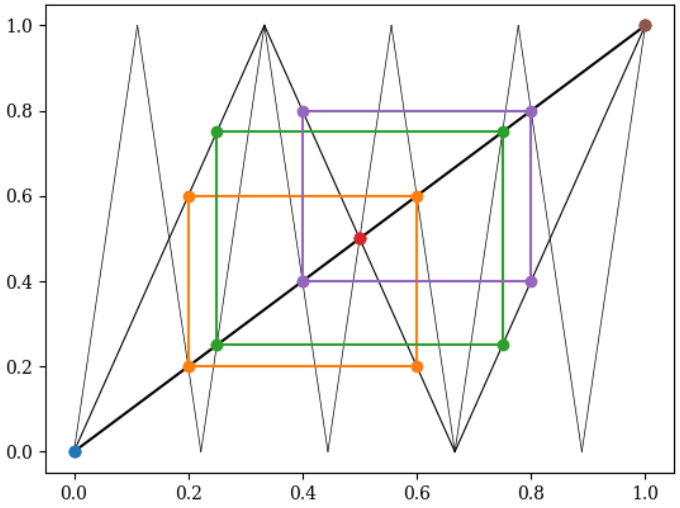
Example: Function g_2^4 with its fixed points.

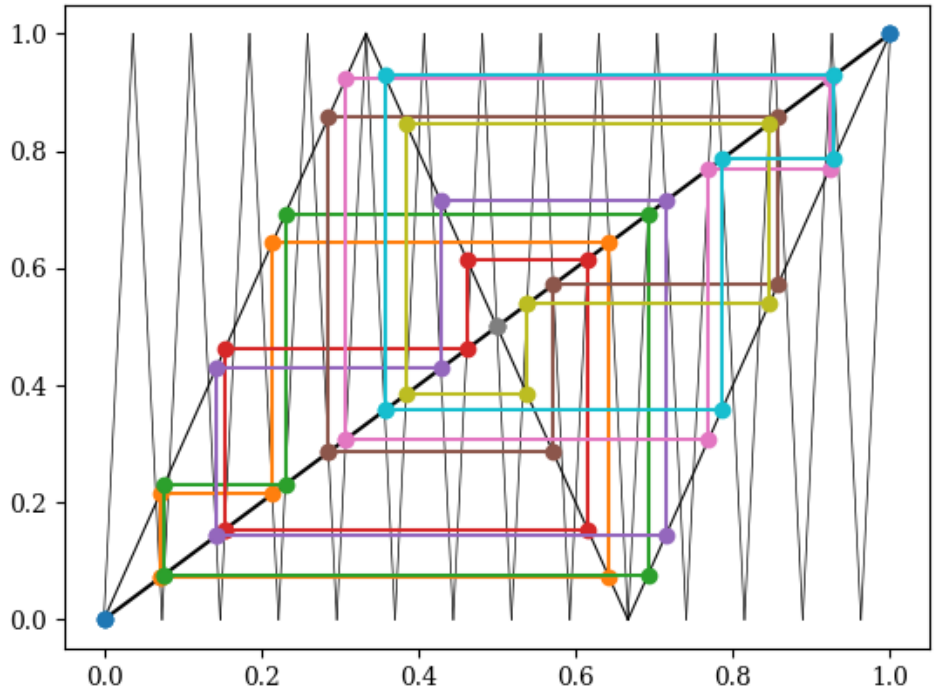
More examples: p=3

π_{3^1} : 0, 1, 2

π_{3^2} : 0, 1, 2, 5, 4, 3, 6, 7, 8

π_{3^3} : 0, 1, 2, 5, 4, 3, 6, 7, 8, 17, 16, 15, 12, 13, 14, 11, 10, 9,
18, 19, 20, 23, 22, 21, 24, 25, 26





Example: Functions g_3, g_3^2 and g_3^3 with its fixed points.

g_p and its periodic points in base p

To prove the main theorem we need to understand g_p, its periodic points, and π_{p^n} in *base p*, .

Proposition

If $x = \sum_{i\geq 1} \frac{a_i}{p^i}$ *for* $0 \leq a_i \leq p-1$, *then*

$$g_p(x) = \begin{cases} \sum_{i\geq 1} \frac{a_{i+1}}{p^i}, & \text{if } a_1 \text{ is even.} \\ \sum_{i\geq 1} \frac{p-1-a_{i+1}}{p^i}, & \text{if } a_1 \text{ is odd.} \end{cases}$$

Proposition

Let $x_0 < x_1 < \ldots < x_{p^n-1}$ *be the fixed points of the function* g_p^n. *Then, for* $k < p^n$ *we have that* $x_k = \frac{k}{p^n-1}$ *if* k *is even, or* $x_k = \frac{k+1}{p^n+1}$ *if* k *is odd.*

g_p and its periodic points in base p

Proposition

The expression in base p of $x = \frac{k}{p^n-1}$ is periodic, where the first n digits represent k in base p and $a_{n+i} = a_i$. Also, if $x = \frac{k+1}{p^n+1}$, its expression in base p has period $2n$, where the first n digits represent k in base p, while the next n digits are complementary of the first n digits, that is $a_{n+i} = p - 1 - a_i$.

Example

The function g_2 permutes the fixed points in $\mathrm{FP}(g_2^3)$ as follows:

$$\begin{gathered} 0.\overline{000}_2 \to 0.\overline{000}_2 \\ 0.\overline{001110}_2 \to 0.\overline{011100}_2 \to 0.\overline{111000}_2 \to 0.\overline{001110}_2 \\ 0.\overline{010}_2 \to 0.\overline{100}_2 \to 0.\overline{110}_2 \to 0.\overline{010}_2 \\ 0.\overline{101010}_2 \to 0.\overline{101010}_2 \end{gathered}$$

π_{p^n} in base p

Proposition

If $k = (a_1 a_2 \dots a_n)_p$ *(where* $k < p^n$*), then* $\pi_{p^n}(k) = (b_1 b_2 \dots b_n)_p$ *where* $b_1 = a_1$*, and* $b_i = a_i$ *when* $b_1 + \cdots + b_{i-1}$ *is even or* $b_i = p - 1 - a_i$ *when* $b_1 + \cdots + b_{i-1}$ *is odd.*

Notice that for p odd, taking $p - 1$ complement doesn't change the parity of the digits, and therefore we could use the parity of $a_1 + \cdots + a_{i-1}$ to create the two cases in the previous proposition. In case $p = 2$ things are a little bit different.

Proposition

If $p = 2$ *and* $i = (a_1 a_2 \dots a_n)_2$*, then* $\pi_{p^n}(i) = (b_1 b_2 \dots b_n)_2$ *where* $b_1 = a_1$*, and* $b_i = 0$ *if* $a_{i-1} = a_i$ *or* $b_i = 1$ *otherwise.*

Sketch of Proof of the Main Theorem

We need to check that $(B_\alpha(x_k))^p = B_\alpha(g_p(x_k))$ for any $x_k \in \mathrm{FP}(g_p^n)$. We can analyze every step in base p to check that the following diagram commutes, where fr is the Frobenius Map $fr : x \mapsto x^p$.

$$\begin{array}{ccc} \mathrm{FP}(g_p^n) & \xrightarrow{B_\alpha} & \mathcal{F}_{p^n} \\ \Big\downarrow{\scriptstyle g_p} & & \Big\downarrow{\scriptstyle fr} \\ \mathrm{FP}(g_p^n) & \xrightarrow[B_\alpha]{} & \mathcal{F}_{p^n} \end{array}$$

If $k = (a_1 \dots a_n)_p < p^n$, then $x_k = (0.\overline{a_1 \dots a_n a_1' \dots a_n'})_p$ where $a_i' = a_i$ in case that $(a_1 \dots a_n)_p$ is even, or $a_i' = p - 1 - a_i$ if $(a_1 \dots a_n)_p$ is odd.
Denote $\pi_{p^n}(k) = (b_1 b_2 \dots b_n)_p$,. Then $b_1 = a_1$, and $b_i = a_i$ when $b_1 + \dots + b_{i-1}$ is even or $b_i = p - 1 - a_i$ when $b_1 + \dots + b_{i-1}$ is odd.
Also $(B_\alpha(x_k))^p = \alpha^{p(b_1 \dots b_n)_p} = \alpha^{(b_1 \dots b_n 0)_p} = \alpha^{(b_2 \dots b_n b_1)_p}$, since $\alpha^{p^n} = \alpha$ in the finite field $\mathcal{F}_{p^n}$.
We need to consider two cases, depending if a_1 is even or odd.
Also cases depending if $p = 2$ or p is odd...

Generalizations: Other functions going up and down

We want to extend our Theorem to other functions $f_p : [u, v] \to [u, v]$ going up and down p times. We restrict to the case that there is a continuous bijection $h : [0, 1] \to [u, v]$ that is a homeomorphism between both functions g_p and f_p (so that $f_p \circ h = h \circ g_p$). In this case we can easily extend our results to f_p as well.

Theorem

If $f_p : [u, v] \to [u, v]$ is such that there is a continuous bijection $h : [0, 1] \to [u, v]$ so that $f_p \circ h = h \circ g_p$, then there is a bijection $B_f : \mathrm{FP}(f_p^n) \to \mathcal{F}_{p^n}$ so that $(B_f(y_i))^p = B_f(f_p(y_i))$ for any fixed point $y_i \in \mathrm{FP}(f_p^n)$.

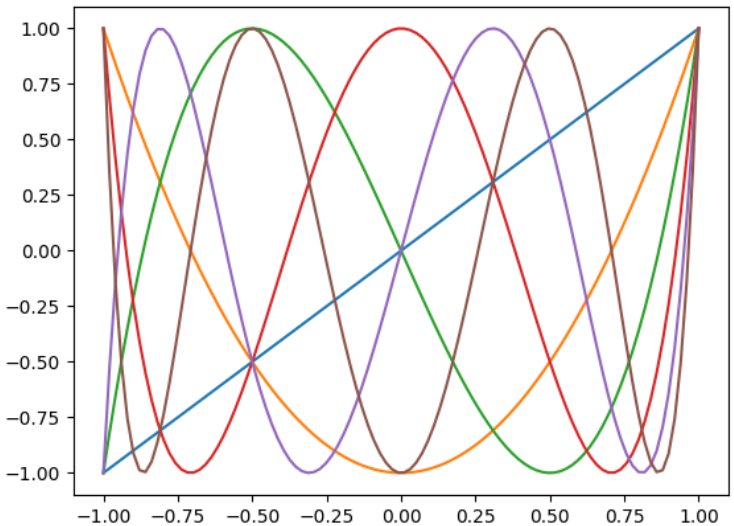
As an example we will see how to extend our results to *Chebyshev Polynomials.*

Example: Chebyshev Polynomials

Definition (Chebyshev polynomials)

The family of Chebyshev polynomials $T_k \in \mathbb{R}[x]$ can be defined recursively by $T_0(x) = 1$, $T_1(x) = x$, and $T_{k+1}(x) = 2xT_k(x) - Tk - 1(x)$.

The next polynomials in the sequence are $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$, and $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$.



Proposition

The Chebyshev polynomial T_n is the expressions for $\cos(n\theta)$ in terms of $\cos(\theta)$, namely it holds that $\cos(n\theta) = T_n(cos(\theta))$.

Example: Chebyshev Polynomials

Then $T_m \circ T_n = T_{mn}$ and also $T_p^n = T_{p^n}$.
T_m goes up and down m times in the interval [-1,1]

Proposition

The continuous bijection $h : [0,1] \to [-1,1]$ *is given by* $h(x) = \cos(\pi x)$. *It holds that* $T_p \circ h = h \circ g_p$ *for any value of* p.

Theorem

There is a bijection $B_f : \mathrm{FP}(T_p^n) \to \mathcal{F}_{p^n}$ *so that* $(B_f(y_i))^p = B_f(T_p(y_i))$ *for any fixed point* $y_i \in \mathrm{FP}(T_p^n)$.

If $x_i \in FP(g_{p^n})$ then $y_i = h(x_i) = \cos(\pi x_i) \in FP(T_{p^n}(x) - x)$.

Corollary

If p *is odd, then* $T_p^n(x) - x =$

$$2^{p^n-1}(x-1)\prod_{k=1}^{(p^n-1)/2}\left(x-\cos\left(\frac{(2k-1)\pi}{p^n+1}\right)\right)\left(x-\cos\left(\frac{2k\pi}{p^n-1}\right)\right).$$

Generalization: piecewise linear functions going up or down

Definition

Let p be a prime number, and $I \subseteq \{0, 1, 2, \ldots, p-1\}$. The function $g_{p,I} : [0,1] \to [0,1]$ given by

$$g_{p,I}(x) = \begin{cases} px - k, & \text{for } \frac{k}{p} \leq x < \frac{k+1}{p} \text{ with } k \in I. \\ k + 1 - px, & \text{for } \frac{k}{p} \leq x < \frac{k+1}{p} \text{ with } k \notin I. \end{cases}$$

In this case $g_{p,I}$ is a piece-wise linear function where I denotes the set of indices k where $g_{p,I}$ is increasing.
If I are all even numbers, then $g_{p,I} = g_p$. If I includes all indices, then $g_{p,I} = \{px\}$ (the fractionary part of px).

Definition

The set $I_{p^n} \subseteq \{0, 1, 2, \ldots, p^n - 1\}$ is the set of indices where $g^n_{p,I}$ is increasing on the interval $(\frac{k}{p^n}, \frac{k+1}{p^n})$.

We can generalize our theorems for all functions $g_{p,I}$.

Generalization for functions $g_{p,I}$

Definition

The permutation $\pi_{p^n,I}$ is defined recursively as follows: take $\pi_{p^1,I}$ the identity map from 0 to $p-1$. Then to define $\pi_{p^n,I}$, if $ap^{n-1} \leq k < (a+1)p^{n-1}$ take

$$\pi_{p^n,I}(k) = \begin{cases} ap^{n-1} + \pi_{p^{n-1},I}(k - ap^{n-1}) \text{ for } a \in I \\ ap^{n-1} + \pi_{p^{n-1},I}(p^{n-1} - (k - ap^{n-1}) - 1) \text{ for } a \notin I \end{cases}$$

Let α be a primitive root and let $x_0 < x_1 < \cdots < x_{p^n-1}$ be the fixed points of $g^n_{p,I}$. We define $B_{\alpha,I} : \mathrm{FP}(g^n_{p,I}) \to \mathcal{F}_{p^n}$ by $B_\alpha(x_{\pi_{p^n,I}(0)}) = 0 \in \mathcal{F}_{p^n}$ or $B_\alpha(x_k) = \alpha^{\pi_{p^n,I}(k)}$ for all other $k > 0$.

Theorem

The function $B_{\alpha,I}$ is a bijection and satisfy that $B_{\alpha,I}(x_k)^p = B_{\alpha,I}(g_{p,I}(x_k))$ for any fixed point $x_k \in \mathrm{FP}(g^n_{p,I})$.

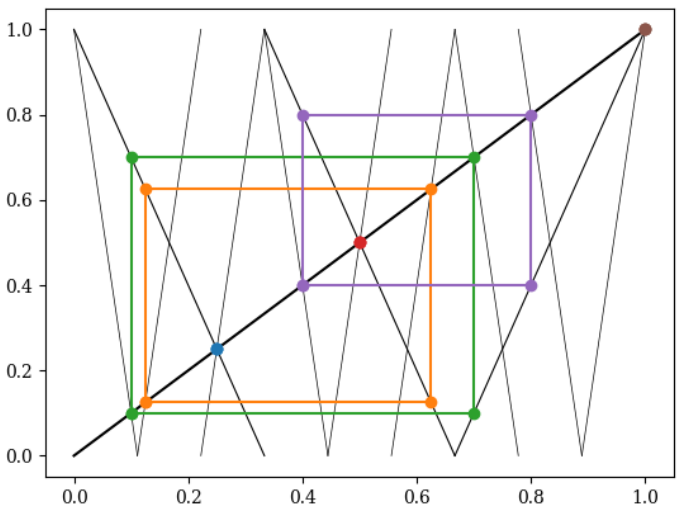
Example: $p = 3$, $I = \{2\}$

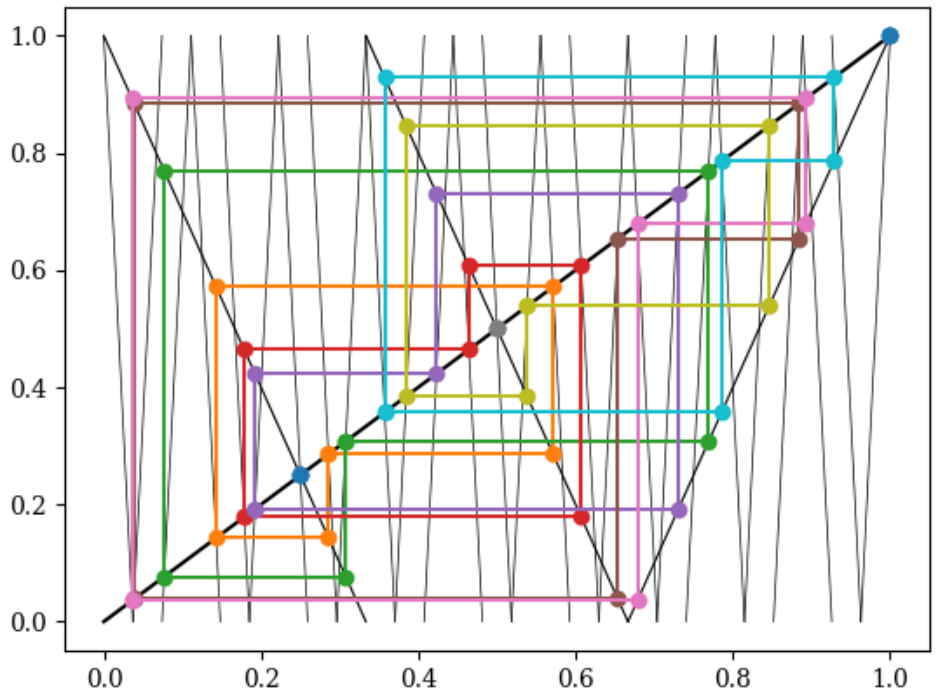
For $p = 3$, and $I = \{2\}$, then $\pi_{p^n,I}$ permutes as follows:

$\pi_{3^1,I}$: $0, 1, 2$

$\pi_{3^2,I}$: $2, 1, 0, 5, 4, 3, 6, 7, 8$

$\pi_{3^3,I}$: $8, 7, 6, 3, 4, 5, 0, 1, 2, 17, 16, 15, 12, 13, 14, 9, 10, 11,$
$20, 19, 18, 23, 22, 21, 24, 25, 26$





Fixed points of $g_{3^2,I}$ and $g_{3^3,I}$, with orbits under $g_{3,I}$.