CS184 Assignment 2

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1. 1. There exists a k such that $k \cdot k = 1$ and $k \cdot j = 0$. k can be $-i$, and there's only one vector like that in the 2D plane.

   2. No, because we're only in 2D space. k cannot be a new vector orthogonal to both j and i if j and i are orthogonal to each other.

   3. In 3D space there would be more possibilities for k in part **1**, and the answer to part **2** is yes.

2. For the matrix $\begin{bmatrix} x_1 & y_1 & 1 & \\ x_2 & y & 2 & 1 \\ x_3 & y_3 & 1 & \end{bmatrix}$, the determinant is $x_1y_2 + y_1x_3 + x_2y_3 - x_3y_2 - x_1y_3 - y_1x_2$. The area of the triangle given by the three points can be modeled by two vectors $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$ and $\begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \end{bmatrix}$. The area of the triangle is given as half the cross product, which is $\frac{1}{2}\big(x_1y_2 + y_1x_3 + x_2y_3 - x_3y_2 - x_1y_3 - y_1x_2\big)$, which is proportional the determinant of the 3×3 matrix. If the area of the triangle is 0, then the points are in a line.

3. $f(x, y) = ax + by + c = 0$, so we can uniquely determine the line by using the gradient $\nabla f(x, y) = (a, b)$. Because $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = 0$, then we can say $a = y_0 - y_1$ and $b = x_1 - x_0$, which gives us

$$f(x, y) = (y_0 - y_1)x + (x_1 - x_0)y + C = 0$$

   at which point we can plug in (x_0, y_0) and solve for C.

4. For $\{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\} : e_i \times e_j = e_k$ so

$$e_1 \times e_2 = \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ 1 - 0 \end{bmatrix} = e_3 \quad e_2 \times e_3 = \begin{bmatrix} 1 - 0 \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = e_1 \quad e_3 \times e_1 = \begin{bmatrix} 0 - 0 \\ 1 - 0 \\ 0 - 0 \end{bmatrix} = e_2$$

   For $\{i, j, k\} = \{1, 3, 2\}, \{3, 2, 1\}, \{2, 1, 3\} : e_i \times e_j = -e_k$ so

$$e_1 \times e_3 = \begin{bmatrix} 0 - 0 \\ 0 - 1 \\ 0 - 0 \end{bmatrix} = -e_2 \quad e_3 \times e_2 = \begin{bmatrix} 0 - 1 \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = -e_1 \quad e_2 \times e_1 = \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ 0 - 1 \end{bmatrix} = -e_3$$

5. $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_1 & z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, and the determinant of this matrix is $\texttt{det}\begin{pmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \end{pmatrix}$. If we have the 4 points of the matrix p_1, p_2, p_3, p_4, then we can define vectors $v_1 = p_2 - p_1 \quad v_2 = p_3 - p_1 \quad v_3 = p_4 - p_1$, and because $\texttt{det}\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ = vol-

ume of the parallelpiped shaped by the vectors, then $\frac{1}{6}$ of that is the volume of the tetrahedron: $\frac{1}{6}\texttt{det}\begin{pmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \end{pmatrix} = \frac{1}{6}\texttt{det}\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

6. We can have vectors $\bar{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$ and $\bar{AC} = \begin{bmatrix} x_2 - x_0 \\ y_2 - y_0 \\ z_2 - z_0 \end{bmatrix}$, and take the cross product to find the normal vector that helps define the plane: $\begin{bmatrix} (y_2 - y_0)(z_1 - z_0) - (z_2 - z_0)(y_1 - y_0) \\ (z_2 - z_0)(x_1 - x_0) - (x_2 - x_0)(z_1 - z_0) \\ (x_2 - x_0)(y_1 - y_0) - (y_2 - y_0)(x_1 - x_0) \end{bmatrix}$.

   We can then take this (a, b, c) and plug into $f(x, y, z) = ax + by + cz = d$, then plug in a point then solve for D to find the full equation of the plane.

7. If p_0, p_1, p_2 are three distinct points in space then the cross product $n = (p_0 - p_1) \times (p_0 - p_2)$, then the resulting vector is orthogonal to the vectors $(p_0 - p_1)$ and $(p_0 - p_2)$. If p is any point on the plane formed by those three points, then n and $p - p_0$ are also orthogonal (dot product is 0). So because all points p in relation to p_0 are orthogonal to n, then the equation of the plane must be of the form $n \cdot (p - p_0) = 0$

8. $M(\theta)^-1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^- 1 = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = M(\theta)^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ so $M(\theta)$ is orthonormal.

   To show that $M(\theta_1+\theta_2) = M(\theta_1)M(\theta_2)$, we consider that $\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2)$ and $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ and we can do the matrix multiplication to prove the identity.

   If $\theta_1 = \theta$ and $\theta_2 = -\theta$, so $M(\theta_1 + \theta_2) = M(0) = I$. $M(\theta)M(-\theta) = I$, so $M(\theta)^{-1} = M(-\theta)$

   For $p = (0, 1)$, considering $M(\theta)p$ for every possible θ gives you a circle ($M(\theta)p = \sin\theta + \cos\theta$, which is the equation for a circle).

9. We have 4 vectors in a plane: x_1, x_2, b_1, b_2 and $Mx_1 = b_1$ and $Mx_2 = b_2$. We need an orthogonal basis, so we define $y_1 = x_1$ and $y_2 = x_2 - \frac{x_2 \cdot x_1}{|x_1|} \cdot \frac{x_1}{|x_1|}$. Then we can write

$$x_3 = ay_1 + by_2 \to Mx_3 = aMy_1 + bMy_2 = ab_1 + b(b_2 - \frac{x_2 \cdot x_1}{|x_1|} \cdot \frac{b_1}{|x_1|})$$

   which gives us $a = x_3 \cdot \hat{y_1}$ and $b = x_3 \cdot \hat{y_2}$, and we can plug these values in to get $Mx_3 = (x_3 \cdot \hat{y_1})Mx_1 + (x_3 \cdot \hat{y_2})M(x_2 - \frac{x_2 \cdot x_1}{|x_1|} \cdot \frac{x_1}{|x_1|})$

10. Parametric eqn for line through p_1 and p_2:

$$\begin{cases} x(t) = (1-t)x_1 + tx_2 \\ y(t) = (1-t)y_1 + ty_2 \end{cases}$$

    For a plane with p_1, p_2, p_3:

$$\begin{cases} x(t_1, t_2) = (1 - t_1 - t_2)x_1 + t_1(1 - t_2)x_2 + (1 - t_1)t_2x_3 \\ y(t_1, t_2) = (1 - t_1 - t_2)y_1 + t_1(1 - t_2)y_2 + (1 - t_1)t_2y_3 \\ z(t_1, t_2) = (1 - t_1 - t_2)z_1 + t_1(1 - t_2)z_2 + (1 - t_1)t_2z_3 \end{cases}$$

    The equalities for

    - points on line between p_1, p_2: $0 \le t_1 \le 1, \quad t_2 = 0$

- triangle between p_1, p_2, p_3: $0 \leq t_1 + t_2 \leq 1, \quad 0 \leq t_1 \leq 1, \quad 0 \leq t_2 \leq 1$

11. **M** is symmetric 2×2 with eigenvalues λ_0 and λ_1.

   - Show $0 \leq x^T Mx \leq (\mathtt{max}(\lambda_0, \lambda_1))x^T x$:

     $x = av_1 + bv_2$, where v_1, v_2 are eigenvalues corresponding to $\lambda_{0,1}$. We can then say
     $$Mx = Mav_1 + Mbv_2 = \lambda_1 av_1 + \lambda_2 bv_2$$
     $$x^T Mx = (av_1 + bv_2)(\lambda_1 av_1 + \lambda_2 bv_2) = \lambda_1 a^2 |v_1|^2 + \lambda_2 b^2 |v_2|^2$$
     So, because $x^T x = a^2|v_1|^2 + b^2|v_2|^2$, then scaling both these terms by $\mathtt{max}(\lambda_1, \lambda_2)$ will mean that the result will be **at least as big** as $x^T Mx$, in which one of the terms will be multiplied by the larger eigenvalue and one by the smaller (or equal).

   - If one eigenvalue is 0, then the corresponding eigenvector may not be 0 by definition of an eigenvector. Thus, if $\lambda_1 = 0$, then $Mx = \lambda_1 x = 0$ with a non-zero x.
   - if $\lambda_0 < 0$ and $\lambda_1 > 0$ with corresponding eigenvalues v_0, v_1, then we can say:

     $$Mx = Mav_0 + Mbv_1 = \lambda_0 av_0 + \lambda_1 bv_1$$
     $$x^T MX = \lambda_0 a^2 |v_0|^2 + \lambda_1 b^2 |v_1|^2 = 0, \text{ so}$$
     $$-\lambda_0 a^2 |v_0|^2 = \lambda_1 b^2 |v_1|^2, \text{ so if } a = 1, \text{ then } b = \sqrt{\frac{-\lambda_0}{\lambda_1} \frac{|v_0|^2}{|v_1|^2}}, \text{ which we can plug into}$$
     $$x = av_0 + bv_1 \neq 0, \text{ but still satisfies } x^T Mx = 0.$$

   - The eigenvalues are positive.

12. M is a 2×2 matrix

$$Mx = \lambda I x \text{ (where I is an identity matrix)}$$
$$Mx - \lambda I x = 0 = (M - \lambda I)x, \text{ so } \mathtt{det}(M - \lambda I) = 0$$
$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - \mathtt{trace}(M)\lambda + \mathtt{det}(M)$$