1A. X^TX is not invertible.

non-zero v s.t. $X^TXv = 0$, or $X^T(Xv) = 0$. Therefore, $Xv = 0$

At least one v_i is zero, so we have linearly dependent columns in X.

B. $Y = X\beta + \varepsilon$.

Then $\hat{\beta} = X^+Y = (X^TX)^{-1}X^TY$ can't be used b/c $(X^TX)^{-1}$ does not exist.

if we have 2 solns $\hat{\beta}_1$, $\hat{\beta}_2$, say

$\hat{\beta}_2 = \hat{\beta}_1 + v$, where v is non-zero

X is rank-deficient → cols are dependent → it's possible to find v That this marks far ($Xv = 0$) → non-unique soln.

a. $P(X_1=0)=P(X_2=0)=P(X_1=1)=P(X_2=1)=\frac{1}{2}$.

$X_1 \perp X_2 \Rightarrow \rho_{X_1X_2}=0$

$P(Y=0)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$ $\quad P(Y=1)=2\cdot\left(\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{1}{2}$

$=P(Y=2)$

$$\rho_{X,Y}=\frac{Cov(X,Y)}{Std(X)\cdot Std(Y)}$$

$$\rho_{X_1X_2\cdot Y}=\frac{\rho_{X_1X_2}-\rho_{X_1Y}\rho_{X_2Y}}{\sqrt{1-\rho^2_{X_1Y}}\sqrt{1-\rho^2_{X_2Y}}}$$

$Cov(X_1,Y)=E[X_1\cdot Y]-E[X_1]E[Y]$

$E[X_1X_2+X_1^2]-E[X_1]^2-E[X_1]E[X_2]$

$E[X_1X_2]+E[X_1^2]-E[X_1]^2-E[X_1]E[X_2]$

$=E[X_1^2]-E[X_1]^2$

$\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

by symmetry $Cov(X_2,Y)=\frac{1}{4}$.

Thus, since Std. is non zero, it must be that ρ_{X_1Y} and ρ_{X_2Y} are non-zero, thus $\rho_{X_1X_2\cdot Y}$ is non-zero.

b.

$$\rho_{XY\cdot Z}=\frac{\rho_{XY}-\rho_{XZ}\rho_{ZY}}{\sqrt{1-\rho^2_{XZ}}\cdot\sqrt{1-\rho^2_{ZY}}}$$

$\rho_{XZ}=0$

$$\rho_{XY}=\frac{Cov(X,Y)}{Std_X\ Std_Y}$$

$Cov(X,Y)=E[XY]-E[X]E[Y]$

$=E[X^3+XZ]-E[X]E[X^2+Z]$

$=E[X^3]+E[XZ]-E[X]E[X^2]-E[Z]$

$E[X]E[Z]$

$\frac{1}{2}+\frac{1}{4}-\frac{1}{4}-\frac{1}{2}=0$ ($\frac{1}{2}$ and $\frac{1}{2}$ above)

$\rho_{XY}=0$

$$\rho_{XY\cdot Z}=\frac{0-0\cdot\rho_{ZY}}{\sqrt{1-\rho^2_{XZ}}\cdot\sqrt{1-\rho^2_{ZY}}}$$

$=0$ ✓

Problem3 Link

Problem4 Link

Problem5 Link