Set 6

1.a. $Var[f(x)] \approx (f'(\mu))^2 Var[x] \quad \mu = E[x]$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$x = \mu = E[x]$

$$f(X) \approx f(\mu) + f'(\mu)(X-\mu)$$

b. $Var(f(X)) \approx Var[f(\mu) + f'(\mu)(X-\mu)]$

$$\approx \underbrace{Var[f(\mu)]}_{0} + Var[f'(\mu)(X-\mu)] + \underbrace{2cov[f'(\mu), f'(\mu)(X-\mu)]}_{0}$$

$$\approx Var[f'(\mu)(X-\mu)]$$

$$\approx E[(f'(\mu)(x-\mu))^2] - E[f'(\mu)(x-\mu)]^2$$

$$\approx E[(f'(\mu)(x-\mu))^2] - f'(\mu)^2 \underbrace{E[x-\mu]}_{0}{}^2$$

$$\approx E[(f'(\mu)(x-\mu))^2]$$

$$\approx E[f'(\mu)^2(x-\mu)^2]$$

$$\approx E[f'(\mu)^2(x-E[x])^2]$$

$$\approx f'(\mu)^2 E[(x-E[x])^2]$$

$$\approx f'(\mu)^2 Var(X) \checkmark$$

2 a. $K|Q = q \sim \text{Poisson}(q)$
$Q \sim \text{Gamma}(k, \theta)$

$E[Q] = k\theta$
$\text{Var}[Q] = k\theta^2$
Show $E[K] = k\theta$, $\text{Var}[K] = k\theta + k\theta^2$

$E[K] = E[E[K|Q]]$
$= E[Q]$ ← Gamma dist.
$= k\theta$

$\text{Var}[K] = E[\underbrace{\text{Var}[K|Q]}_{q}] + \text{Var}[\underbrace{E[K|Q]}_{Q}]$
$E[Q] + \text{Var}[Q]$
$= k\theta + k\theta^2$

B. $K'|Q, s \sim \text{Poisson}(sQ)$
$Q \sim \text{Gamma}(\frac{1}{\phi}, \mu\phi)$

$E[K'] = E[E[K'|Q,s]]$
$= E[sq]$
$= sE[Q]$
$= s\mu$
$= \mu'$

$\text{Var}[K'] = E[\text{Var}[K'|Q,s]] + \text{Var}[E[K|Q]]$
$= E[sQ] + \text{Var}[sQ]$
$= s\mu + s^2\text{Var}(Q)$
$= s\mu + s^2(\phi\mu^2)$
$= \mu' + \mu'^2\phi$

C. $Y = \frac{K'}{s}$

$E[Y] = E[\frac{K'}{s}] = \frac{E[K']}{s} = \frac{\mu'}{s} = \mu$

$\text{Var}[Y] = \text{Var}[\frac{K'}{s}]$
$= \frac{1}{s^2}\text{Var}[K']$
$= \frac{1}{s^2}(s\mu + s^2\mu^2\phi)$

D. the variance of Y is dependent on the size factor still, so if size factor varies, then Y is still not stabilized with the delta method.

3.

https://colab.research.google.com/drive/1yb9vWt69AJKundEwY-82mhCvT8jmvS8F?usp=sharing