Set 2

3 a i) $\pi(n)\ \frac{1}{n^3}$

$\omega(\pi(n))$

$p > \frac{1}{p}$

$\boxed{E[T]} = \frac{n(n-1)(n-2)}{6} \cdot p^3 \to \infty$

$\Rightarrow 0$

$p < \frac{1}{n}$

$O(\pi(n))$

take $\pi(n) = \frac{1}{n}$. for $\omega(\pi(n))$, we take $p > \frac{1}{n}$ in the limit of $n \to \infty$. Thus, since we have $E[T] = \frac{n(n-1)(n-2)}{6} p^3$,

we know in the limit of $n \to \infty$, $p^3 > n^3$, Which means we have an unbounded $E[T] \Rightarrow E[T] \to \infty$

for $O(\pi(n))$, take $p < \frac{1}{n}$ in the limit of $n \to \infty$. Thus, since we have $E[T] = \frac{n(n-1)(n-2)}{6} p^3$, in the limit of $n \to \infty$, our function is dominated by p^3, which by O notation means $E[T] \to 0$ in the limit of $n \to \infty$

ii) Enumerate all $\triangle$

$$\mathrm{Var}[T] = E[T^2] - E[T]^2$$

$$= \underbrace{\binom{n}{3}p^3 + \binom{6}{3}\binom{n}{6}p^6 + \binom{4}{3}\binom{3}{2}\binom{n}{4}p^5 + \binom{n}{5}\binom{5}{3}\binom{3}{1}p^6}_{E[T^2]} - \underbrace{\binom{n}{3}^2 p^6}_{E[T]^2}$$

$$\pi(n) = \frac{1}{n}$$

So if $p(n) = \pi(n)\log(n)$, we can cancel $\binom{6}{3}\binom{n}{6}p^6$ and $-\binom{n}{3}^2 p^6$ in the limit of n. the p^5 and p^6 terms left are then dominated by $\frac{1}{n}$, so they $\to 0$ as $n \to \infty$. Then, all that is left is the term w/ p^3, which in the limit of n, the n terms cancel leaving $\log^3(n)$. Thus, it must be that $\mathrm{var}[T] \in \Theta(\log^3(n))$

$$\Pr(T=0) \le P(|T-\mu| \ge k) \qquad k = \mu$$

$$\le \frac{\sigma^2}{k^2}$$

$$\le \frac{\log^3(n)}{\left(\binom{n}{3}p^3\right)^2}$$

$$\le \frac{\log^3(n)}{\left(\frac{n^3}{n^3}\log^3\right)^2} \text{ in the limit of } n \to \infty$$

$$\le \frac{\log^3(n)}{(\log^6(n))}$$

$$\le 0 \text{ in limit } n \to \infty$$

So $\Pr(T=0) \in o(1)$

$$\Pr(T>0) = 1 - \Pr(T=0)$$

b/c $\Pr(T=0) \in o(1)$, as $n \to \infty$, $\Pr(T=0) \to 0$, so

$$\lim_{n\to\infty} \Pr(T>0) = 1 - \lim_{n\to\infty} \Pr(T=0) = 1 - 0 = 1 \quad \checkmark$$

3b. $\Pr(d=2) = 1$

$$\lim_{n\to\infty} \Pr(d=1) + \Pr(d=2) + \Pr(d>2) = 1$$

$p^{\binom{n}{2}} \to 0$

Probability of fully connected G.

$\binom{n}{2}(1-p)(1-p^2)^{n-2} \to 0$

pick 2 vertices | prob they don't share edge | prob both the other $n-2$ vertices aren't connected to both chosen vertices

Thus, the 2 other possibilities' $\Pr \to 0$ as $n \to \infty$, so $\Pr(d=2) = 1$ must be true.