Using the normal distribution for inference

Overview

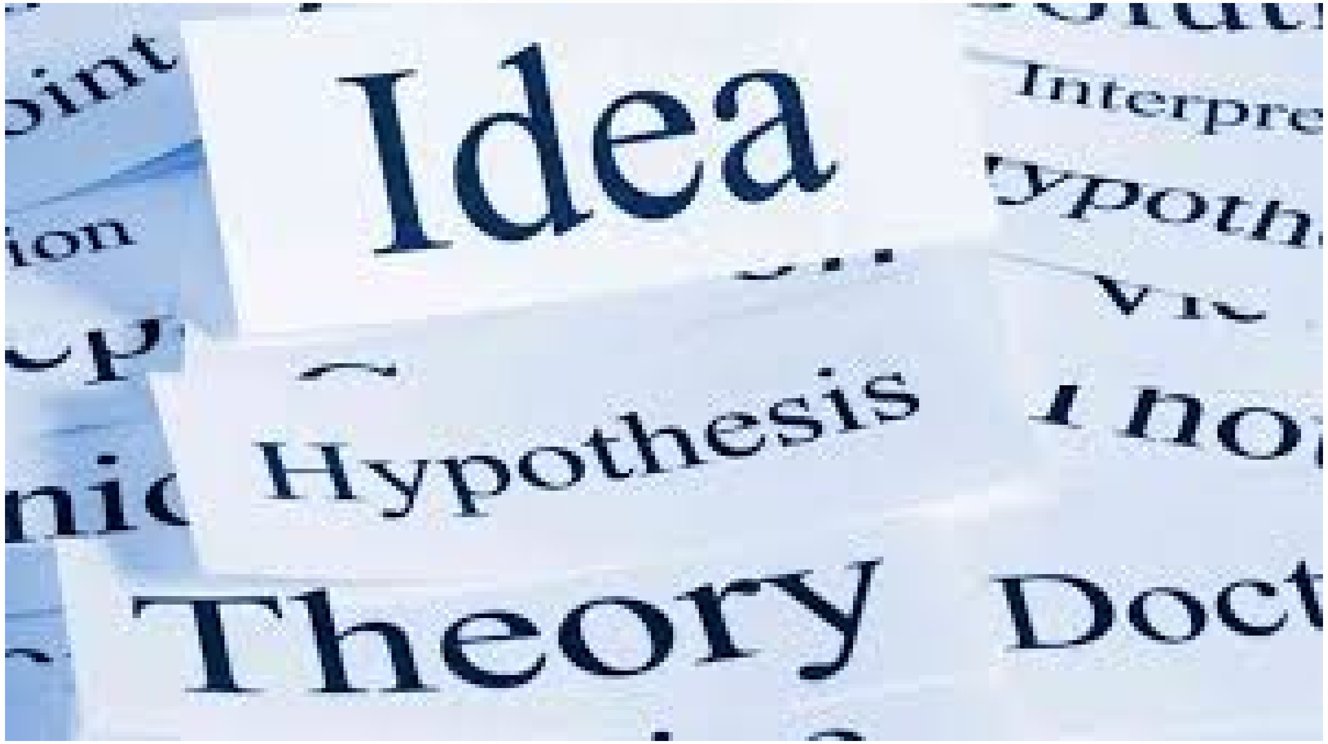
Quick review of theories/concepts in hypothesis testing

Review and continuation of the normal distribution

Using the normal distribution for inference

- Hypothesis tests
- Confidence intervals

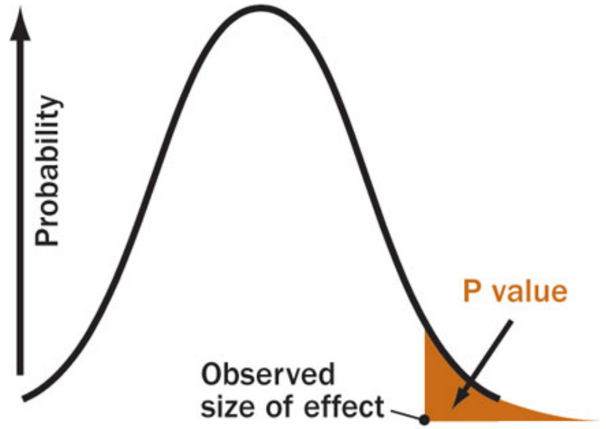
Quick review of theories of hypothesis tests



Two theories of hypothesis testing

1. **Significance testing** of Ronald Fisher
   - p-value as strength of evidence against the null hypothesis

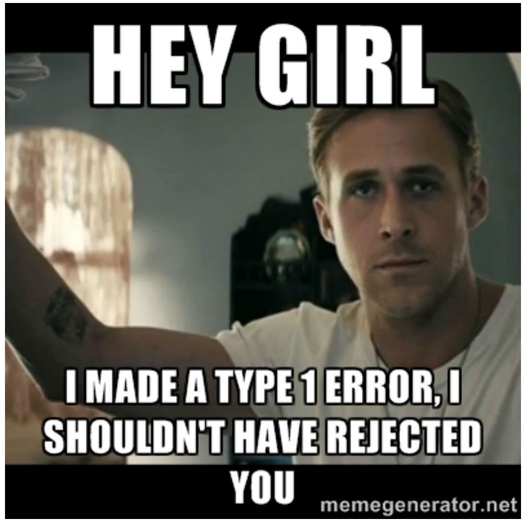
2. **Hypothesis testing** of Jezy Neyman and Egon Pearson
   - Make a formal decision of whether to reject H_0 (if p-value < predefined α value)

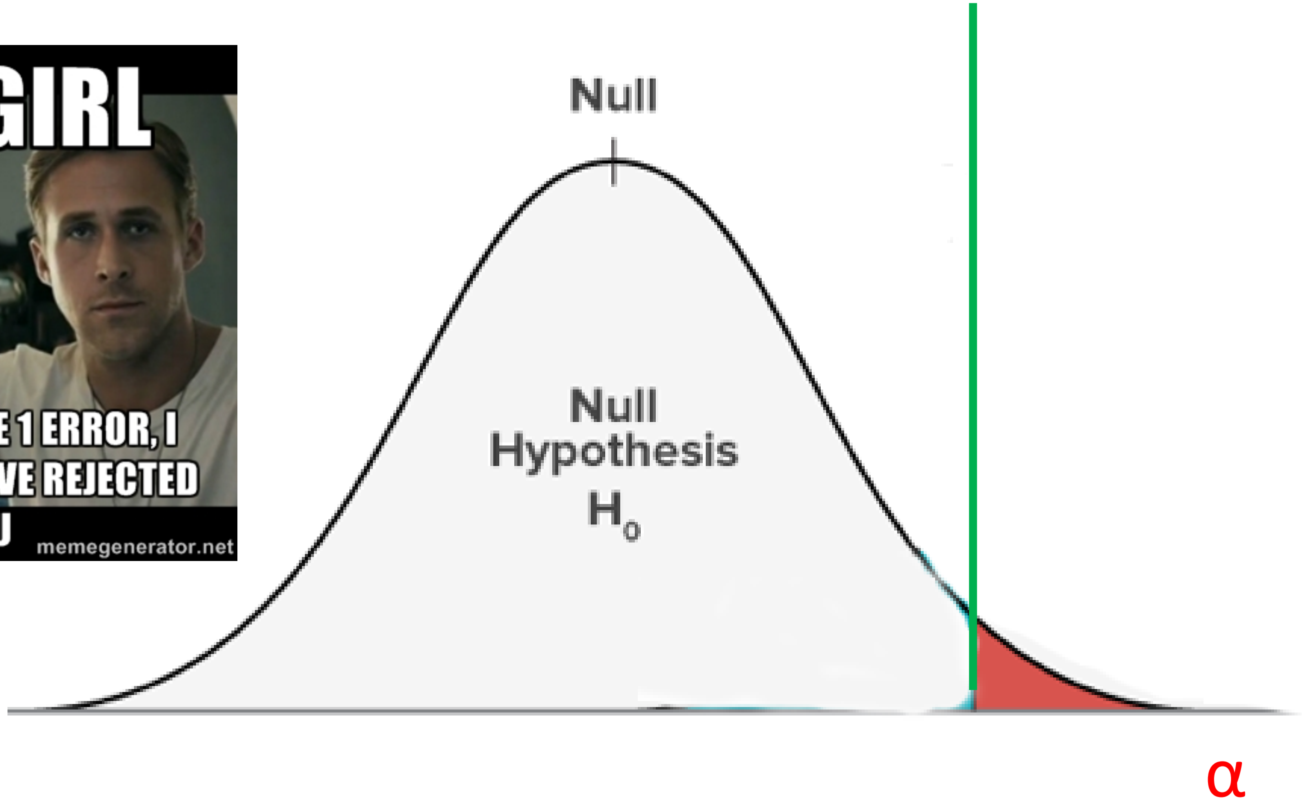


Significance testing

Hypothesis testing

Neyman-Pearson Frequentist logic

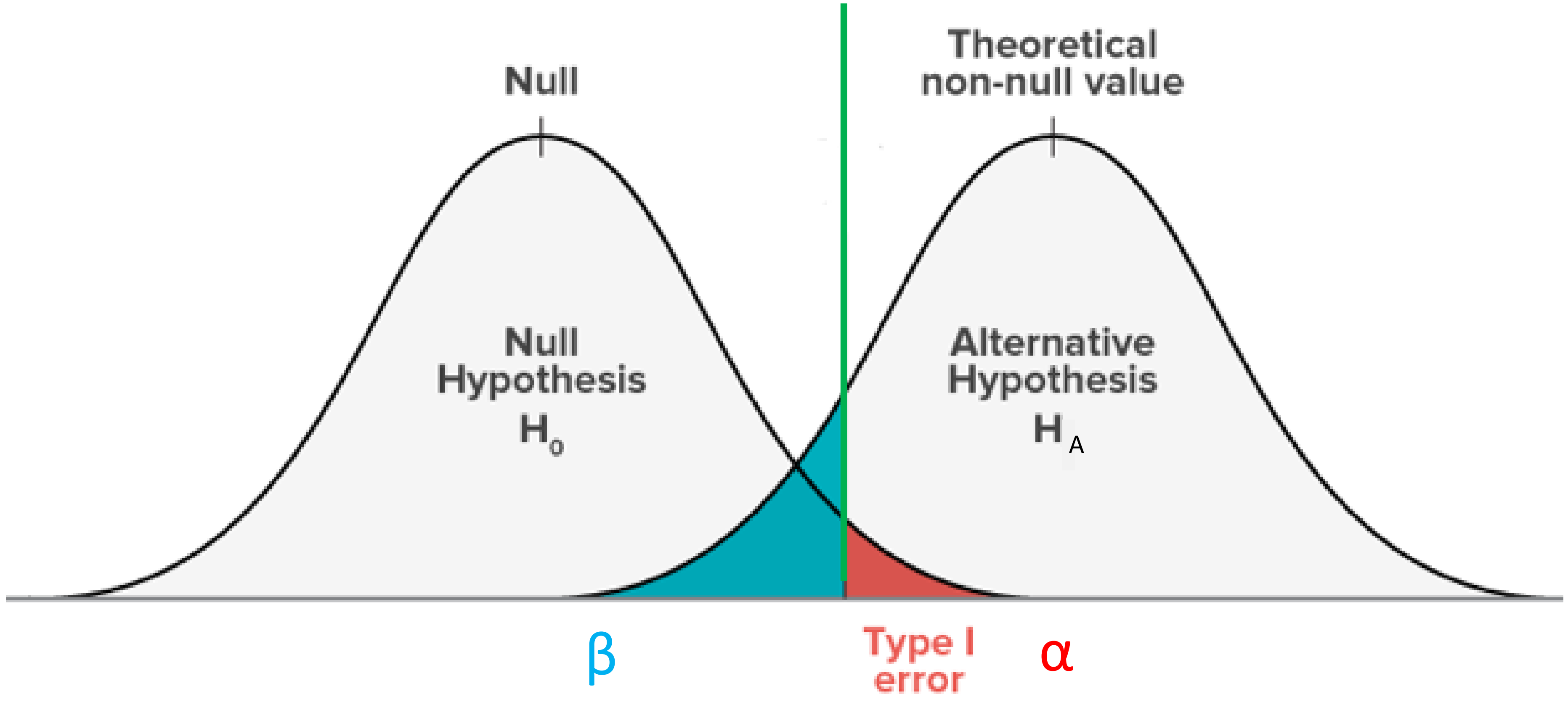




If Neyman-Pearson null hypothesis testing paradigm was followed perfectly, then only ~5% of all published research findings should be wrong (for $\alpha = 0.05$)

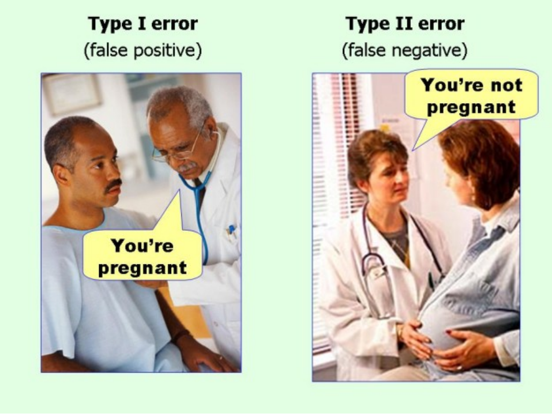
Type I error: incorrectly rejecting the null hypothesis when it is true

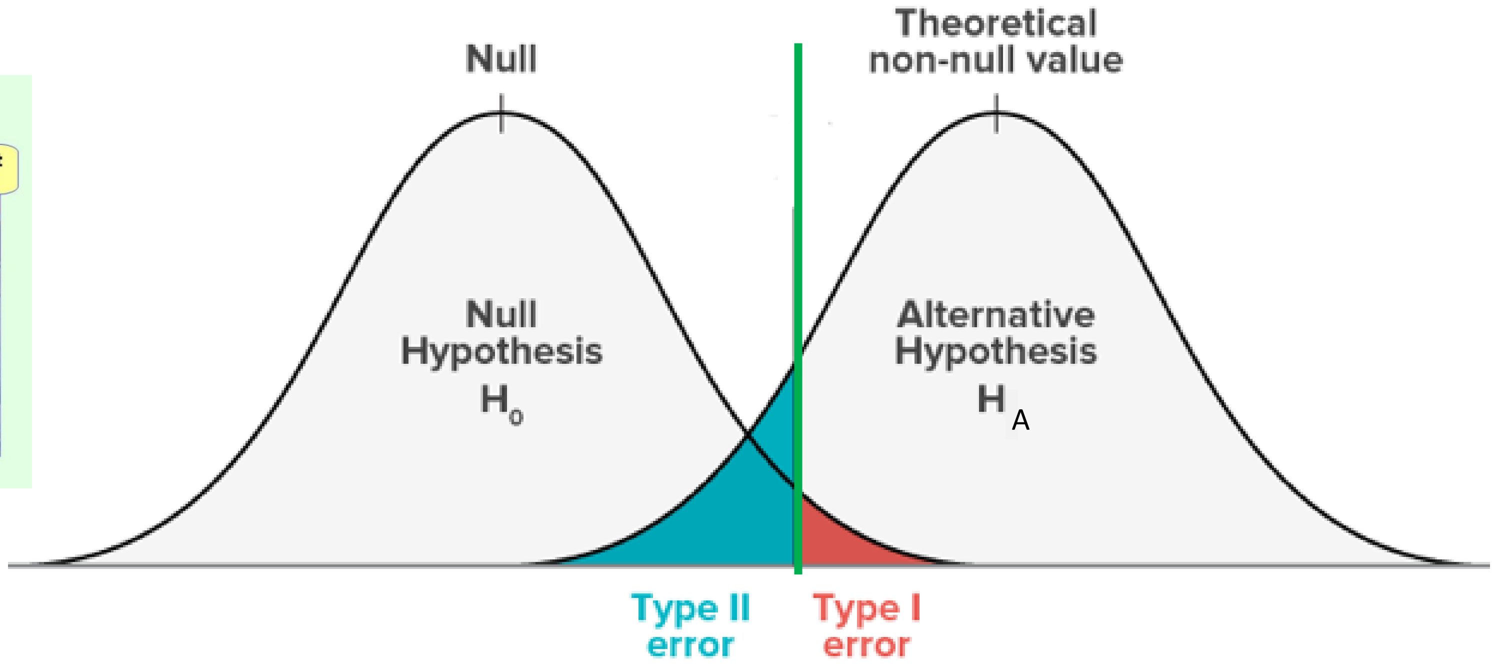
Neyman-Pearson Frequentist logic



Type 2 error: incorrectly rejecting failing to reject H_0 when it is false

Type I and Type II Errors





	Reject H_0	Do not reject H_0
H_0 is true	Type I error (α) (false positive)	No error

Problems with the NP hypothesis tests

Problem 1: we are interested in the results of a specific experiment, not whether we are right most of the time

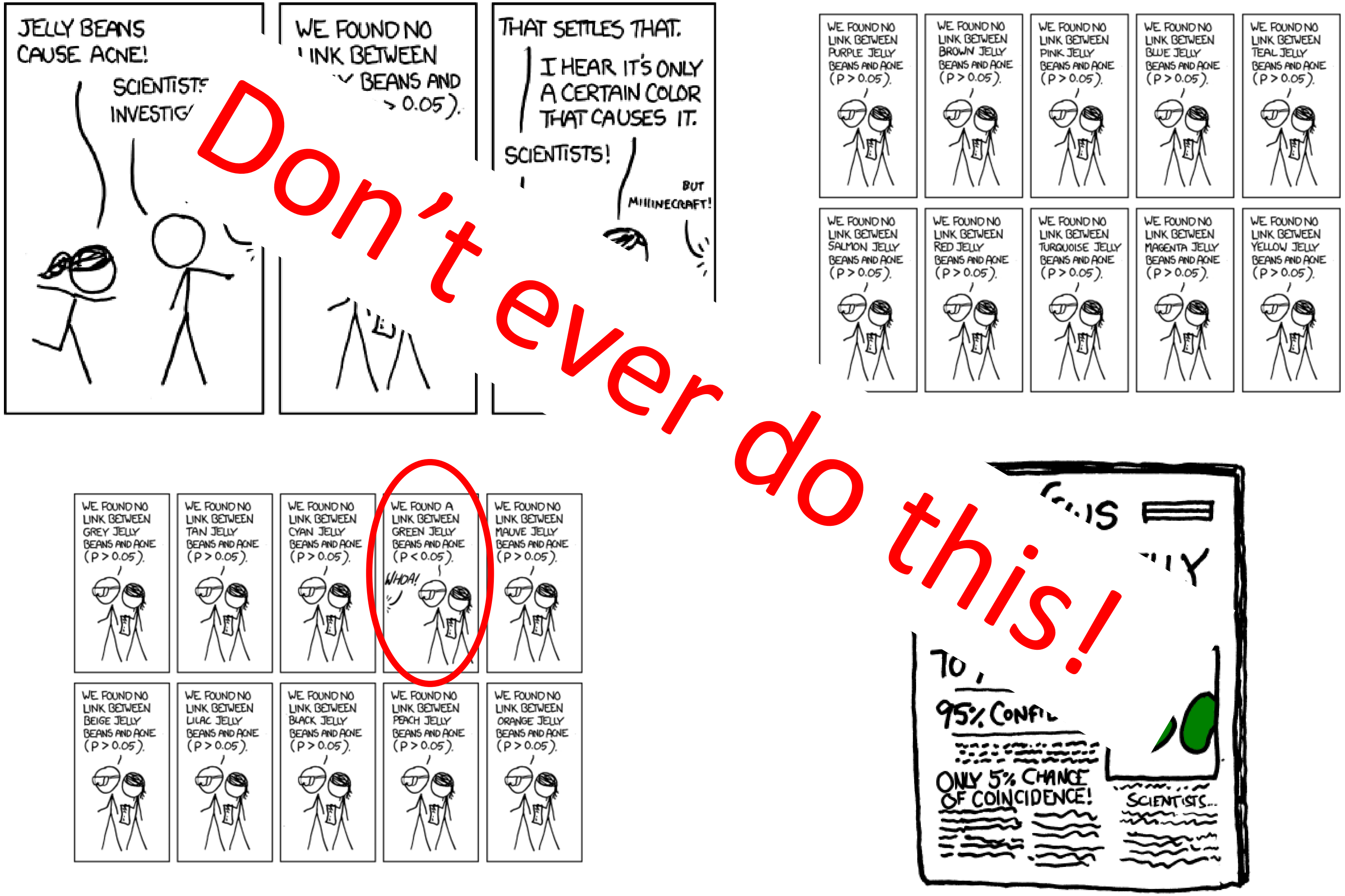
- E.g., 95% of these statements are true:
  - Calcium is good for your heart, Paul is psychic, Buzz and Doris can communicate, …

Problem 2: Arbitrary thresholds for alpha levels

- P-value = 0.051, we don't reject H_0?

Problem 3: running many tests can give rise to a high number of type 1 errors

Multiple hypothesis tests



Don't ever do this!

Replication crisis

Essay

Why Most Published Research Findings Are False

John P. A. Ioannidis

The file drawer effect

[American Statistical Association's ' Statement on p-values](#)

Some thoughts…

Better to have hypothesis tests than none at all. Just need to think carefully and use your judgment.

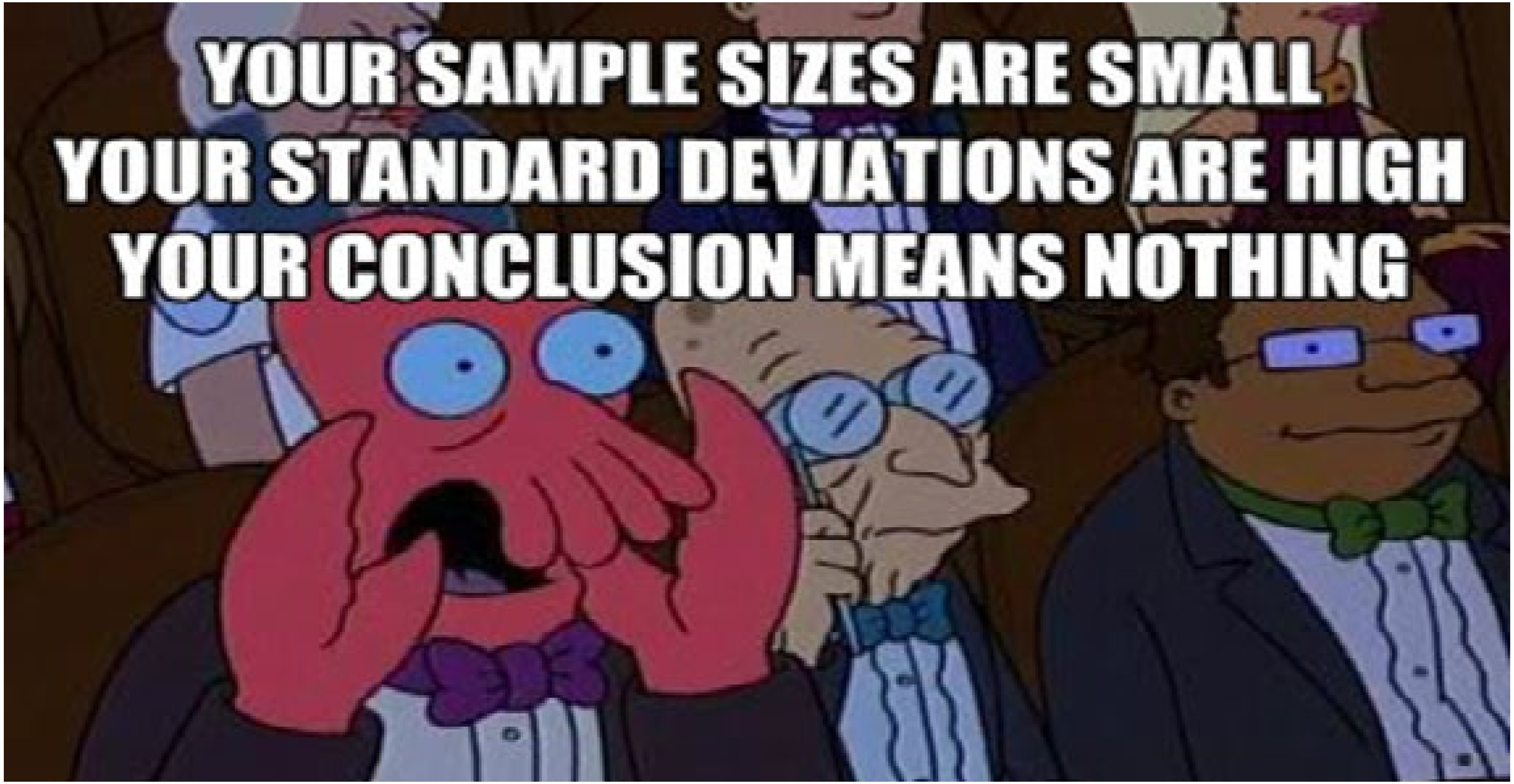
Report effect size in most cases – i.e., confidence intervals

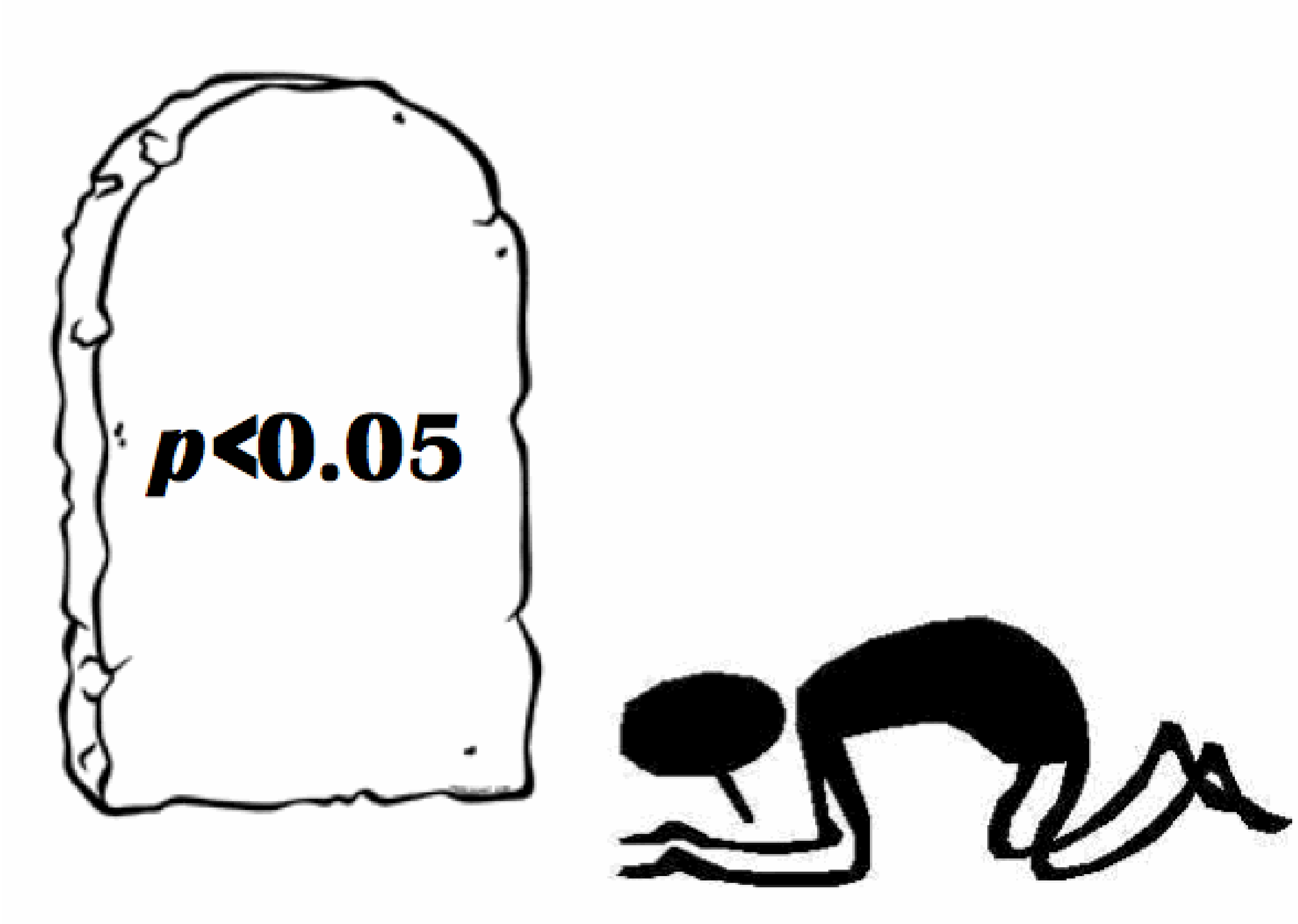
Report the p-values rather than accept/reject H_0

- i.e., report $p = 0.023$ not $p < 0.05$

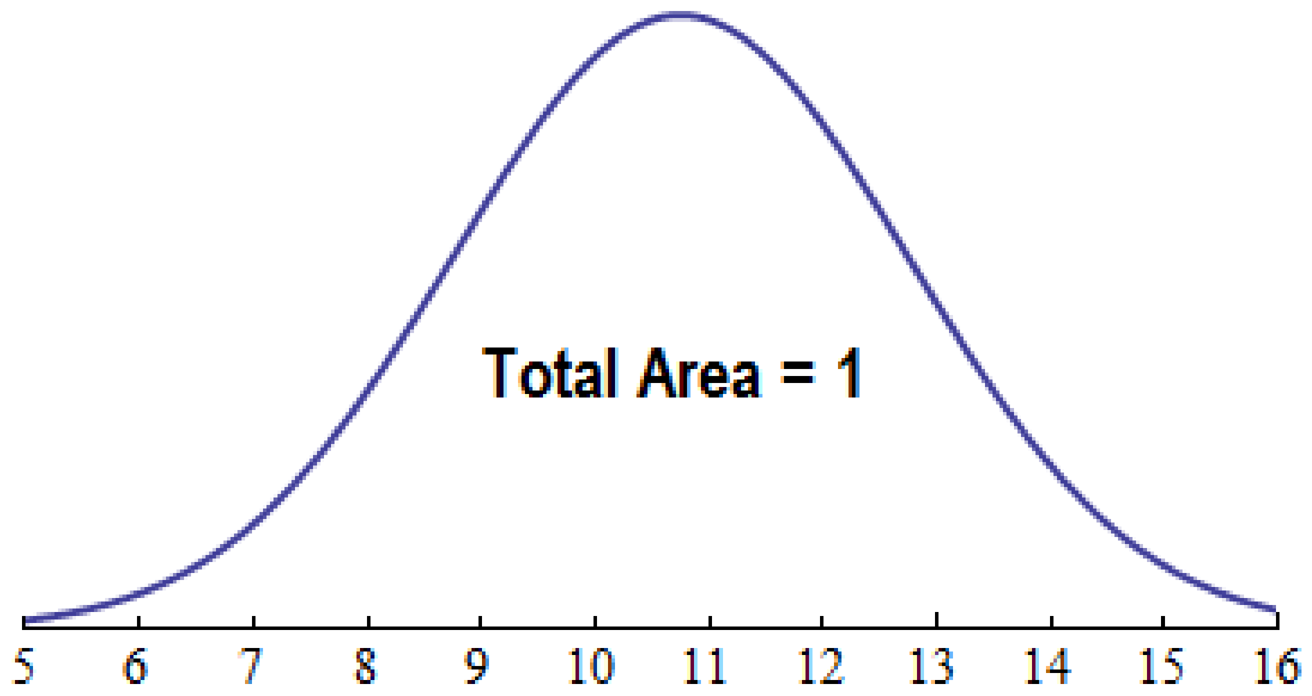
Replicate findings (perhaps in different contexts) to make sure you get the same results

Be a good/honest scientists and try to get at the Truth!


YOUR SAMPLE SIZES ARE SMALL
YOUR STANDARD DEVIATIONS ARE HIGH
YOUR CONCLUSION MEANS NOTHING


p<0.05

Inference using parametric probability distributions



Inference using parametric probability distributions

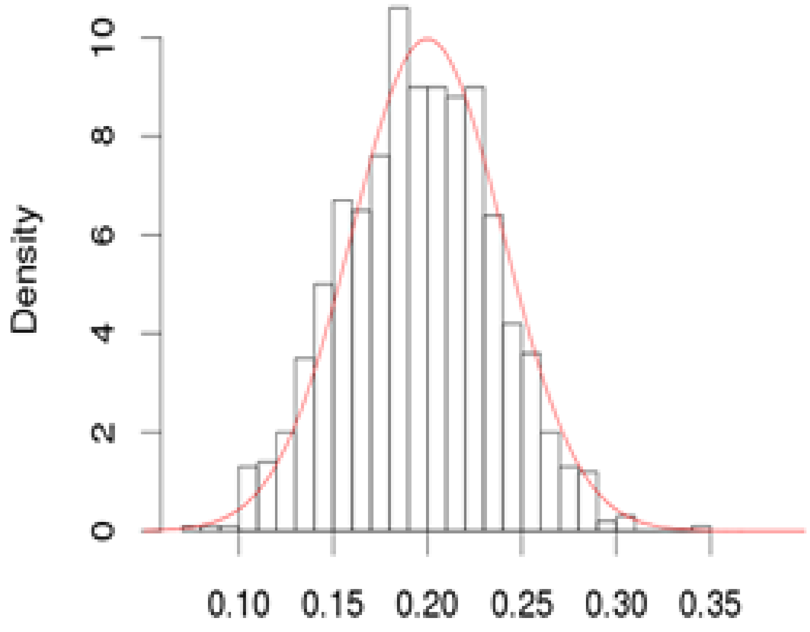
We can use mathematical functions called **probability distributions** to do inference

- e.g. instead of running computer simulations to create null distributions we can just mathematical probability distributions

A **density curve** is a mathematical function f(x) that has two important properties:

1. The total area under the curve f(x) is equal to 1

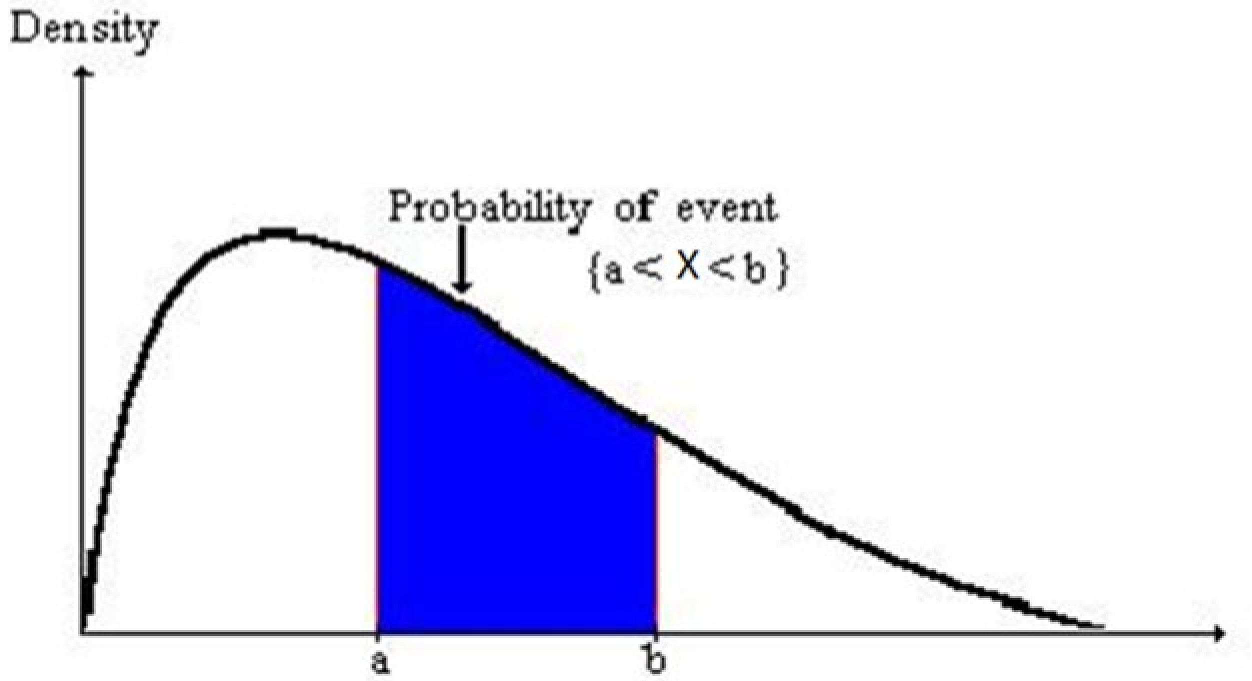
2. The curve is always ≥ 0



Density Curves

The area under the curve in an interval [a, b] models the probability that a random number X will be in the interval

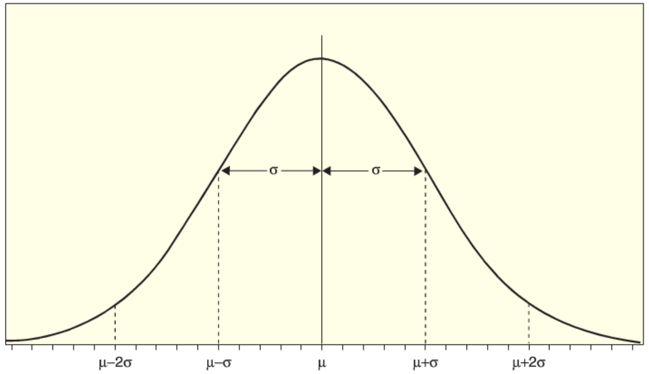
Pr(a < X < b) is the area under the curve from a to b



The Normal Density Curve

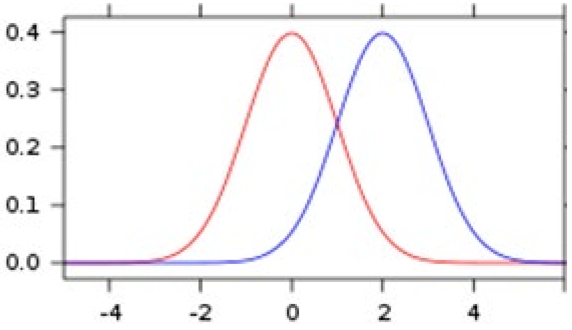
Normal distributions are a family of bell-shaped curves with two parameters

- The mean: μ
- The standard deviation: σ

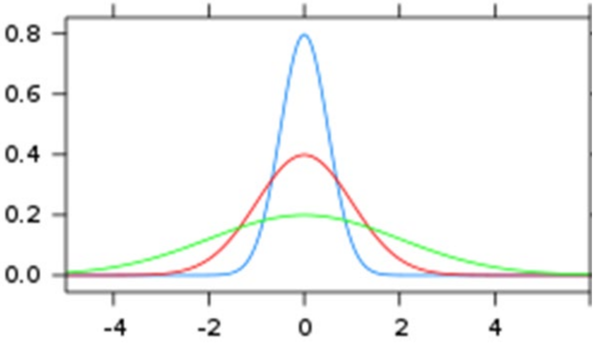


Notation: X ~ N(μ, σ)

Changing μ



Changing σ



Graphing Normal Curves

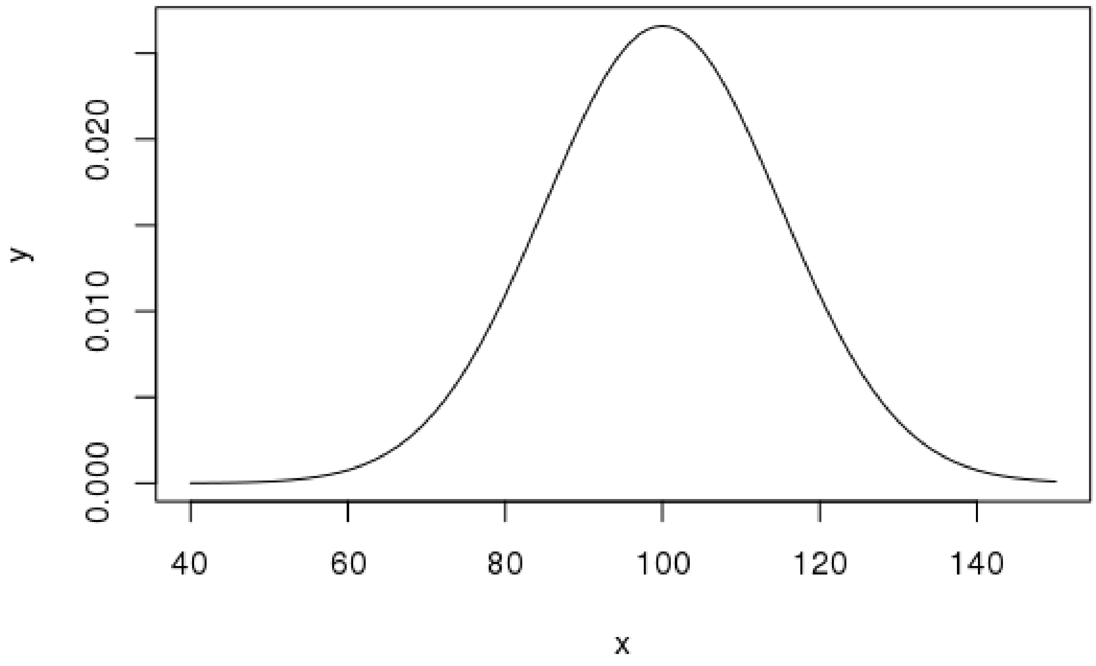
Plotting IQ scores

μ σ

```
x <- 40:150

y <- dnorm(x, 100, 15)

plot(x, y, type = "l")
```



Finding normal probabilities of a normal curve

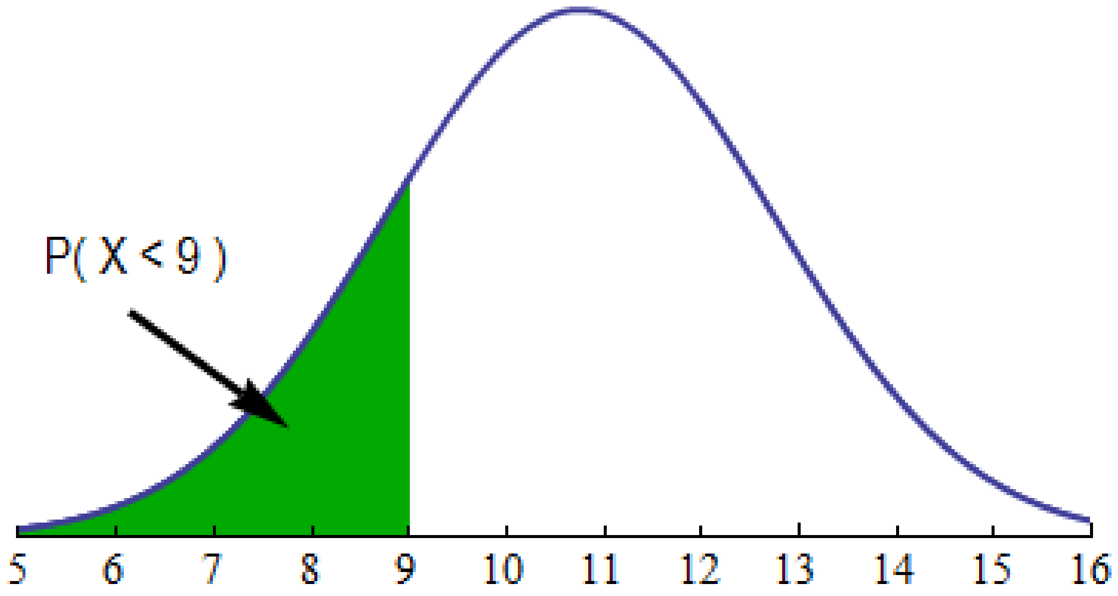
To get the probability (area) from a normal distribution we can use the pnorm function

```
pnorm(x, mu, sigma)
```

Pr(X < 9; 11, 3)

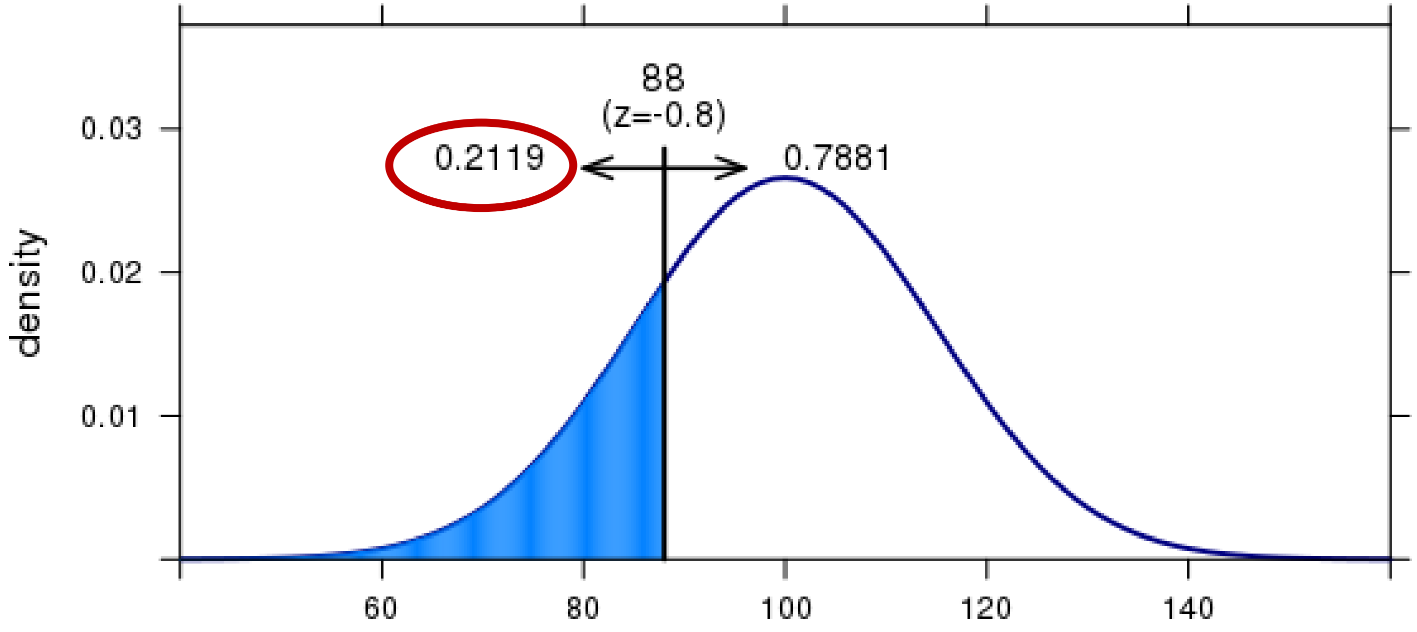
μ σ

```
pnorm(9, 11, 3)
```



Calculate the probability a random person you meet has an IQ less than 88

```
pnorm(88, 100, 15)
```



Normal area Pr(X ≤ x) app

Normal area Pr(a < X < b) app

Probability practice questions

1. What is probability a randomly chosen person will have an IQ greater than 96?

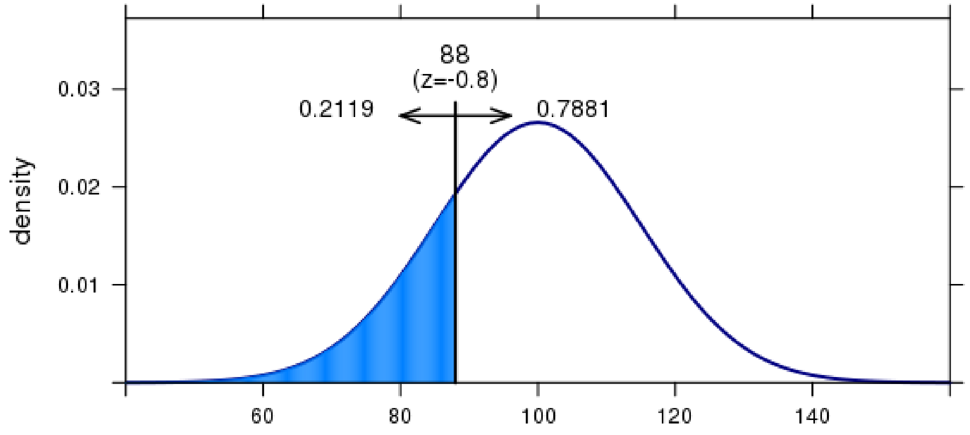
   pnorm(96, 100, 15, lower.tail = FALSE)

- Answer: 0.605

2. What is the probability a randomly chosen person will have an IQ between 88 and 96?

   pnorm(96, 100, 15) - pnorm(88, 100, 15)

- Answer: 0.183



Calculating quantiles

To find quantiles of the normal distribution we can use the quantile function:

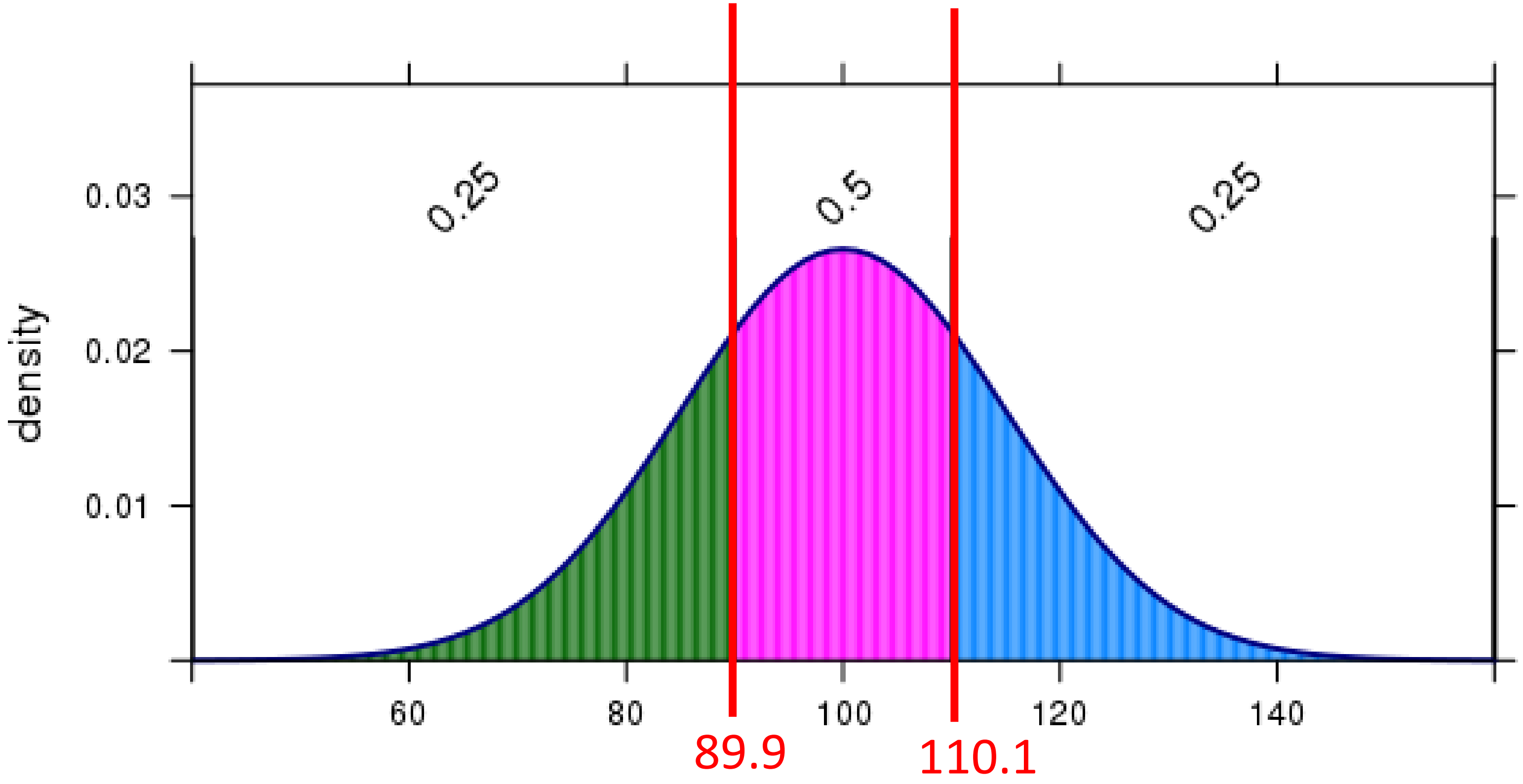
```
qnorm(quantile, mu, sigma)
```

What are the IQ scores (interval) that demark the middle 50% of the IQ range?

What about the middle 95%?

Normal quantile app

Middle 50% of IQ scores




```
qnorm(c(.25, .75), 100, 15)
```

Middle 50%: 89.9 to 110.1

Middle 95%: 70.6 to 129.3

Summary of R functions

Plot the actually density curve

- dnorm(x_vec, mu, sigma)

Get the probability that we would get a random value less than x

- pnorm(x_vec, mu, sigma)

Get the quantile value for a given proportion of the distribution

- qnorm(area, mu, sigma)

Note: pnorm and qnorm are inverses of each other

- y = pnorm(x, mu, sigma)
- qnorm(y, mu, sigma) # the output value here is x

The Standard Normal distribution and the Central Limit Theorem

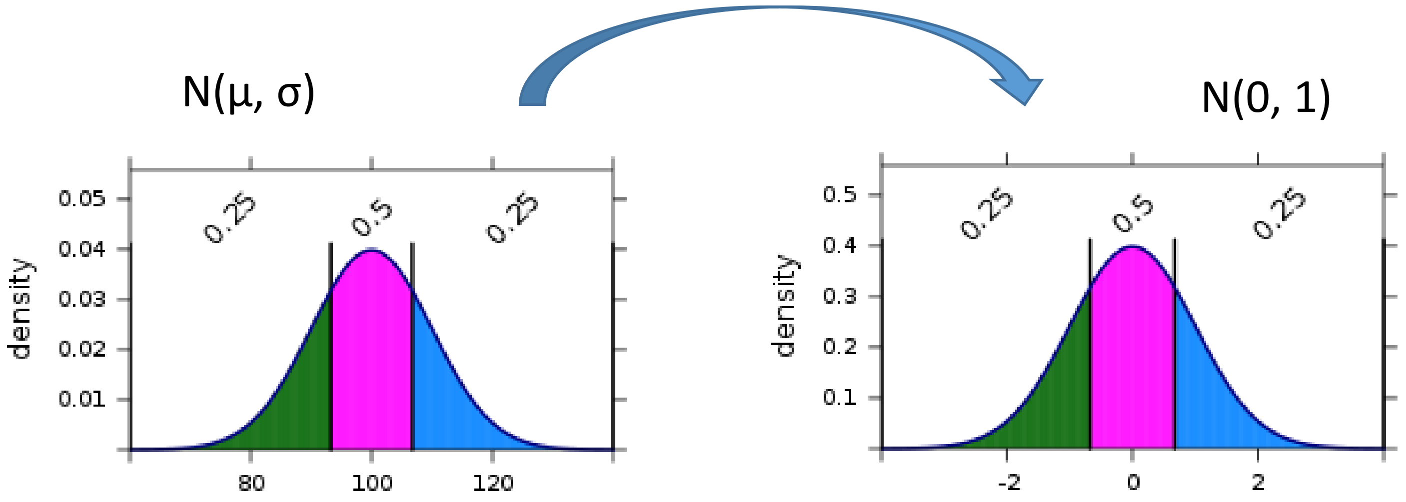
Standard Normal N(0, 1)

Since all normal distributions have the same shape, it is convenient to convert them to a standard scale with:

$$\mu = 0, \quad \sigma = 1$$

This is called the **standard normal** distribution:

$$Z \sim N(0, 1)$$



Converting to the standard normal distribution

We can use a z-score transformation to any normally distributed random variable $X \sim N(\mu, \sigma)$ to the standard normal distribution $Z \sim N(0, 1)$:

$$Z = (X - \mu)/ \sigma$$

To convert from $Z \sim N(0, 1)$ to any $X \sim N(\mu, \sigma)$, we reverse the standardization with:

$$X = \mu + Z \cdot \sigma$$

Converting to the standard normal distribution

1. What is the Z-score of someone who has an IQ score of 112?

$$Z = (X - \mu)/ \sigma$$

2. What if someone has an Z-score of 2.2, what is their IQ score?

$$X = \mu + Z \cdot \sigma$$

Answer 1: $Z = (112 - 100)/15 = .8$

Answer 2: $IQ = 100 + 2.2 * 15 = 133$

Central limit theorem

For random samples with a sufficiently large sample size (n), the distribution of sample statistics for a mean ($\bar{x}$) or a proportion ($\hat{p}$) is:

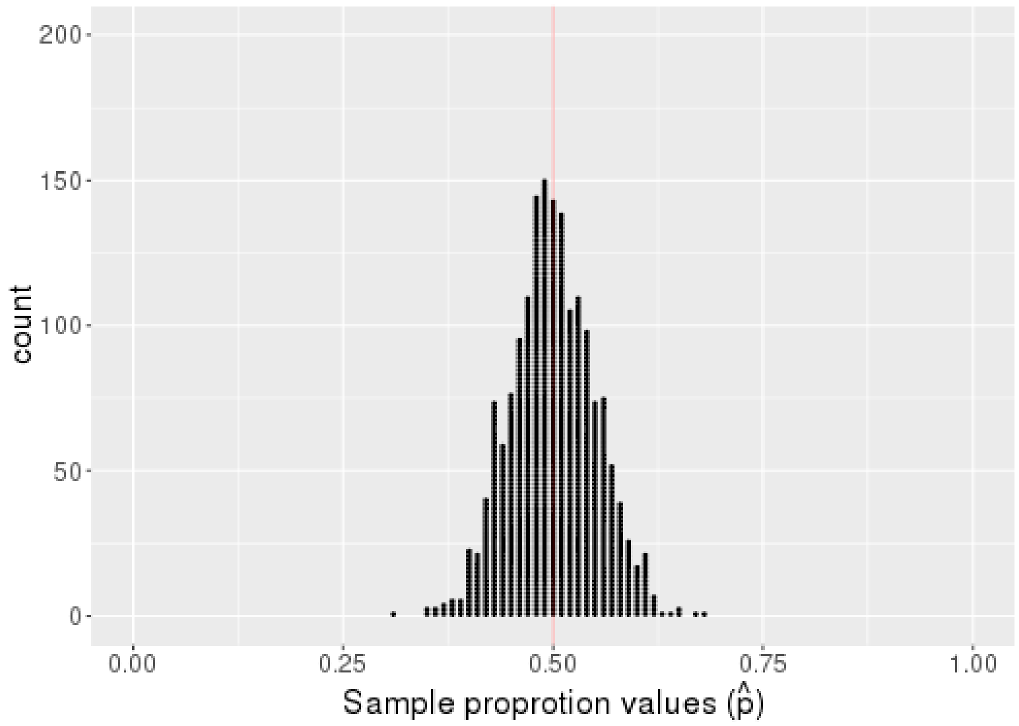
- normally distributed
- centered at the value of the population parameter

Stated again: the sampling distribution for means or proportions will be a normal distribution

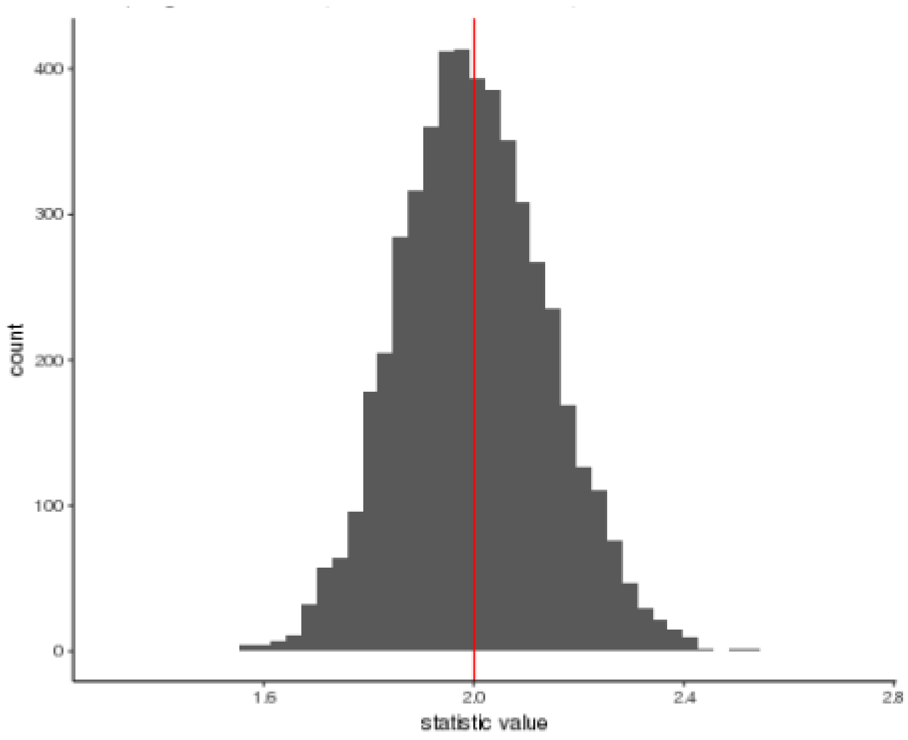
- so we don’t need to do resampling to get a bootstrap or null distribution!

Central limit theorem

proportion ($\hat{p}$)



mean ($\bar{x}$)



[Proportion sampling distribution app]

[Sampling/Bootstrap distribution app]

Summary of standard normal and CLT

For large n, the sampling distributions of $\bar{x}$ and $\hat{p}$ are normal

We can convert any normal distribution $N(\mu, \sigma)$, into a standard normal distribution $N(0, 1)$

We are now (almost) ready to run hypothesis tests and compute confidence intervals for $\bar{x}$ and $\hat{p}$ using normal distributions

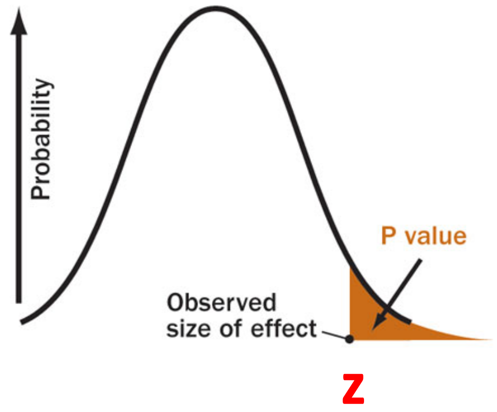
Hypothesis tests using a normal distribution

Hypothesis tests based on a Normal Distribution

When the null distribution is normal, it is often convenient to use a standard normal test statistic using:

$$z = \frac{\textit{Sample Statistic} - \textit{Null Parameter}}{SE}$$

The p-value for the test is the probability a standard normal value is beyond this standardized test statistic



$$\Pr(Z \geq z_{obs}\ ;\ \ \mu = 0,\ \ \sigma = 1)$$

pnorm(z, 0, 1, lower.tail = FALSE)

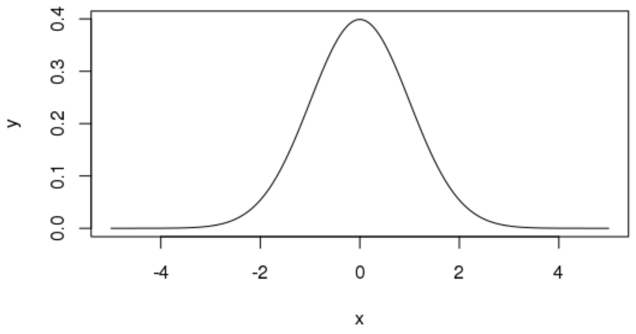
Hypothesis tests based on a Normal Distribution

To repeat what was on the last slide: we can transform our obs_stat to a z-statistic that comes from a standard normal distribution N(0, 1) using:

$$z = \frac{stat_{obs} \text{ - } param_0}{SE}$$

The p-value is then the probability of obtaining a value from a standard normal distribution beyond this z statistic

> pnorm(z, 0, 1)	if	H_A: $\mu < param_0$
> 1 - pnorm(z, 0, 1)	if	H_A: $\mu > param_0$
> 2 * (1 - pnorm(abs(z), 0, 1))	if	H_A: $\mu \neq param_0$



Do greater than 40% of Americans go without using cash in a typical week?

A survey of 1,000 Americans reported that 43% said they went an entire week without using cash, with a SE = 0.016

Assuming the distribution of the statistic is normal, calculate whether the proportion of all Americans going a week without using cash is greater than 40%

1. Start by stating H_0 and H_A

H_0: $\pi = .4$

H_A: $\pi > .4$

Do greater than 40% of Americans go without using cash in a typical week?

A survey of 1,000 Americans reported that 43% said they went an entire week without using cash, with a SE = 0.016

Assuming the distribution of the statistic is normal, calculate whether the proportion of all Americans going a week without using cash is greater than 40%

2. Can you compute the z statistic?

$$z = \frac{Sample\ \ Statistic\ -\ Null\ \ Parameter}{SE}$$

Do greater than 40% of Americans go without using cash in a typical week?

A survey of 1,000 Americans reported that 43% said they went an entire week without using cash, with a SE = 0.016

Assuming the distribution of the statistic is normal, calculate whether the proportion of all Americans going a week without using cash is greater than 40%

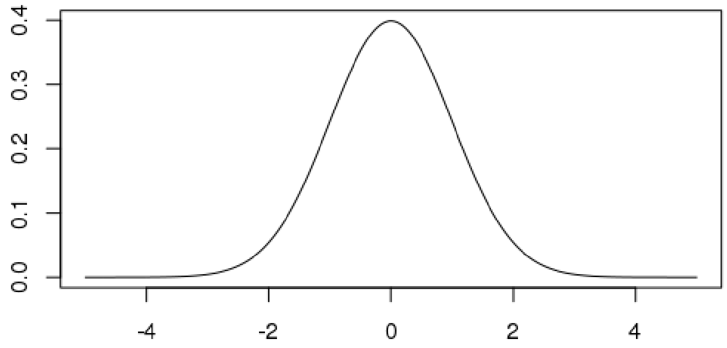
2. Can you compute the z statistic?

$$z = (.43 - .4)/.016 = 1.875$$

Do greater than 40% of Americans go without using cash in a typical week?

Steps: 3-4. What is the probability one would get a z-statistic as larger or larger than 1.875 from a standard normal distribution?

Standard normal null distribution




```
> pnorm(1.875, 0, 1, lower.tail = FALSE)
> 1 – pnorm(1.875, 0, 1)
```

p-value = .0304

Step 5?

Normal area app Pr(X ≤ x)

Confidence intervals using a normal distribution

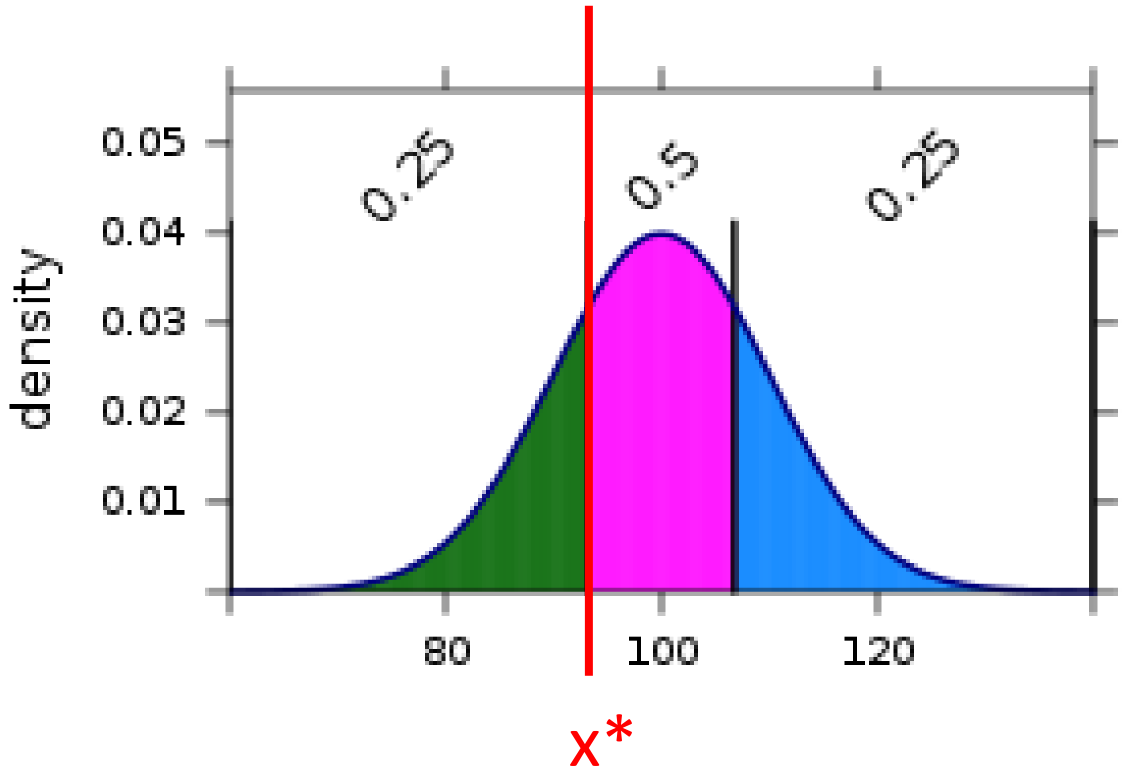
Finding quantile values

We can find the quantile value from a normal distribution with:

qnorm(q, mu, sigma)

The 'q' in qnorm stands for quantile

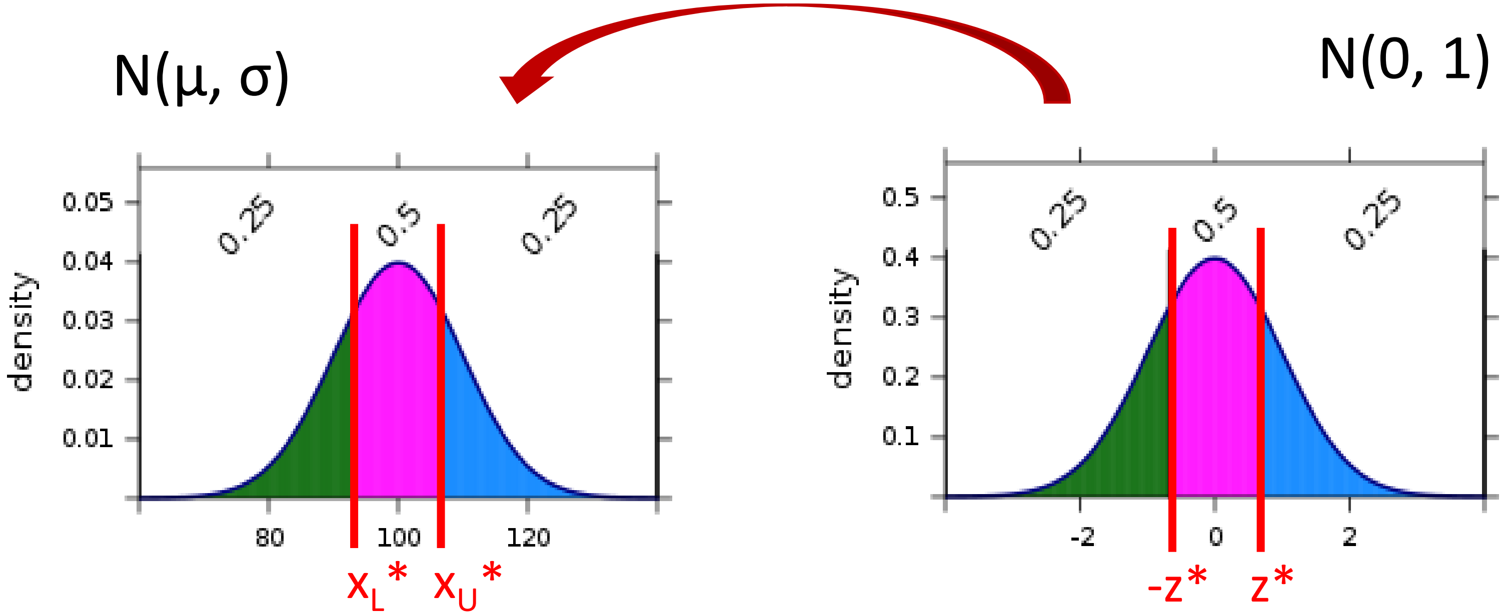
What is the max and min that q can be?



Standard Normal N(0, 1)

It is often convenient to find quantiles on the standard normal distribution $Z \sim N(0, 1)$ and then to transform them to an arbitrary normal distribution $X \sim N(\mu, \sigma)$, using :

$$X = \mu + Z \cdot \sigma$$



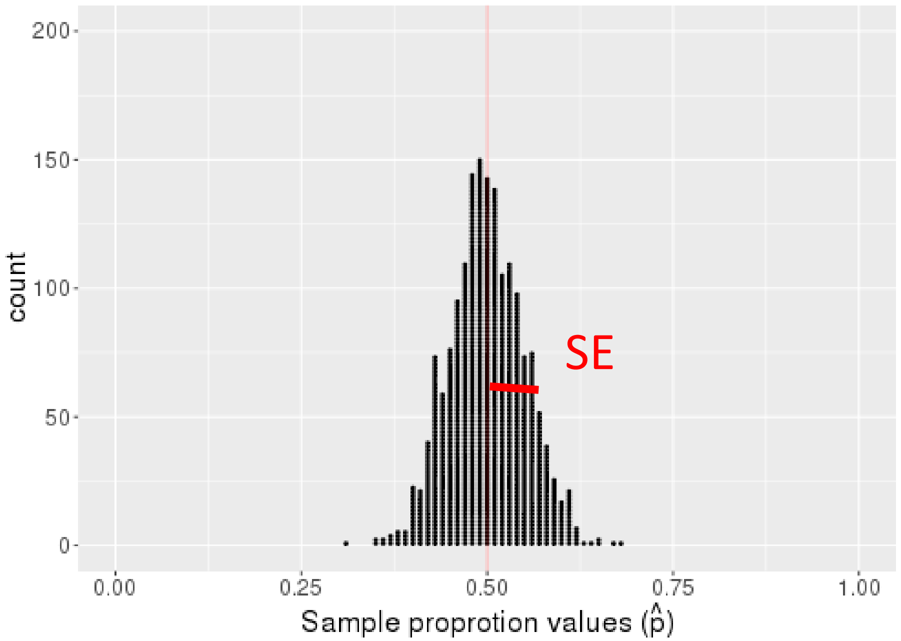
Central limit theorem

Questions:

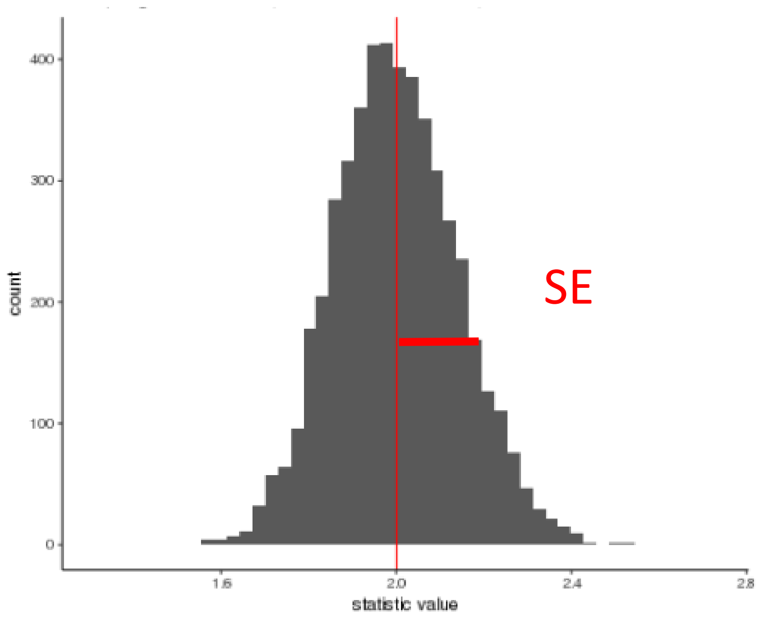
1. What is the standard deviation of these sampling distributions called?
2. Suppose we have a $\hat{p}$ or $\bar{x}$ and know the SE, how can we create a 95% CI?

For a proportion π: $CI_{95} = \hat{p} \pm 2 \cdot SE$ For a mean μ: $CI_{95} = \bar{x} \pm 2 \cdot SE$

proportion ($\hat{p}$)



mean ($\bar{x}$)



Confidence intervals based on a Normal Distribution

If the distribution for a statistic is normal with a standard error SE, we can find a confidence interval for the parameter using:

$$\text{sample statistic} \pm z^* \times SE$$

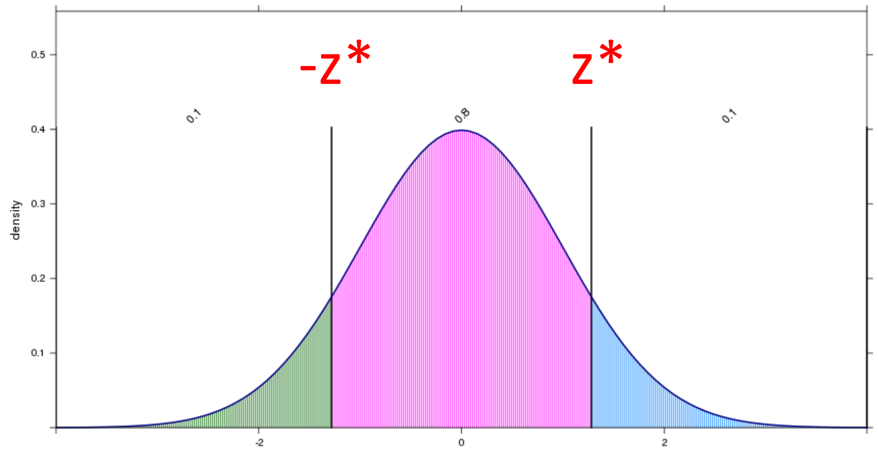
where z^* is chosen so that the area between $-z^*$ and $+z^*$ in the standard normal distribution is the desired confidence level

- i.e., z^* is chosen such that say 95% of the distribution is between $\pm z^*$

Confidence intervals based on a Normal Distribution

Suppose we are interested in 80% confidence intervals for μ

We calculate the $\pm\, z_{80}$ that has 80% of the data on N(0, 1)



Let’s assume we know the SE but don’t know μ. If we have an observed statistic from:

$$x_{obs} \sim N(\mu, SE)$$

We can create an interval that will capture μ 80% of the time using:

$$x_{obs} \pm z_{80} \cdot SE$$

Normal percentiles for common confidence levels

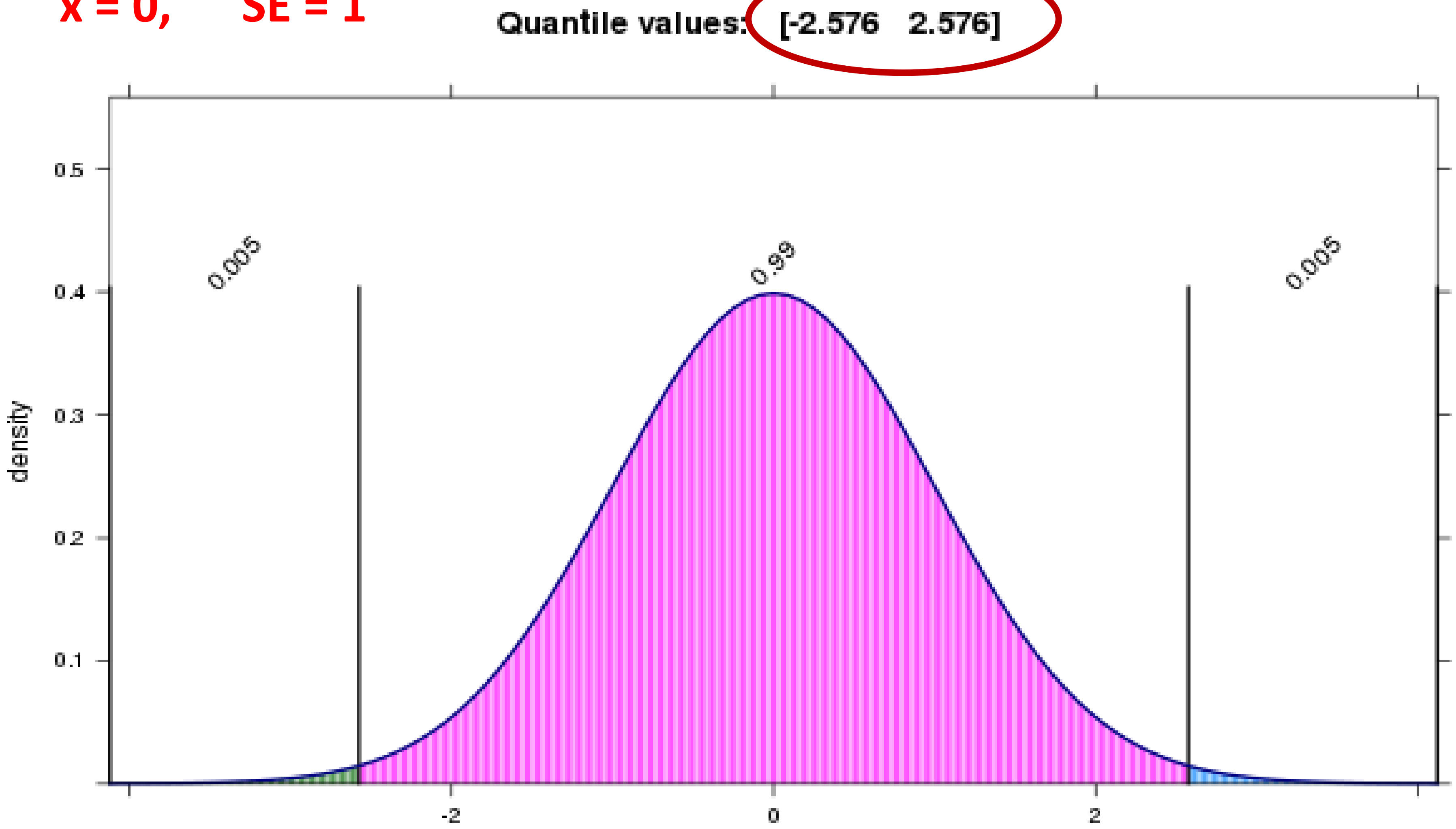
Confidence level	80%	90%	95%	98%	99%
Z*	1.282	1.645	1.960	2.326	2.576

```
z_stars <- qnorm(c(.90, .95, .975, .99, .995), 0, 1)
```

[Normal quantile app]

.99 quantile values

$\bar{x} = 0$, SE = 1



$\bar{x} = 0$

[Normal quantile app]

What is the most preferred seat?

A survey of 1,000 air travelers found that 60% prefer a window seat, with a bootstrap standard error of SE = 0.015

Use the normal distribution to compute a 90%, 95% and 99% CIs for the proportion of people who prefer a window seat

$$\text{sample statistic} \pm z^* \times \text{SE}$$

Confidence level	80%	90%	95%	98%	99%
Z*	1.282	1.645	1.960	2.326	2.576

What is the most preferred seat?

A survey of 1,000 air travelers found that 60% prefer a window seat, with a bootstrap standard error of SE = 0.015.

90% CI = $.6 \pm 1.645 \times .015 = [.575 \quad .625]$

95% CI = $.6 \pm 1.96 \times .015 = [.571 \quad .629]$

99% CI = $.6 \pm 2.576 \times .015 = [.569 \quad .638]$

Sample statistics $\pm z^* \times$ SE

Confidence level	80%	90%	95%	98%	99%
Z*	1.282	1.645	1.960	2.326	2.576