Using the normal distribution for inference

Overview

Quick review of theories/concepts in hypothesis testing

Review and continuation of the normal distribution

Using the normal distribution for inference

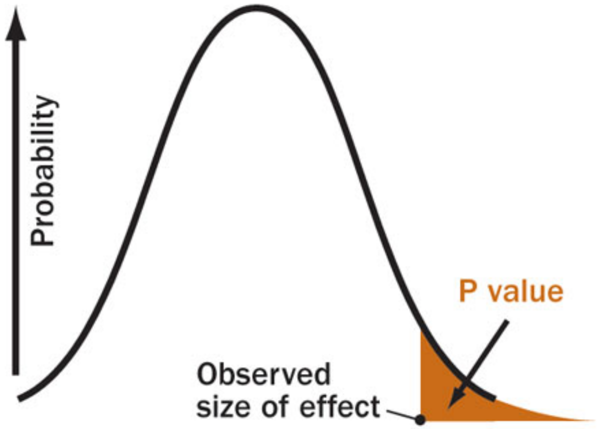
- Hypothesis tests
- Confidence intervals

Quick review of theories of hypothesis tests

Two theories of hypothesis testing

1. **Significance testing** of Ronald Fisher
   - p-value as strength of evidence against the null hypothesis

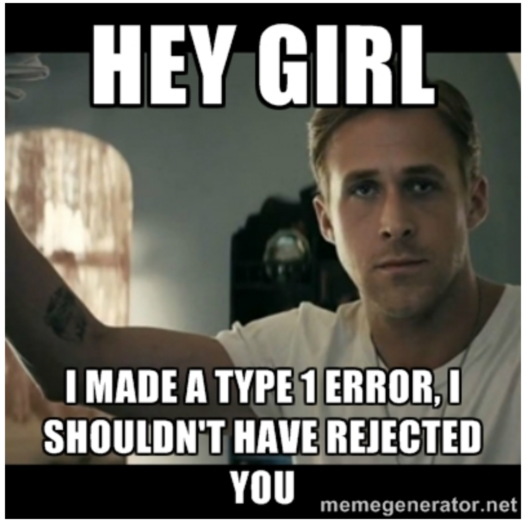
2. **Hypothesis testing** of Jezy Neyman and Egon Pearson
   - Make a formal decision of whether to reject H_0 (if p-value < predefined α value)

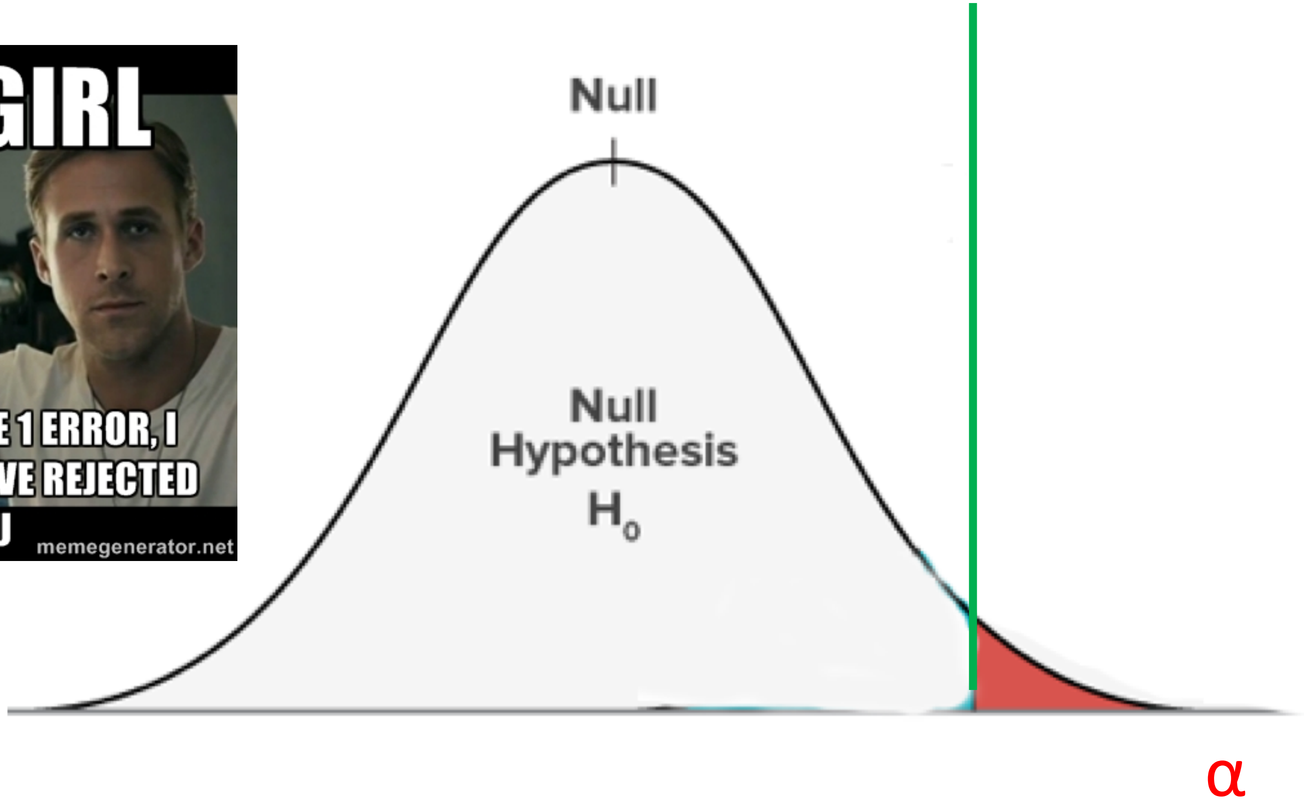


Significance testing

Hypothesis testing

Neyman-Pearson Frequentist logic

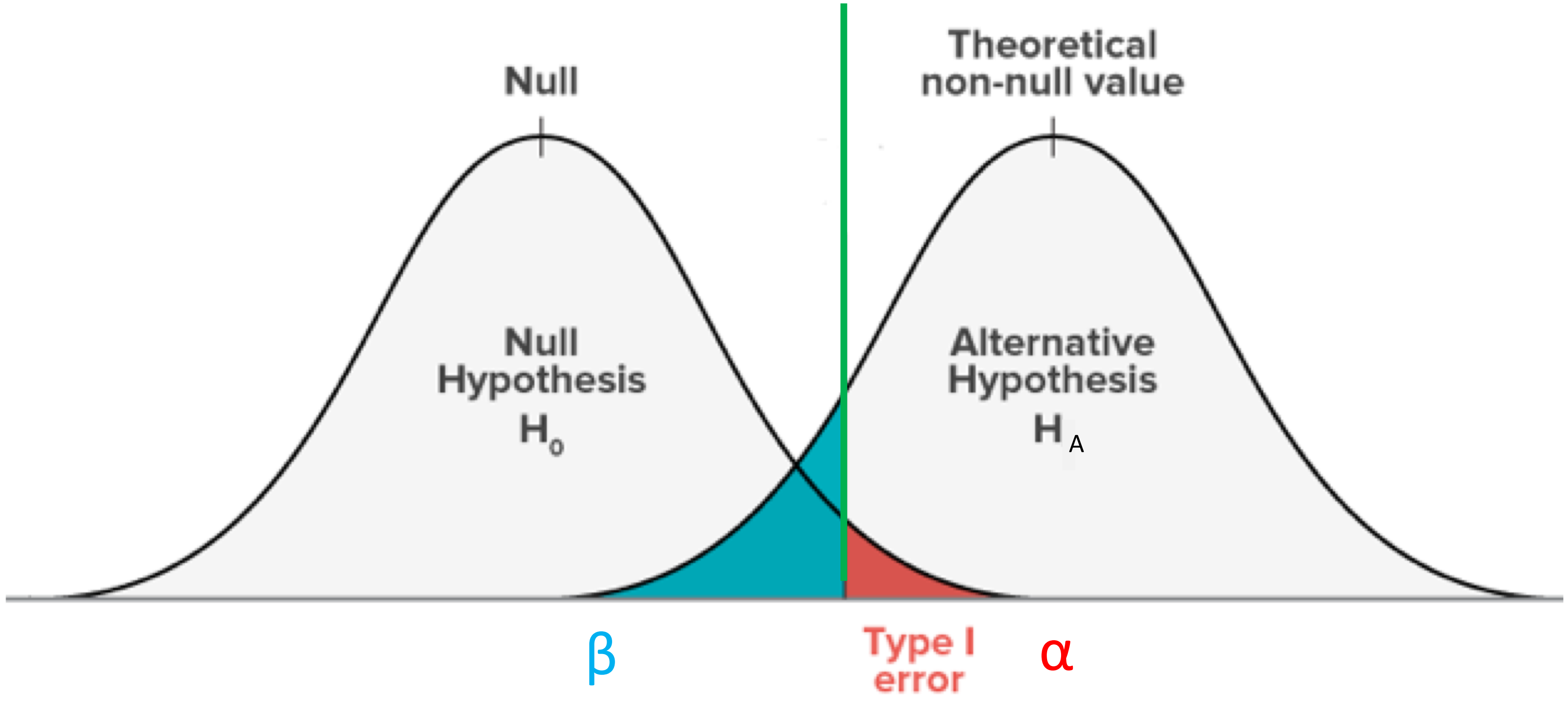




If Neyman-Pearson null hypothesis testing paradigm was followed perfectly, then only ~5% of all published research findings should be wrong (for α = 0.05)

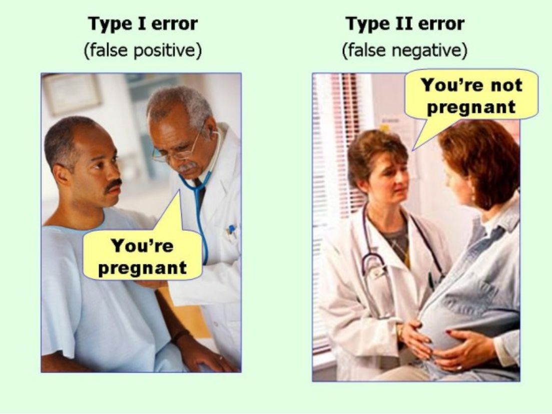
Type I error: incorrectly rejecting the null hypothesis when it is true

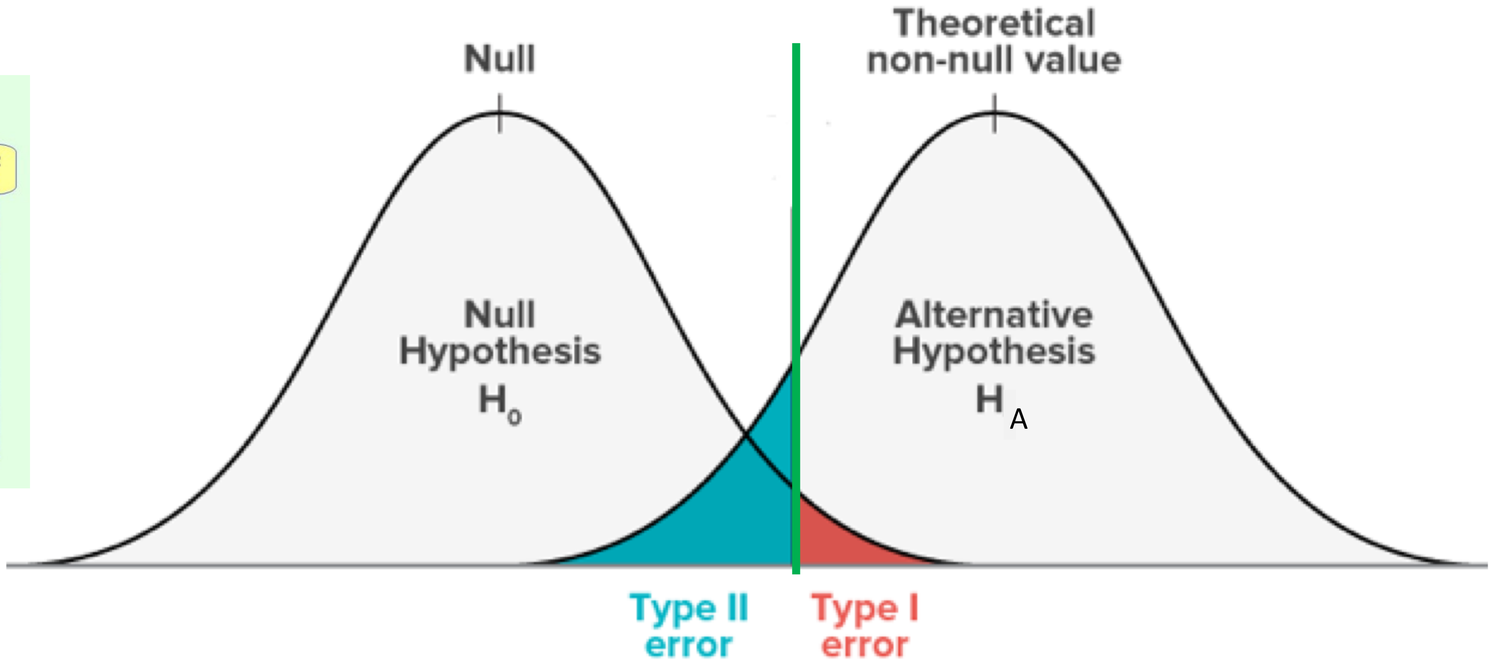
Neyman-Pearson Frequentist logic



Type 2 error: incorrectly rejecting failing to reject H_0 when it is false

Type I and Type II Errors





	Reject H_0	Do not reject H_0
H_0 is true	Type I error (α) (false positive)	No error

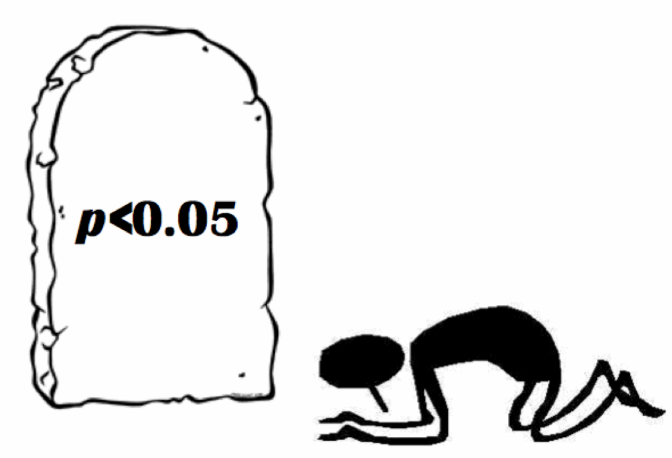
Problems with the NP hypothesis tests

Problem 1: we are interested in the results of a specific experiment, not whether we are right most of the time

- E.g., 95% of these statements are true:
  - Calcium is good for your heart, Paul is psychic, Buzz and Doris can communicate, …

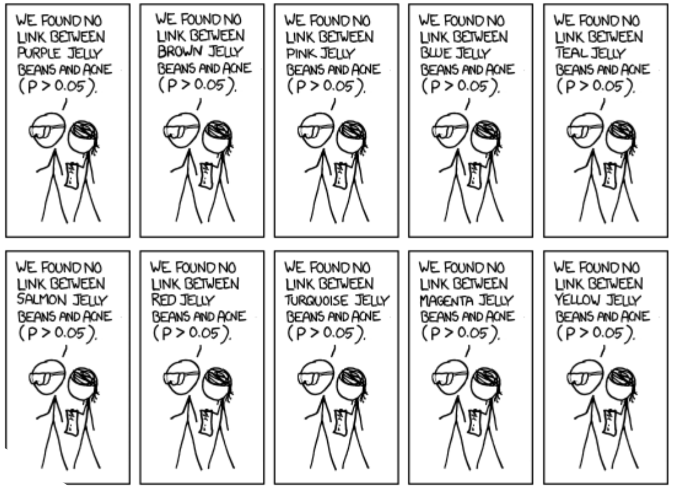
Problem 2: Arbitrary thresholds for alpha levels

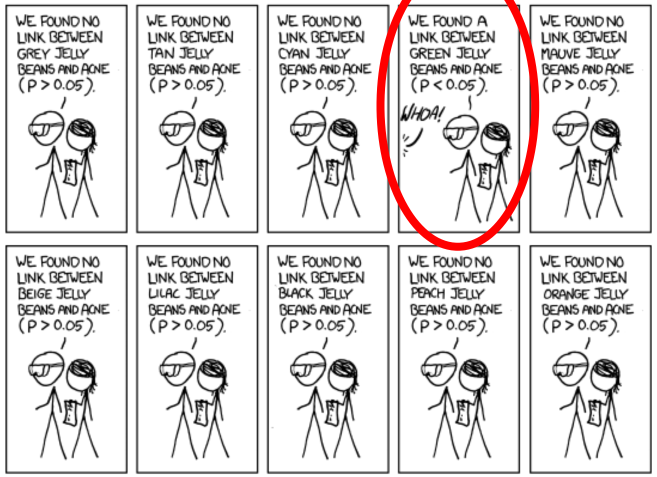
- P-value = 0.051, we don't reject H_0?

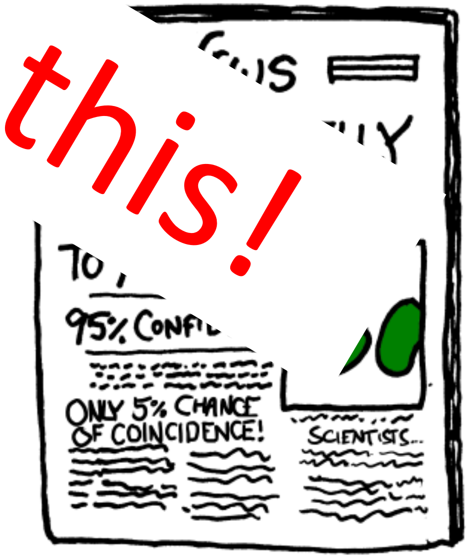


Problem 3: running many tests can give rise to a high number of type 1 errors

Multiple hypothesis tests







Don't ever do this!

Replication crisis

Essay

Why Most Published Research Findings Are False

John P. A. Ioannidis

The file drawer effect

American Statistical Association's 'Statement on p-values

Some thoughts…

Better to have hypothesis tests than none at all. Just need to think carefully and use your judgment.

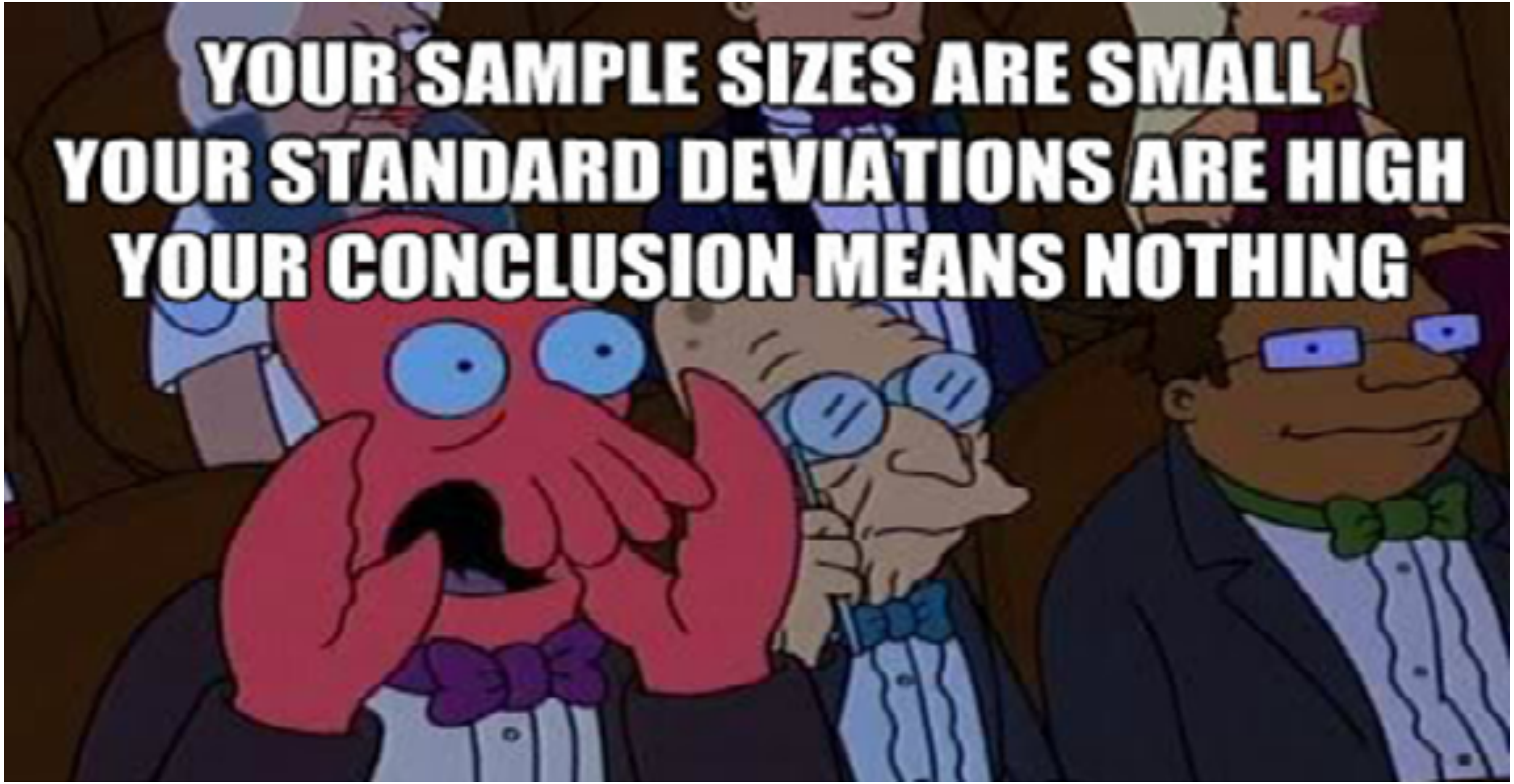
Report effect size in most cases – i.e., confidence intervals

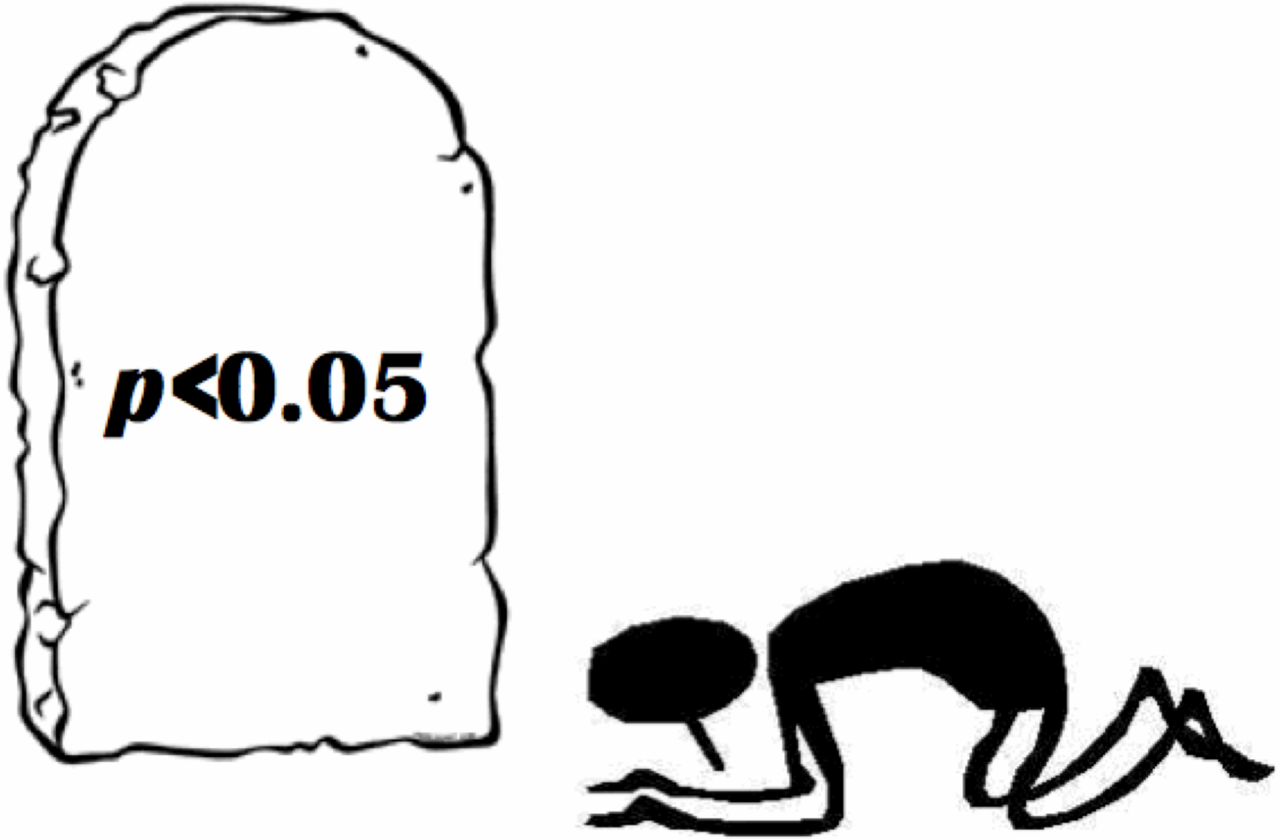
Report the p-values rather than accept/reject H_0

- i.e., report $p = 0.023$ not $p < 0.05$

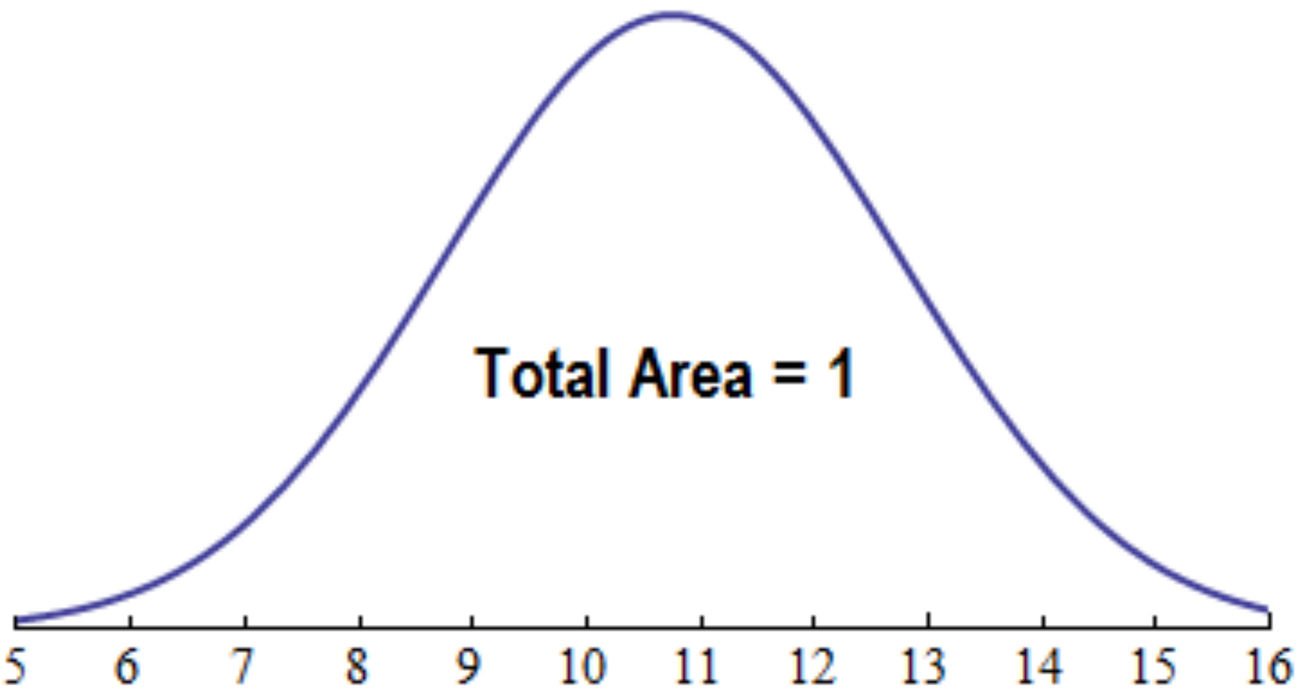
Replicate findings (perhaps in different contexts) to make sure you get the same results

Be a good/honest scientists and try to get at the Truth!


YOUR SAMPLE SIZES ARE SMALL
YOUR STANDARD DEVIATIONS ARE HIGH
YOUR CONCLUSION MEANS NOTHING


p<0.05

Inference using parametric probability distributions



Inference using parametric probability distributions

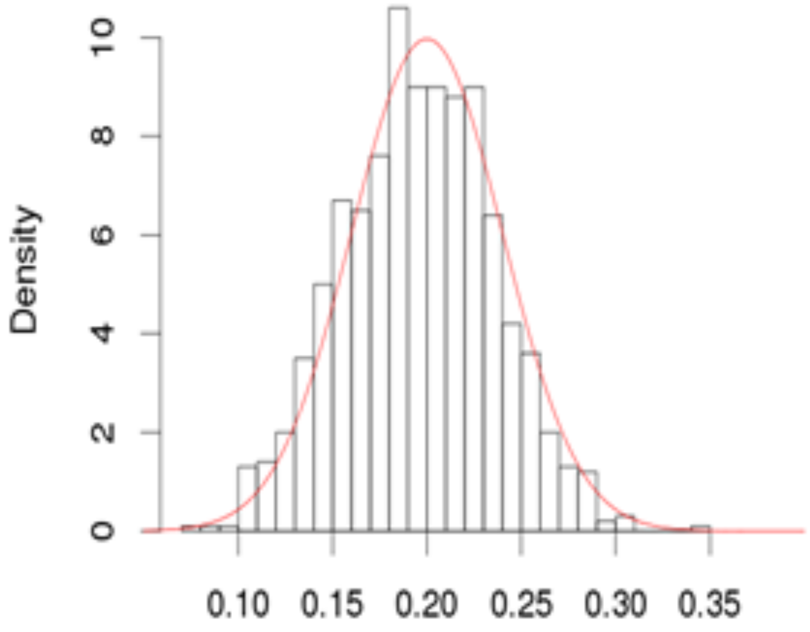
We can use mathematical functions called **probability distributions** to do inference

- e.g. instead of running computer simulations to create null distributions we can just mathematical probability distributions

A **density curve** is a mathematical function f(x) that has two important properties:

1. The total area under the curve f(x) is equal to 1

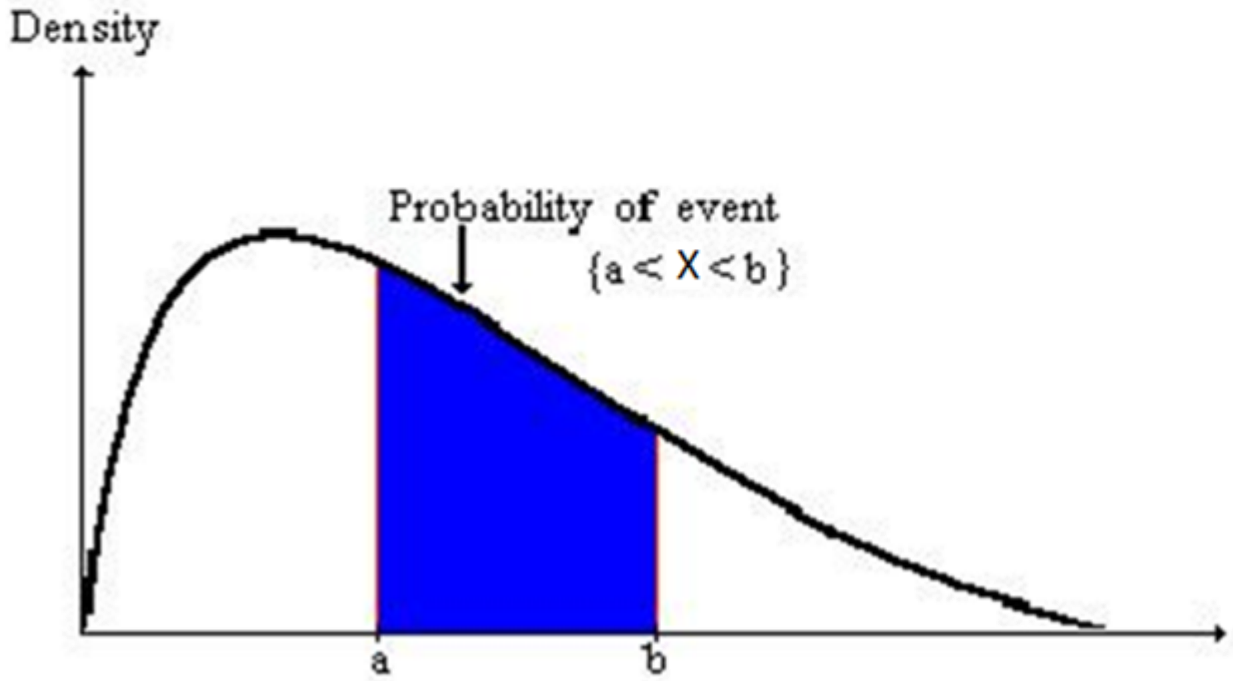
2. The curve is always ≥ 0



Density Curves

The <u>area under the curve</u> in an interval [a, b] models the probability that a random number X will be in the interval

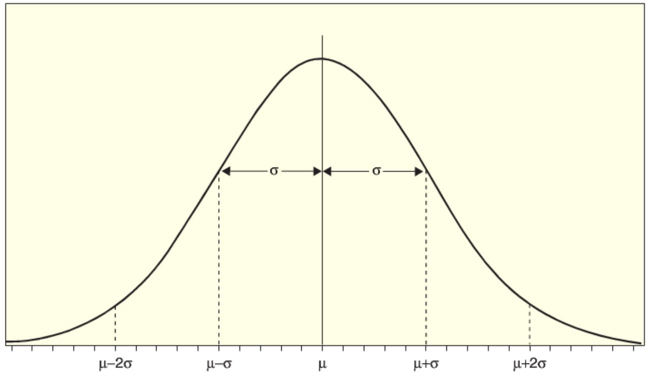
Pr(a < X < b) is the area under the curve from a to b



The Normal Density Curve

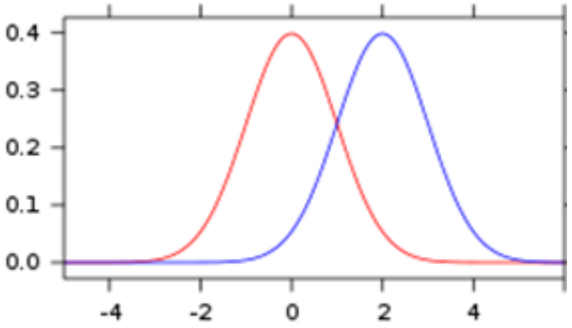
Normal distributions are a family of bell-shaped curves with two parameters

- The mean: μ
- The standard deviation: σ

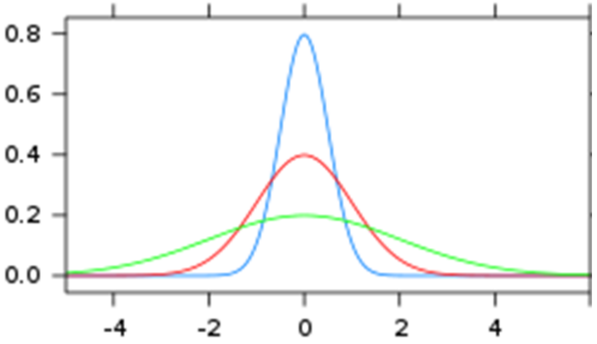


Notation: $X \sim N(\mu, \sigma)$

Changing μ



Changing σ



Graphing Normal Curves

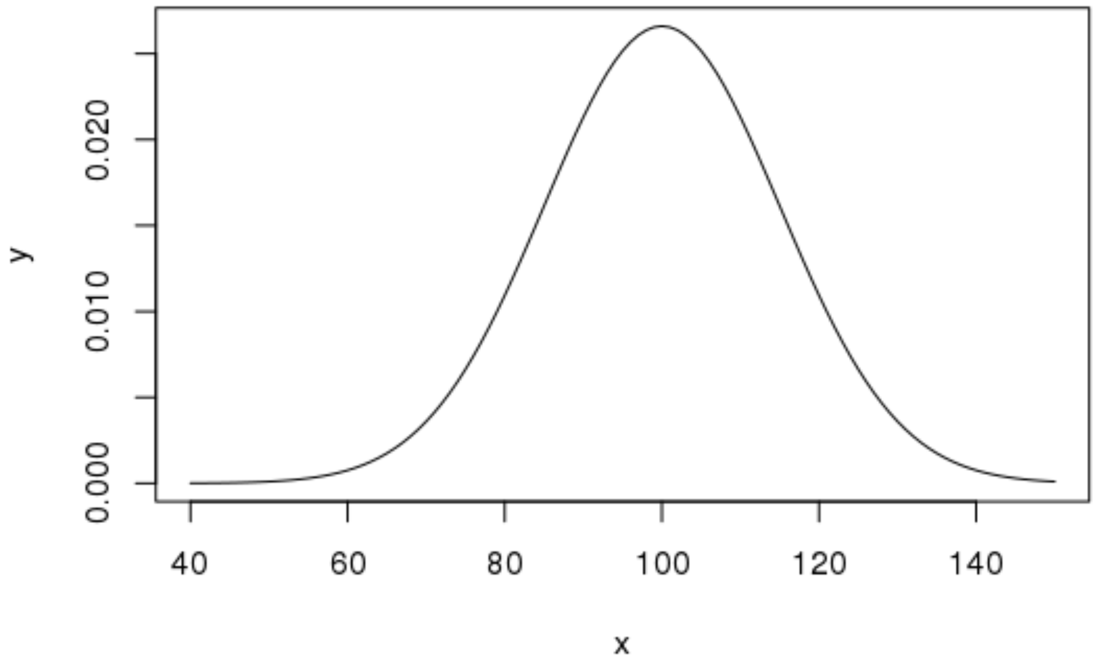
Plotting IQ scores

μ σ

```
x <- 40:150

y <- dnorm(x, 100, 15)

plot(x, y, type = "l")
```



Finding normal probabilities of a normal curve

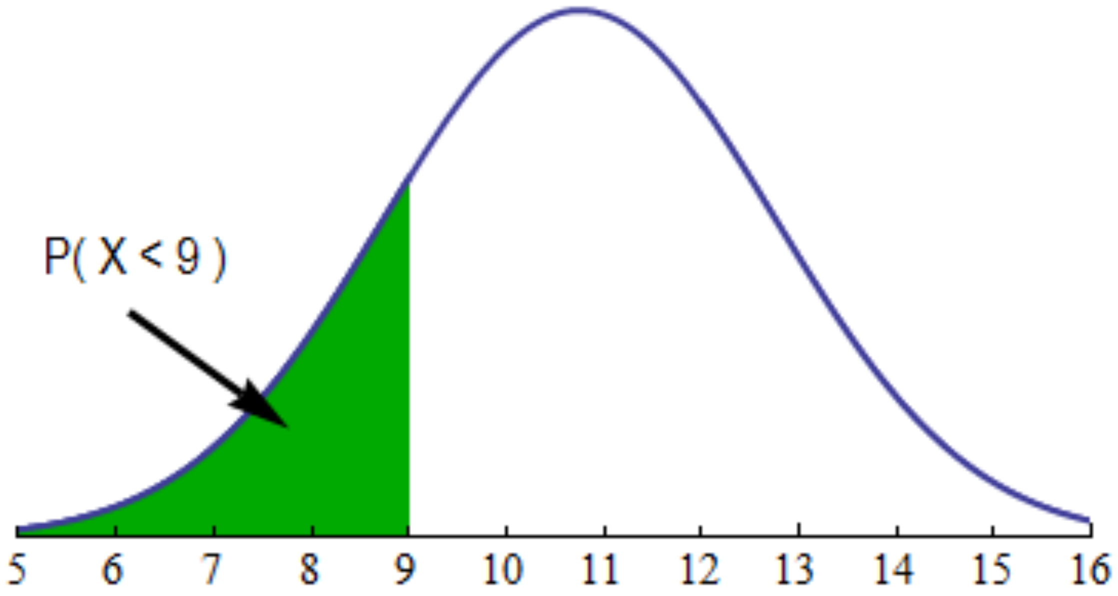
To get the probability (area) from a normal distribution we can use the pnorm function

```
pnorm(x, mu, sigma)
```

Pr(X < 9; 11, 3)

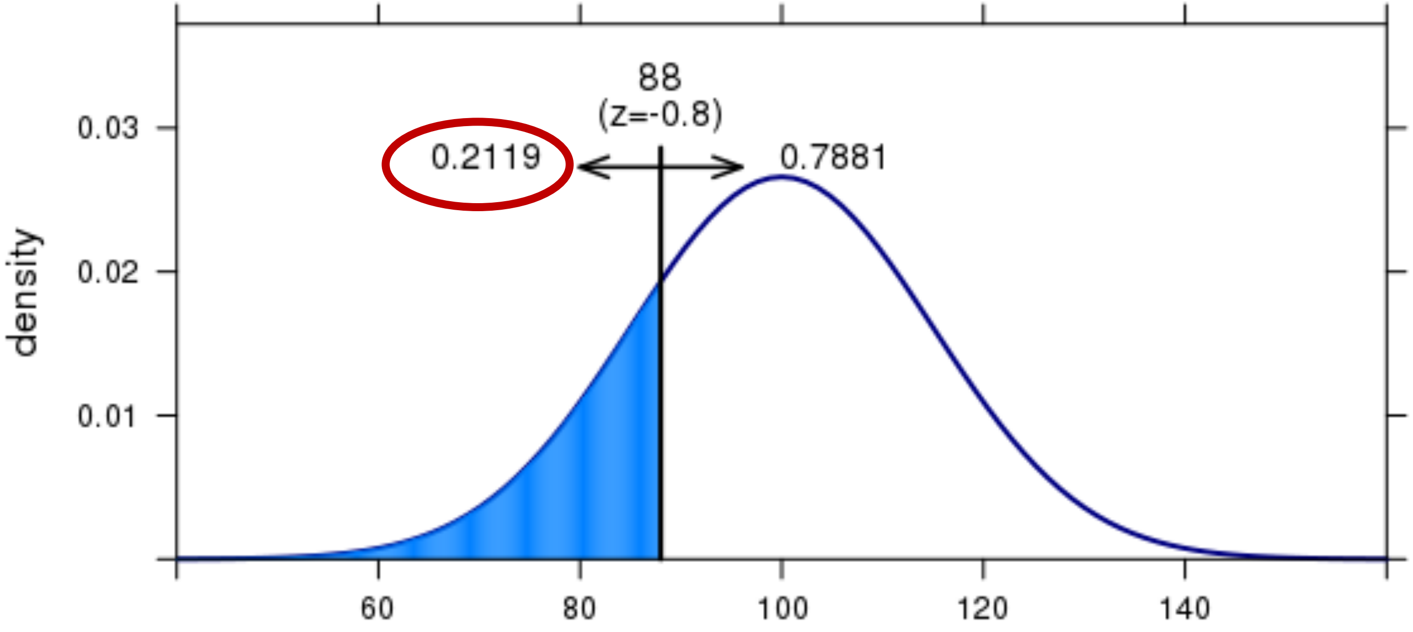
μ σ

```
pnorm(9, 11, 3)
```



Calculate the probability a random person you meet has an IQ less than 88

```
pnorm(88, 100, 15)
```



[Normal area Pr(X ≤ x) app]()

[Normal area Pr(a < X < b) app]()

Probability practice questions

1. What is probability a randomly chosen person will have an IQ greater than 96?

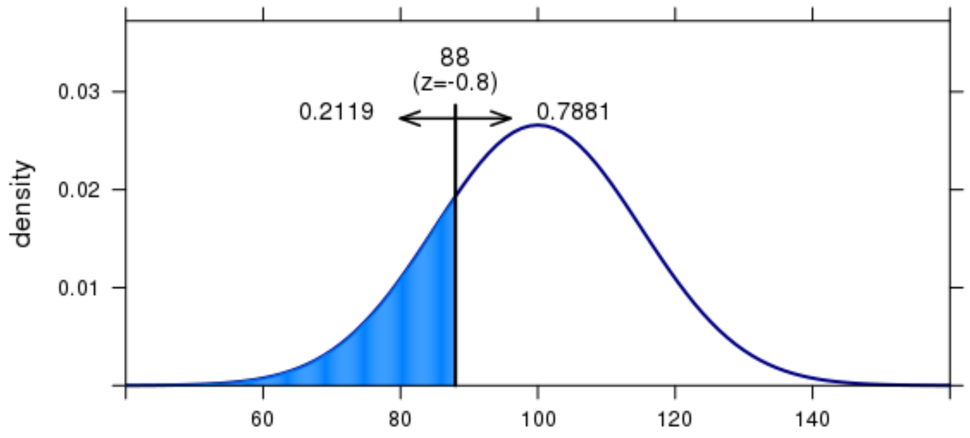
   pnorm(96, 100, 15, lower.tail = FALSE)

   - Answer: 0.605

2. What is the probability a randomly chosen person will have an IQ between 88 and 96?

   pnorm(96, 100, 15) - pnorm(88, 100, 15)

   - Answer: 0.183



Calculating quantiles

To find quantiles of the normal distribution we can use the quantile function:

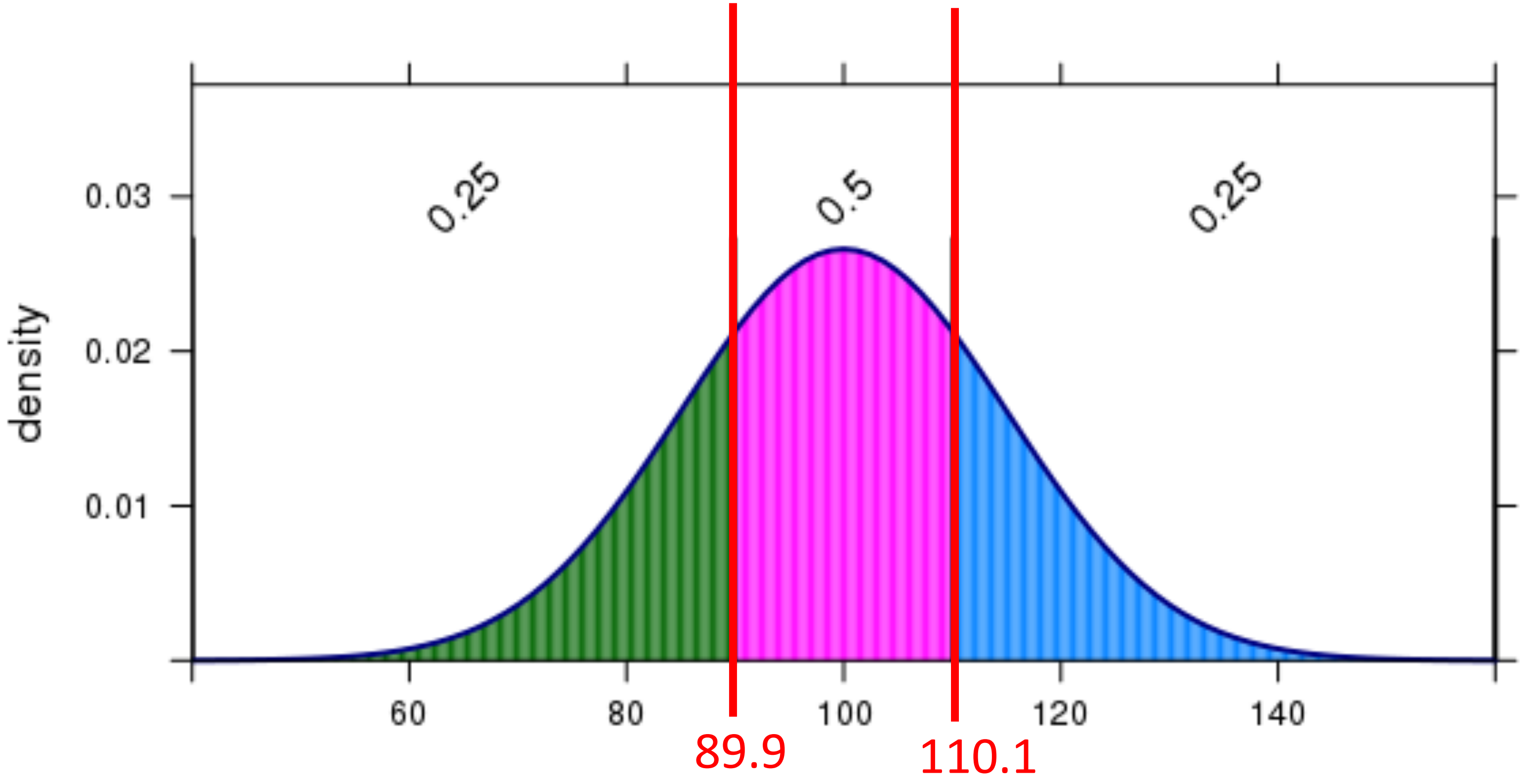
```
qnorm(quantile, mu, sigma)
```

What are the IQ scores (interval) that demark the middle 50% of the IQ range?

What about the middle 95%?

Normal quantile app

Middle 50% of IQ scores




```
qnorm(c(.25, .75), 100, 15)
```

Middle 50%: 89.9 to 110.1

Middle 95%: 70.6 to 129.3

Summary of R functions

Plot the actually density curve

- dnorm(x_vec, mu, sigma)

Get the probability that we would get a random value less than x

- pnorm(x_vec, mu, sigma)

Get the quantile value for a given proportion of the distribution

- qnorm(area, mu, sigma)

Note: pnorm and qnorm are inverses of each other

- y = pnorm(x, mu, sigma)
- qnorm(y, mu, sigma) # the output value here is x

The Standard Normal distribution and the Central Limit Theorem

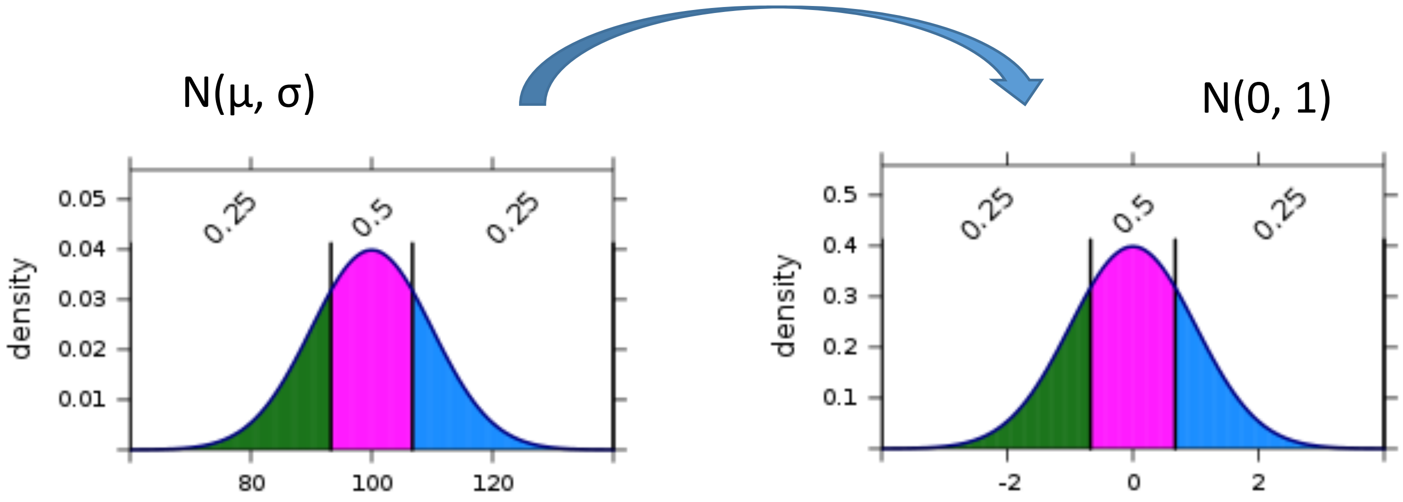
Standard Normal N(0, 1)

Since all normal distributions have the same shape, it is convenient to convert them to a standard scale with:

$$\mu = 0, \quad \sigma = 1$$

This is called the **standard normal** distribution:

$$Z \sim N(0, 1)$$



Converting to the standard normal distribution

We can use a z-score transformation to any normally distributed random variable $X \sim N(\mu, \sigma)$ to the standard normal distribution $Z \sim N(0, 1)$:

$$Z = (X - \mu)/\sigma$$

To convert from $Z \sim N(0, 1)$ to any $X \sim N(\mu, \sigma)$, we reverse the standardization with:

$$X = \mu + Z \cdot \sigma$$

Converting to the standard normal distribution

1. What is the Z-score of someone who has an IQ score of 112?

$$Z = (X - \mu)/\ \sigma$$

2. What if someone has an Z-score of 2.2, what is their IQ score?

$$X = \mu + Z \cdot \sigma$$

Answer 1: $Z = (112 - 100)/15 = .8$

Answer 2: $IQ = 100 + 2.2 * 15 = 133$

Central limit theorem

For random samples with a sufficiently large sample size (n), the distribution of sample statistics for a mean ($\bar{x}$) or a proportion ($\hat{p}$) is:

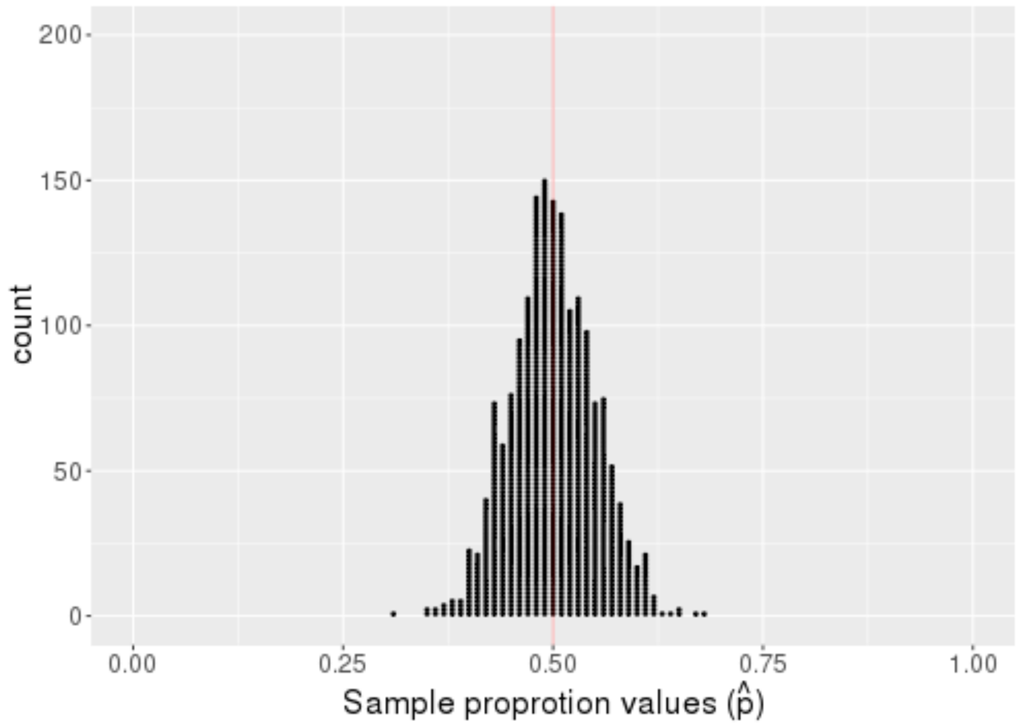
- normally distributed
- centered at the value of the population parameter

Stated again: the sampling distribution for means or proportions will be a normal distribution

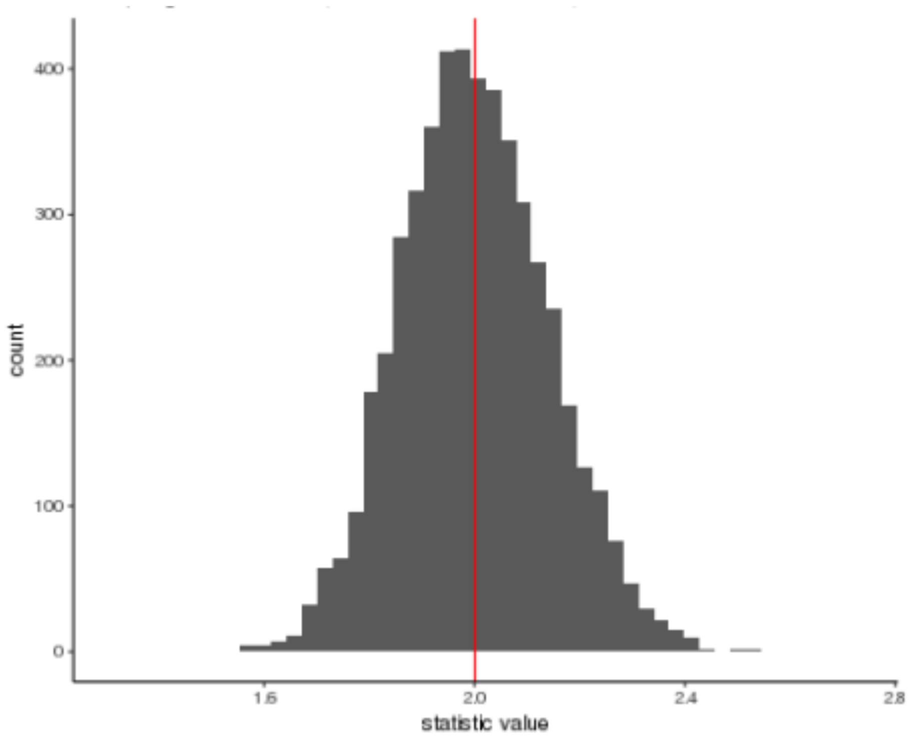
- so we don't need to do resampling to get a bootstrap or null distribution!

Central limit theorem

proportion ($\hat{p}$)



mean ($\bar{x}$)



Proportion sampling distribution app

Sampling/Bootstrap distribution app

Summary of standard normal and CLT

For large n, the sampling distributions of $\bar{x}$ and $\hat{p}$ are normal

We can convert any normal distribution $N(\mu, \sigma)$, into a standard normal distribution $N(0, 1)$

We are now (almost) ready to run hypothesis tests and compute confidence intervals for $\bar{x}$ and $\hat{p}$ using normal distributions

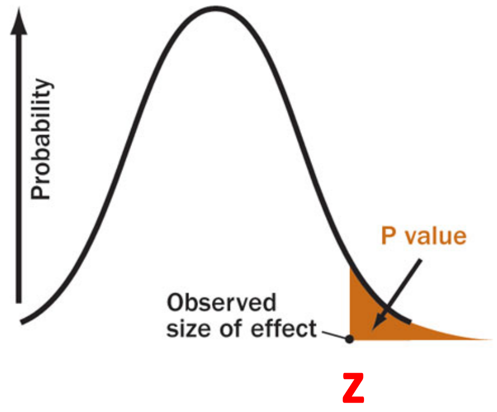
Hypothesis tests using a normal distribution

Hypothesis tests based on a Normal Distribution

When the null distribution is normal, it is often convenient to use a standard normal test statistic using:

$$z = \frac{\textit{Sample Statistic} - \textit{Null Parameter}}{SE}$$

The p-value for the test is the probability a standard normal value is beyond this standardized test statistic



$\Pr(Z \geq z_{obs} ;\quad \mu = 0,\ \sigma = 1)$

pnorm(z, 0, 1, lower.tail = FALSE)

Hypothesis tests based on a Normal Distribution

To repeat what was on the last slide: we can transform our obs_stat to a z-statistic that comes from a standard normal distribution N(0, 1) using:

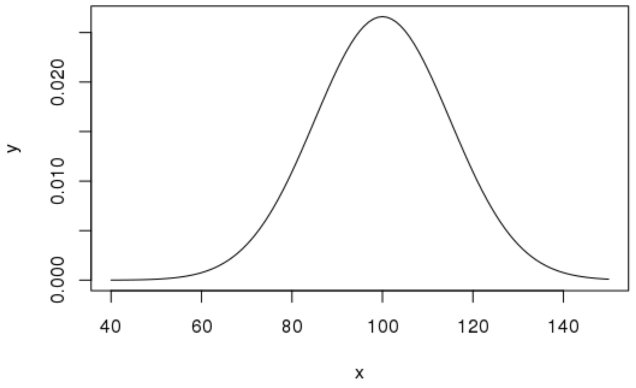
$$z = \frac{stat_{obs} \text{ - } param_0}{SE}$$

The p-value is then the probability of obtaining a value from a standard normal distribution beyond this z statistic

> pnorm(z, 0, 1) if H_A: $\mu < param_0$

> 1 - pnorm(z, 0, 1) if H_A: $\mu > param_0$

> 2 * (1 - pnorm(abs(z), 0, 1)) if H_A: $\mu \neq param_0$



Do greater than 40% of Americans go without using cash in a typical week?

A survey of 1,000 Americans reported that 43% said they went an entire week without using cash, with a SE = 0.016

Assuming the distribution of the statistic is normal, calculate whether the proportion of all Americans going a week without using cash is greater than 40%

1. Start by stating H_0 and H_A

H_0: $\pi = .4$

H_A: $\pi > .4$

Do greater than 40% of Americans go without using cash in a typical week?

A survey of 1,000 Americans reported that 43% said they went an entire week without using cash, with a SE = 0.016

Assuming the distribution of the statistic is normal, calculate whether the proportion of all Americans going a week without using cash is greater than 40%

2. Can you compute the z statistic?

$$z = \frac{\text{Sample Statistic} - \text{Null Parameter}}{SE}$$

Do greater than 40% of Americans go without using cash in a typical week?

A survey of 1,000 Americans reported that 43% said they went an entire week without using cash, with a SE = 0.016

Assuming the distribution of the statistic is normal, calculate whether the proportion of all Americans going a week without using cash is greater than 40%

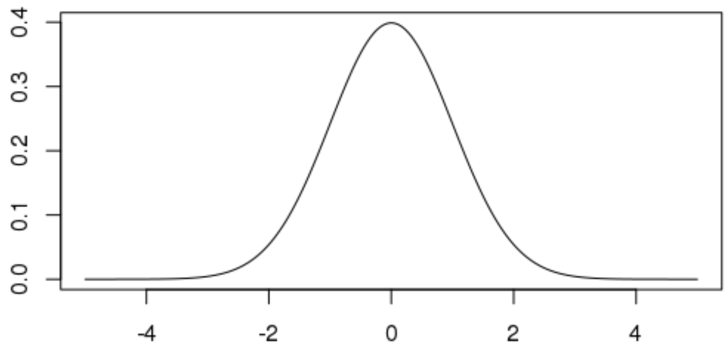
2. Can you compute the z statistic?

$$z = (.43 - .4)/.016 = 1.875$$

Do greater than 40% of Americans go without using cash in a typical week?

Steps: 3-4. What is the probability one would get a z-statistic as larger or larger than 1.875 from a standard normal distribution?

Standard normal null distribution




```
> pnorm(1.875, 0, 1, lower.tail = FALSE)
> 1 – pnorm(1.875, 0, 1)
```

p-value = .0304

Step 5?

Normal area app Pr(X ≤ x)

Confidence intervals using a normal distribution

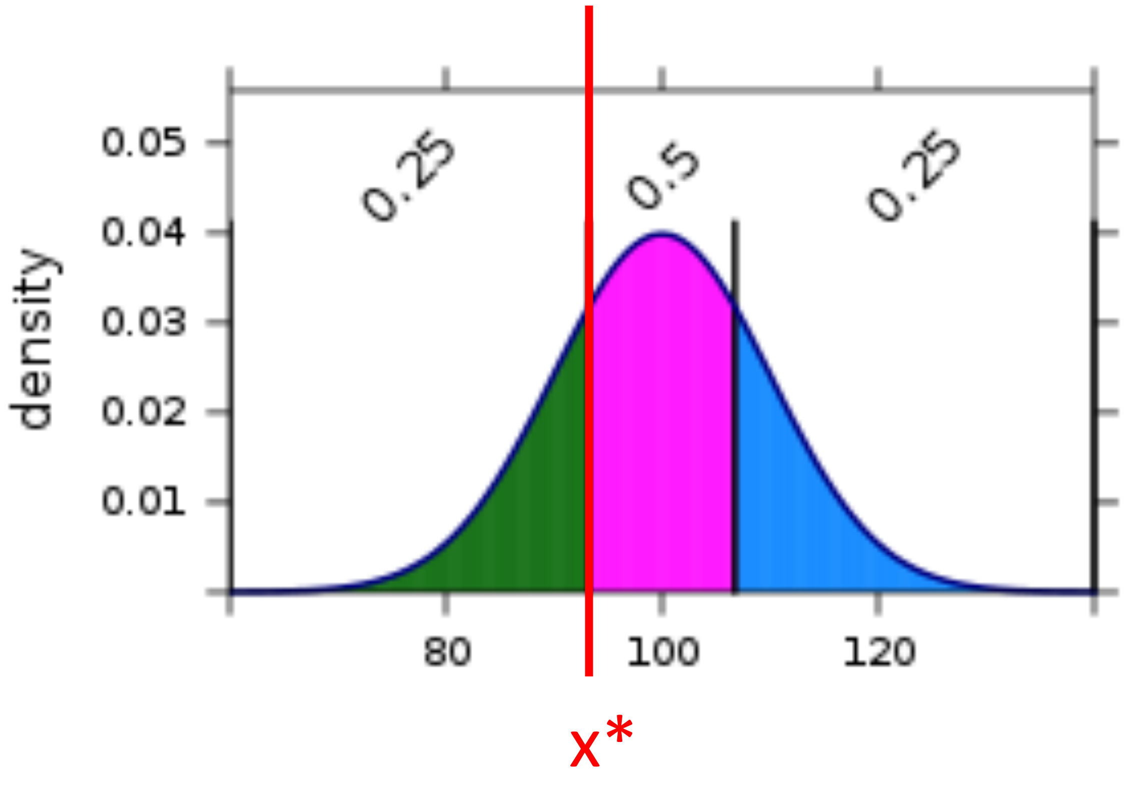
Finding quantile values

We can find the quantile value from a normal distribution with:

qnorm(q, mu, sigma)

The ‘q’ in qnorm stands for quantile

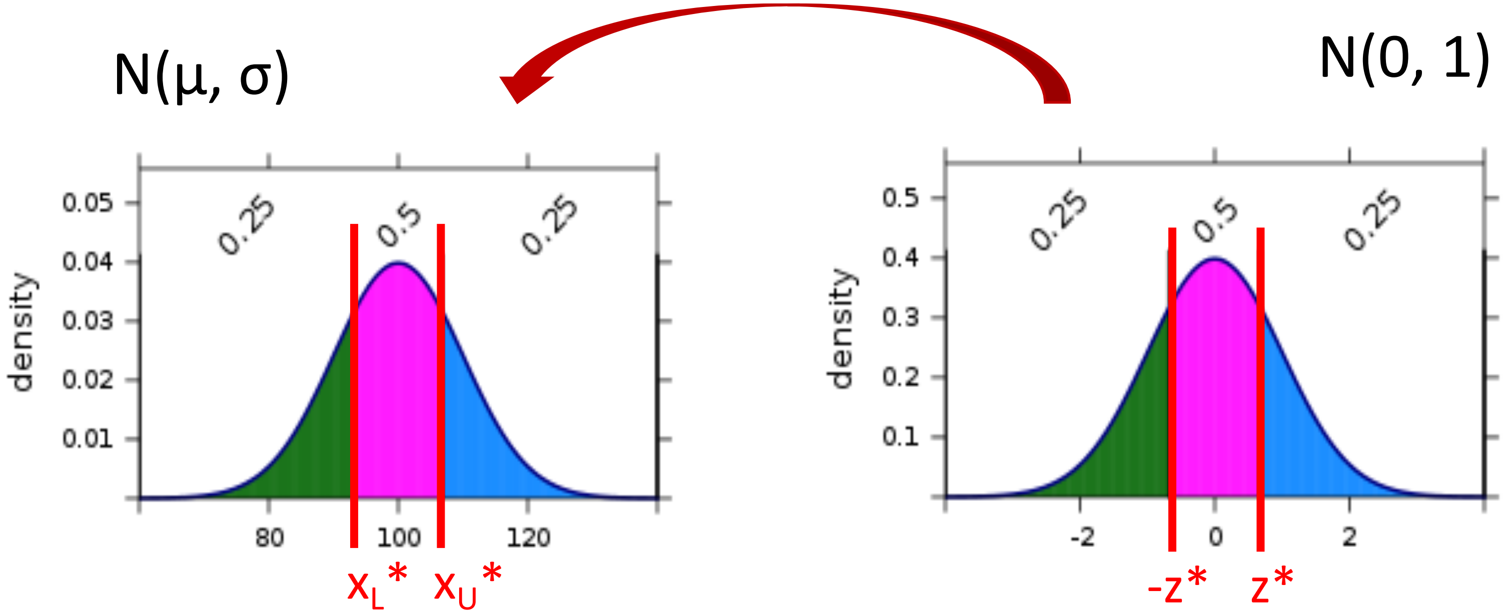
What is the max and min that q can be?



Standard Normal N(0, 1)

It is often convenient to find quantiles on the standard normal distribution $Z \sim N(0, 1)$ and then to transform them to an arbitrary normal distribution $X \sim N(\mu, \sigma)$, using :

$$X = \mu + Z \cdot \sigma$$



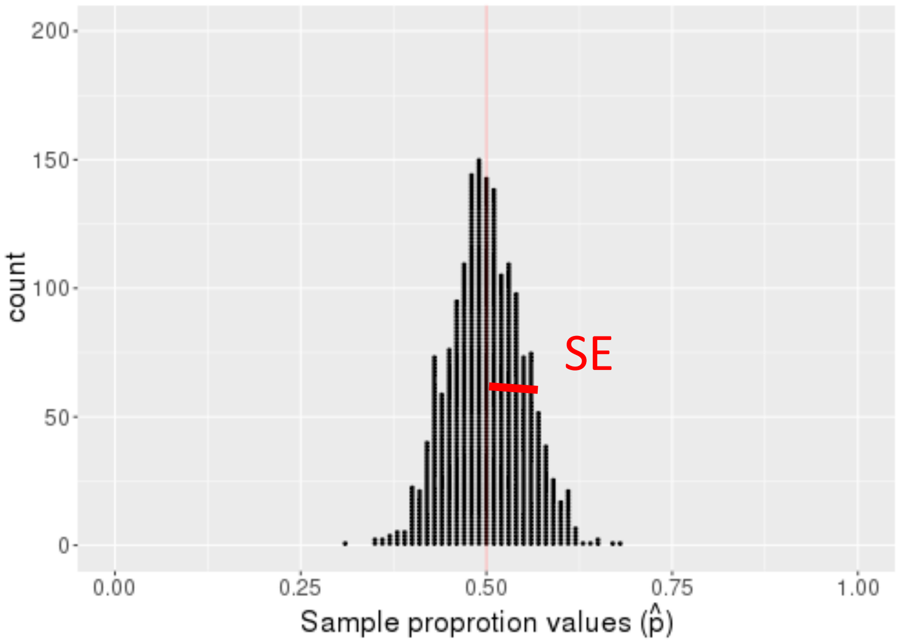
Central limit theorem

Questions:

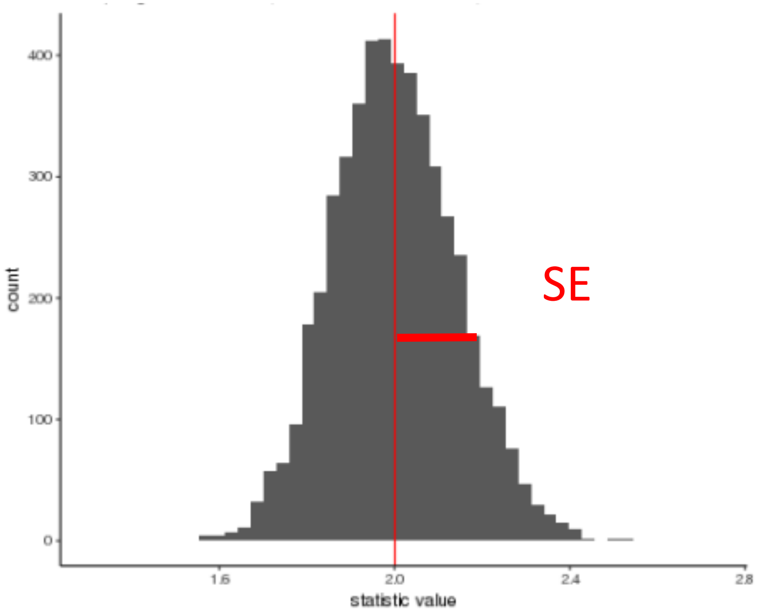
1. What is the standard deviation of these sampling distributions called?
2. Suppose we have a $\hat{p}$ or $\bar{x}$ and know the SE, how can we create a 95% CI?

For a proportion π: $CI_{95} = \hat{p} \pm 2 \cdot SE$ For a mean μ: $CI_{95} = \bar{x} \pm 2 \cdot SE$

proportion ($\hat{p}$)



mean ($\bar{x}$)



Confidence intervals based on a Normal Distribution

If the distribution for a statistic is normal with a standard error SE, we can find a confidence interval for the parameter using:

$$\text{sample statistic} \;\pm\; z^* \times \text{SE}$$

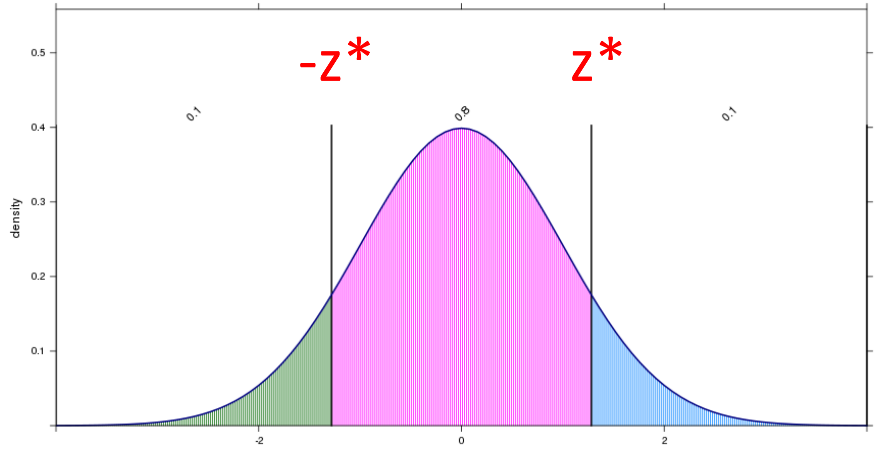
where z^* is chosen so that the area between $-z^*$ and $+z^*$ in the standard normal distribution is the desired confidence level

- i.e., z^* is chosen such that say 95% of the distribution is between $\pm z^*$

Confidence intervals based on a Normal Distribution

Suppose we are interested in 80% confidence intervals for μ

We calculate the $\pm z_{80}$ that has 80% of the data on N(0, 1)



Let's assume we know the SE but don't know μ. If we have an observed statistic from:

$$x_{obs} \sim N(\mu, SE)$$

We can create an interval that will capture μ 80% of the time using:

$$x_{obs} \pm z_{80} \cdot SE$$

Normal percentiles for common confidence levels

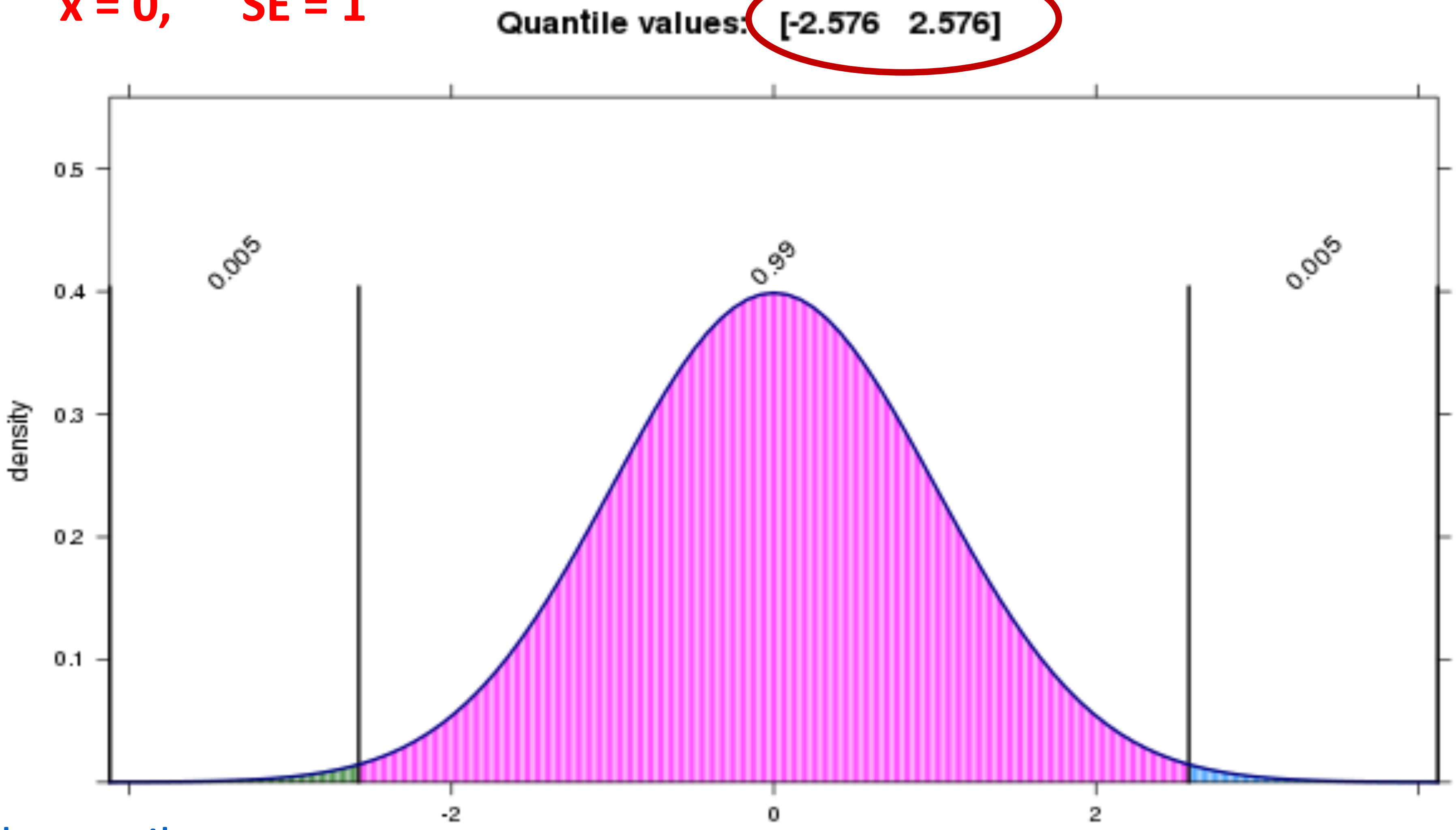
Confidence level	80%	90%	95%	98%	99%
Z*	1.282	1.645	1.960	2.326	2.576

```
z_stars <- qnorm(c(.90, .95, .975, .99, .995), 0, 1)
```

Normal quantile app

.99 quantile values

$\bar{x}$ = 0, SE = 1



$\bar{x}$ = 0

Normal quantile app

What is the most preferred seat?

A survey of 1,000 air travelers found that 60% prefer a window seat, with a bootstrap standard error of SE = 0.015

Use the normal distribution to compute a 90%, 95% and 99% CIs for the proportion of people who prefer a window seat

$$\text{sample statistic} \pm z^* \times \text{SE}$$

Confidence level	80%	90%	95%	98%	99%
Z*	1.282	1.645	1.960	2.326	2.576

What is the most preferred seat?

A survey of 1,000 air travelers found that 60% prefer a window seat, with a bootstrap standard error of SE = 0.015.

90% CI = .6 ± 1.645 × .015 = [.575 .625]

95% CI = .6 ± 1.96 × .015 = [.571 .629]

99% CI = .6 ± 2.576 × .015 = [.569 .638]

Sample statistics ± z* × SE

Confidence level	80%	90%	95%	98%	99%
Z*	1.282	1.645	1.960	2.326	2.576

Homework 8

Is on Canvas

Due Sunday April 5^{th} at 11:30pm