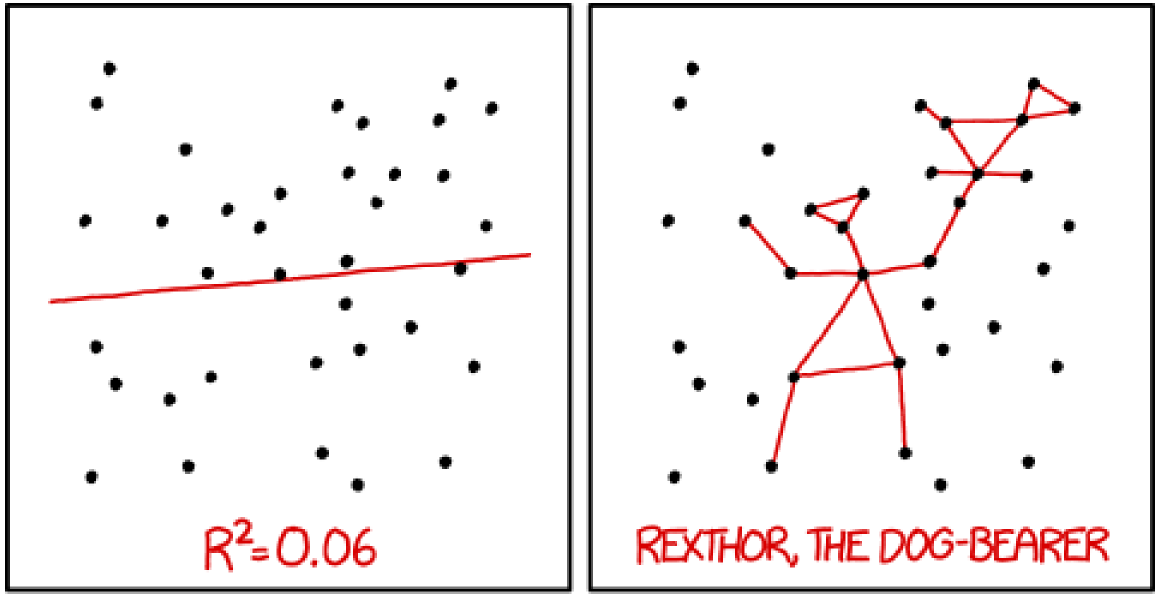
Multiple regression continued



I DON'T TRUST LINEAR REGRESSIONS WHEN IT'S HARDER TO GUESS THE DIRECTION OF THE CORRELATION FROM THE SCATTER PLOT THAN TO FIND NEW CONSTELLATIONS ON IT.

Overview

Quick review

- Multiple regression with categorical predictors
- Interaction effects

Log transformations of the response variable y

Multicollinearity

If there is time: Polynomial regression

Quick review

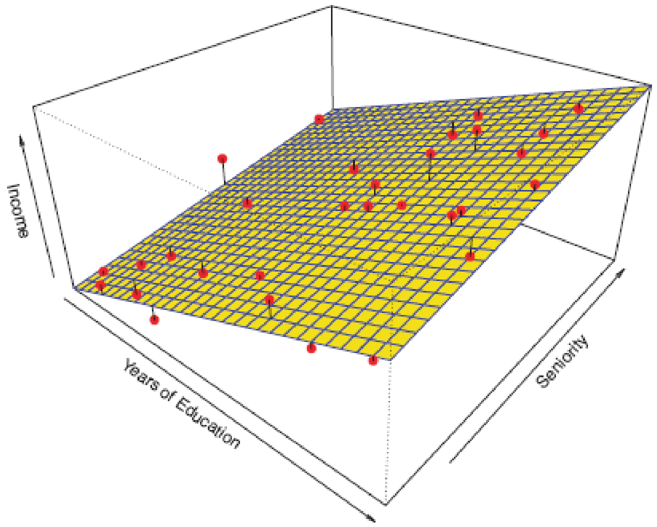
Multiple regression

In multiple regression we try to predict a quantitative response variable *y* using several predictor variables $x_1, x_2, \ldots, x_k$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

Goals:

- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)

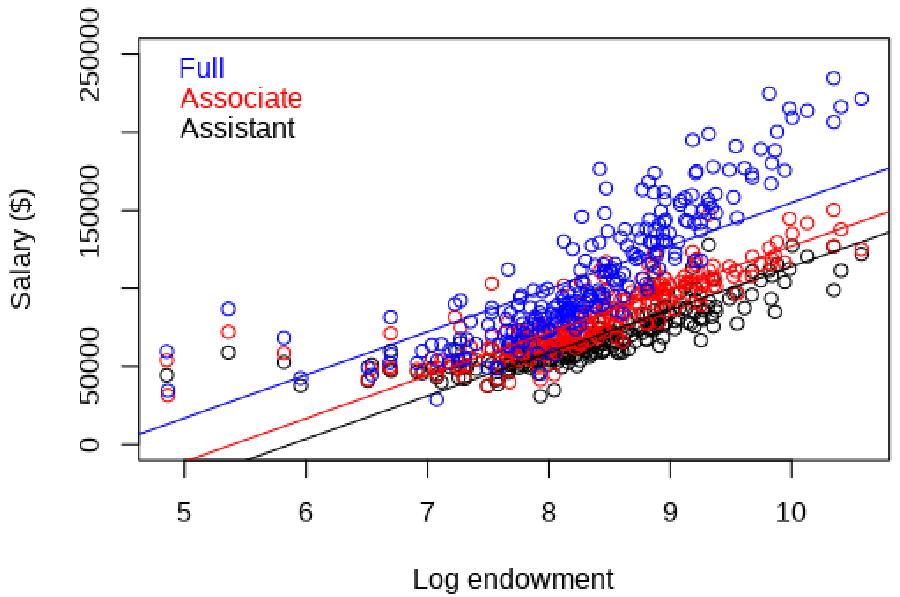


Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it

Suppose we want to predict faculty salary *y* as a function of endowment x_1, with separate intercepts for faculty rank



$$x_{i1} = \log(\text{endowment})$$

$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{i3} = \begin{cases} 1 & \text{if associate professor} \\ 0 & \text{otherwise} \end{cases}$$

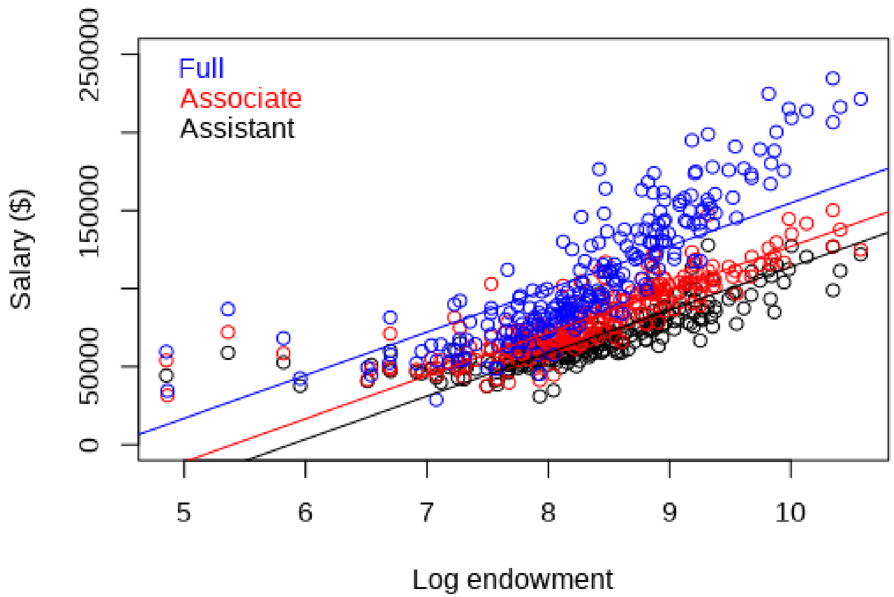
$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} \cdot$$

$$= \begin{cases} \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 & \text{if assistant professor} \\ \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_3 & \text{if associate professor} \\ \hat{\beta}_0 + \hat{\beta}_1 x_{i1} & \text{if full professor} \end{cases}$$

Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it



Suppose we want to predict faculty salary *y* as a function of endowment x_1, with separate intercepts for faculty rank

```
> summary(fit_prof_rank_offset)

Call:
lm(formula = salary_tot ~ log_endowment + rank_name, data = IPED_2)

Residuals:
   Min     1Q Median     3Q    Max 
-52464 -10844  -2703   6936  74994 

Coefficients:
                     Estimate Std. Error t value             Pr(>|t|)    
(Intercept)         -120822.1     6713.9  -18.00 <0.0000000000000002 ***
log_endowment         27569.9      791.7   34.82 <0.0000000000000002 ***
rank_nameAssociate   -27855.4     1685.5  -16.53 <0.0000000000000002 ***
rank_nameAssistant   -40973.7     1685.5  -24.31 <0.0000000000000002 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 18370 on 707 degrees of freedom
Multiple R-squared:  0.7192,	Adjusted R-squared:  0.718 
F-statistic: 603.7 on 3 and 707 DF,  p-value: < 0.00000000000000022
```

$$\hat{y}_i = \begin{cases} \boxed{\hat\beta_0} + \boxed{\hat\beta_1} x_{i1} + \boxed{\hat\beta_2} & \text{if assistant professor} \\ \boxed{\hat\beta_0} + \boxed{\hat\beta_1} x_{i1} + \boxed{\hat\beta_3} & \text{if associate professor} \\ \boxed{\hat\beta_0} + \boxed{\hat\beta_1} x_{i1} & \text{if full professor} \end{cases}$$

$$\hat{y}_i = \hat\beta_0 + \hat\beta_1 x_{i1} + \hat\beta_2 x_{i2} + \hat\beta_3 x_{i3}$$

$$= -120,822 + 27,570 x_{i1} - 40,973 x_{i2} - 27,855 x_{i3}$$

Let’s quickly try it in R…

Interaction terms

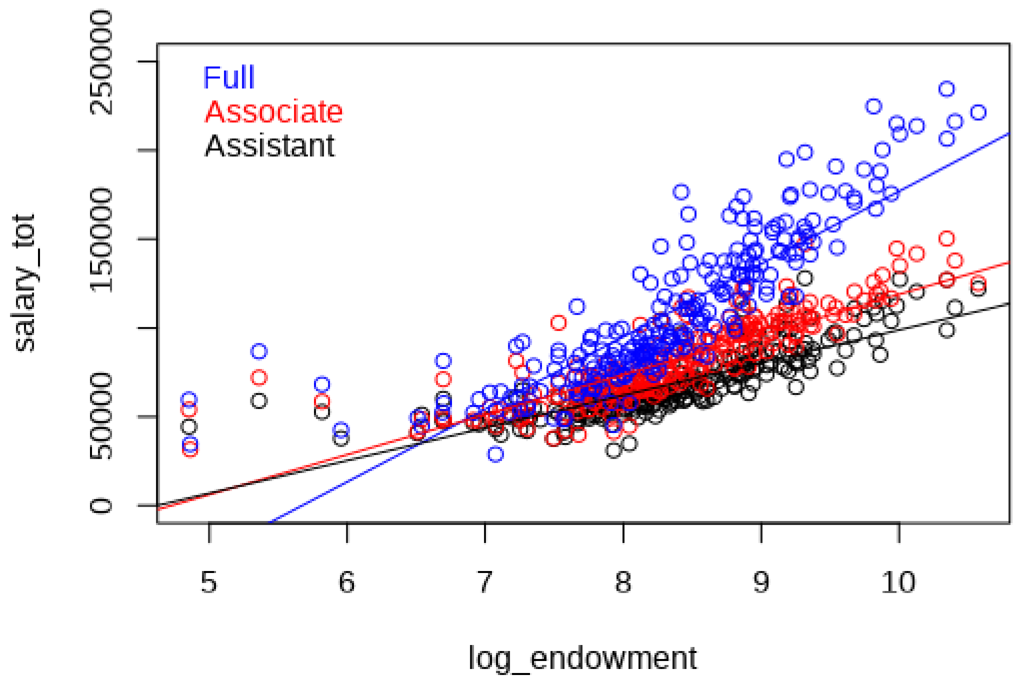
An ***interaction effect*** occurs when the response variable y is influenced by the levels of two or more predictors in a non-additive way

We can model this using an equation with an interaction term

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_3 (x_1 \cdot x_2) + \epsilon$$

An interaction term between a quantitative and categorical variable corresponds to different slopes depending for the quantitative variable depending on the value of the categorical variable

Interaction terms



If Full Professor:

$$\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment}$$

If Assistant Professor:

$$\text{salary} \approx (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{endowment}$$

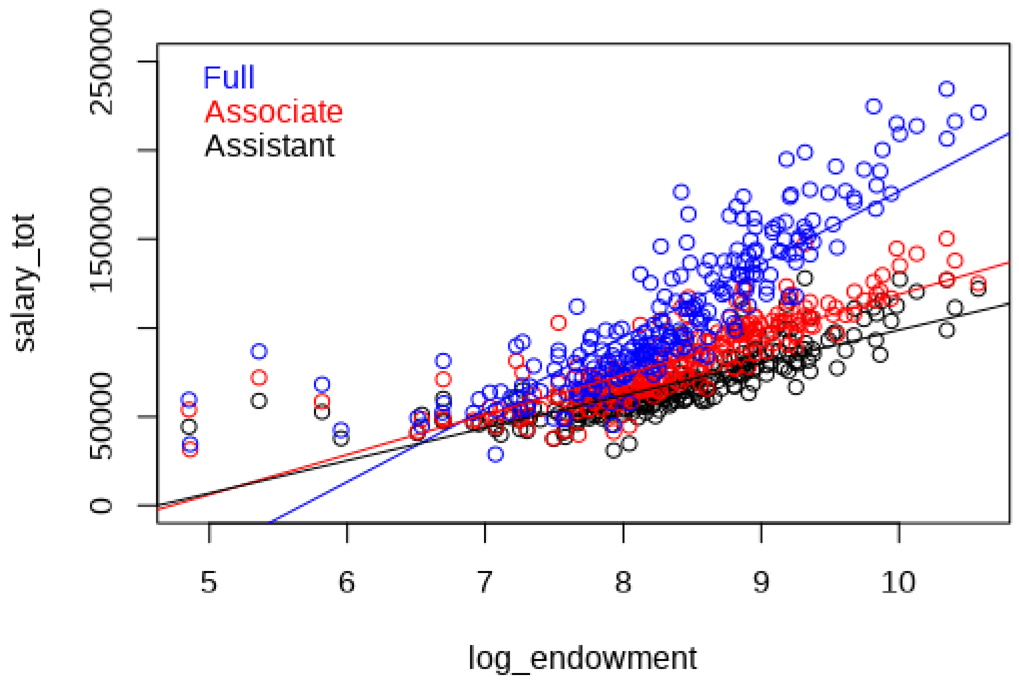
Modification to intercept if Assistant Professor

Modification to slope if Assistant Professor

$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i1} \cdot x_{i2}$$

Interaction terms




```
Call:
lm(formula = salary_tot ~ log_endowment + rank_name + log_endowment:rank_name,
    data = IPED_2)

Residuals:
   Min      1Q Median      3Q     Max
-46914   -9554  -2263    6233   99678

Coefficients:
                                 Estimate Std. Error t value Pr(>|t|)
(Intercept)                       -231986       9989 -23.224   <2e-16 ***
log_endowment                       40888       1190  34.357   <2e-16 ***
rank_nameAssociate                 125551      14289   8.786   <2e-16 ***
rank_nameAssistant                 146880      14429  10.180   <2e-16 ***
log_endowment:rank_nameAssociate   -18369       1701 -10.800   <2e-16 ***
log_endowment:rank_nameAssistant   -22482       1717 -13.094   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 16260 on 705 degrees of freedom
Multiple R-squared:  0.7806,    Adjusted R-squared:  0.7791
F-statistic: 501.7 on 5 and 705 DF,  p-value: < 2.2e-16
```

Intercept for full professor

Slope for full professor

Modification to intercept for assistant prof

Modification to slope for assistant prof

x_{i1}: Log endowment (continuous)

x_{i2}: Assistant prof (indicator/dummy variable)

$$\hat{y}_i = \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{i1} + \boxed{\hat{\beta}_2} x_{i2} + \boxed{\hat{\beta}_3} x_{i1} \cdot x_{i2}$$

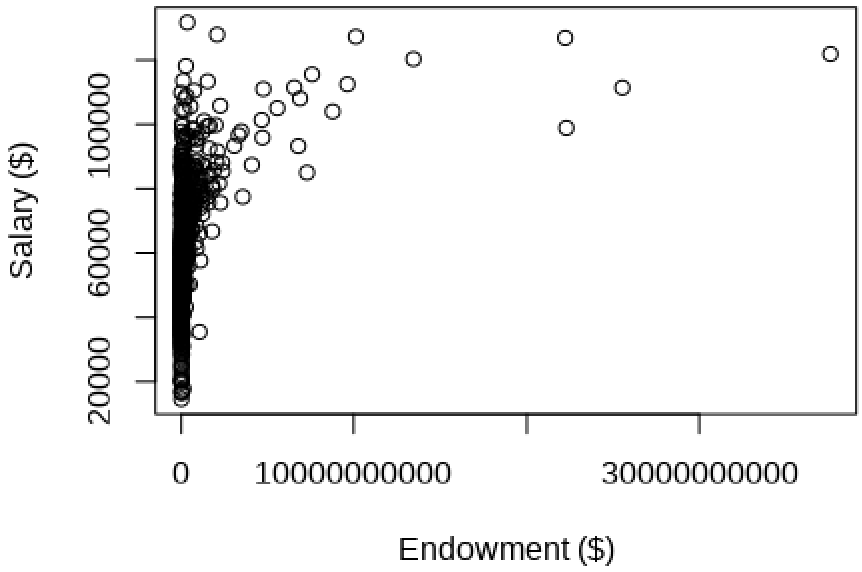
Questions?

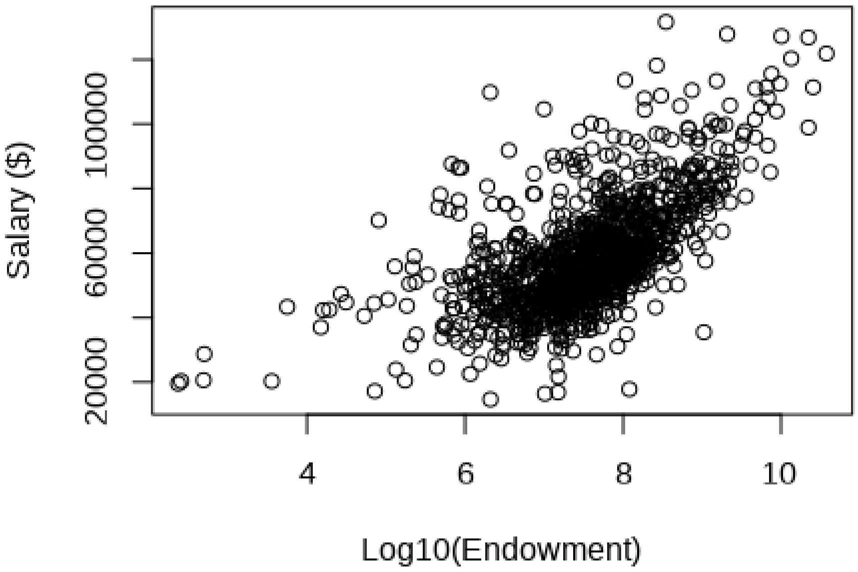
Let’s quickly explore this in R

Transformations of the response variable (y)

Log transformation of the response variable y

As we've seen, we can take a log transformation of an *explanatory x* variable to make a non-linear relationship more linear

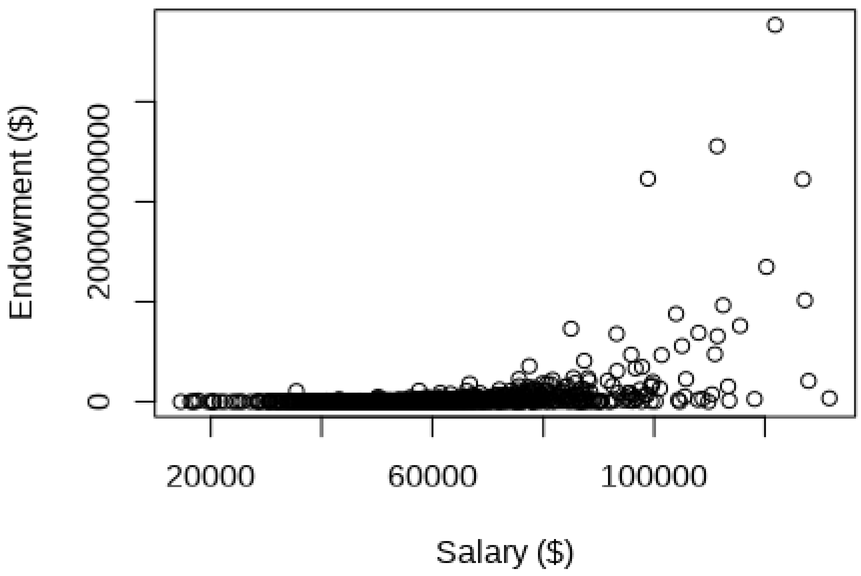


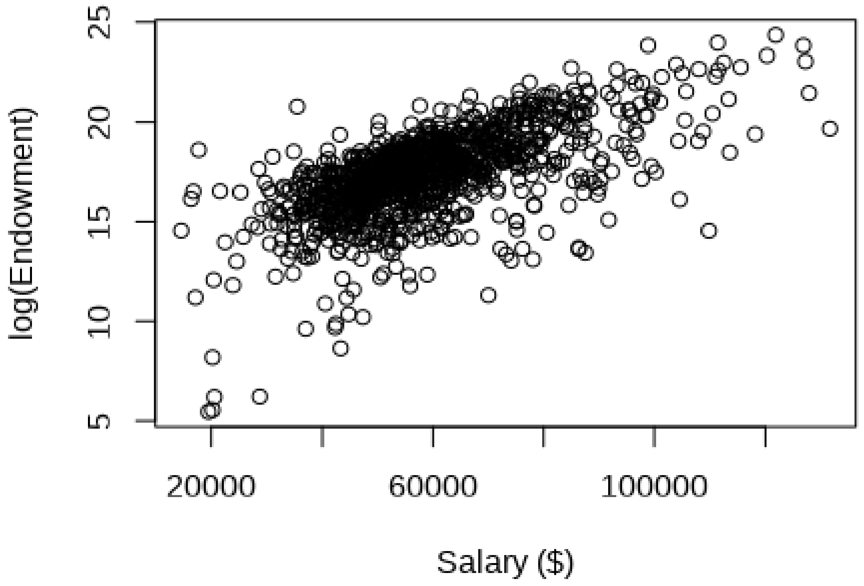


Log transformation of the response variable y

Often, it can be useful to take log transformation of a **response variable y** to make the relationship more linear

- This can also be useful to deal with heteroskedasticity





Log transformation of the response variable y

How can we interpret the regression coefficients when we have taken a log transformation of the response variable y?

$$log(\hat{y}) = \hat{\beta}_0 + \hat{\beta}_1 x$$

If we exponentiate both sides we get:

$$\hat{y} = e^{\hat{\beta}_0 + \hat{\beta}_1 x} \quad = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x}$$

If we increase x by 1, we multiply the previous predicted value of ŷ by $e^{\hat{\beta}_1}$

$$\hat{y} = \hat{f}(x+1) = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x + 1} = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x} \cdot e^{\hat{\beta}_1} = \hat{f}(x) \cdot e^{\hat{\beta}_1}$$

Log transformation of the response variable y

Side note: Often the natural (base e) log of y is used because for small values of $\hat{\beta}$

$$e^{\hat{\beta}} \approx 1 + \hat{\beta}$$

This is used as a justification for using the natural log, since this allows one to directly see what $e^{\hat{\beta}}$ approximately is from just looking at $\hat{\beta}$

- Although it's not very hard to use the exp() on the regression coefficients in R

Questions?

Let's try it in R…

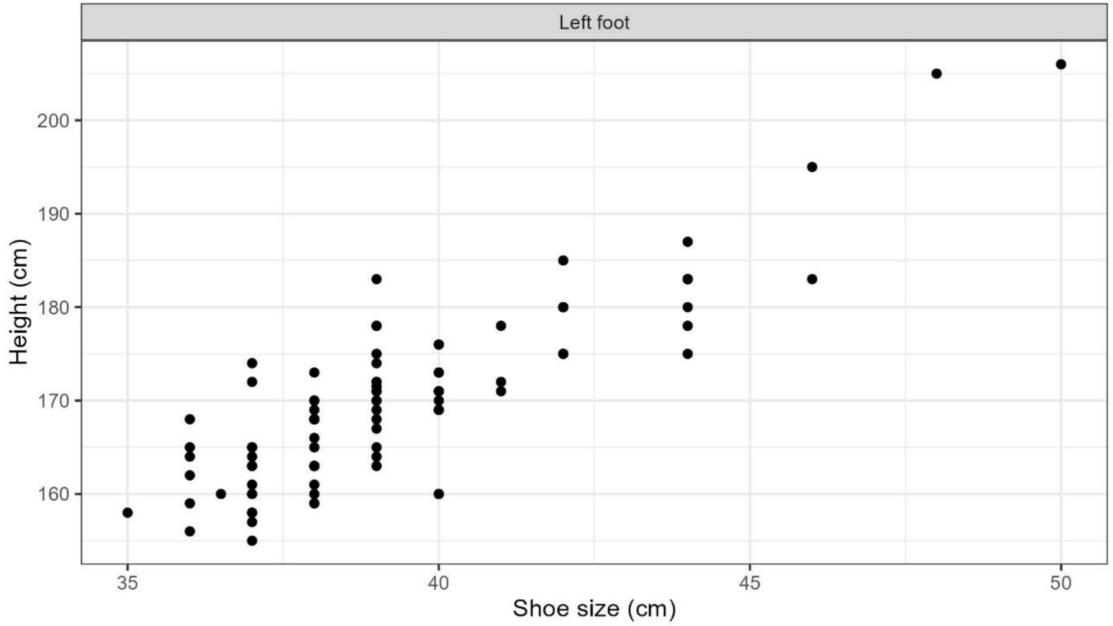
Multicollinearity

Multicollinearity

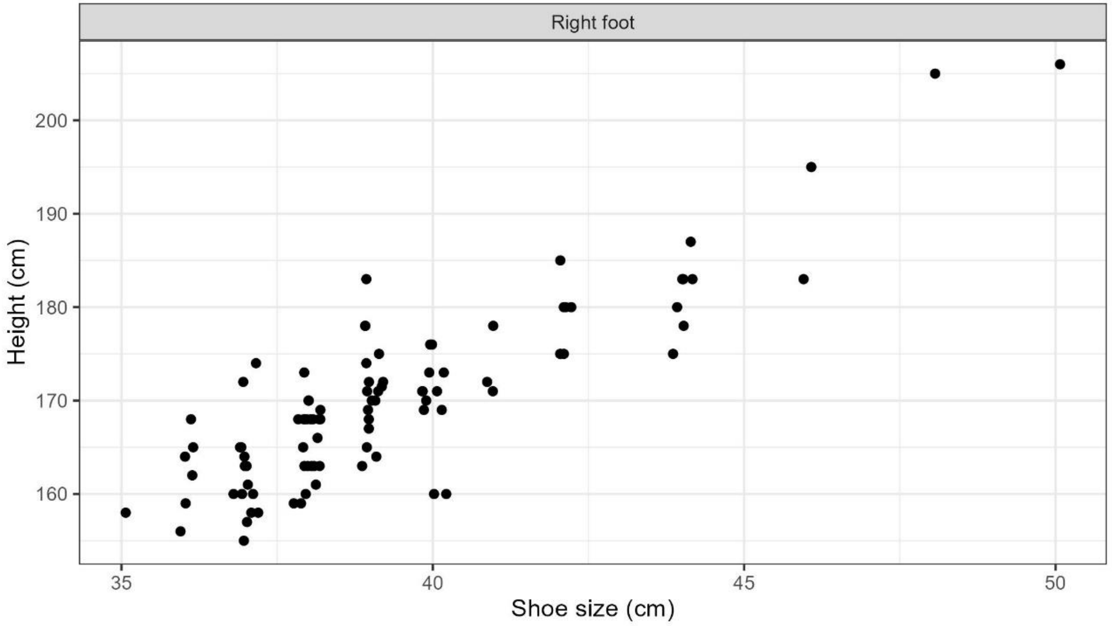
Multicollinearity occurs when two or more variables are closely related to each other

- E.g., if they have a high correlation

Left foot



Right foot

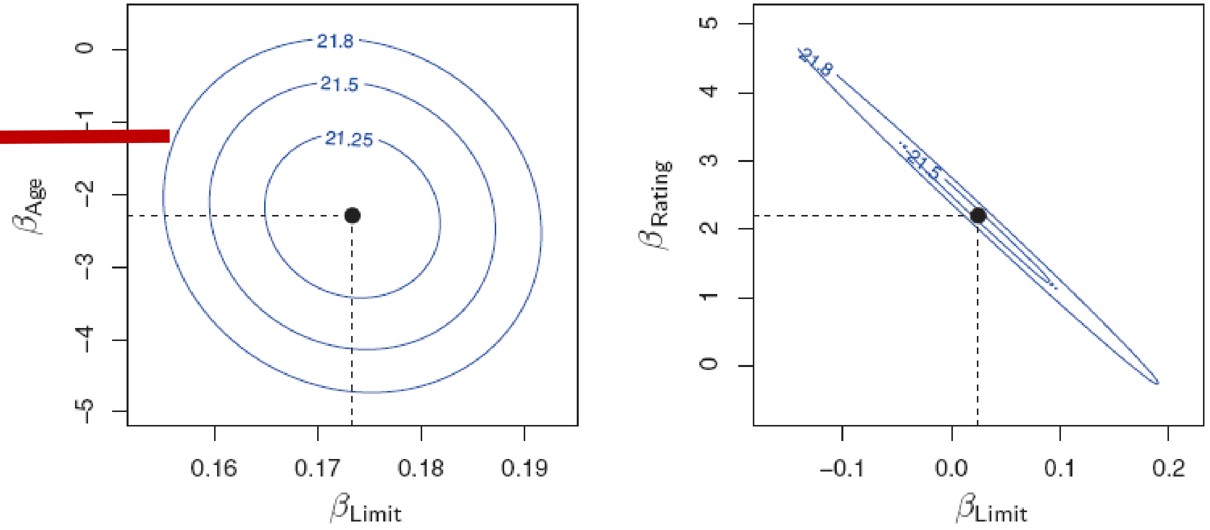


Multicollinearity

Multicollinearity can make our estimate of the regression coefficients unstable

- i.e., a large range of coefficient β-hat values give the same SSResidual and $\hat{\sigma}_e$

Contours of equal $\hat{\sigma}_e$ value



This increases our estimate of the variance of the coefficients we measure and hence can decrease the power to detect a statistically significant predictor

Multicollinearity

The **variance inflated factor** is a statistic that can be computed to test for multicollinearity for the j^{th} explanatory variable:

$$VIF_j = \frac{1}{1-R_j^2}$$

where R_j^2 is the coefficient of determination for a model to predict x_j using the other explanatory variables in the model ($x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p$)

- i.e., the R^2 value for this model:

$$\hat{x}_j = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \ldots + \hat{\beta}_{j-1} x_{j-1} + \hat{\beta}_{j+1} x_{j+1} + \ldots + \hat{\beta}_p x_p$$

<u>Rule of thumb</u>: suspect multicollinearity for VIF > 5

car::vif(lm_fit)

Are any of the predictors x_i related to y?

We can set this up as a hypothesis test:

H_0: $\beta_1 = \beta_2 = \ldots = \beta_p = 0$

H_A: At least one $\beta_j \neq 0$

We can run a parametric hypothesis test based on an F statistic to test this hypothesis

summary(lm_fit)

Left foot

```
Call:
lm(formula = height ~ left_shoe, data = height_shoe)

Residuals:
     Min       1Q   Median       3Q      Max 
-12.0750  -3.1323  -0.1036   2.6320  13.8964 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  53.2219     7.1640   7.429 5.19e-11 ***
left_shoe     2.9713     0.1818  16.343  < 2e-16 ***
```

Right foot

```
Call:
lm(formula = height ~ right_shoe, data = height_shoe)

Residuals:
     Min       1Q   Median       3Q      Max 
-12.6462  -3.2368   0.0896   2.3655  14.1697 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  53.0717     7.1283   7.445  4.8e-11 ***
right_shoe    2.9734     0.1808  16.446  < 2e-16 ***
```

Left and right foot

```
lm(formula = height ~ left_shoe + right_shoe, data = height_shoe)

Residuals:
     Min       1Q   Median       3Q      Max 
-12.9453  -3.3197   0.1906   2.3335  14.3130 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   53.141      7.165   7.416 5.78e-11 ***
left_shoe     -1.573      4.591  -0.343    0.733    
right_shoe     4.544      4.586   0.991    0.324    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.808 on 92 degrees of freedom
Multiple R-squared:  0.7445,	Adjusted R-squared:  0.7389 
F-statistic:   134 on 2 and 92 DF,  p-value: < 2.2e-16
```

Neither coefficient is significant

Overall H_0: $\beta_1 = \beta_2 = 0$ is highly significant

This can happen when there is multicolinearity

Let’s try it in R

Polynomial regression

Polynomial regression extends linear regression to non-linear relationships by including nonlinear transformations of predictors

$$
\begin{aligned}
\text{salary} = \; & \beta_0 + \beta_1 \cdot \text{endowment} \\
& + \beta_2 \cdot (\text{endowment})^2 + \\
& + \beta_3 \cdot (\text{endowment})^3 + \varepsilon
\end{aligned}
$$

Still a linear equation but non-linear in original predictors

Polynomial regression

Polynomial regression extends linear regression to non-linear relationships by including nonlinear transformations of predictors

We can compare model fits by:

- Assessing if higher order terms are statistically significant
- Looking at the r^2 values
- Running hypothesis tests comparing nested models
- Etc.

Let’s try it in R…