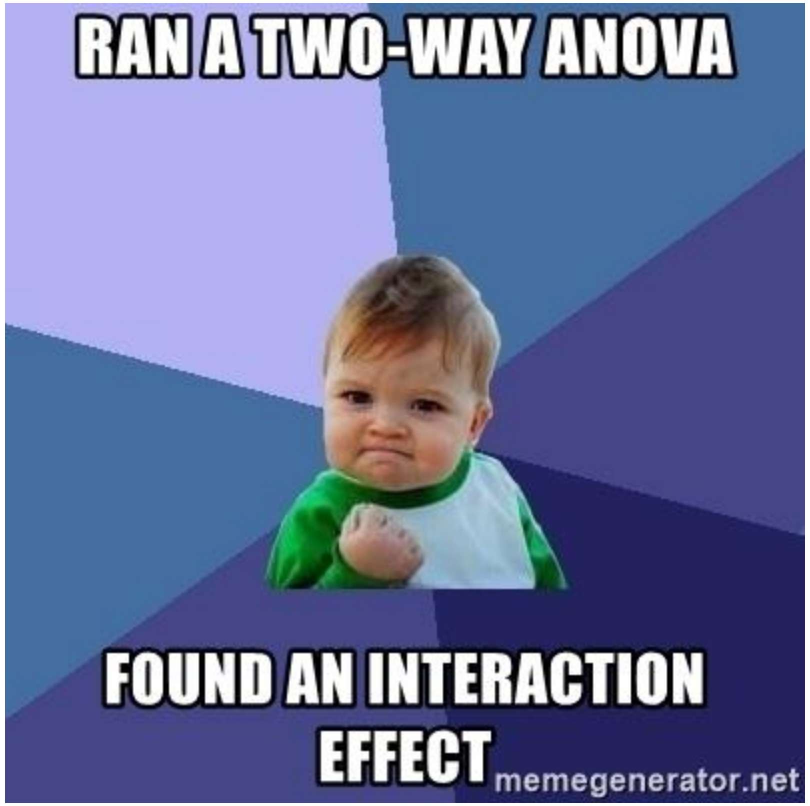
Analysis of Variance continued



Overview

Review of one-way ANOVA

Pairwise comparisons after running an ANOVA

Factorial ANOVAs and interaction effects

If there is time: unbalanced designs

Review: One-way ANOVA

A **one-way analysis of variance (ANOVA)** is a parametric hypothesis test that can be used to examine if a set of means are all the same

H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$

H_A: $\mu_i \neq \mu_j$ for some i, j

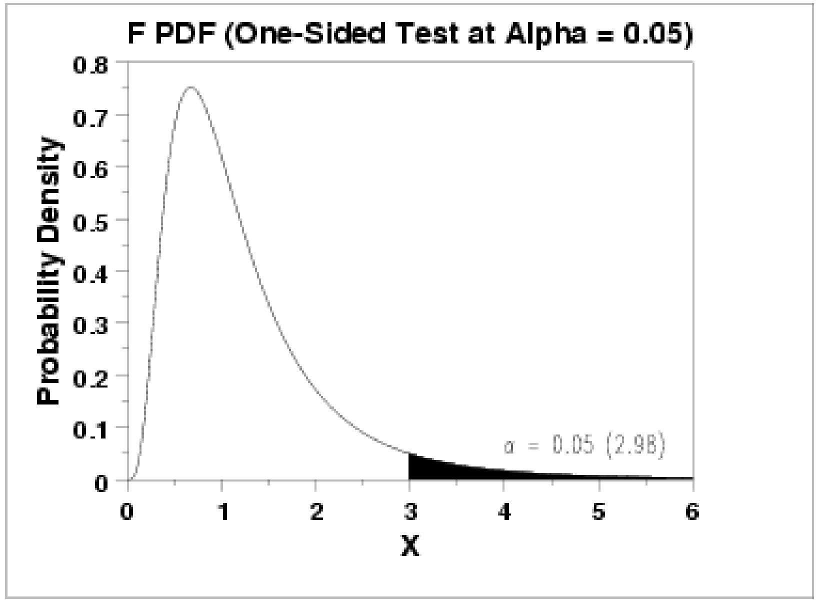
The statistic we use for a one-way ANOVA is the F-statistic

$$F = \frac{\text{between-group variability}}{\text{within-group variability}} = \frac{\frac{1}{K-1}\sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y}_{tot})^2}{\frac{1}{N-K}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2}$$

Review: One-way ANOVA

If H_0 is true, the F-statistic will come from an F distribution with parameters

- $df_1 = K - 1$
- $df_2 = N - K$



The F-distribution is valid if these conditions are met:

- The data in each group should follow a normal distribution
  - Check this with a Q-Q plot
- The variances in each group should be approximately equal
  - Check that $s_{max}/s_{min} < 2$

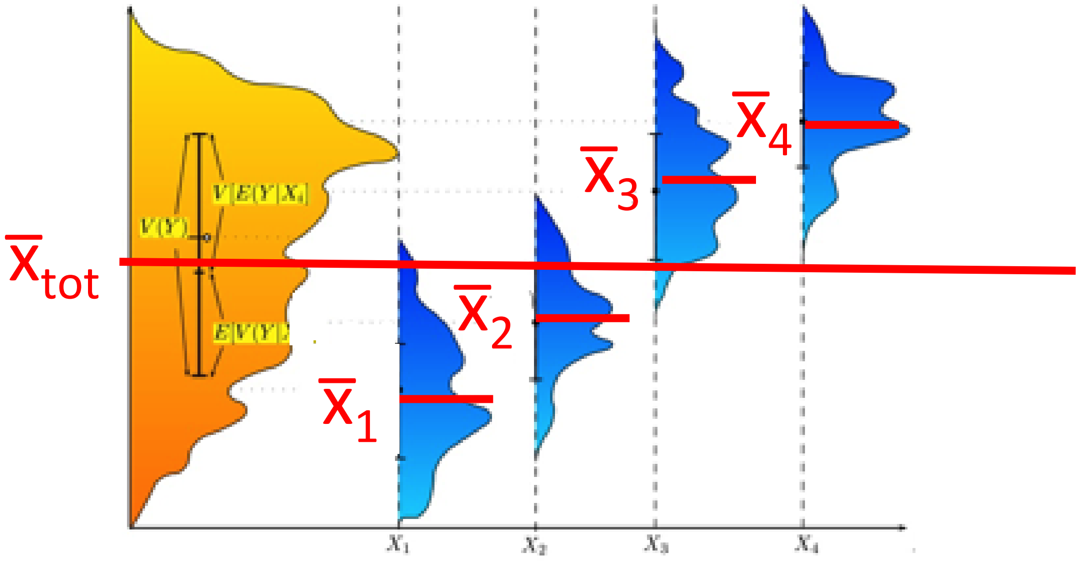
ANOVAs are fairly robust to these assumptions

Review: The F-statistic

$$F = \frac{\text{between-group variability}}{\text{within-group variability}} = \frac{\frac{1}{K-1}\sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y}_{tot})^2}{\frac{1}{N-K}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2}$$

The F statistic measures a fraction of:

$$F = \frac{\text{variability between group means}}{\text{variability within each group}}$$



ANOVA table

$$F = \frac{\text{between-group variability}}{\text{within-group variability}} = \frac{\frac{1}{K-1}\sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y}_{tot})^2}{\frac{1}{N-K}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2}$$

Source	df	Sum of Sq.	Mean Square	F-statistic	p-value
Groups	$k-1$	SSG	$MSG = \frac{SSG}{k-1}$	$F = \frac{MSG}{MSE}$	Upper tail $F_{k-1,n-k}$
Error	$n-k$	SSE	$MSE = \frac{SSE}{n-k}$		
Total	$n-1$	$SSTotal$			

Where:

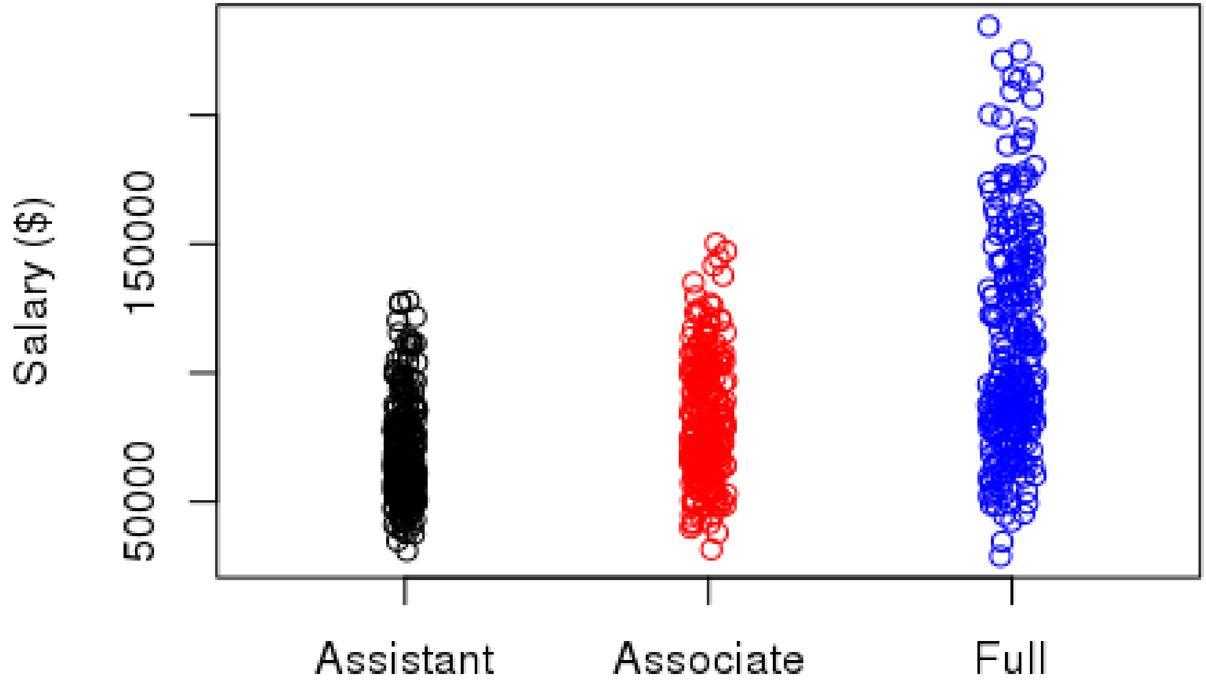
$$SST = \sum_{i-1}^{k}\sum_{j-1}^{n_i}(y_{ij} - \bar{y}_{tot})^2$$

$$SSG = \sum_{i-1}^{k} n_i(\bar{y}_i - \bar{y}_{tot})^2$$

$$SSE = \sum_{i-1}^{k}\sum_{j-1}^{n_i}(y_{ij} - \bar{y}_i)^2$$

SST = SSG + SSE

ANOVA as regression with only categorical predictors

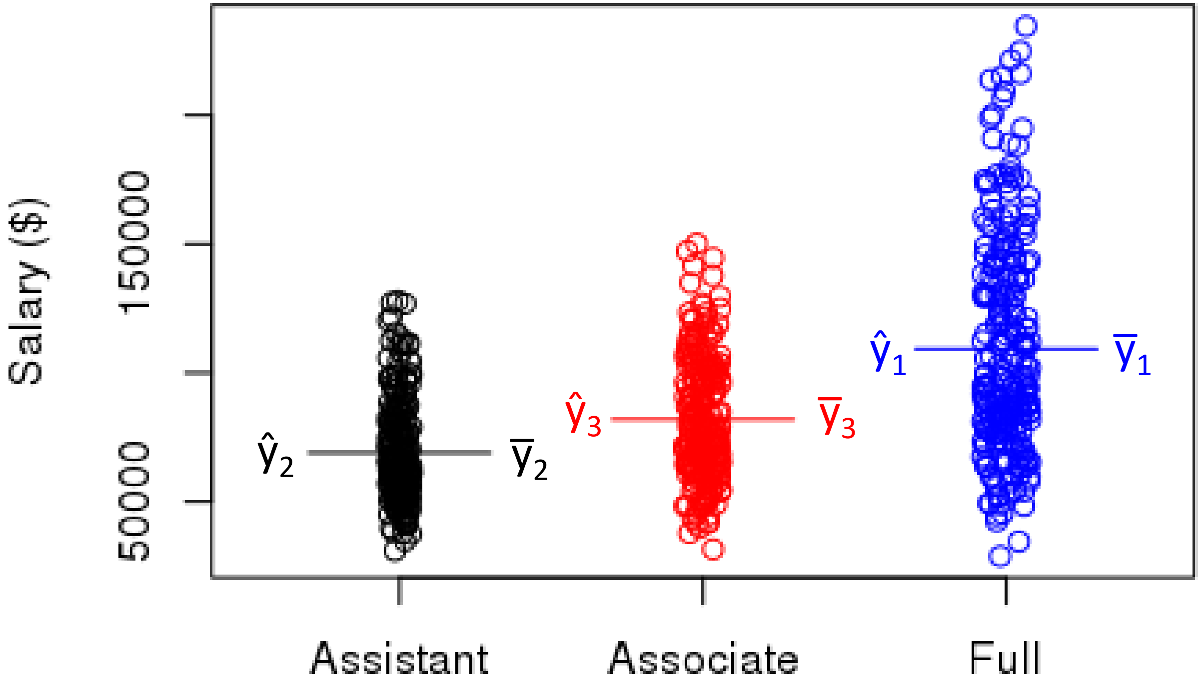


$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if Assistant Professor} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if Associate Professor} \\ \beta_0 + \epsilon_i & \text{if Full Professor} \end{cases}$$

ANOVA as regression with only categorical predictors

If we use least squares, our predicted value $\hat{y}_i$ is $\bar{y}_k$

- i.e., of x_i belongs to category k, our prediction is the mean of the y-values of points in category k



$$\hat{y}_i = \bar{y}_k = \begin{cases} \bar{y}_2 = \hat{\beta}_0 + \hat{\beta}_1 & \text{if Assistant professor} \\ \bar{y}_3 = \hat{\beta}_0 + \hat{\beta}_2 & \text{if Associate professor} \\ \bar{y}_1 = \hat{\beta}_0 & \text{if Full} \end{cases}$$

Kruskal-Wallis (non-parametric) test

There are **non-parametric** tests which don’t make assumptions about normality

The **Kruskal-Wallis** test compares several groups to see if one of the groups ‘stochastically dominates’ another

- Does not assume normality
- Tests if one group stochastically dominates another group
- Also tests whether the median for all the groups are the same
    - (if you assume groups have the same shaped and scale)
- The test is based on ranks so it is not influenced by outliers

Let's quickly review this in R…

Silly question: Do Assistant, Associate and Fully Professors get paid the same on average?

Planned comparisons/posthoc tests

Planned comparisons/posthoc tests

Suppose we run a one-way ANOVA and we are able to reject the null hypothesis.

H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$

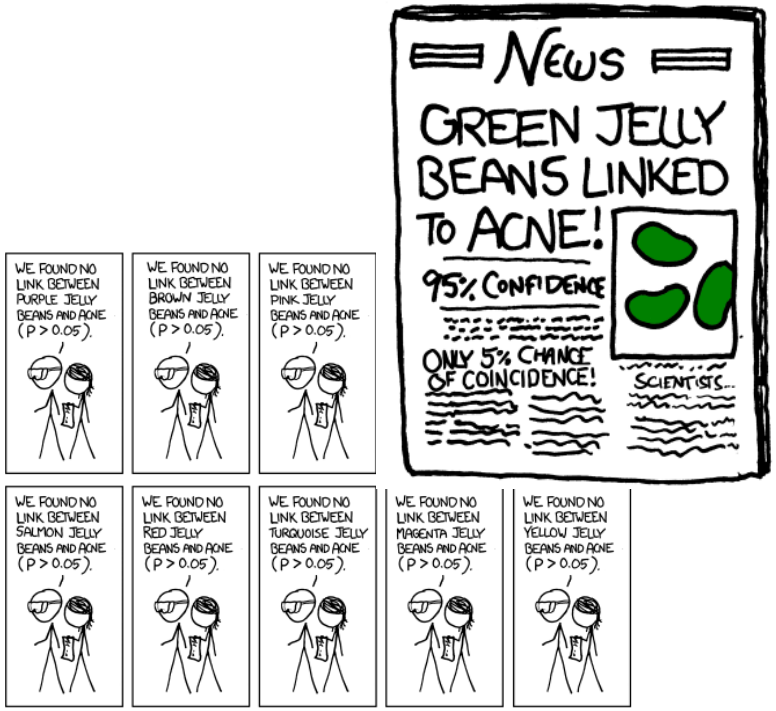
H_A: $\mu_i \neq \mu_j$ for some i, j

Q: What else would we like to know?

A: We would like to know which groups actually differed!

Q: What would be a problem if we ran two sample tests on all pairs?

A: The problem of multiplicity



Pairwise comparisons

There are several tests that can be used to examine which pairs of means differed; i.e., to test:

- H_0: $\mu_i = \mu_j$
- H_A: $\mu_i \neq \mu_j$

These tests include:

- Fisher’s Least Significant Difference
- Bonferroni procedure/correction
- Tukeys Honest significantly different

Fisher's Least Significant Difference (LSD)

1. Perform the ANOVA

2. If the ANOVA F-test is not significant, stop

3. If the ANOVA F-test is significant, then you can test H_0 for a pairwise comparisons using:

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{MSE \cdot (\frac{1}{n_i} + \frac{1}{n_j})}}$$

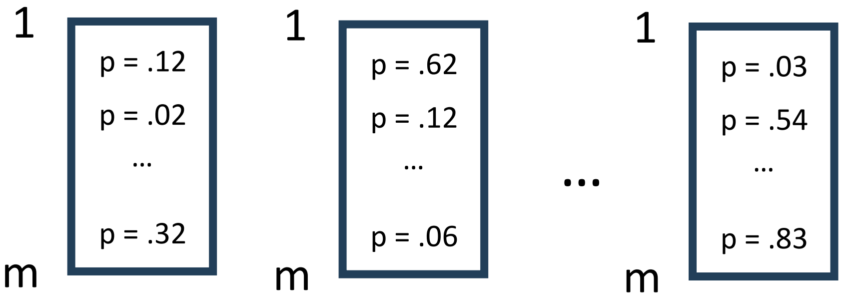
Estimate of the SE
Uses the MSE as a pooled estimate of the σ^2

Use a t-distribution with n-k degrees of freedom

Very 'liberal' tests

- Likely to make Type I errors (lots of false rejections of H_0)
- Less likely to make Type II errors (highest chance of detecting effects)

Bonferroni correction



Controls for the ***family-wise error rate***

- i.e., α = 0.05 for making ***any*** Type I error ***over all pairs of "m" comparisons***

1. Choose an α-level for the family-wise error rate α
2. Decide how many comparisons you will make. Call this m
3. Reject any hypothesis tests that have p-values less than α/m
   - Pairwise tests typically done using a t-statistic, where the MSE is used in the estimate of the SE

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{MSE \cdot (\frac{1}{n_i} + \frac{1}{n_j})}}$$

Use a t-distribution with n-k degrees of freedom

Very 'conservative' tests

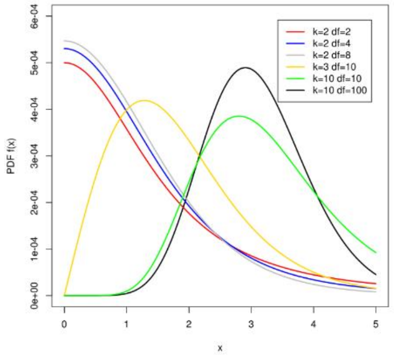
- Unlikely to make Type I errors (few false rejections of H_0)
- Likely to make Type II errors (insensitive at detecting real effects)

Tukey's Honest Significantly Different Test

Tukey's Honest Significantly Different test controls for the family-wise error rate but is less conservative than the Bonferroni correction

If the null hypothesis was true, *q* comes from a ***studentized range distribution***

$$q = \frac{\sqrt{2}(\bar{x}_{max} - \bar{x}_{min})}{\sqrt{MSE \cdot (\frac{1}{n_{max}} + \frac{1}{n_{min}})}}$$



We can compare $q = \frac{\sqrt{2}(\bar{x}_i - \bar{x}_j)}{\sqrt{MSE \cdot (\frac{1}{n_i} + \frac{1}{n_j})}}$ for a pair of means i, j, to a studentized range distribution with parameters k, and n-k, to get a p-value

- Still based on assumptions that the data in each group is normal with equal variance

Let’s try pairwise comparisons in R…

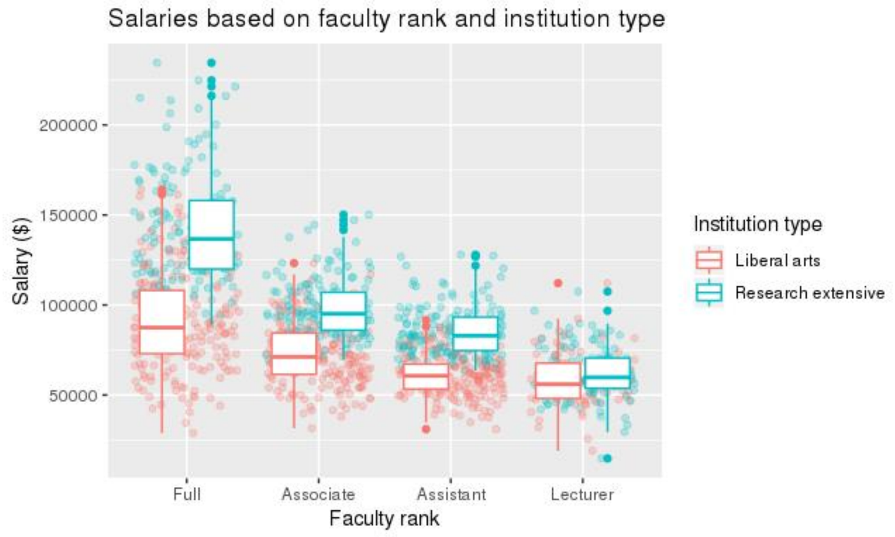
Factorial ANOVA

Factorial ANOVA

In a **factorial ANOVA**, we model the response variable y as a function of **more than one** categorical predictor

Example 1: Do faculty salaries depend on faculty rank, and the type of college/university

- Factors are:
  - **Rank**: Lecturer, Assistant, Associate, Full
  - **Institute**: liberal arts college, research university
  - 4 x 2 design



Factorial ANOVA

Example 2: A student at Queensland University of Technology conducted an experiment to determine what types of sandwiches ants prefer

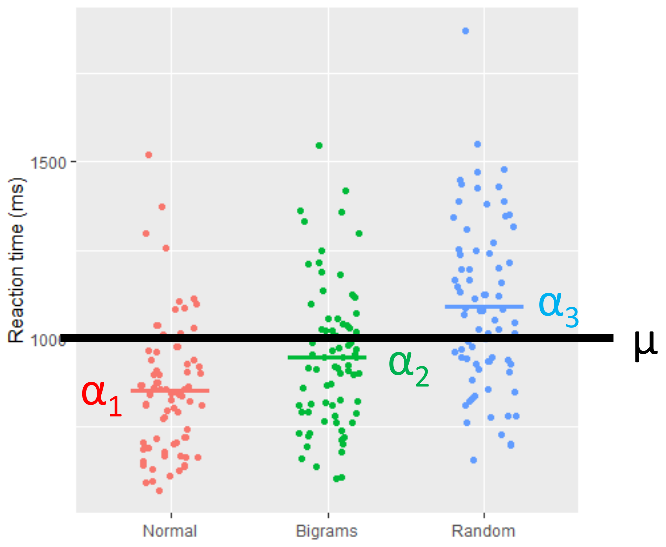
- Factors he looked at were:
  - **Bread**: rye, whole wheat multigrain, white
  - **Filling**: peanut better, ham and pickle, and vegemite
  - 4 x 3 design

The student creating 4 sandwiches of all combinations of bread and filling (48 sandwiches total) and randomly left pieces in front of ant nests.

He then measured how many ants were on the sandwiches 5 minutes later

Factorial ANOVA

It is useful to think of running an ANOVA as running a linear regression with only categorical predictors



The value for the i^{th} data point can be written as:

$$y_{ijk} = \mu + \alpha_j + \beta_k + \gamma_{jk} + \varepsilon_{ijk}$$

Where:

Main effects

- μ: is the overall mean — e.g., overall mean number of ants
- α_j : is how the j^{th} level of factor α affects y — e.g., how bread type affects ants
- β_k: is how the k^{th} level of factor β affects y — e.g., how filling affects ants
- γ_{jk}: is how the particular combination affects y — e.g., combination of bread and filling
- ε_{jki}: is the "error" not explained by the model. Comes from a 0 mean normal distribution

Interaction effects

Two-way ANOVA hypotheses

Main effect for A (bread type doesn't matter)

H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_J = 0$

H_A: $\alpha_j \neq 0$ for some j

Main effect for B (filling doesn't matter)

H_0: $\beta_1 = \beta_2 = \ldots = \beta_K = 0$

H_A: $\beta_k \neq 0$ for some k

Interaction effect (exact bread-filling combo):

H_0: All $\gamma_{jk} = 0$

H_A: $\gamma_{jk} \neq 0$ for some j, k

Where:

α_j: main effect for factor A at level j

β_k: main effect for factor B at level k

γ_{jk} : interaction between level j of factor A, and level k of factor B.

$$y_{ijk} = \mu + \alpha_j + \beta_k + \gamma_{jk} + \varepsilon_{ijk}$$

Two-way ANOVA in R with interaction

Source	df	Sum of Sq.	Mean Square	F-stat	p-value
Factor A	K - 1	SSA	MSA = SSA/(K-1)	MSA/MSE	$F_{K-1,KJ(c-1)}$.
Factor B	J - 1	SSB	MSB = SSB/(J-1)	MSB/MSE	$F_{J-1,KJ(c-1)}$
A x B	(K-1)(J-1)	SSAB	MSAB = SSAB/(K-1)(J-1)	MSAB/MSE	$F_{(K-1)(J-1),KJ(c-1)}$
Error	KJ(c - 1)	SSE	MSE = SSE/(K-1)(J-1)		
Total	N - 1	SSTotal			

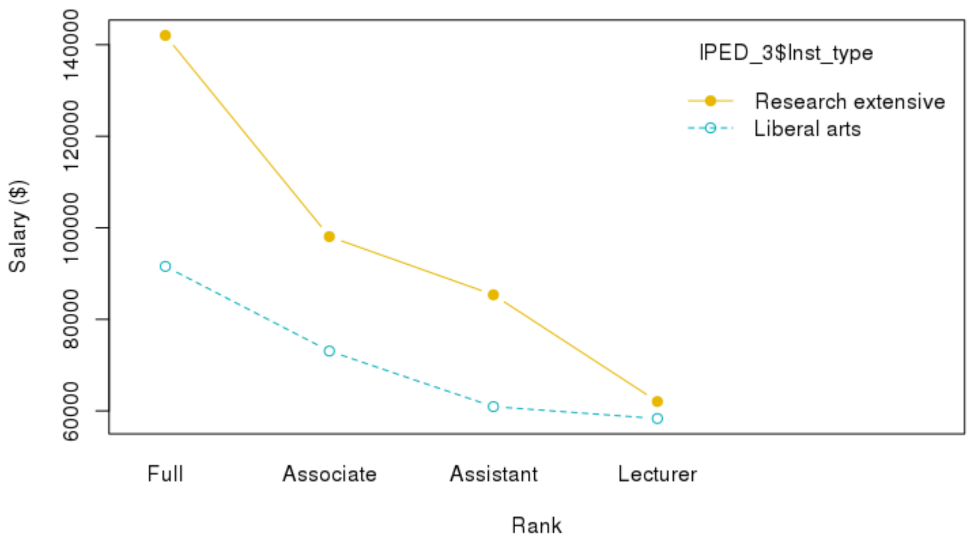
For balanced design: SSTotal = SSA + SSB + SSAB + SSE

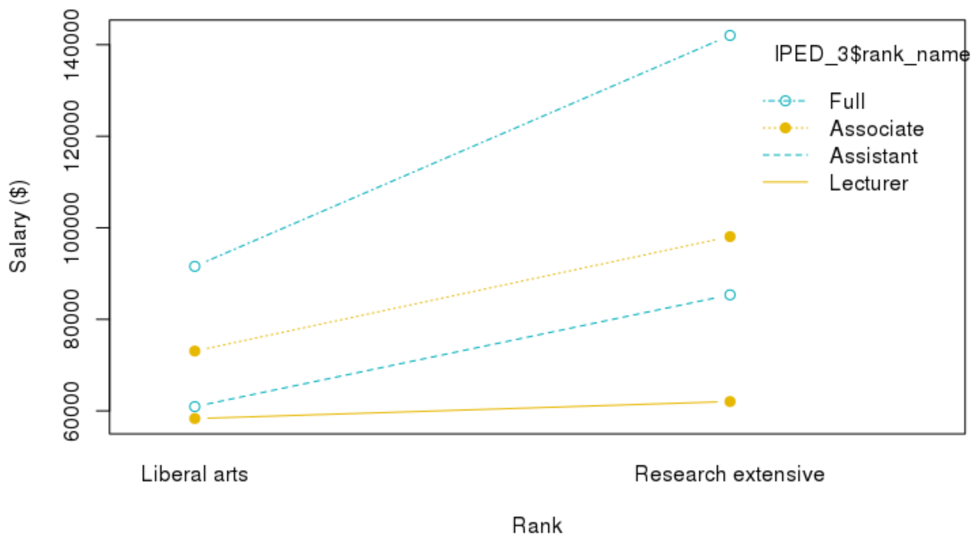
ANOVA table for a balanced design with c replicates in each group

Interaction plots

Interaction plots can help us visualize main effects and interactions

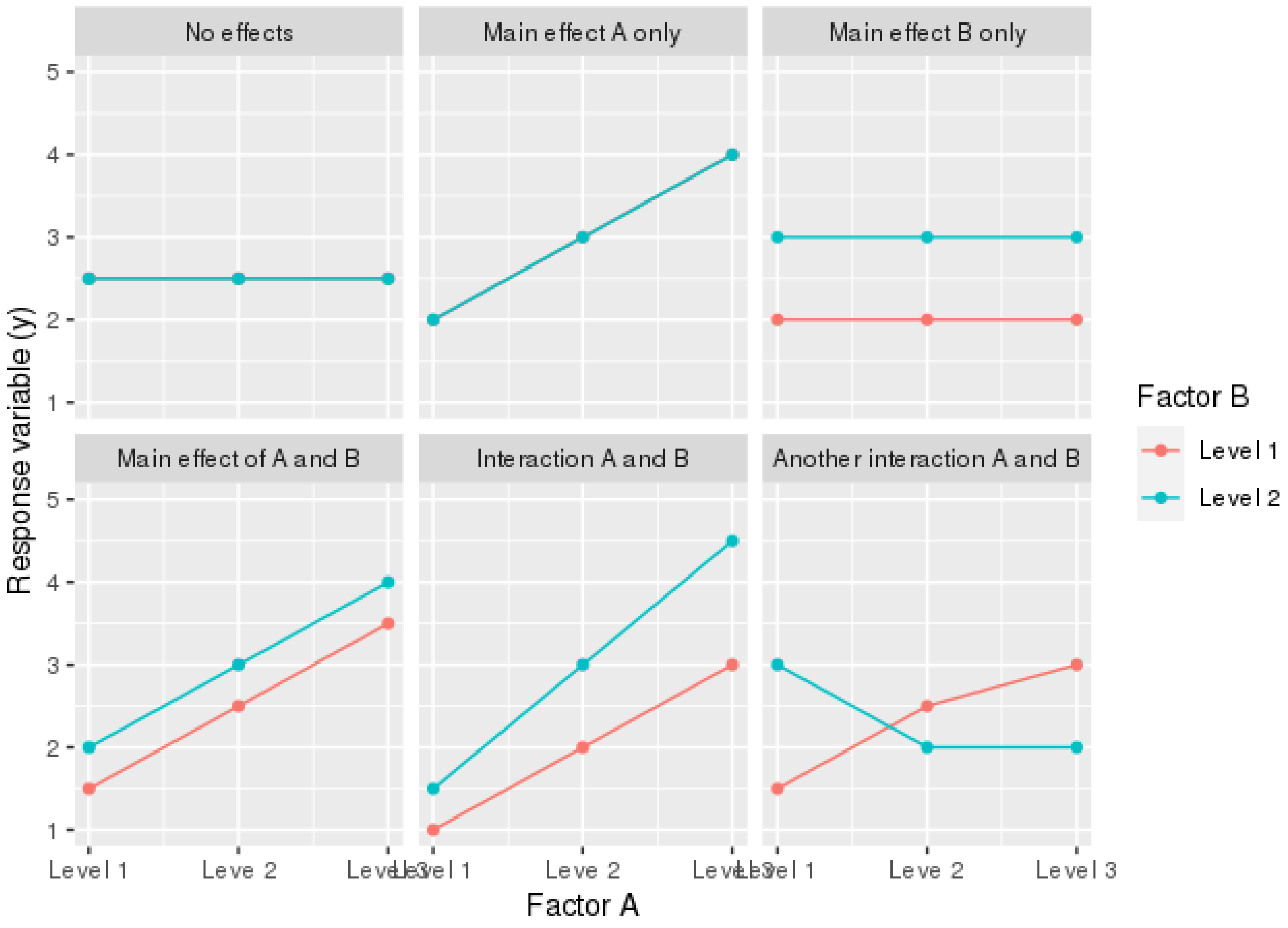
- Plot the levels of one of the factors on the x-axis
- Plot the levels of the other factor as separate lines





Either factor can be on the x-axis although sometimes there is a natural choice

Interpreting interaction plots



Interpreting interactions

When interactions are present, one must be careful interpreting main effects

- i.e., the value of one factor A, depends on the value of second factor B

For example, suppose you want to determine which condiment is the most enjoyable, chocolate sauce or mustard

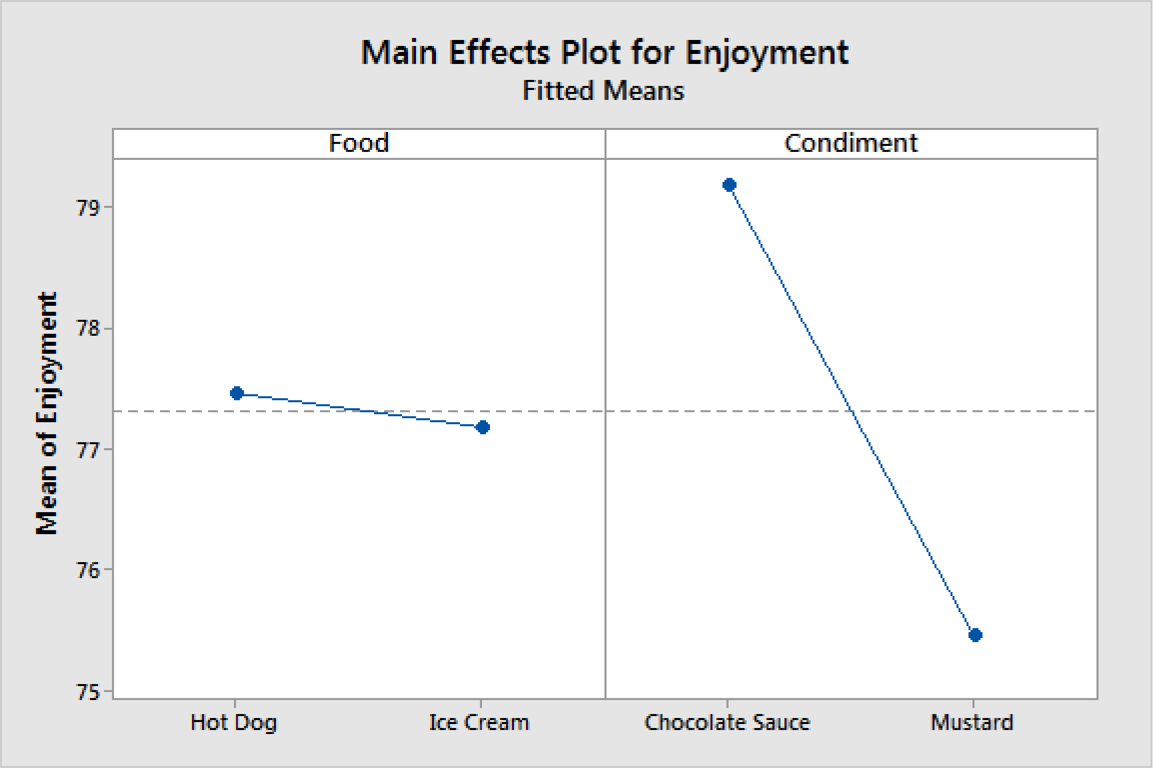
- Run a 2 x 2 ANOVA, 20 people each condition
- Get rating of enjoyment

Number of participants	Ice cream	Hot dog
Chocolate sauce	20	20
Mustard	20	20

Example taken from [Statistics by Jim]

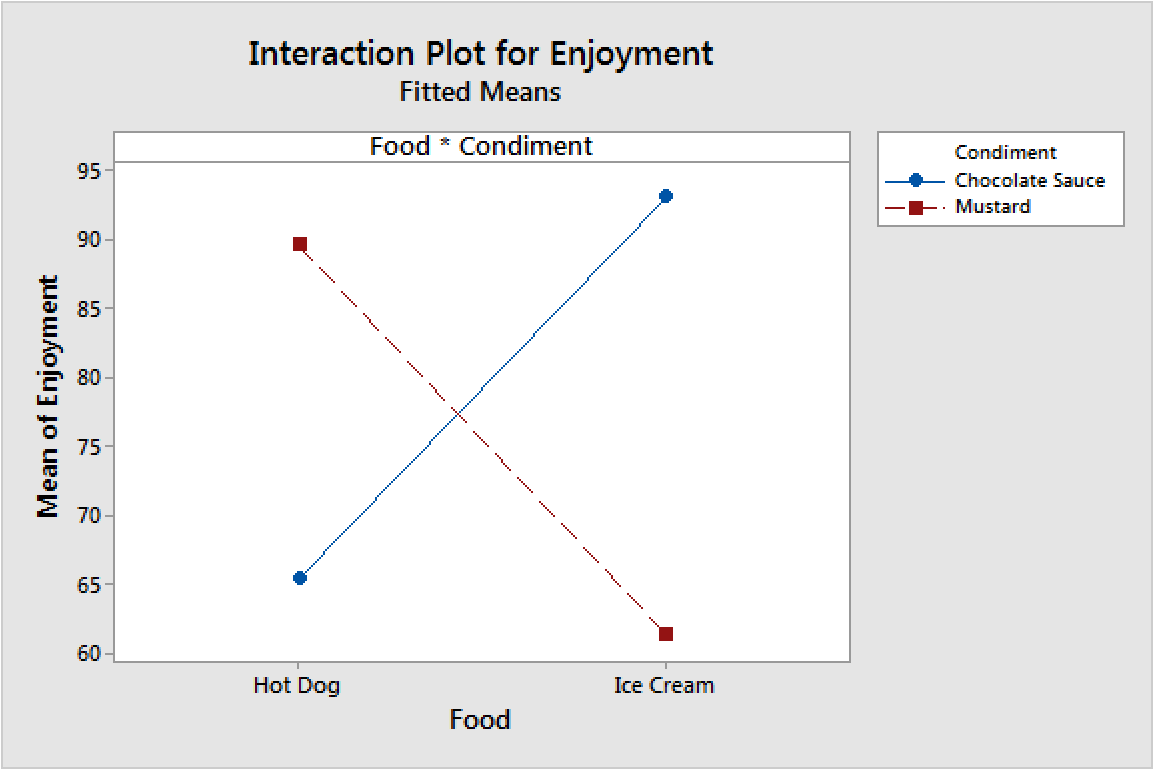
Model with only main effects

```
> summary(aov(Enjoyment ~ Food + Condiment, data = condiments_food ))
            Df Sum Sq Mean Sq F value Pr(>F)
Food         1      2     1.6   0.007  0.934
Condiment    1    278   277.5   1.214  0.274
Residuals   77  17601   228.6
```



Model with interactions

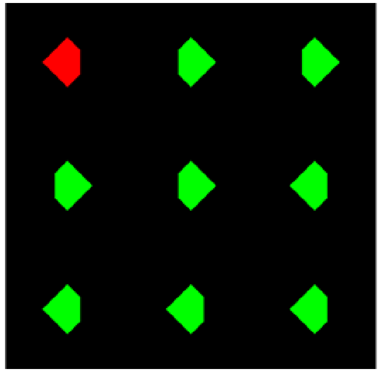
```
> summary(aov(Enjoyment ~ Food*Condiment, data = condiments_food ))
               Df Sum Sq Mean Sq F value                Pr(>F)    
Food            1      2       2   0.064               0.80136    
Condiment       1    278     278  11.071               0.00135 ** 
Food:Condiment  1  15696   15696 626.153 < 0.0000000000000002 ***
Residuals      76   1905      25                                  
```



Let’s examine two-way ANOVAs in R…

Homework 10

Homework 10



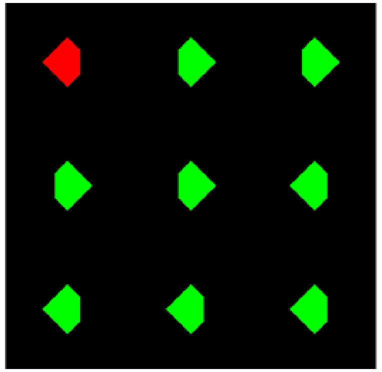
The experiment had a 9 x 2 x 2 x 2 ***factorial*** design:

1. Position (9 levels): 9 locations where the target stimulus could appear
2. Isolated/distractor condition (2 levels): isolated or cluttered display
3. Target color (2 levels): red or green target
    - For cluttered displays, the distractors always had the opposite color of the target
4. Cut direction (2 levels): left or right side of the target diamond was cut off
    - Corresponds to pressing the "z" or "/" key

The experiment had 10 blocks where all 72 (9 x 2 x 2 x 2) stimuli were shown

8 volunteer participants participated in the experiment

Homework 10



On homework 10 you will run:

- A one-way ANOVA to see if the mean reaction time is the same at all target positions
- A two-way ANOVA to look at how both position and isolated/cluttered displays affect mean reaction times
- Explore another question using this data

Questions?

Homework 10 is due 11pm on Monday December 2nd

- There is no late credit for this homework, so all submissions must be in by 11pm on Monday

Complete and balanced designs

Complete and balanced designs

Complete factorial design: at least one measurement for each possible combination of factor levels

- E.g., in a two-way ANOVA for factors A and B, if there are K levels for factor A, and J levels for factor B, then there needs to be at least one measurement for each of the KJ levels

Balanced design: the sample size is the same for all combination of factor levels

- E.g., there are the same number of samples in each of the KJ level combinations
- The computations and interpretations for non-balanced designs are a bit harder

Unbalanced designs

We can get p-values by from an F-distribution with the appropriate degrees of freedom

Two-way ANOVA table with interaction

Source	SS = Sum of Squares	df	MS = Mean Square	F
A (row factor)	SS_A	$a-1$	$MS_A = \frac{SS_A}{df_A}$	$F_A = \frac{MS_A}{MSE}$
B (column factor)	SS_B	$b-1$	$MS_B = \frac{SS_B}{df_B}$	$F_B = \frac{MS_B}{MSE}$
A×B (interaction)	$SS_{A\times B}$	$(a-1)(b-1)$	$MS_{A\times B} = \frac{SS_{A\times B}}{df_{A\times B}}$	$F_{A\times B} = \frac{MS_{A\times B}}{MSE}$
Error (within)	SSE	$ab(n-1)$	$MSE = \frac{SSE}{df_E}$	
Total	SST	$N-1$		

```
df1 <- a - 1
df2 <- ab(n-1)
pf(F_stat,  df1, df2,
            lower.tail = FALSE)
```

where “a” is the number of levels for factor A, etc.

For unbalanced designs, there are different ways to compute the sum of squares, and hence one can get different p-values

- The problem is analogous to multicollinearity. If two explanatory variables are correlated either can account for the variability in the response data

Unbalanced designs

Type I sum of squares: (also called sequential sum of squares) the order that terms are entered in the model matters

- SS(A) is taken into account before SS(B) is considered etc.
- anova(lm(y ~ A*B)) gives different results than using anova(lm(y ~ B*A))

Type II and Type III sum of squares: the order that that terms are entered into the model does not matter.

- For each factor, SS(A), SS(B), SS(AB) is taken into account after all other factors are added
- Car::Anova(lm(y ~ A*B) , type = "III") is the same as car::Anova(lm(y ~ B*A) , type = "III")

Let’s examine it R…

Bonus material: Understanding sum of squares for unbalanced designs

Unbalanced designs

We can get p-values by from an F-distribution with the appropriate degrees of freedom

Two-way ANOVA table with interaction

Source	SS = Sum of Squares	df	MS = Mean Square	F
A (row factor)	SS_A	$a-1$	$MS_A = \frac{SS_A}{df_A}$	$F_A = \frac{MS_A}{MSE}$
B (column factor)	SS_B	$b-1$	$MS_B = \frac{SS_B}{df_B}$	$F_B = \frac{MS_B}{MSE}$
A×B (interaction)	$SS_{A\times B}$	$(a-1)(b-1)$	$MS_{A\times B} = \frac{SS_{A\times B}}{df_{A\times B}}$	$F_{A\times B} = \frac{MS_{A\times B}}{MSE}$
Error (within)	SSE	$ab(n-1)$	$MSE = \frac{SSE}{df_E}$	
Total	SST	$N-1$		

```
df1 <- a - 1
df2 <- ab(n-1)
pf(F_stat,  df1, df2,
            lower.tail = FALSE)
```

where "a" is the number of levels for factor A, etc.

For unbalanced designs, there are different ways to compute the sum of squares, and hence one can get different p-values

- (for balanced designs, all methods give the same results)

Understanding sum of squares for unbalanced designs

Let's define:

- SS(A, B, AB) is the sum of squares explained by a model with interactions: lm(y ~ A*B)
- SS(A, B) is the sum of squares explained by a model w/o interactions: lm(y ~ A + B)
- SS(A) is the sum of squares explained by a model with only factor A: lm(y ~ A)
- Etc.

We can define incremental sums of squares to represent differences:

- SS(AB | A, B) = SS(A, B, AB) – SS(A, B)
- SS(A | B, AB) = SS(A, B, AB) – SS(B, AB)
- SS(B | A, AB) = SS(A, B, AB) – SS(A, AB)
- SS(A | B) = SS(A, B) – SS(B)
- SS(B | A) = SS(A, B) – SS(A)

SS accounted by interaction after SS of main effects have been subtracted

Type I sum of squares

Type I sum of squares for a fit lm(y ~ A*B) is then defined using:

- Factor A: SS(A)
- Factor B: SS(B|A) =   SS(A, B) - SS(A)
- Interaction AB: SS(AB| A, B) = SS(A, B, AB) - SS(A, B)

The advantage of this method is that SST = SSA + SSB + SSAB + SSE

The disadvantage is that the order you specify terms affects which factors are determined to be statistically significant

Type II sum of squares

Type II sum of squares for a fit lm(y ~ A*B) is then defined using:

- Factor A: SS(A | B) = SS(A, B) - SS(B)
- Factor B: SS(B | A) = SS(A, B) - SS(A)
- Interaction AB: SS(AB| A, B) = SS(A, B, AB) - SS(A, B)

The advantage is that the order you specify terms does not effect which factors are determined to be statistically significant

The disadvantage is that the relationship SST = SSA + SSB + SSAB + SSE does not hold

Type III sum of squares

Type III sum of squares for a fit lm(y ~ A*B) is then defined using:

- Factor A: SS(A | B, AB) = SS(A, B, AB) - SS(B, AB)
- Factor B: SS(B | A, AB) = SS(A, B, AB) - SS(A, AB)
- Interaction AB: SS(AB| A, B) = SS(A, B, AB) - SS(A, B)

The advantage is that the order you specify terms does not effect which factors are determined to be statistically significant

The disadvantage is that the relationship SST = SSA + SSB + SSAB + SSE does not hold