Parametric hypothesis tests

Overview

Review: hypothesis test for 2 means – using a t-statistic

Probability functions

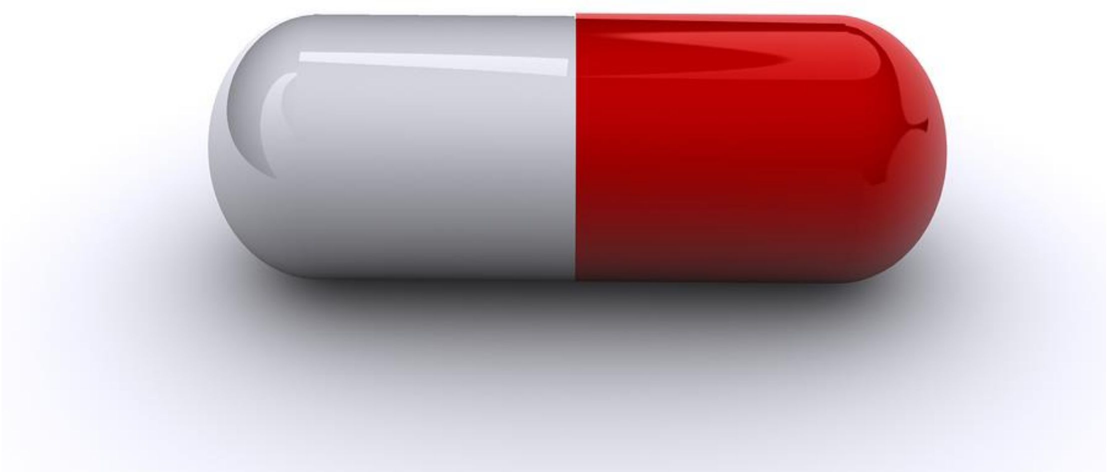
Parametric tests – the t-tests

Where we are in the plan for the semester

			Analysis	R
1	Aug 29	Course overview, introduction to R, descriptive statistics		base R
2	Sep 3-6	Review of central statistical concepts and exploratory analysis using R	resampling methods	
3	Sep 10-12	Confidence Intervals and the bootstrap		
4	Sep 17-19	Review of hypothesis tests and permutation tests in R		
5	Sep 24-26	Parametric hypothesis tests and theories of hypothesis testing		

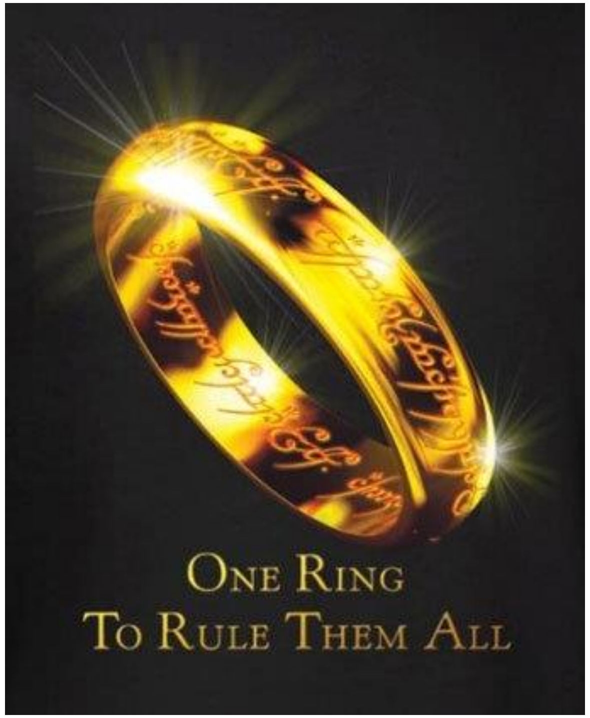
t-tests	78 respondents	69 %	
confidence intervals	87 respondents	77 %	
the bootstrap	28 respondents	25 %	
permutation tests	17 respondents	15 %	
one-way ANOVA	41 respondents	36 %	

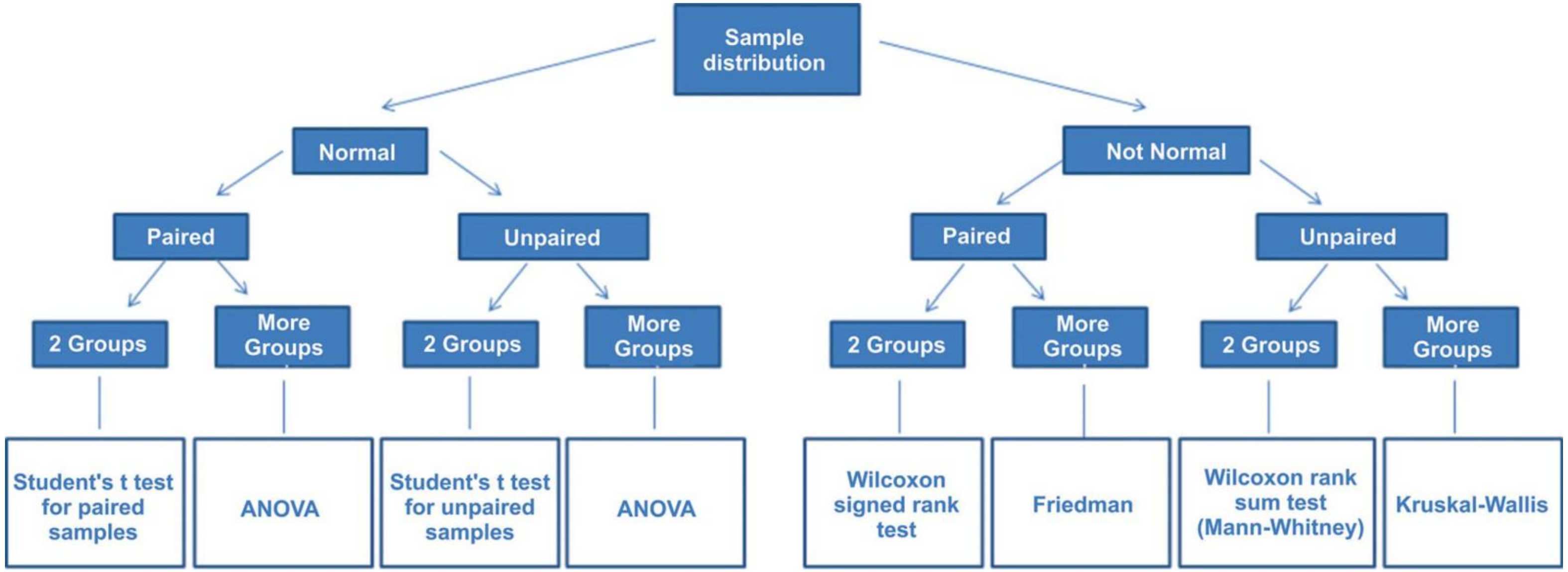
Very quick review of randomization test for two means



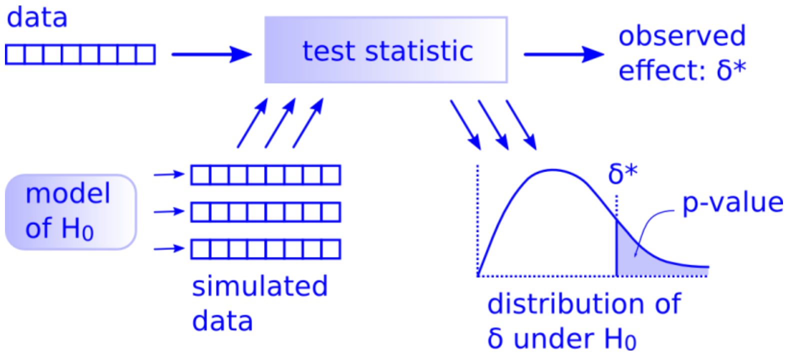
Question: Is this pill effective?

The big picture: There is only one hypothesis test!

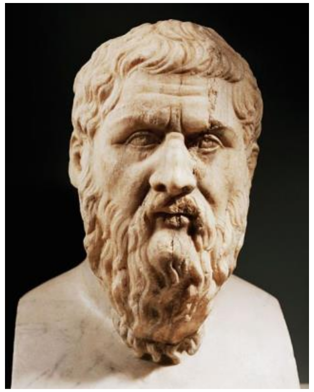




Just need to follow 5 steps!



Review: Five steps of hypothesis testing

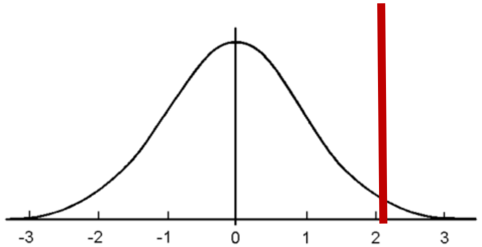


1. State H_0 and H_A
    - Assume Gorgias (H_0) was right
    - α = .05 of the time he will be right, but we will say he is wrong

2. Calculate the actual observed statistic

3. Create a distribution of what statistics would look like if Gorgias is right
    - Create the **null distribution** (that is consistent with H_0)

4. Get the probability we would get a statistic more than the observed statistic from the null distribution
    - p-value

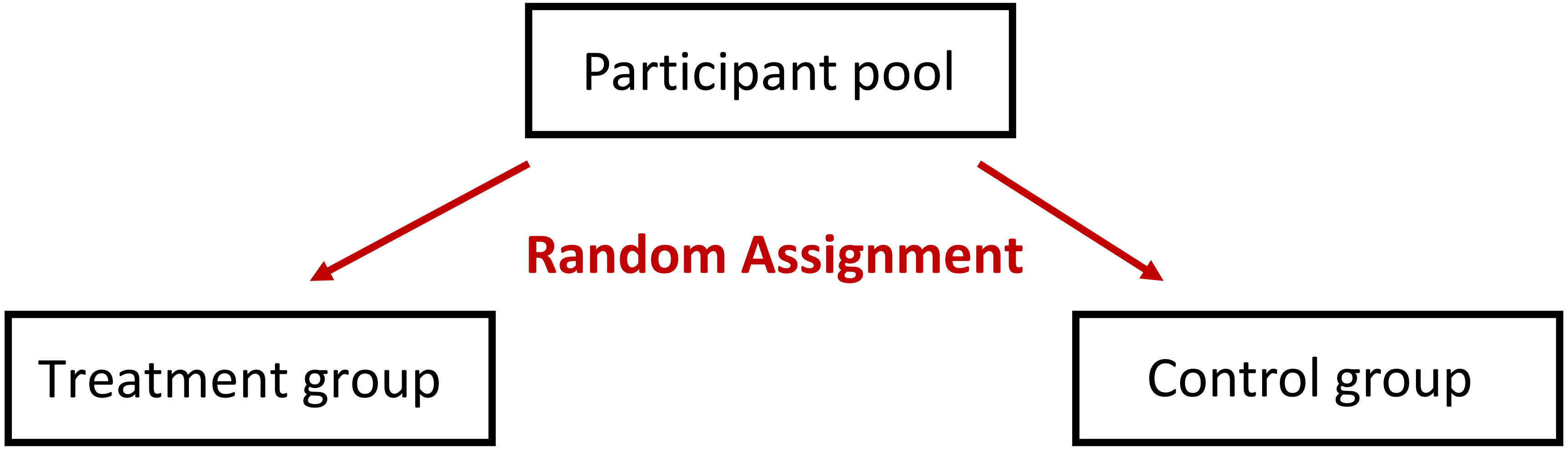


5. Make a judgement
    - Assess whether the results are statistically significant

Review: Drug study experimental design

Take a group of participant and ***randomly assign***:

- Half to a *treatment group* where they get the pill
- Half in a *control group* where they get a fake pill (placebo)
- See if there is more improvement in the treatment group compared to the control group



Hypothesis tests for differences in two group means

1. State the null and alternative hypothesis
   - H_0: $\mu_{Treatment} = \mu_{Control}$ or $\mu_{Treatment} - \mu_{Control} = 0$
   - H_A: $\mu_{Treatment} > \mu_{Control}$ or $\mu_{Treatment} - \mu_{Control} > 0$

2. Calculate statistic of interest
   - For randomization/permutation tests we have a choice of the statistic to use

The statistic used before: $\bar{x}_{Effect} = \bar{x}_{Treatment} - \bar{x}_{Control}$

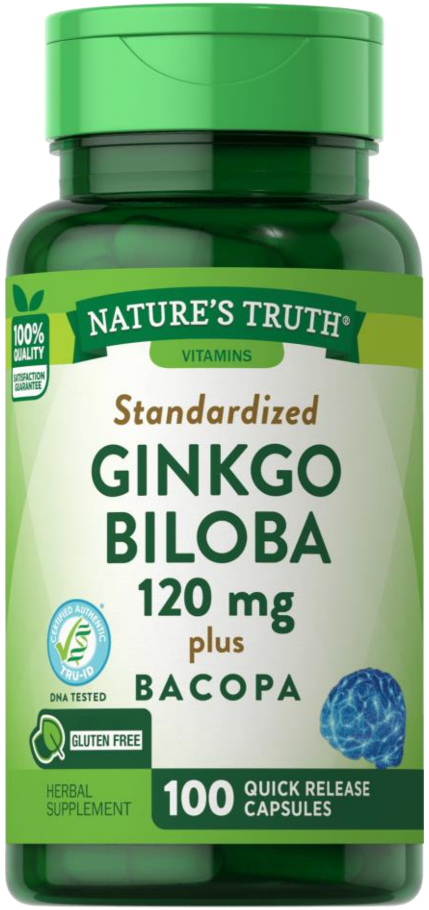
Let's try Welch's t-statistic instead: $$t = \frac{\bar{x}_t - \bar{x}_c}{\sqrt{\frac{s_t^2}{n_t} + \frac{s_c^2}{n_c}}}$$

Example: Does Gingko improve memory?

A double-blind randomized controlled experiment by Solomon et al (2002) investigated whether taking a Ginkgo supplement could improve memory

- A treatment group of n = 104 participants took a Gingko supplement 3 times per day for 6 weeks
- A control group of n = 99 participants took a placebo 3 times per day for 6 weeks

Question: Was there a difference in the mean cognitive score between the treatment and control groups?



2. Visual the data can calculate the observed statistic

Last class we used a difference of means as our observed statistic:

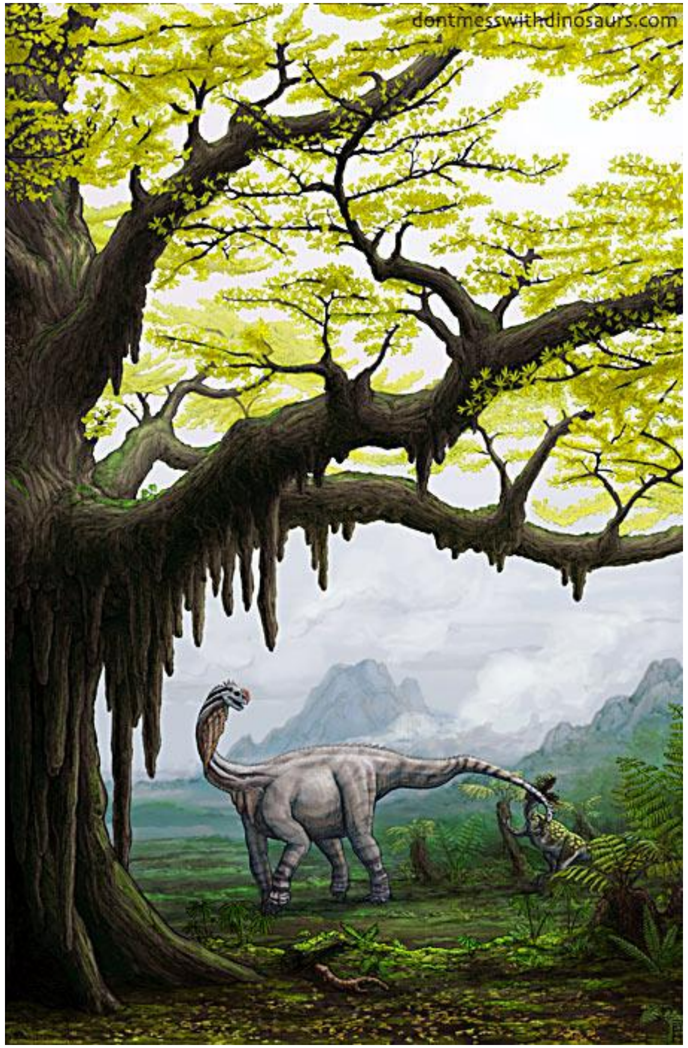
- $\bar{x}_{Effect} = \bar{x}_{Gingko} - \bar{x}_{Placebo}$

With randomization/permutation tests we have the freedom to choose any statistic

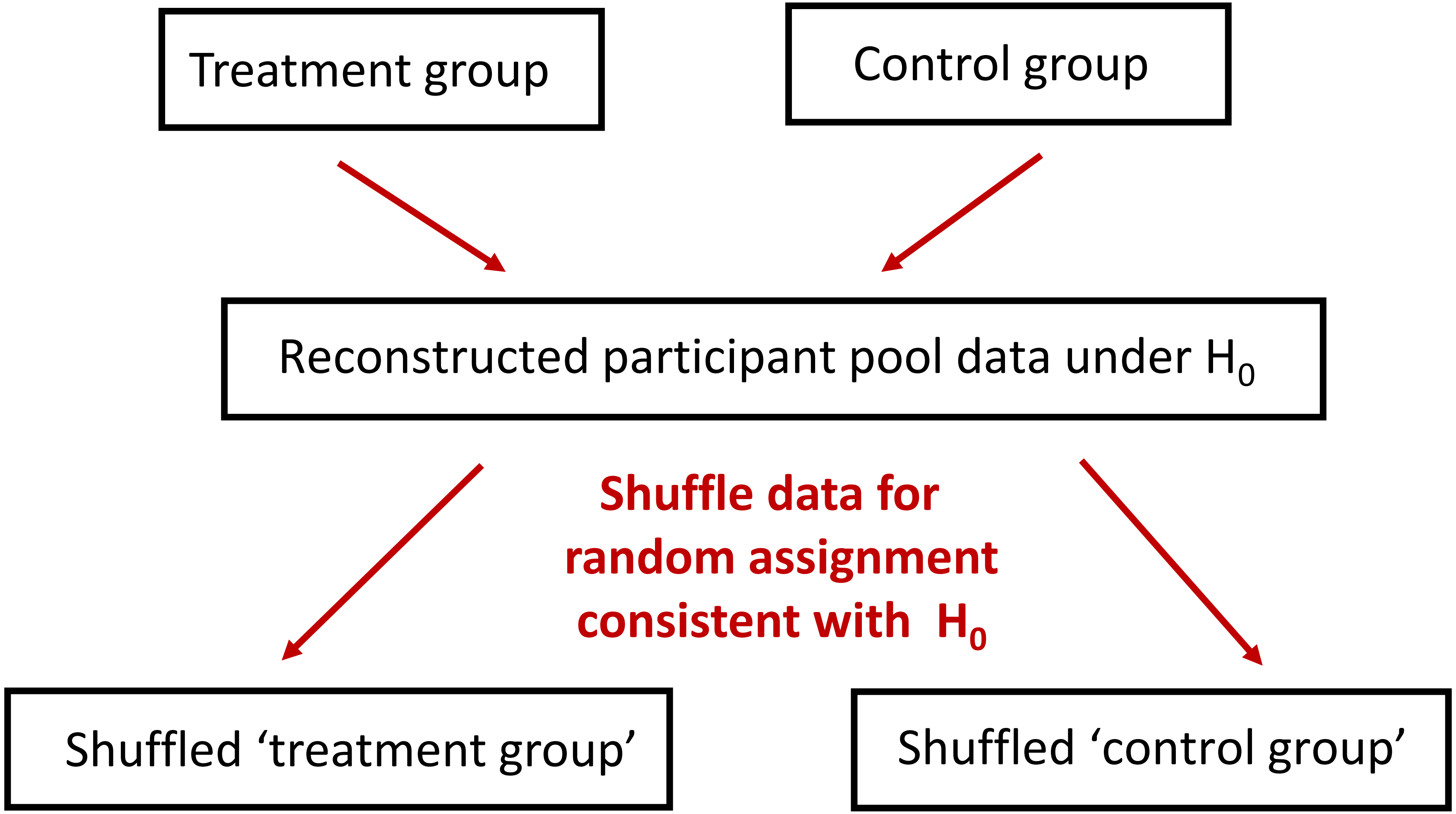
Let's try using a t-statistic!

- t = -1.53

$$t = \frac{\bar{x}_t - \bar{x}_c}{\sqrt{\frac{s_t^2}{n_t} + \frac{s_c^2}{n_c}}}$$



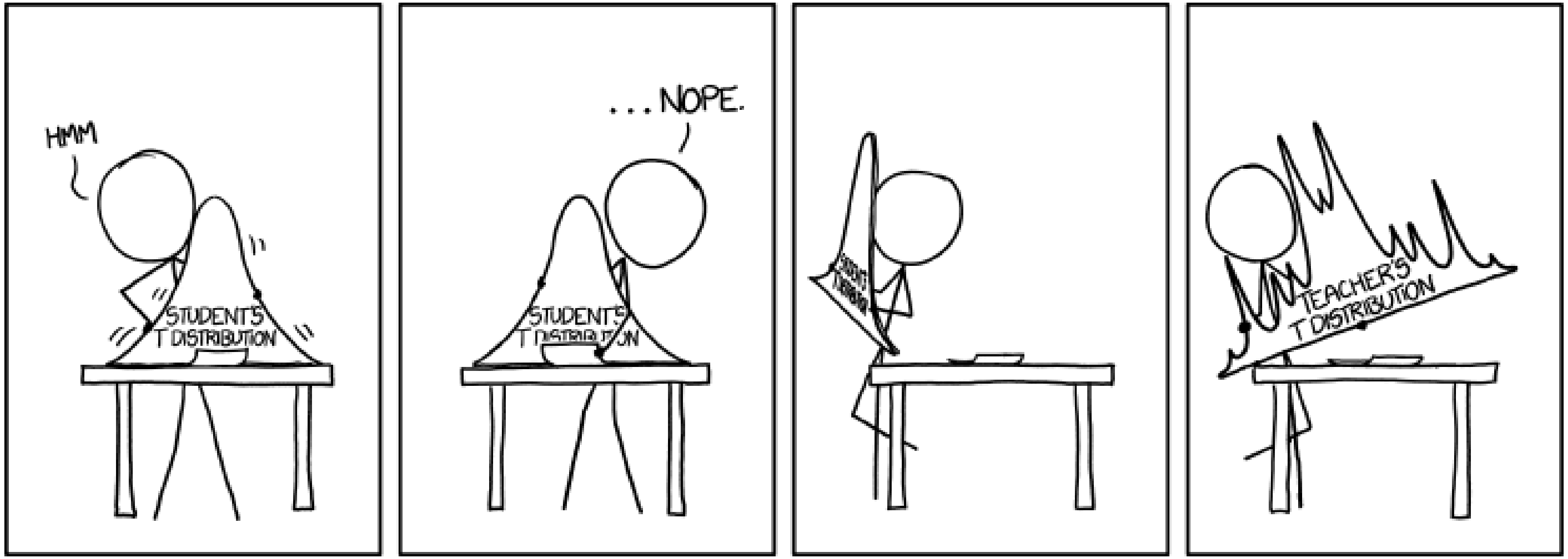
3. Create the null distribution!



One null distribution statistic: t_{shuff}

Repeat 10,000 times for null distribution

Let's quickly try the rest of the hypothesis test in R…



Permutation/randomization tests for other parameters

We can run permutation tests to test other parameters

For example, we could test if ***more than*** 2 means are equal

What are the null and alternative hypothesis for this?

- **H_0**: $\mu_1 = \mu_2 = \ldots = \mu_k$
- **H_A**: $\mu_i \neq \mu_j$ for at least one pair i, j

What is a statistic we could use?

- max $\bar{x}$ - min $\bar{x}$

How could we generate a null distribution?

Group 1 | Group 2 | . . . | Group k

Reconstructed participant pool data under H_0

Shuffle data for random assignment consistent with H_0

Shuffled 'group 1' | Shuffled 'group 2' | . . . | Shuffled 'group k'

Compute statistics from shuffled groups

$$\max \bar{x}_{shuff} - \min \bar{x}_{shuff}$$

Try it at home!

Permutation/randomization tests for other parameters

Suppose we wanted to test whether there is an association between to variables

- E.g. is there a correlation between the number of pages in a book and the books price?

Q_1: What are the null and alternative hypotheses?

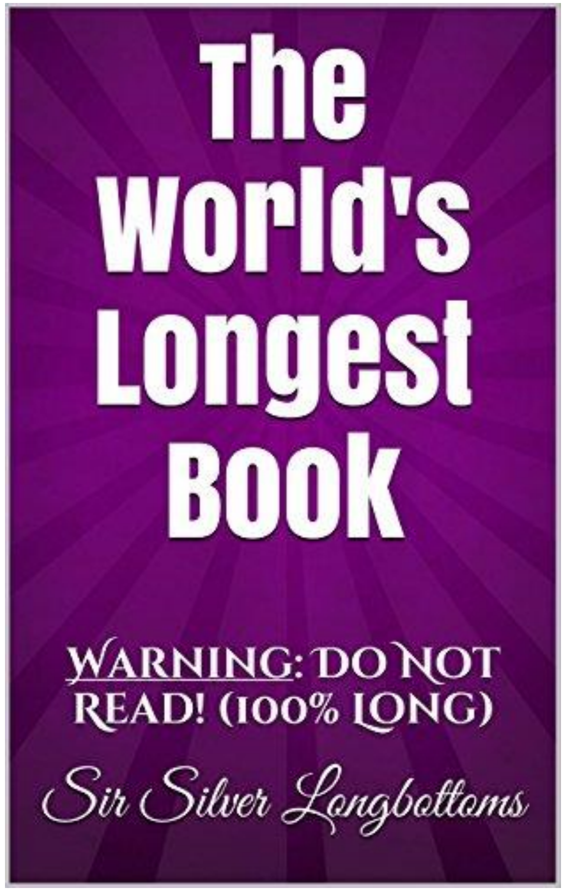
- H_0: $\rho = 0$
- H_A: $\rho > 0$

Q_2: What is the observed statistic?

- SDS230::download_data("amazon.rda")
- r_correlation <- cor(amazon$NumPages, amazon$List.Price)

Q_3: How can we create a null distribution?

- Try it at home!



Parametric probability distributions

Parametric hypothesis tests

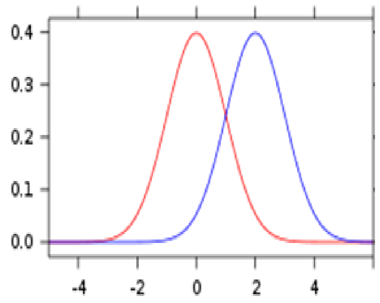
In **parametric hypothesis tests**, the null distribution is given by a density function.

These density functions have a finite set of ***parameters*** that control the shape of these functions

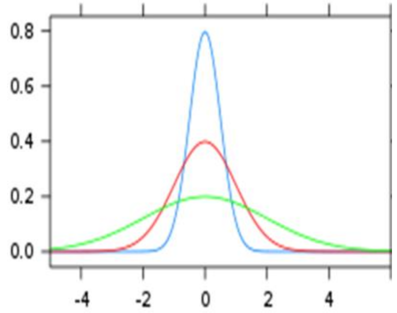
- Hence the name “parametric hypothesis tests”
- Example: the normal density function has two parameters: μ and ***σ***

Remember density curves?

Changing μ



Changing σ



Density Curves

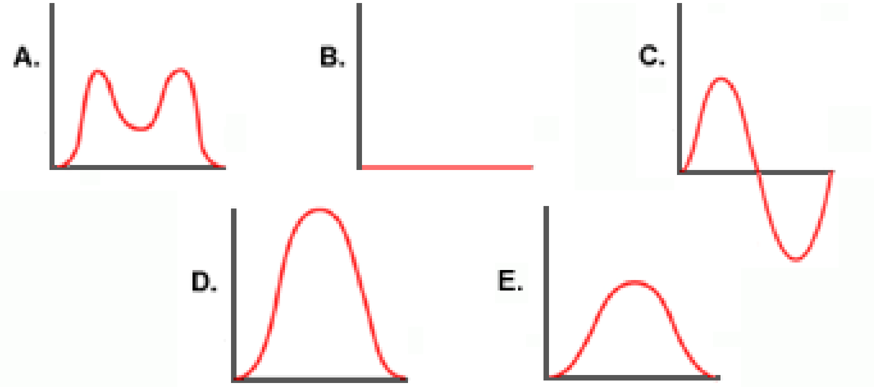
A **density curve** is a mathematical function f(x) that can be used to model data

- We can imagine density curves as histograms that have:
  - Infinitely large data sample
  - With infinitely small bins sizes
  - Normalized to have an area of 1

Density curves have two defining properties:

1. The total area under the curve f(x) is equal to 1
2. The curve is always ≥ 0

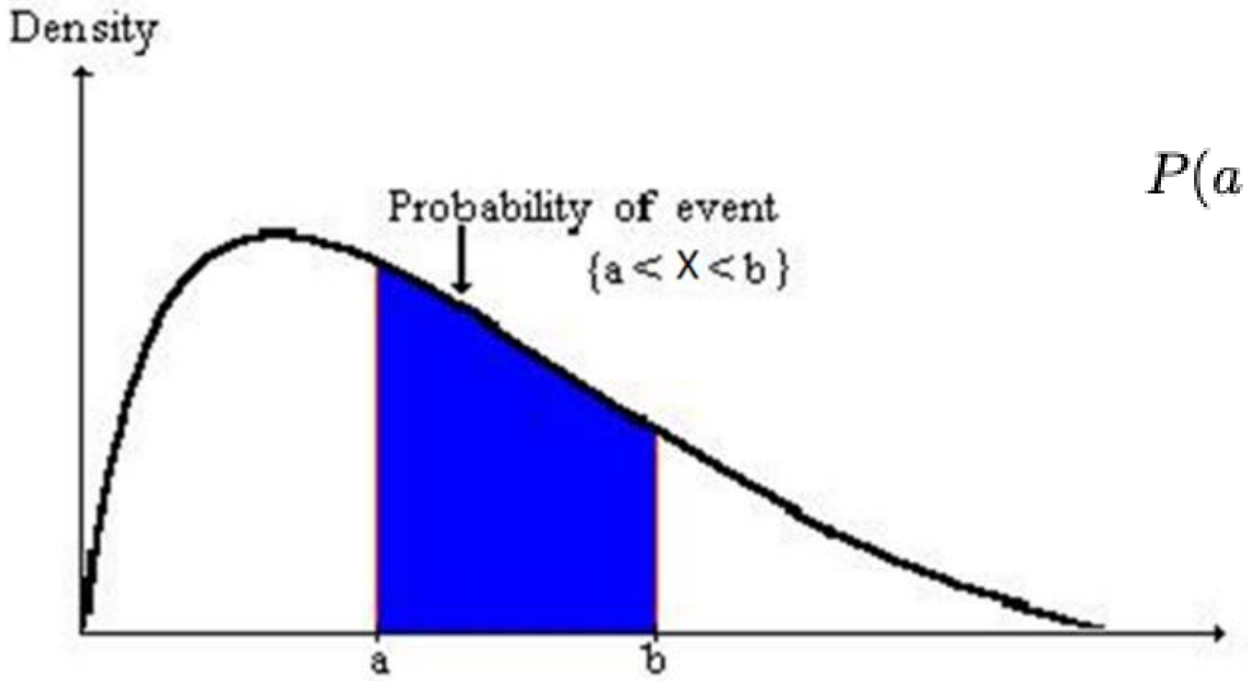
Which of these could ***not*** be a density curve?



Density Curves

The <u>area under the density curve</u> in an interval [a, b] models the probability that a random number X will be in the interval

P(a < X < b) is the area under the curve from a to b



$$P(a < X < b) = \int_a^b f(x)dx$$

Examples of density curves

R has built in functions to create density curves

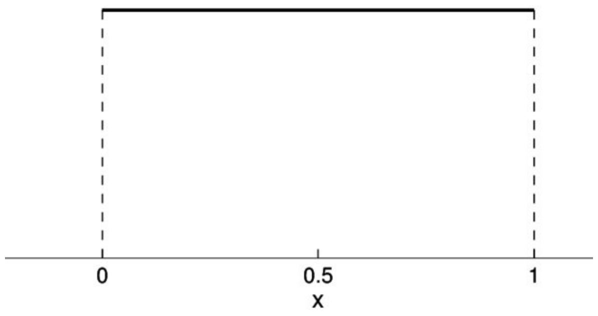
- All these functions start with the letter ***d***

The uniform distribution

- (here b = 1, a = 0)

```
> x <- seq(-.2, 1.2, by = .001)
> y <- dunif(x)
>  plot(x, y, type = "l")
```

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

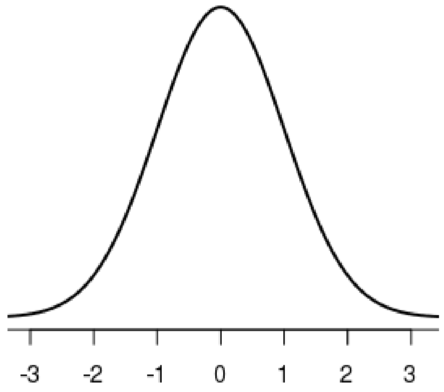


The normal distribution

- (here μ = 0, σ = 1)

```
>  x <- seq(-3, 3, by = .001)
>  y <- dnorm(x)
> plot(x, y, type = "l")
```

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Examples of density curves

R has built in functions to create density curves

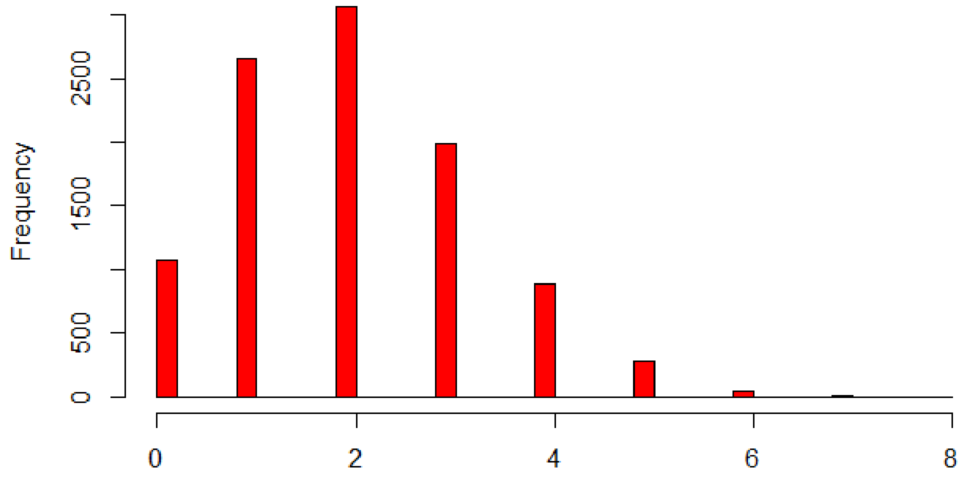
- All these functions start with the letter ***d***

The binomial distribution

- (actually a probability mass function)

$$f(k; n, \pi) = \binom{n}{k} \pi^k (1-\pi)^{n-k}$$

```
> x <- 0:8
> y <- dbinom(x, 8, .2)
> names(y) <- x
> barplot(y)
```



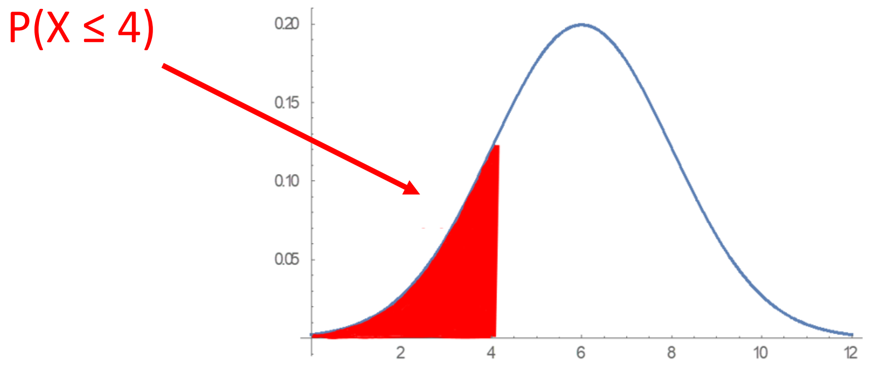
Cumulative distribution functions

Cumulative distribution functions give the probability of getting a random value X less than or equal to a value x: P(X ≤ x)

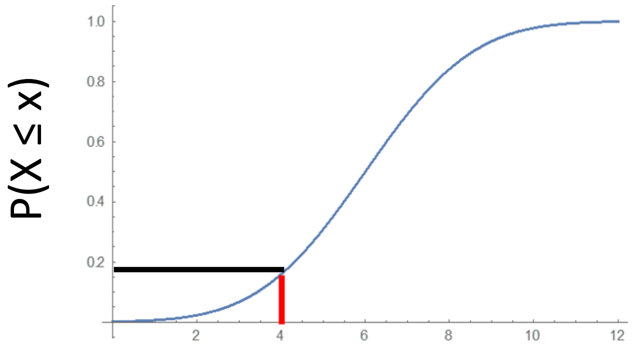
- For example, we would write the probability of getting a random number X less than 2 as: P(X ≤ 2)

Cumulative distribution functions are obtained by calculating the area under a probability density function

Density: f(x)



Cumulative distribution F(x)



$$P(X \leq x)$$

$$= F(x)$$

$$= \int_{-\infty}^{x} f(x)dx$$

Examples of cumulative distributions in R

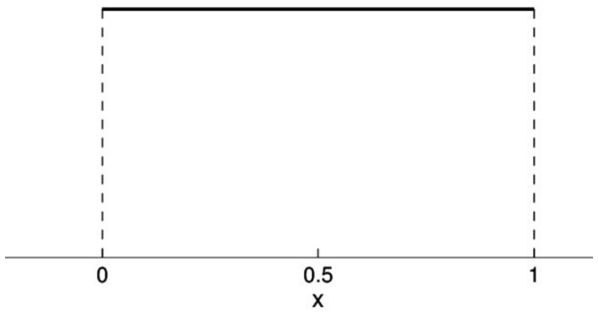
R has built in functions to get probabilities from different distributions

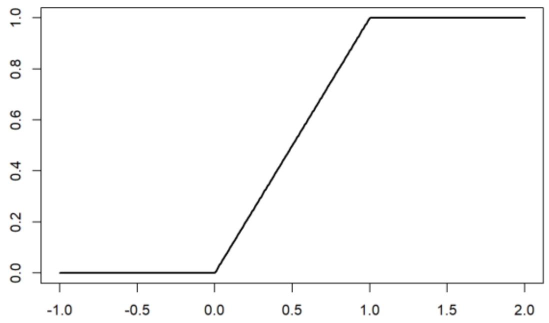
- All these functions start with the letter ***p***

The uniform distribution

P(X ≤ .25)

punif(.25)

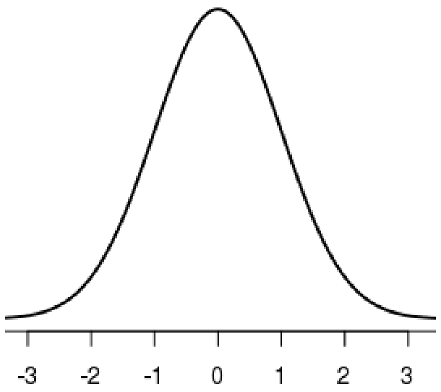


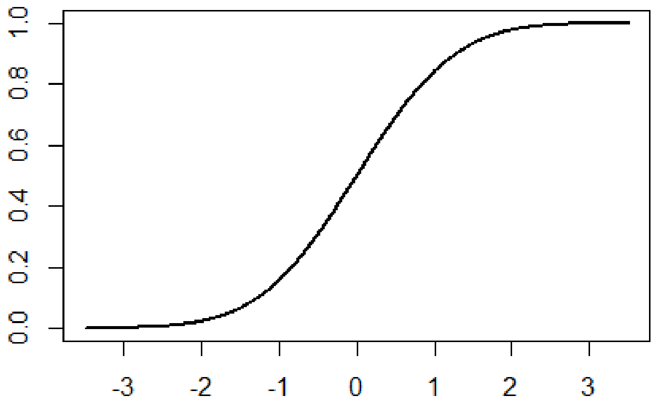


The normal distribution

P(X ≤ 2)

pnorm(2)





Examples of cumulative distributions in R

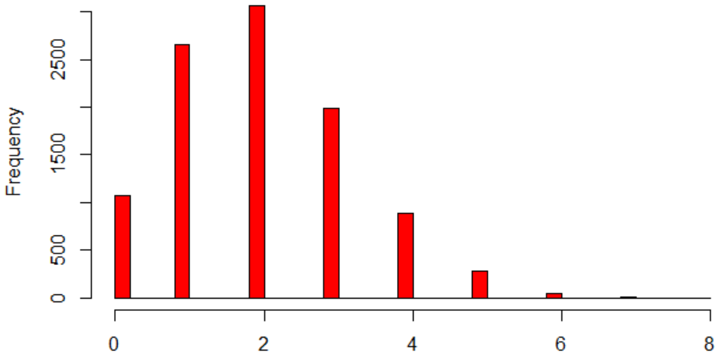
R has built in functions to get probabilities from different distributions

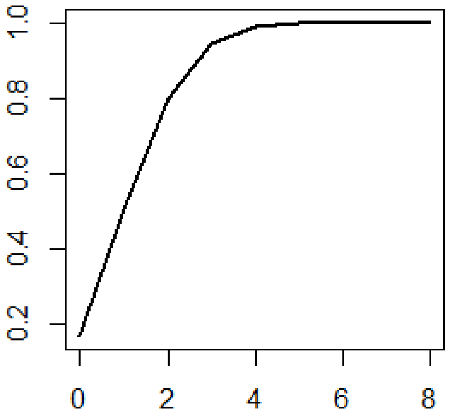
- All these functions start with the letter ***p***

The binomial distribution

P(X ≤ 2; n = 8, π = .2)

pbinom(2, 8, .2)





Let’s try it in R…

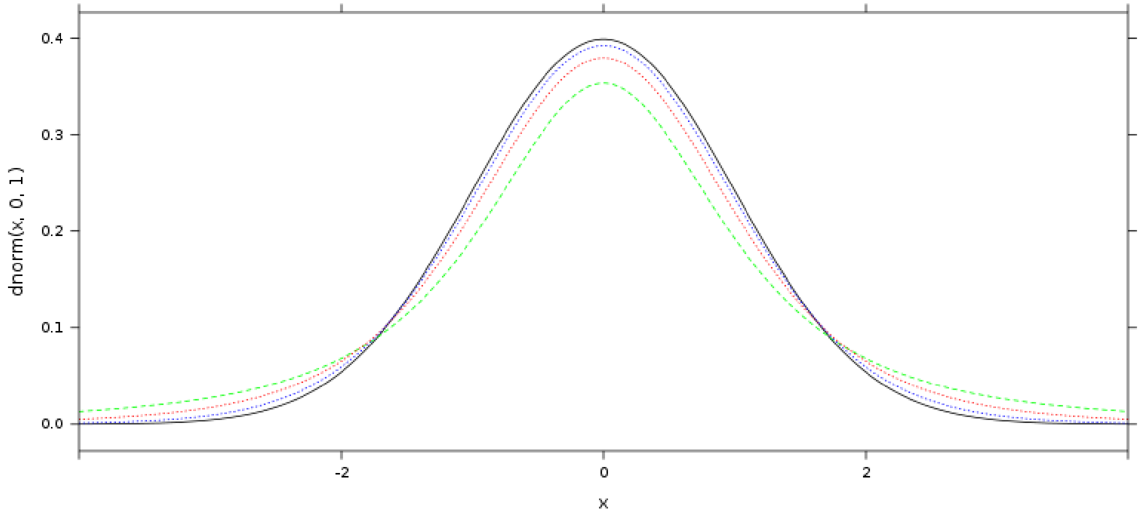
Parametric hypothesis tests

t-distributions

A commonly used density function (distribution) used for statistical inference is the t-distribution

- In R: rt(), dt(), pt() and qt()

t-distributions have one parameter called “degrees of freedom”



df = 2

df = 5

df = 15

N(0, 1)

t-distributions

When using t-distributions for statistical inference, each point in our t-distribution is a t-statistic

- i.e., we use t-distributions as null distributions for hypothesis tests and as sampling distributions when creating confidence intervals

t-statistics are a ratio of:

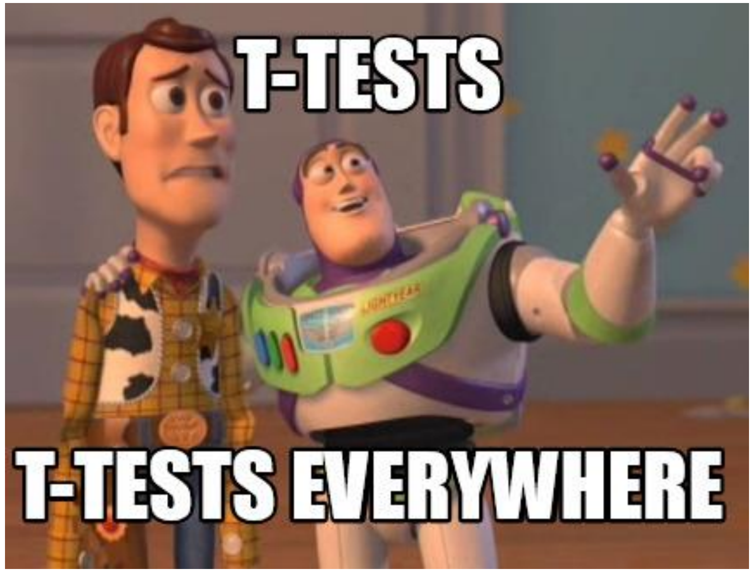
- The departure of an estimated value from a hypothesized parameter value
- Divided by an estimate of the standard error

$$t = \frac{estimate\ \text{-}\ param_0}{\hat{SE}}$$

If the SE did not involve in intermediate estimate (i.e., $\bar{x}$) this would be a "z-statistic" that comes from a standard normal distribution

t-tests

t-tests are parametric hypothesis tests where the null distribution is a density function called a t-distribution



t-tests can be used to test:

- If a mean is equal to a particular value: H_0: $\mu = 7$
- If two means are equal: H_0: $\mu_t = \mu_c$
- If a regression coefficient is equal to a particular value: H_0: $\beta = 2$
- etc.

t-tests for comparing two means

Let's examine t-tests for comparing **two means**

Step 1: what is the null hypotheses?

- H_0: $\mu_t - \mu_c = 0$

Step 2a: What is the numerator of the t-statistic?

$(\bar{x}_t - \bar{x}_c)$ $\quad$ 0

$$t = \frac{estimate - param_0}{\hat{SE}} \leftarrow = \frac{(\bar{x}_t - \bar{x}_c) - 0}{\hat{SE}} = \frac{\bar{x}_t - \bar{x}_c}{\hat{SE}}$$

t-tests for comparing two means

Step 2b: What is the denominator of the t-statistic?

$$t = \frac{stat \text{ - } param_0}{\hat{SE}}$$

Students' t-test assumes the variance in each population is the same, and uses an SE estimate of:

$$\hat{SE}_{\bar{x}_t\text{-}\bar{x}_c} = s_p \cdot \sqrt{\frac{1}{n_t} + \frac{1}{n_c}} \qquad s_p = \sqrt{\frac{\sum_i^{n_t}(x_i - \bar{x}_t)^2 + \sum_j^{n_c}(x_j - \bar{x}_c)^2}{n_t + n_c \text{ - } 2}}$$

Welch's t-test does **not** assume that the variance in each population is the same and uses an estimate of:

$$\hat{SE}_{\bar{x}_t\text{-}\bar{x}_c} = \sqrt{\frac{s_t^2}{n_t} + \frac{s_c^2}{n_c}}$$

t-tests for comparing two means

Step 2b: What is the denominator of the t-statistic?

$$t = \frac{stat\ \text{-}\ param_0}{\hat{SE}}$$

Students' t-test assumes the variance in each population is the same, and uses an SE estimate of:

$$t = \frac{\bar{x}_t\ \text{-}\ \bar{x}_c}{s_p \cdot \sqrt{\frac{1}{n_t} + \frac{1}{n_c}}}$$

$$s_p = \sqrt{\frac{\sum_i^{n_t}(x_i - \bar{x}_t)^2 + \sum_j^{n_c}(x_j - \bar{x}_c)^2}{n_t + n_c\ \text{-}\ 2}}$$

Welch's t-test does **not** assume that the variance in each population is the same and uses an estimate of:

$$\hat{SE}_{\bar{x}_t\text{-}\bar{x}_c} = \sqrt{\frac{s_t^2}{n_t} + \frac{s_c^2}{n_c}}$$

$$t = \frac{\bar{x}_t - \bar{x}_c}{\sqrt{\frac{s_t^2}{n_t} + \frac{s_c^2}{n_c}}}$$

Side note: t-tests for comparing two means

Question: which statistic/test is better to use?

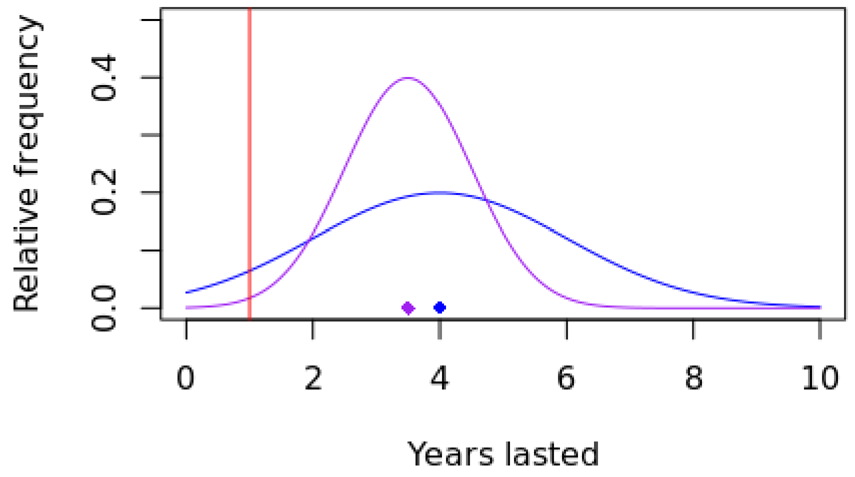
A: generally better to choose the "robust" test

- i.e., Welch's t-test is robust to unequal variances, so generally a better choice

However, we need to be careful with the decisions we make based on differences of means when there are unequal variances

E.g., Which car battery company produces better batteries in terms of how long they last?

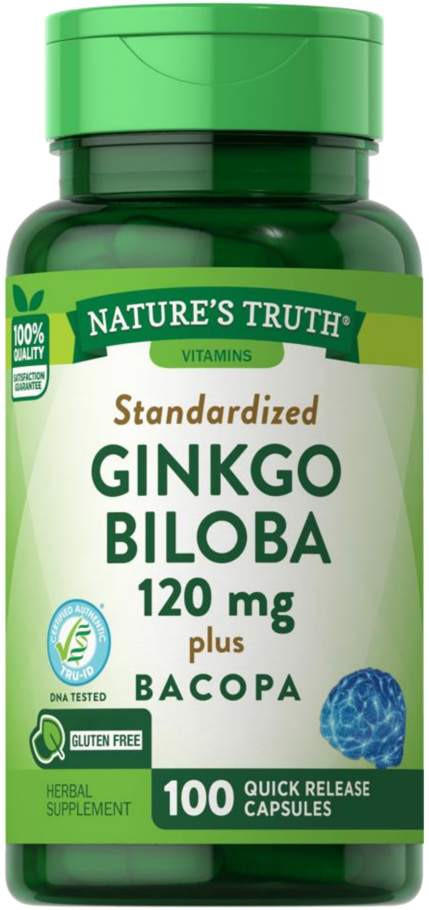
- Company A: μ = 4 years, σ = 2 years
- Company B: μ = 3.5 years, σ = 1 years



- Company A: 7% fail within a year
- Company B: 0.6% fail with a year

Example: Does Gingko improve memory?

A double-blind randomized controlled experiment by Solomon et al (2002) investigated whether taking a Ginkgo supplement could improve memory



Let's try using a t-statistic!

- t = -1.53

$$t = \frac{\bar{x}_t - \bar{x}_c}{\sqrt{\frac{s_t^2}{n_t} + \frac{s_c^2}{n_c}}}$$

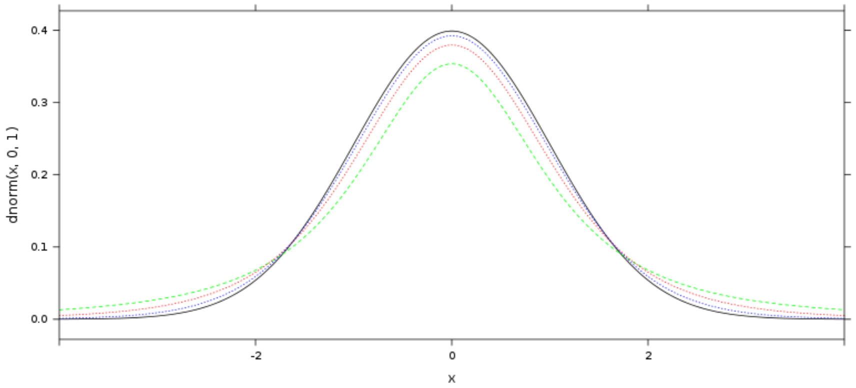
3. What is the null distribution?

- What additional piece of information do we need to create it?

t-tests for comparing two means

When using a t-distribution to compare two means, a conservative estimate of the degrees of freedom is the minimum of the two samples sizes, n_t and n_c, minus 1

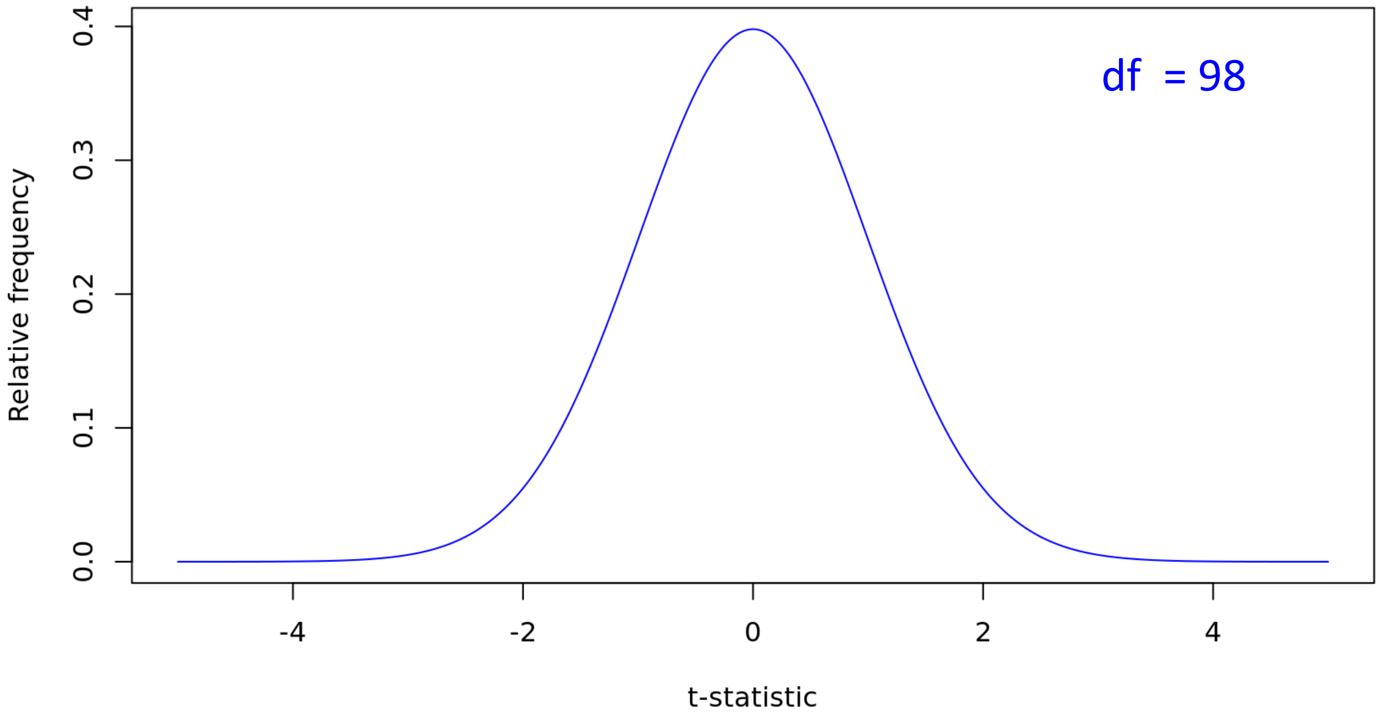
- $df = \min(n_t, n_c) - 1$



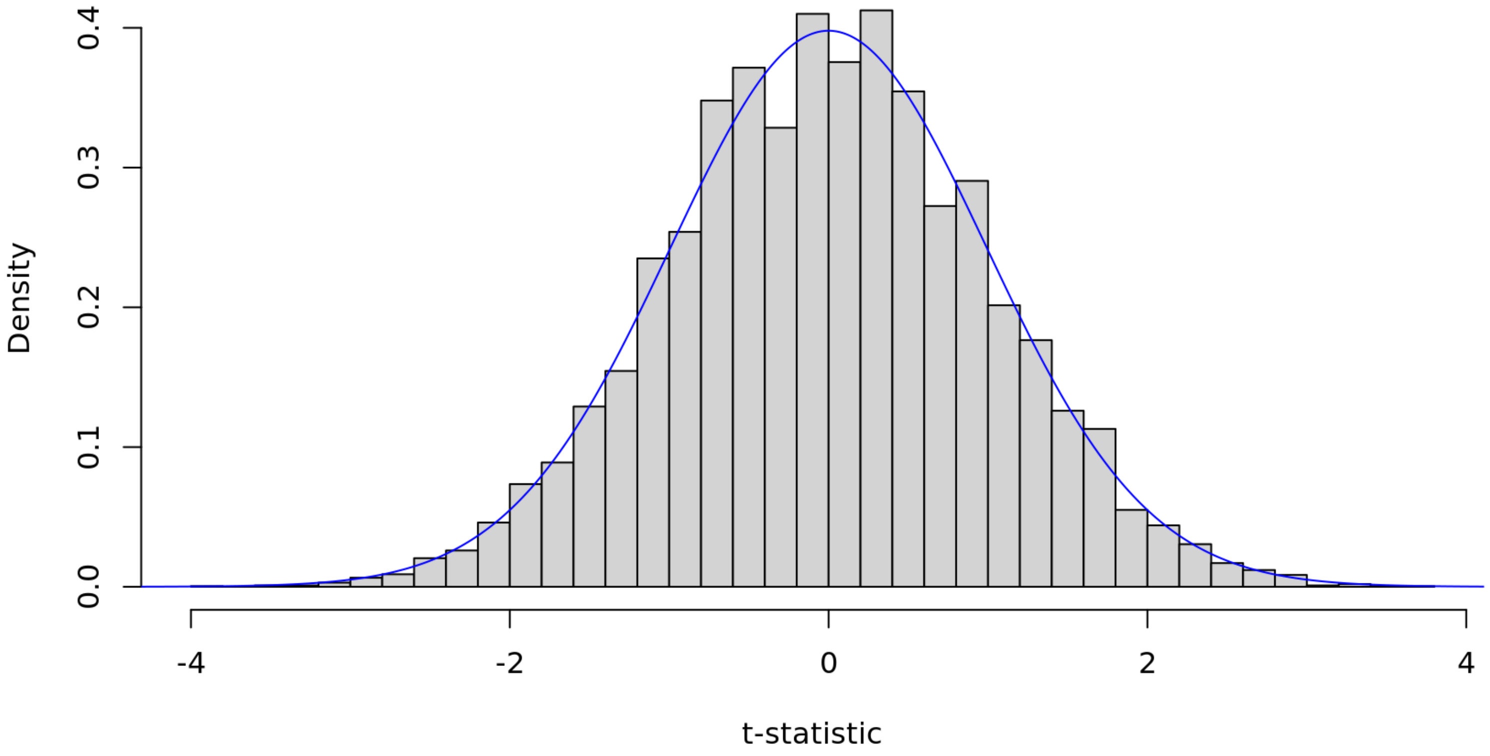
Q: For the Gingko study we had 104 people in the treatment group and 99 people in the control group so the degrees of freedom parameter is?

- 98

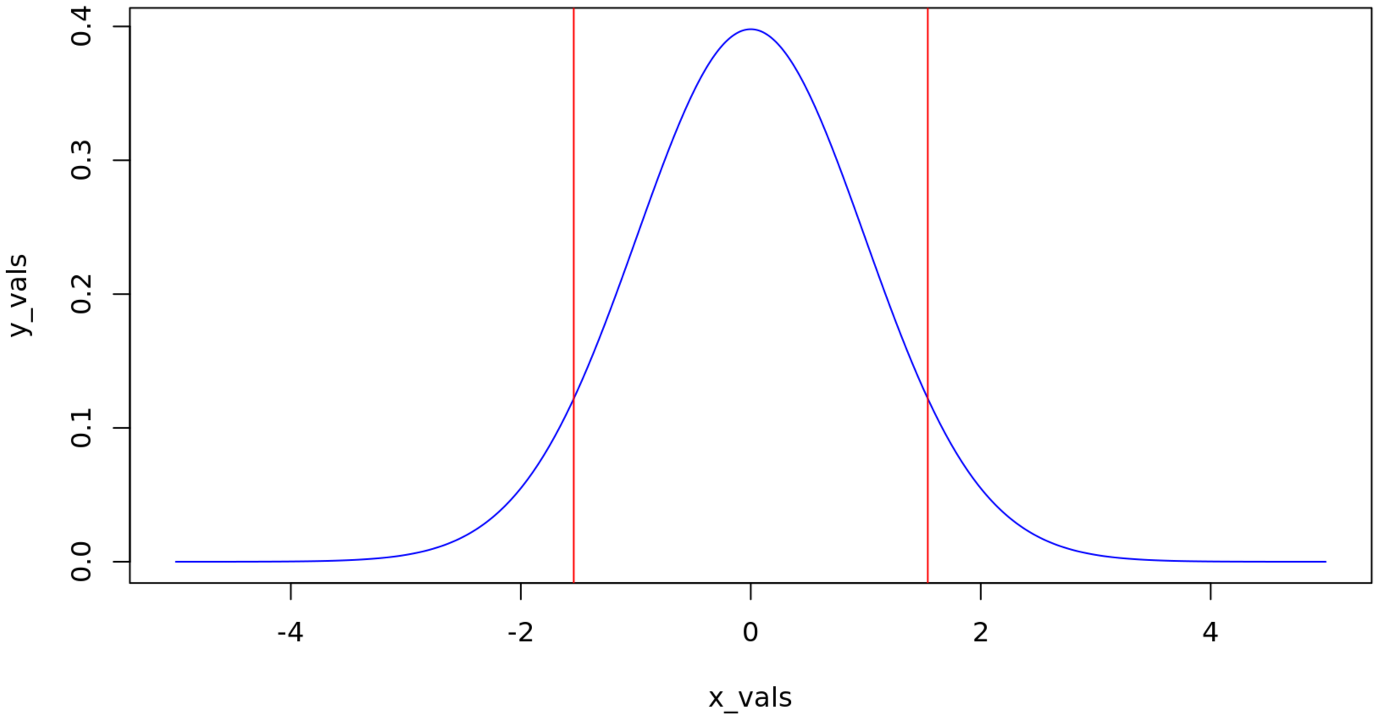
Step 3: Null t-distribution



Step 3: parametric vs. randomization distributions

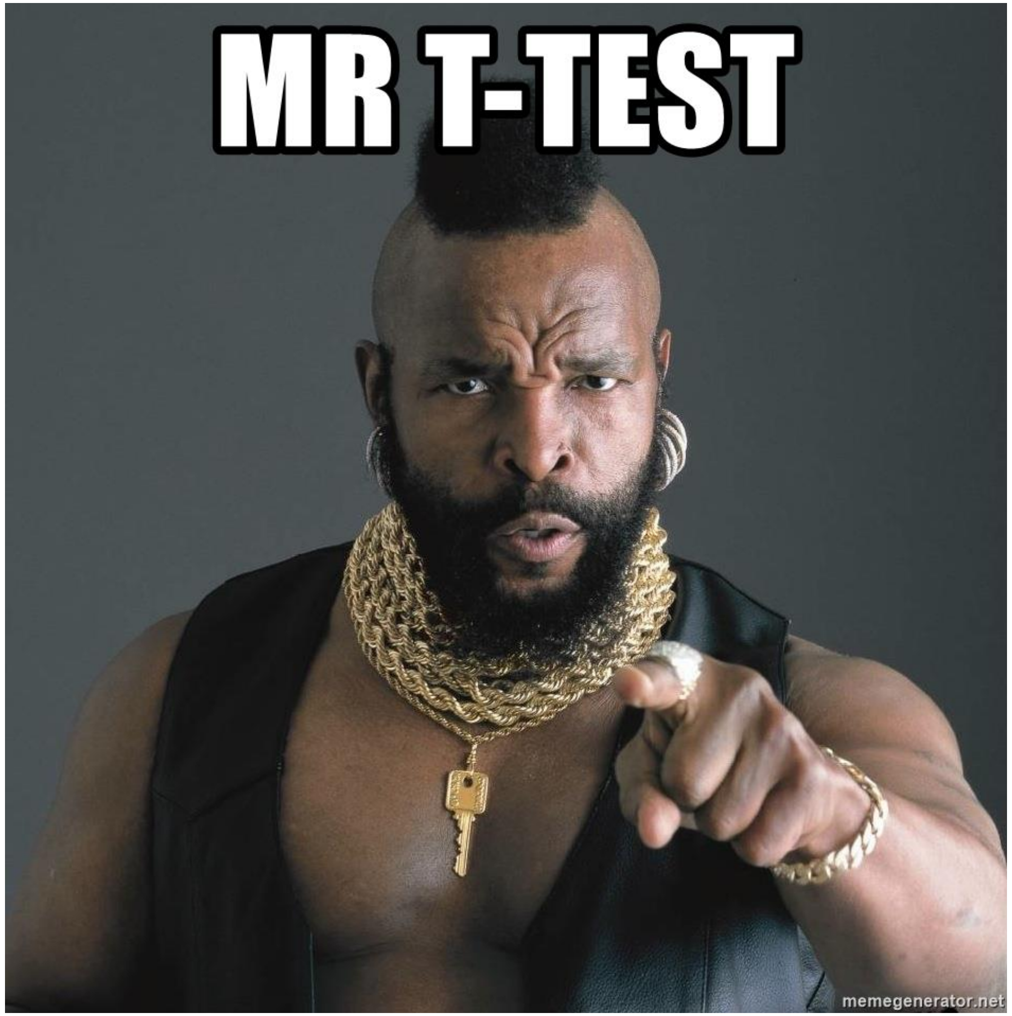


Step 4-5: p-value and conclusion



p-value = 0.127

Conclusion?



I pity the fool who doesn't want to try it in R!