Influential points, ANOVA for regression and multiple regression

Overview

Review of inference for simple linear regression

Examining unusual data points

Analysis of variance for regression

If there is time: multiple regression

- Basic ideas
- Nested model comparison
- Related sampling and multiple regression coefficients

Announcements

Homework 7 has been posted. It is due on Sunday

I have your cheat sheets to give back to you

Quick review of simple linear regression

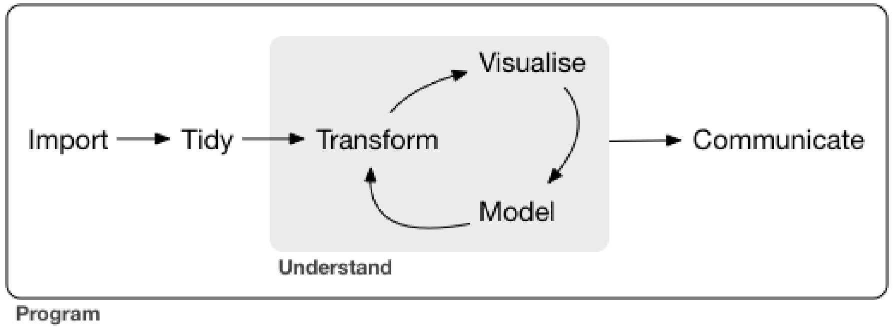
The process of building regression models

Choose the form of the model
- Identify and transform explanatory and response variables

Fit the model to the data
- Estimate model parameters

Assess how well the model describes the data
- Analyze the residuals, evaluate unusual points, etc.

Use the model to address questions of interest
- Make predictions, explore relationships, etc.



All models are wrong, but some models are useful

Simple linear regression concepts

Theoretical model: $Y = \beta_0 + \beta_1 x + \epsilon$

Estimated model: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

Inference for simple linear regression models

- Hypothesis tests for intercept and slope
- Confidence intervals for slope and line; prediction intervals

Inference is valid if these conditions are met:

- Linearity, Independence, Normality, Equal variance of errors

Hypothesis test for regression coefficients

We can run hypothesis tests to assess whether there is a relationship between y and x, and calculate p-values

- H_0: $\beta_1 = 0$ (slope is 0, so no relationship between x and y
- H_A: $\beta_1 \neq 0$

One type of hypothesis test we can run is based on a t-statistic: $t = \frac{\hat{\beta}_1 - 0}{\hat{SE}_{\hat{\beta}_1}}$

- The t-statistic comes from a t-distribution with n - 2 degrees of freedom

$$\hat{SE}_{\hat{\beta}_1} = \frac{\hat{\sigma}_e}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \qquad \hat{SE}_{\hat{\beta}_0} = \hat{\sigma}_e \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

By default the R summary(lm_fit) shows p-values from running a two-sided test

Summary of confidence and prediction intervals

1. CI for slope β

$$\hat{\beta}_1 \pm t^* \cdot \hat{SE}_{\hat{\beta}_1} \qquad \hat{SE}_{\hat{\beta}_1} = \hat{\sigma}_e \sqrt{\frac{1}{\Sigma_{i=1}^n (x_i - \bar{x})^2}}$$

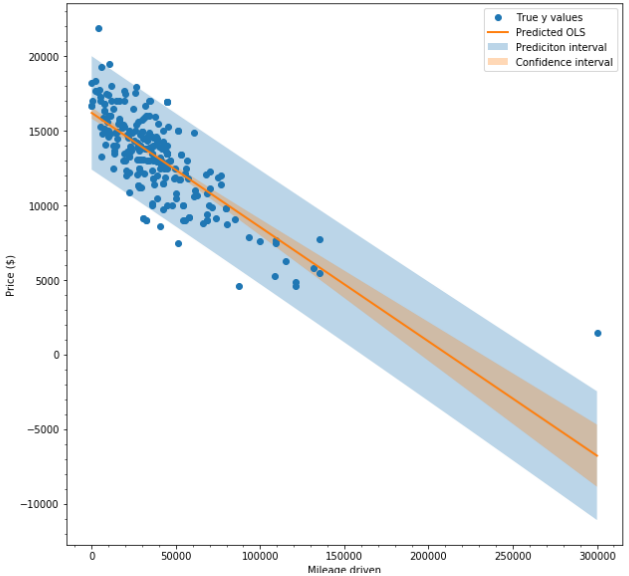
β_1

2. CI for regression line μ_Y at point x*

$$\hat{y}_{(x^*)} \pm t^* \cdot \hat{SE}_{\hat{y}_{(x^*)}} \qquad \hat{SE}_{\hat{y}_{(x*)}} = \hat{\sigma}_e \sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma_{i=1}^n (x_i - \bar{x})^2}}$$

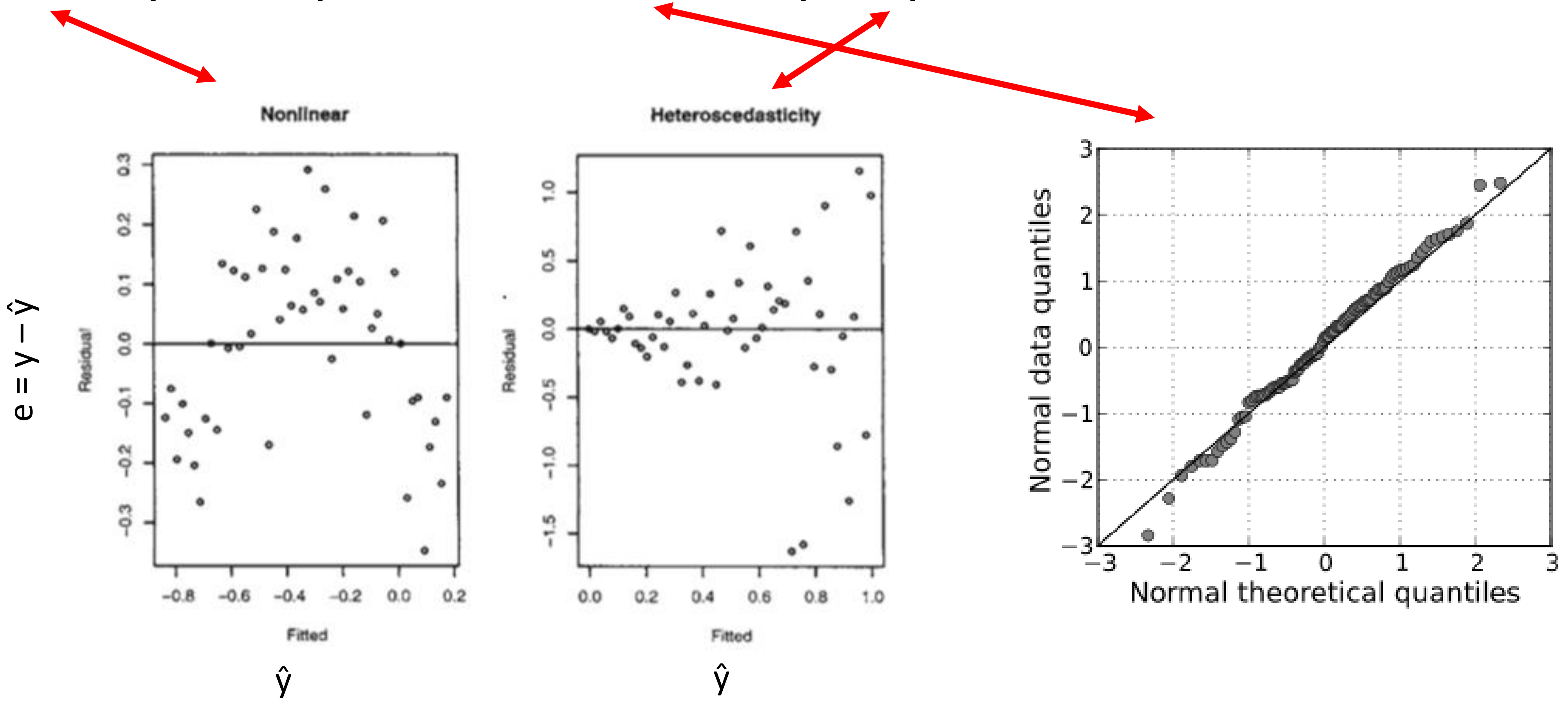
3. Prediction interval y

$$\hat{y}_{(x^*)} \pm t^* \cdot \hat{SE}_{pred} \qquad \hat{SE}_{pred} = \hat{\sigma}_e \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma_{i=1}^n (x_i - \bar{x})^2}}$$



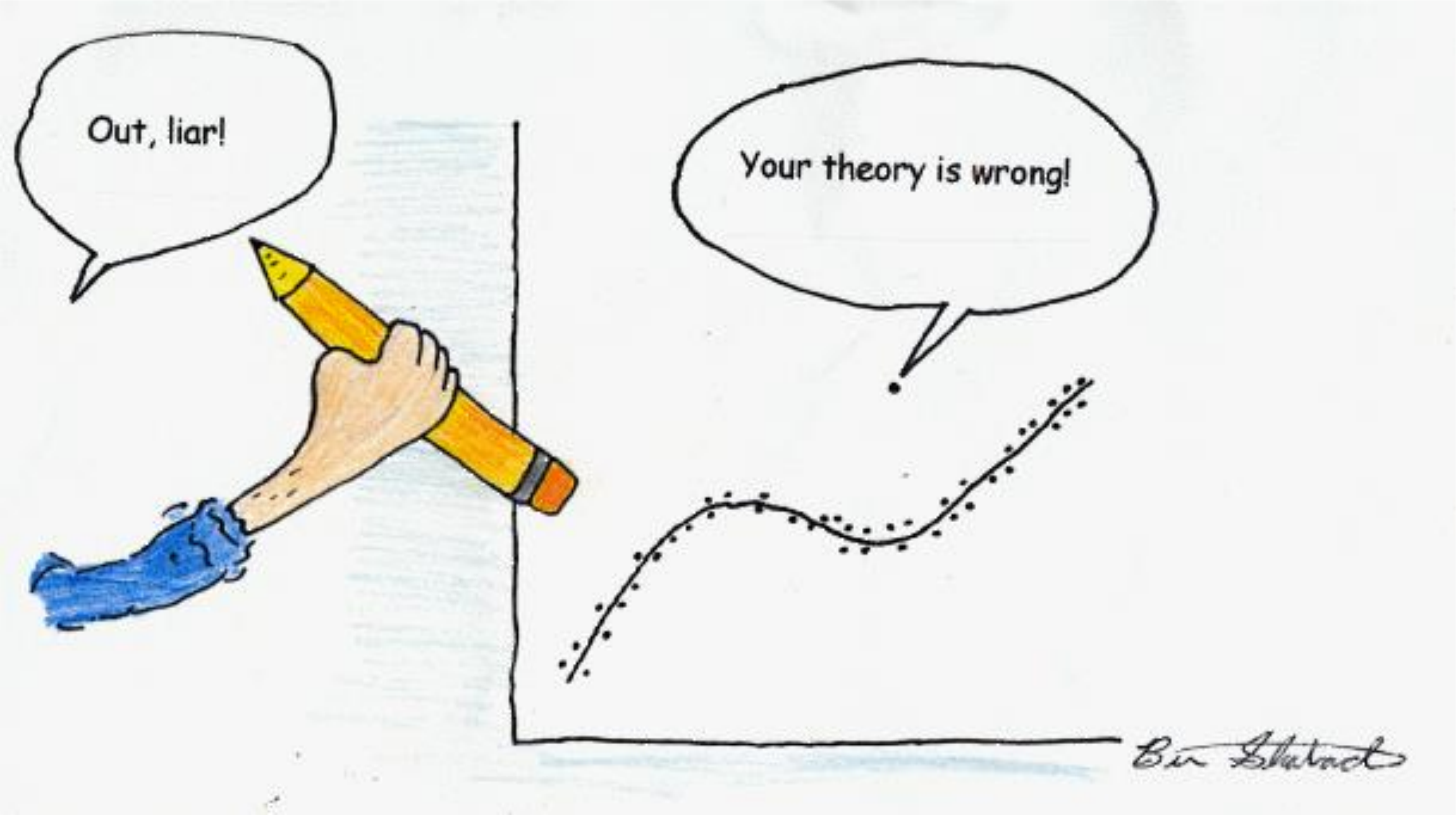
Regression diagnostics

Linearity, Independence, Normality, Equal variance of errors



Questions?

Statistics for unusual observations



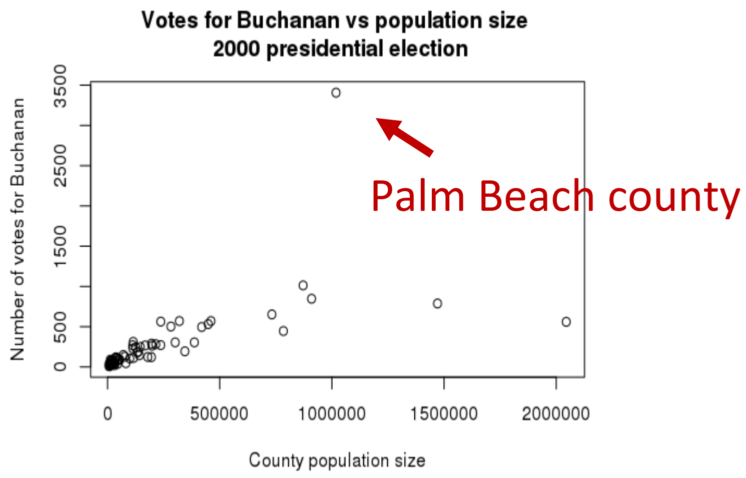
Statistics for unusual observations

There are statistics that are useful for flagging usual observations

- **High leverage points**: usual **x** values
- **Outliers (large residuals):** unusual **y** values
- **Influential points**: both an outlier and a high leverage

Unusual observations can indicate:

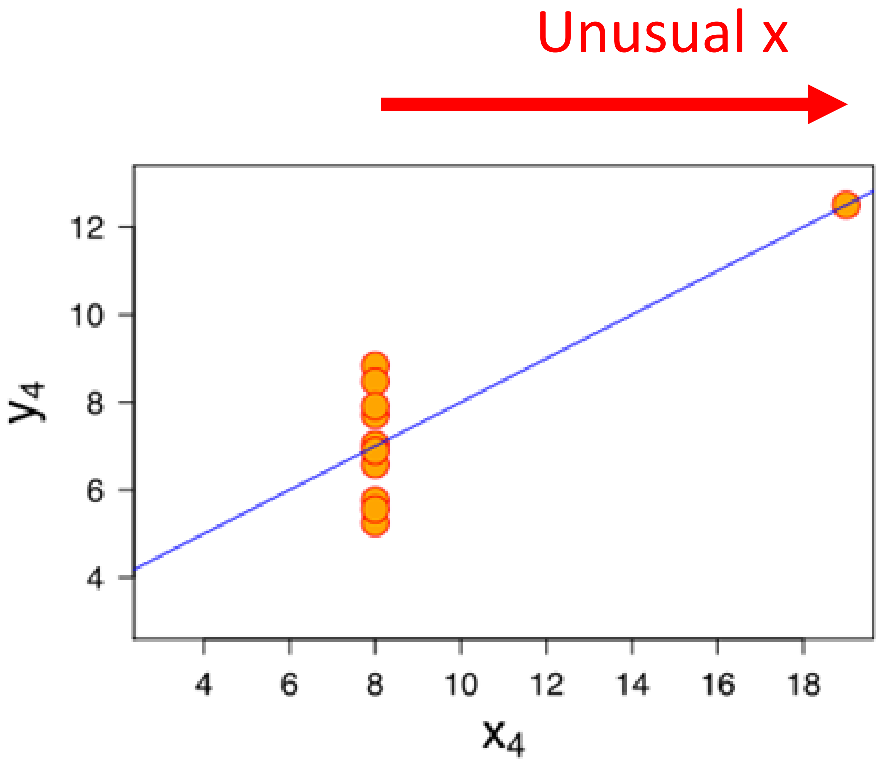
- An error in data processing
- A need to modify the model
- An interesting phenomenon



Unusual observations **can also have a big effect on the model fit**

- E.g., a big effect on $\hat{\beta}_0$ $\hat{\beta}_1$

Leverage: unusual x values



High leverage points are predictors **x** that are far from the mean

We can quantify the leverage a data point x_i has using the statistic:

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{j=1}^{n}(x_j - \bar{x})^2}$$

$$\Sigma_{i=1}^{n} h_i = 2$$

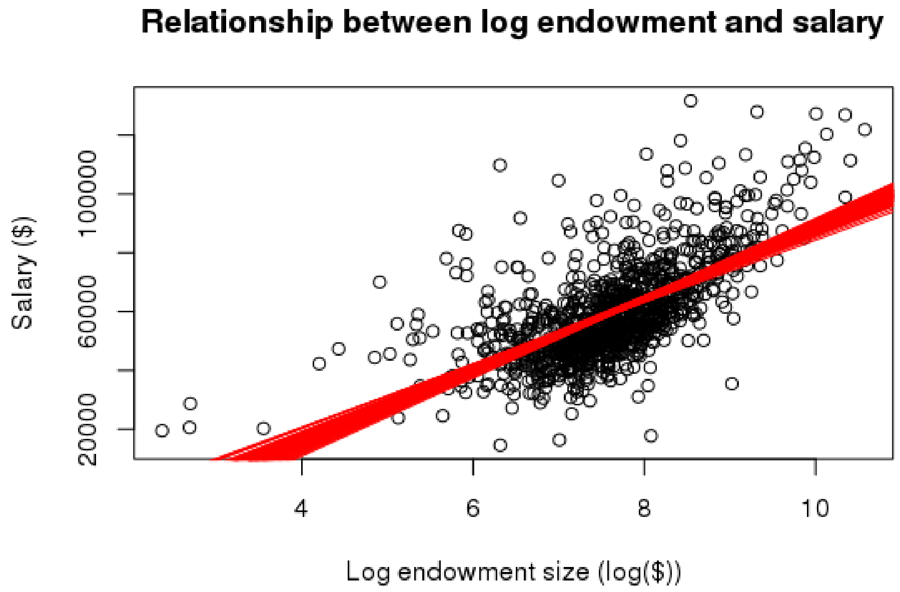
Typical:	$h_i = 2/n$
High:	$h_i = 4/n$
Very high:	$h_i = 6/n$

R: hatvalues()

Outliers (residuals): unusual y values

The **standardized residual** for the ith data point in a regression model can be computed using:

$$stdres_i = \frac{y_i - \hat{y}}{\hat{\sigma}_e \sqrt{1 - h_i}}$$



Puts residuals on a 'normalized' scale

Makes residuals at the ends a bit larger to deal with the fact that they are 'overfit'

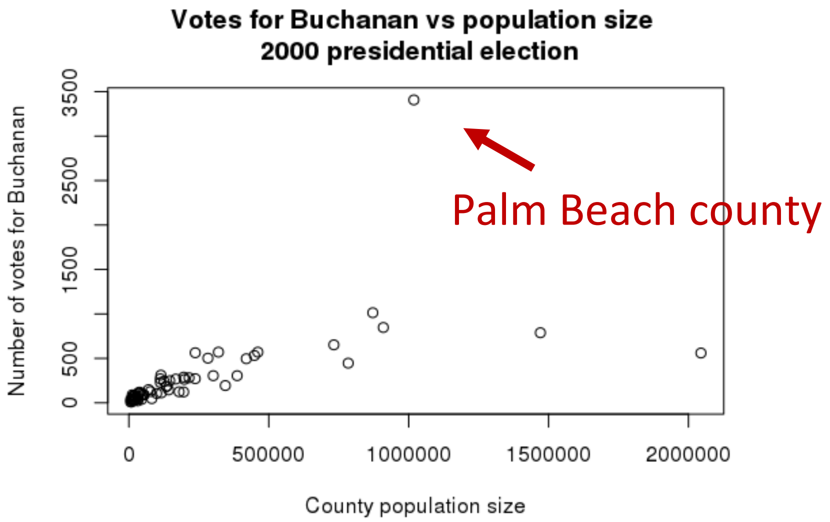
R: rstandard()

Outliers (residuals): unusual y values

The **studentized residual** for the i^{th} data point in a regression model can be computed using:

$$studres_i = \frac{y_i - \hat{y}}{\hat{\sigma}_{e(i)}\sqrt{1-h_i}}$$

Here $\hat{\sigma}_{e(i)}$ is the estimate of $\hat{\sigma}_e$ with the i^{th} point removed



Q: Why might we want to remove the i^{th} point when calculating $\hat{\sigma}_e$?

A: Outliers could have a big effect on our estimate of $\hat{\sigma}_e$

R: rstudent ()

Influential points: unusual x and y values

The amount of influence a point has on a regression line depends on:

- The size of the residual e_i
- The amount of leverage h_i

Cook's distance is a statistic that captures how much influence a point has on a regression line

$$D_i = \frac{(stdres_i)^2}{k+1} \frac{h_i}{1-h_i}$$

Larger for larger residuals (outliers)

Larger for high leverage points

Where *k* is the number of predictors in the model

R: cooks.distance()

- For simple linear regression k = 1 (just a single predictor x)

Influential points: unusual x and y values

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Larger for larger residuals (outliers)

Larger for high leverage points

Rule of thumb:

- Moderately influential: $D_i > 0.5$
- Very influential: $D_i > 1$

R: cooks.distance()

Influential points: unusual x and y values

Cook's distance can also be expressed as the how much the predicted values ŷ's would change if the i^{th} was not used when fitting the model

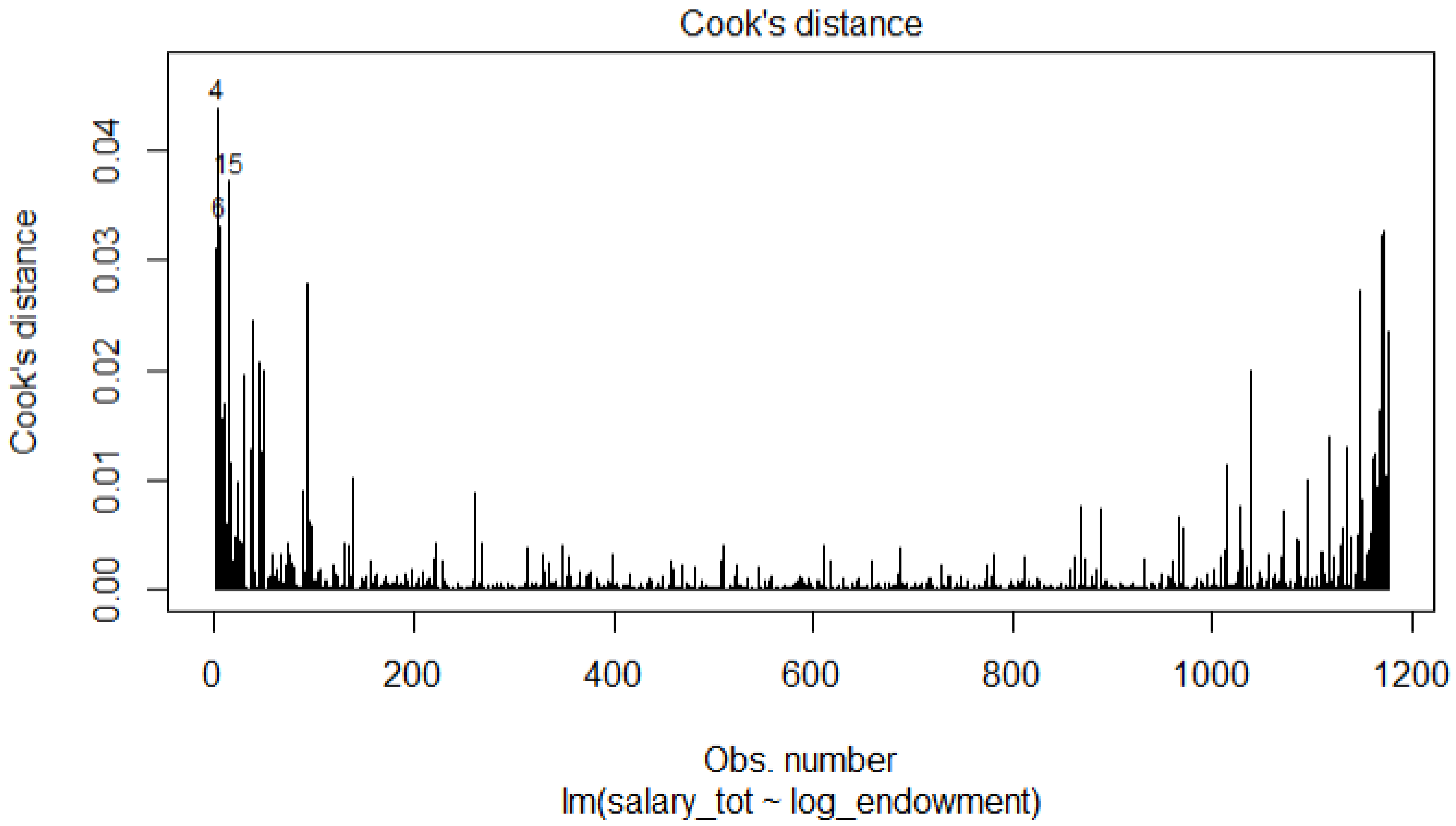
$$D_i = \frac{\sum_{j=1}^{n-1} (\hat{y}_j - \hat{y}_{j(i)})^2}{(k+1) \cdot \hat{\sigma}_e^2}$$

Number of predictors in the model

(i.e., k = 1 for simple linear regression)

The model fit with the i^{th} point removed

Cook’s distances for salary ~ $\log_{10}$(endowment)



plot(lm_fit, 4)

Unusual points rules of thumb

Statistic	Moderately unusual	Very unusual
Leverage, h_i	Above $2(k + 1)/n$	Above $3(k + 1)/n$
Standardized residual	Beyond ± 2	Beyond ± 3
Studentized residual	Beyond ± 2	Beyond ± 3
Cook’s Distance	Above 0.5	Above 1.0

Where:

- k is the number of explanatory variables
- n is the number of data points

From STAT2 by Cannon, Cobb, Hartlaub, Legler, Lock, Moore, Rossman, and Witmer

Questions?

Let’s try it in R!

Analysis of Variance (ANOVA) for regression

ANOVA for regression motivation

Suppose you had to guess a value Y

- E.g., the more accurate your guess the more $ you win

ANOVA for regression motivation

Our guess would be off by about $\hat{\sigma}_y$

Suppose you had to guess a value Y

- E.g., the more accurate your guess the more $ you win

Suppose you had a sample of n = 30 from the distribution that Y came from

- They were values like: 27, 21, …, 36

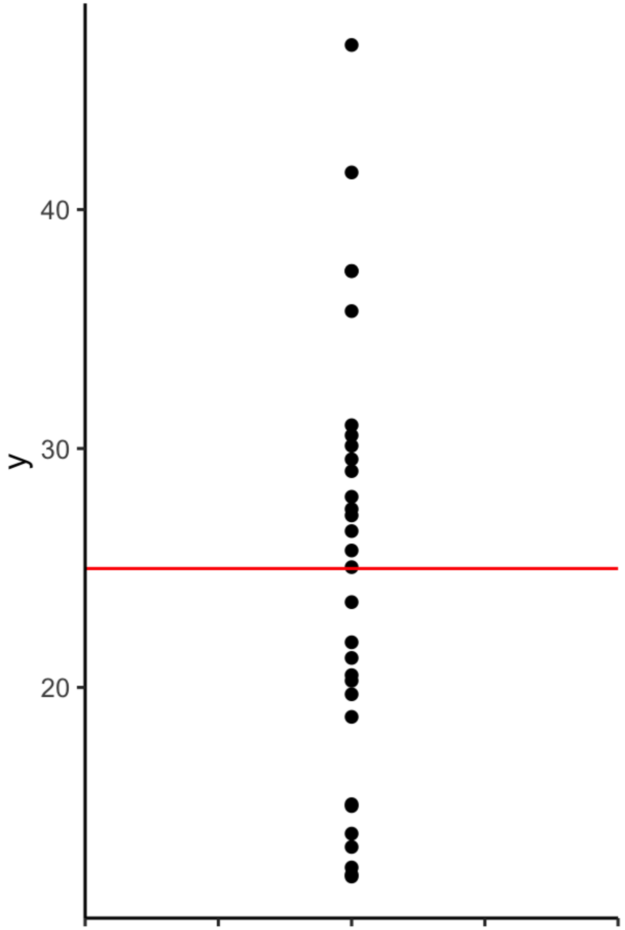
Q: What would your guess be?

A: The average of the data ȳ would be a good guess

- ȳ minimizes the sum of the squared deviations (residuals)

$$SS = \sum_{i=1}^{30} (y_i - c^*)^2$$

c* = ȳ minimizes SS



ANOVA for regression motivation

Our guess would be off by about $\hat{\sigma}_e$

Suppose you had to guess a value Y

- E.g., the more accurate your guess the more $ you win

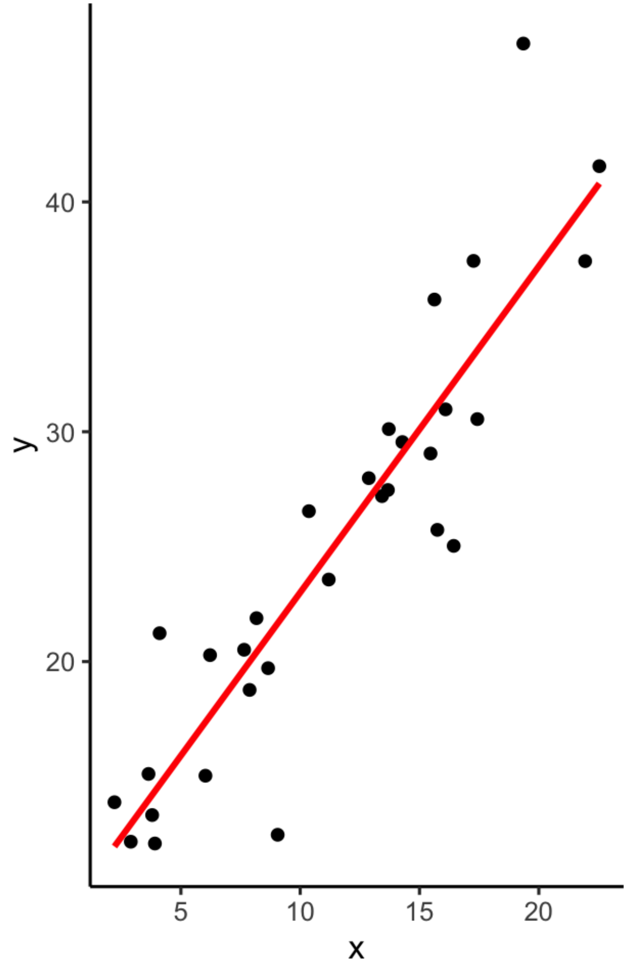
Suppose you also had a sample of n = 30 from the distribution with the following data:

x	10	8	…	16
y	27	21	…	36

and you were told x = 15

Q: What would your guess be?

A: Could fit a linear regression model and predict y



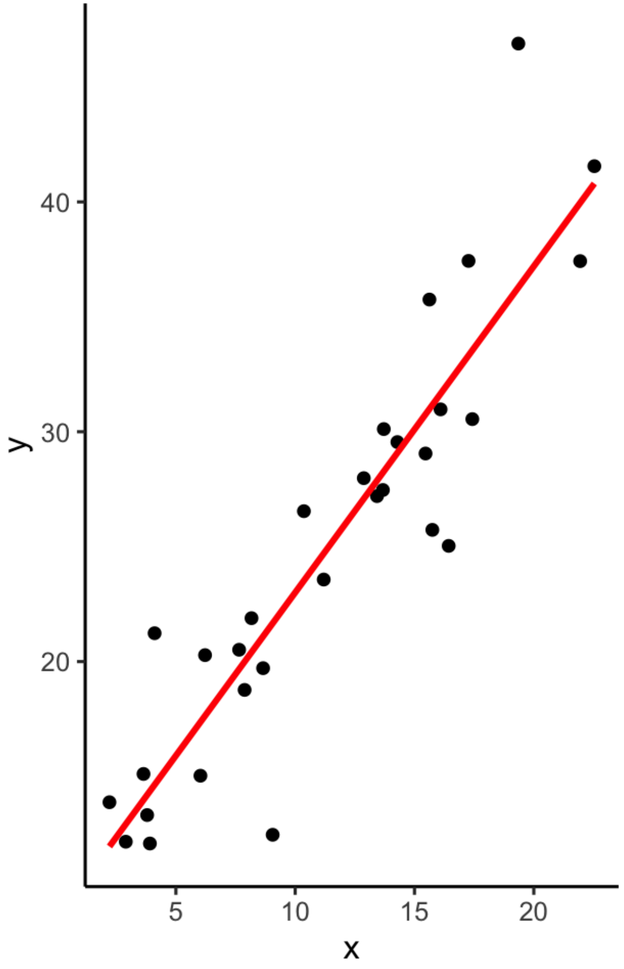
ANOVA for regression motivation

Suppose you had to guess a value Y

- E.g., the more accurate your guess the more $ you win

As we add additional data x (predictors) our predictions become better, and we are able to account for more of the variability in the data y

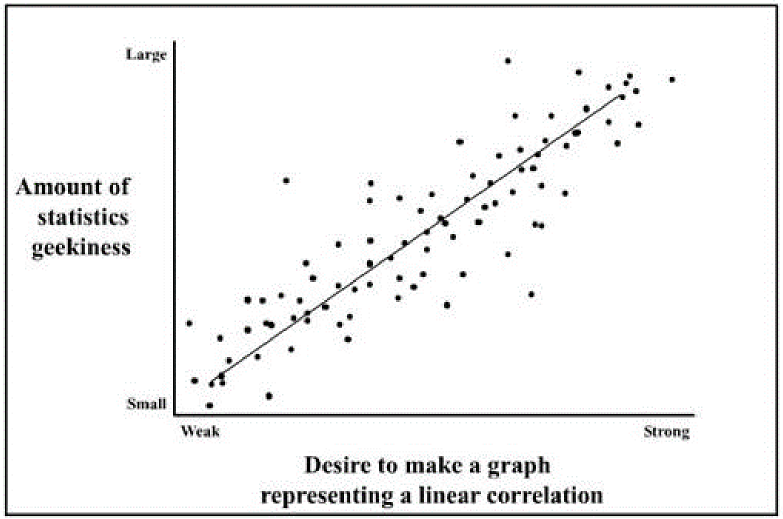
One could view the central goal of statistical analyses as coming up with models that can account for as much of the variability in y as possible



Analysis of Variance (ANOVA) for regression

In an analysis of variance, we break down the **total variability (σ_y)** in a **response variable y** into:

1. the variability explained by the model
2. the variability not explained by the model
   - i.e., the residuals



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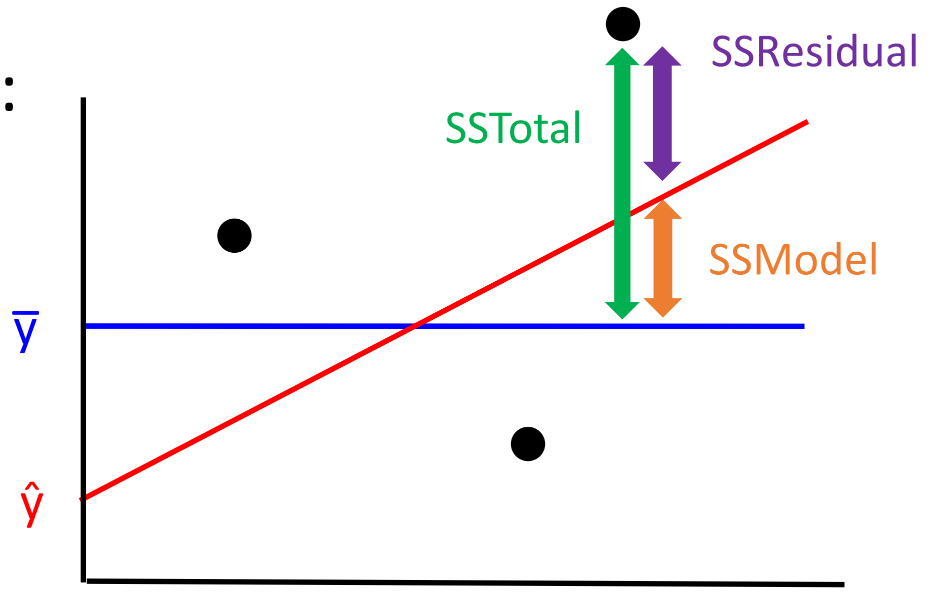


We can express this as:

- SSTotal = SSModel + SSResidual

$$y_i - \bar{y} = (\hat{y}_i - \bar{y}) + (y_i - \hat{y}_i)$$

Added and subtracted $\hat{y}_i$

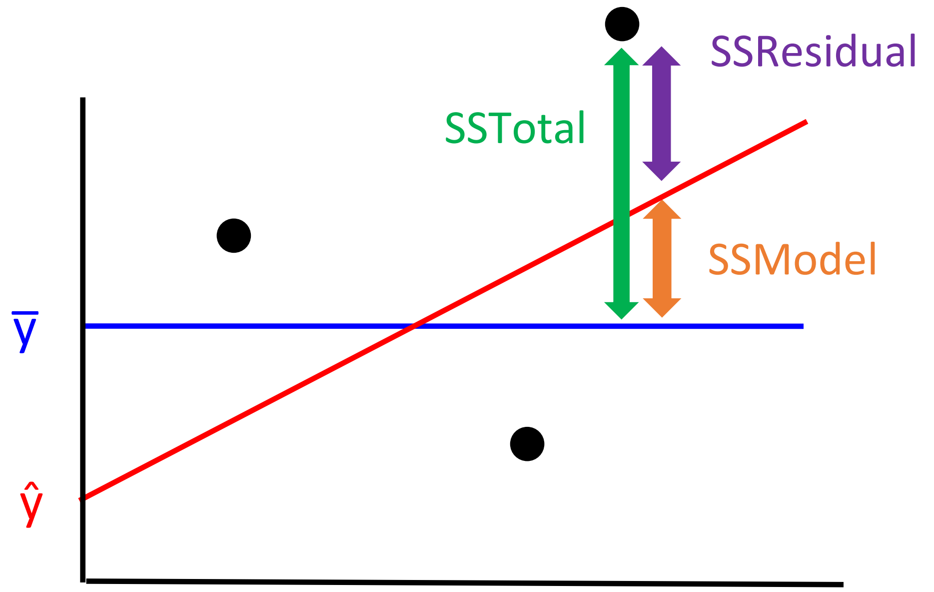
This equal 0 (when using least squares)

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + (y_i - \hat{y}_i)^2 + \cancel{2(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})}$$

The coefficient of determination r^2

The **percentage of the total variability explained by the model** is given by

$$r^2 = \frac{SSModel}{SSTotal} = 1 - \frac{SSResidual}{SSTotal}$$



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This equal 0 (when using least squares)

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + (y_i - \hat{y}_i)^2 + 2(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$$

Hypothesis test based on ANOVA for regression

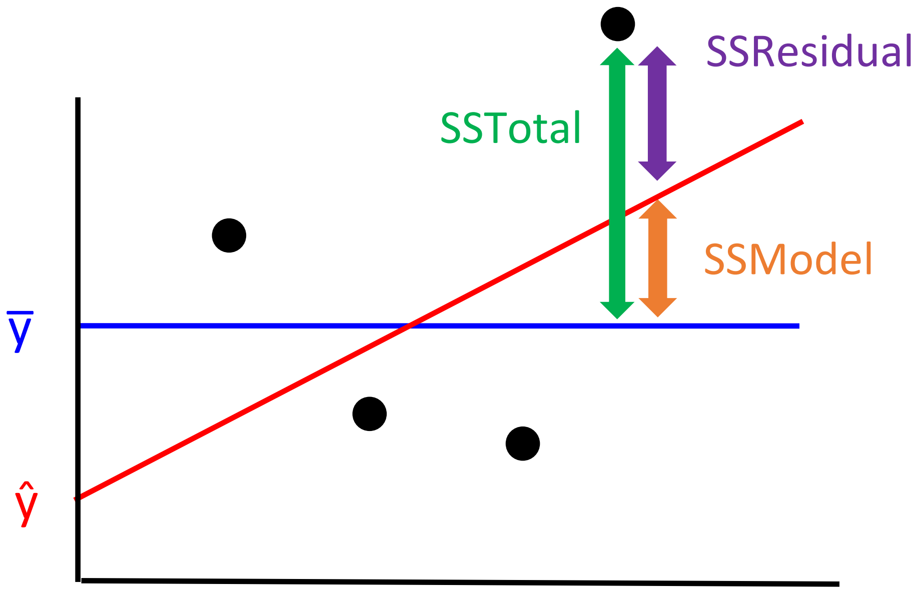
$$F = \frac{SSModel/df_{model}}{SSResidual/df_{error}}$$

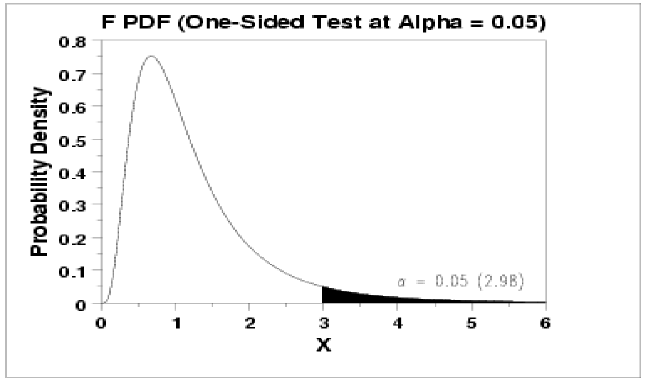
$df_{model} = 1$

$df_{error} = n - 2$

If the null hypothesis is true that $\beta_1 = 0$:

- F comes from an F-distribution with df_{model}, df_{error} degrees of freedom
- For simple linear regression, this gives the same results as running a t-test
  - $F = t^2$





Analysis of Variance (ANOVA) for regression in R

You can create an ANOVA table for regression relationships in R using:

anova(lm_fit)

```
lm_fit <- lm(salary_tot ~ log_endowment, data = assistant_data)

anova(lm_fit)
```
```

SSModel

SSResidual

F

```
Analysis of Variance Table

Response: salary_tot
                Df       Sum Sq      Mean Sq F value                 Pr(>F)
log_endowment    1 132879258586 132879258586  764.29 < 0.00000000000000022 ***
Residuals     1173 203936190958    173858645
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

$\hat{\sigma}_e^2$
```

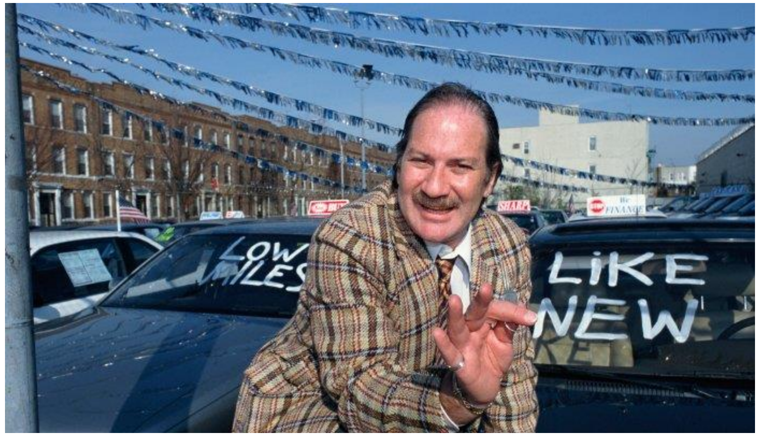
Analysis of Variance (ANOVA) for regression in R

You can create an ANOVA table for regression relationships in R using:

anova(lm_fit)

We can check that the ANOVA relationships holds: SSTotal = SSModel + SSResidual using:

- The original data y values
- lm_fit$residuals
- lm_fit$fitted.values



You can also check that F = t^2 by comparing anova(lm_fit) and summary(lm_fit) values

Homework 7!

edmunds.com

Multiple regression

Multiple regression

In multiple regression we try to predict a quantitative response variable *y* using several predictor variables $x_1, x_2, \dots, x_k$

For multiple linear regression, the underlying model is:

$$y = \beta_0 + \beta_1 \cdot x_1 + \beta_2 \cdot x_2 + \dots \beta_k \cdot x_k + \epsilon$$

We estimate coefficients using a data set to make predictions ŷ

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \dots + \hat{\beta}_k \cdot x_k$$

Multiple regression

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

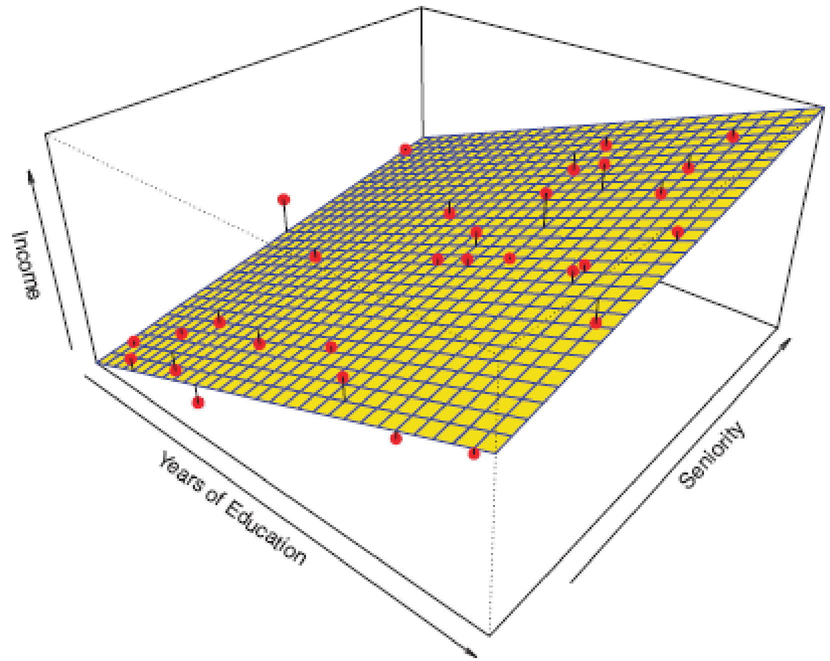
There are many uses for multiple regression models including:

- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)

Multiple regression

$$\text{salary} \;=\; \hat{\beta}_0 \;+\; \hat{\beta}_1 \cdot \text{f(endowment)} \;+\; \hat{\beta}_2 \cdot \text{g(enrollment)}$$

Let's explore this in R…



Nested model comparison

We can also assess whether a particular subset of *q* parameters is 0

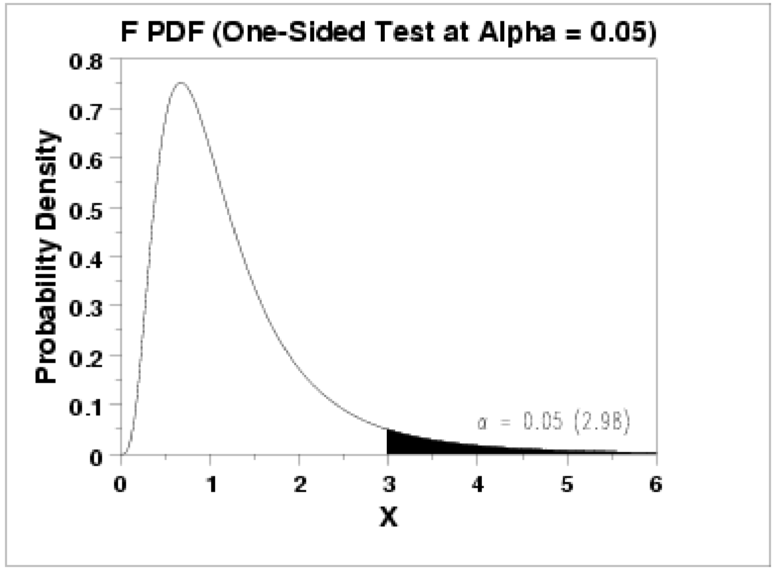
H_0: $\beta_h = \beta_i = \ldots = \beta_g = 0$

To do this we:

1. Fit the model without these features
2. Calculate the $SSRes_{Reduced}$ for the model without these predictors
3. Compare it to the full model $SSRes_{Full}$ with an F-statistic:

$$F = \frac{(SSRes_{Reduced} - SSRes_{Full})/q}{SSRes_{Full}/(n-k-1)}$$

where q is the number of additional terms in the full model



$df_1 = df_{Reduced} - df_{Full}$
$df_2 = df_{Full}$

Relating simple and multiple regression

Relating simple and multiple regression

Suppose we fit both a simple and multiple regression models to the same data.

simple linear regression coefficient

Simple regression model: $\hat{y} = \hat{\beta}_{0(1)} + \hat{\beta}_{1(1)} \cdot x_1$

multiple linear regression coefficient

Multiple regression model: $\hat{y} = \hat{\beta}_{0(2)} + \hat{\beta}_{1(2)} \cdot x_1 + \hat{\beta}_{2(2)} \cdot x_2$

Question: How are the coefficients $\hat{\beta}_{1(1)}$ and $\hat{\beta}_{1(2)}$ related?

Relating simple and multiple regression

Question: How are the simple regression coefficients $\hat{\beta}_{1(1)}$ and the multiple regression coefficient $\hat{\beta}_{1(2)}$ (for a predictor x_1) related?

We can view the multiple regression coefficient $\hat{\beta}_{1(2)}$ as the change in y with the change in x_1 when we **set the predictor x_2 to a fixed value**

- For real data, it might not be possible/realistic to set x_2 to a fixed value while changing x_1

We can view the simple regression coefficient $\hat{\beta}_{1(1)}$ as the change in y **when we let the other predictor x_2 change with the value of x_1**

Relating simple and multiple regression

If the predictor x_1 is correlated with x_2, then changing x_1 will be associated with changes in x_2 which in turn will be associated with changes in y

We can assess the association between x_1 and x_2, using regression:

$$x_2 \;=\; \hat{\delta}_0 \;+\; \hat{\delta}_1 \cdot x_1$$

We can then relate the change in *y* with the change in x_1 in the simple regression coefficient to the multiple regression coefficients as:

$$\hat{\beta}_{1(1)} \cdot x_1 \;=\; \hat{\beta}_{1(2)} \cdot x_1 \;+\; \hat{\beta}_{2(2)} \cdot \hat{\delta}_1 \cdot x_1$$