Logistic regression

Overview

Information on the final project

Brief mention: Visualizing linear models using ggplot

Logistic regression

If there is time: Poisson regression

Final projects!

The final project is a 5-8 page R Markdown report where you analyze your own data to address a question that you find interesting

- It's a chance to practice everything you've learned in class!

The goal of the project is to present a clear and compelling analyses of data showing a few interesting results!

A few sources for data sets are listed on Canvas

- You can use data you collect as well. If you use data for another class your work must be unique for each class.

Final projects!

An R Markdown template describing sections in the project is on the class GitHub site.

- library(SDS230)
- download_any_file("homework/final_project.Rmd")

A challenge is going to be to fit your analyses into 5-8 pages:

- You can include an appendix with additional code that does not count against your 5-8 pages
  - E.g., you can include functions in your appendix and then just call them in the body of your report

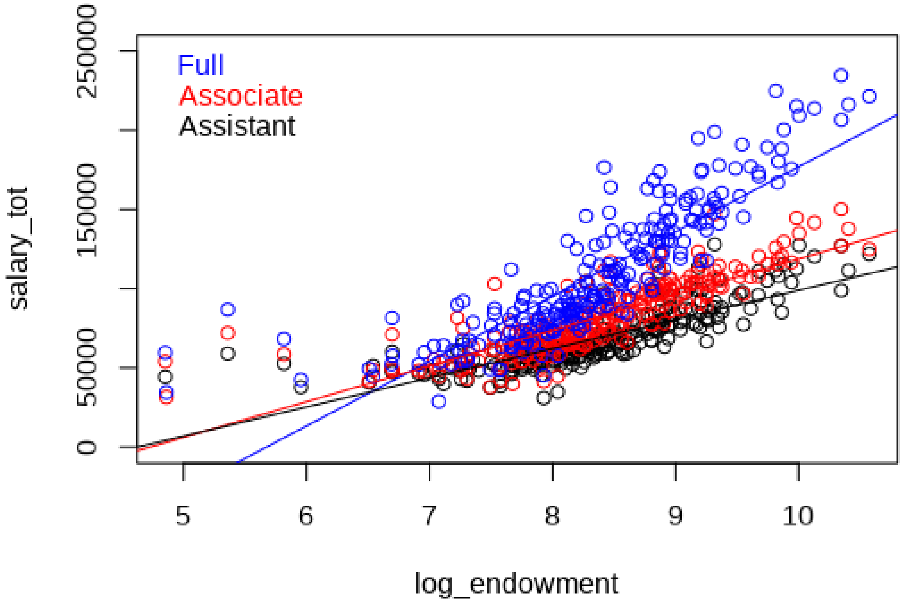
Project is due at 11pm on Sunday December 8th

- i.e., the day before the start of reading period

Very quick review of multiple regression

Very quick review of multiple regression

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_3 (x_1 \cdot x_2) + \epsilon$$



There are many uses for multiple regression models:

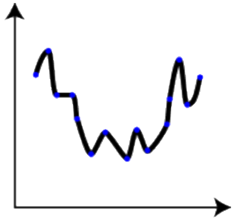
- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)

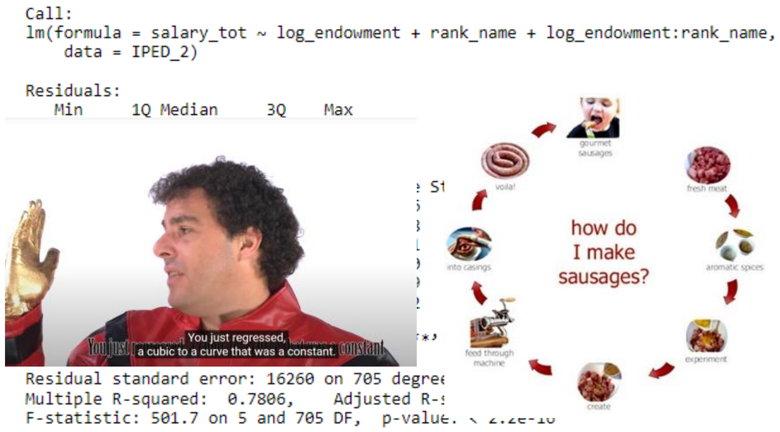
We can have categorical predictors and interactions

We can fit nonlinear functions

There are methods, statistics, and plots that help us:

- **Identify unusual points**: hatvalues, standardized residuals, Cook's distance, VIF
- **Assess the model fit**: diagnostic plots
- **Select models**: ANOVA, Adjusted R^2, AIC, BIC, cross-validation



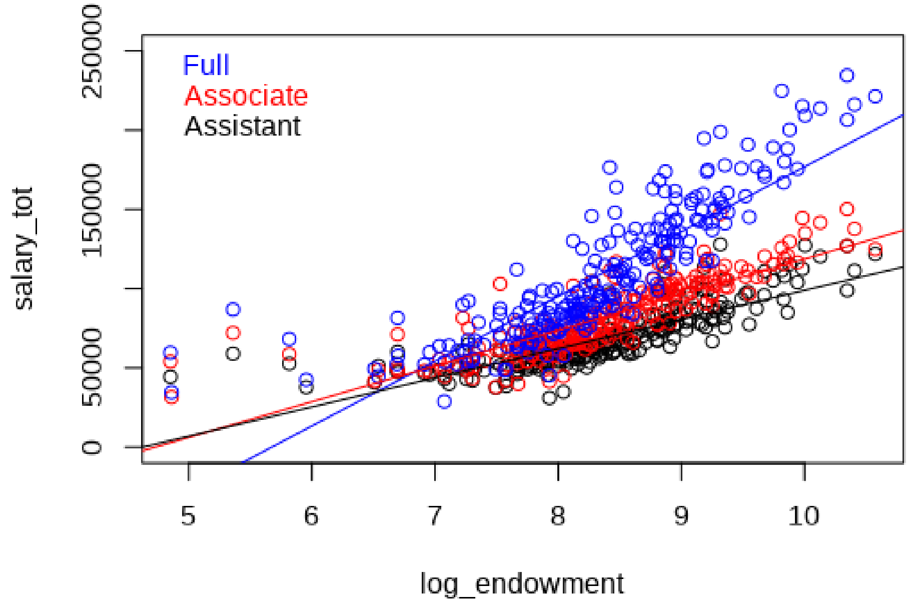


Plotting multiple regression models with ggplot

So far we have plotted our multiple regression models using base R graphics

This was useful for seeing the relationship between how R fits linear models, and what these models represent

However, if you want an easier/prettier way to visualize linear models, we can use ggplot!




```
Call:
lm(formula = salary_tot ~ log_endowment + rank_name + log_endowment:rank_name,
    data = IPED_2)
```

Let’s try it in R!

```
Residual standard error: 16260 on 705 degrees of freedom
Multiple R-squared:  0.7806,    Adjusted R-squared:  0.7791
F-statistic: 501.7 on 5 and 705 DF,  p-value: < 2.2e-16
```

Logistic regression

Logistic regression

In **logistic regression** we try to predict whether a case belongs to one of two categories

- Does a case below to category *1* or category *0*?
- Example: based on the salary level, can we predict if a faculty member is an Assistant of Full professor?

Making predictions for a categorical variable is called **classification**

- The field of Machine Learning has developed many classification methods

In logistic regression we build a conditional probability model:

- P(Class = 1 | x)
- P(Full Professor | salary = $80,000)

Logistic regression

Question: could we use linear regression to make these predictions?

$$P(Y = 1 \mid x_1) = \beta_0 + \beta_1 x_1$$

Logistic regression

Question: what if we transformed the probability to odds?

$$\frac{P(Y = 1 \mid x_1)}{P(Y = 0 \mid x_1)} = \frac{P(Y = 1 \mid x_1)}{1 - P(Y = 1 \mid x_1)} = \beta_0 + \beta_1 x_1$$

Question: what is the range of values odds can take on?

Logistic regression

Instead, we model the log odds as a linear function of our predictors

$$log(\frac{P(Y=1|x)}{1-P(Y=1|x)}) = \beta_0 + \beta_1 \cdot x$$

log-odds or logit

This scales values in the range of [0 1] to values in the range of (-∞ ∞)

Generalized linear models

Generalized linear models use a linear combinations of predictors to predict ***a function of the mean***

Y is a binary response variable (Y = 0 or 1)
P(Y = 1|x) is the mean of Y (given x)

"Linear predictor"

$$log(\frac{P(Y=1|x)}{1-P(Y=1|x)}) = \beta_0 + \beta_1 \cdot x$$

The logit function (log-odds) is a "link function" that links the mean to the linear predictor

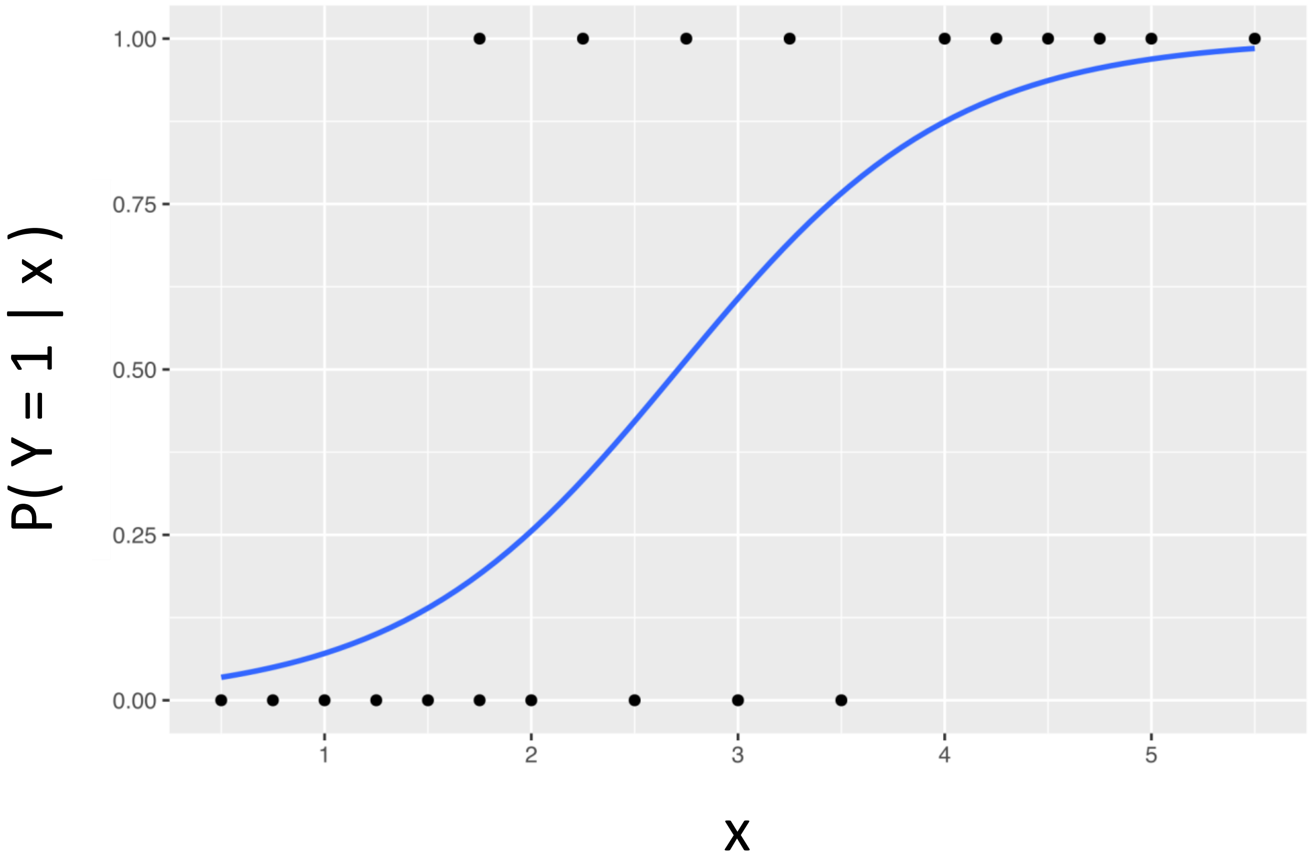
Logistic function

$$log(\frac{P(Y=1|x)}{1-P(Y=1|x)}) = \beta_0 + \beta_1 \cdot x$$

Solving for P(Y = 1| x) we get the "inverse link" function, which in the case of logistic regression is called a ***logistic function***

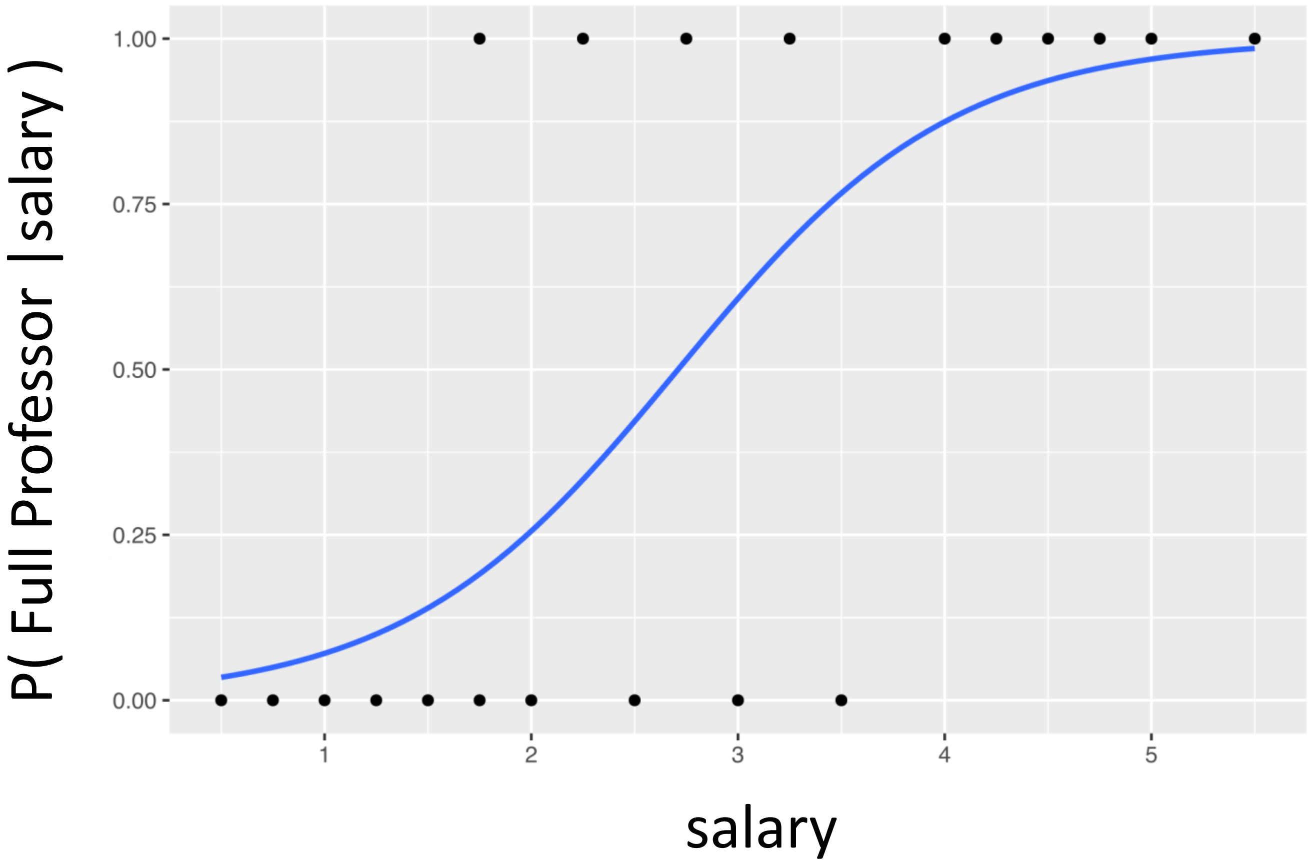
$$P(Y = 1|x) = \frac{\exp(\beta_0+\beta_1 \cdot x_1)}{1+\exp(\beta_0+\beta_1 \cdot x_1)} = \frac{e^{\beta_0+\beta_1 \cdot x_1}}{1+e^{\beta_0+\beta_1 \cdot x_1}}$$

Plotting the logistic function



$$P(Y = 1|x) = \frac{e^{\beta_0 + \beta_1 \cdot x}}{1 + e^{\beta_0 + \beta_1 \cdot x}}$$

Plotting the logistic function



$$P(\text{Full Professor} \mid \text{salary}) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 \cdot \text{salary}}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 \cdot \text{salary}}}$$

Multivariate logistic regression

We can easily extend our logistic regression model to include multiple explanatory variables

$$log(\frac{P(Y=1|x)}{1-P(Y=1|x)}) = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + ... + \hat{\beta}_k \cdot x_k$$

We can also use categorical predictors via dummy variable encoding as we did for regular multiple linear regression

Interpreting categorical predictors

When using a categorical predictor, x_2, in a logistic regression model, the exponential of the regression coefficient $e^{\hat{\beta}_2}$ is the ***odds ratio***

- Tells us how many times greater the odds are when $x_2 = 1$ vs. when $x_2 = 0$

$$log(\frac{P(Y=1|x_1,x_2)}{1-P(Y=1|x_1,x_2)}) = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2$$

Dummy variable

If $x_2 = 1$

If $x_2 = 0$

$$\frac{\dfrac{P(Y|x_1,x_2=1)}{1-P(Y|x_1,x_2=1)}}{\dfrac{P(Y|x_1,x_2=0)}{1-P(Y|x_1,x_2=0)}} = \frac{\cancel{e^{\hat{\beta}_0} e^{\hat{\beta}_1 \cdot x_1}} e^{\hat{\beta}_2}}{\cancel{e^{\hat{\beta}_0} e^{\hat{\beta}_1 \cdot x_1}}} = e^{\hat{\beta}_2}$$

Let’s look at logistic regression in R…

Poisson regression

Summary of linear regression

We can summarize the linear regression model as:

$$Y_i \;=\; \mu_i \;+\; \varepsilon_i \qquad\qquad \text{where} \quad \varepsilon_i \sim N(0, \sigma_\varepsilon)$$

$$\mu_i \;=\; \beta_0 \;+\; \beta_1 x_1 \;+ \ldots +\; \beta_k x_k$$

Equivalently, $\quad Y_i \;\sim\; N(\beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k, \;\; \sigma_\varepsilon)$

Generalized linear models

We can summarize the linear regression model as:

$$Y_i \;=\; \mu_i \;+\; \varepsilon_i \qquad\qquad \text{where} \quad \varepsilon_i \sim N(0, \sigma_\varepsilon)$$

$$\mu_i \;=\; \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k$$

In generalized linear models, we generalize the model to:

$$Y_i \;\sim\; f(y|\theta_i) \qquad\qquad \text{where } f(y|\theta_i) \text{ is some probability distribution}$$

$$\theta_i \;=\; g^{-1}(\beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k)$$

g^{-1} is called an "inverse link function"
Links "linear predictor" to parameters

We choose a particular "family" of distributions (e.g., Poisson, binomial, etc.)

Example: logistic regression

In logistic regression we model whether a case belongs to one of two categories

- P(Y = 0 | **x**)   or   P(Y = 1| **x**)

The logit function (log-odds) is a "link function"

"Linear predictor"

$$log\left(\frac{P(Y=1|x)}{1-P(Y=1|x)}\right) = \beta_0 + \beta_1 \cdot x$$

Solving for P(Y = 1 | x)

Inverse link function (logistic function)

$$P(Y = 1|x_1) = \frac{e^{\beta_0+\beta_1 \cdot x_1}}{1+e^{\beta_0+\beta_1 \cdot x_1}}$$

Family is Bernoulli distribution (binomial with n = 1)

$$Y_i \sim Bernoulli(P(Y = 1|x))$$

R: `glm_fit <- glm(y ~ x, family = binomial(link = logit))`

Poisson regression

In Poisson regression we model counts

- i.e., integer values: 0, 1, 2, 3, …

The log is the "link function"

"Linear predictor"

$$log(\lambda) = \beta_0 + \beta_1 \cdot x$$

Solving for λ

Inverse link function (exponential function)

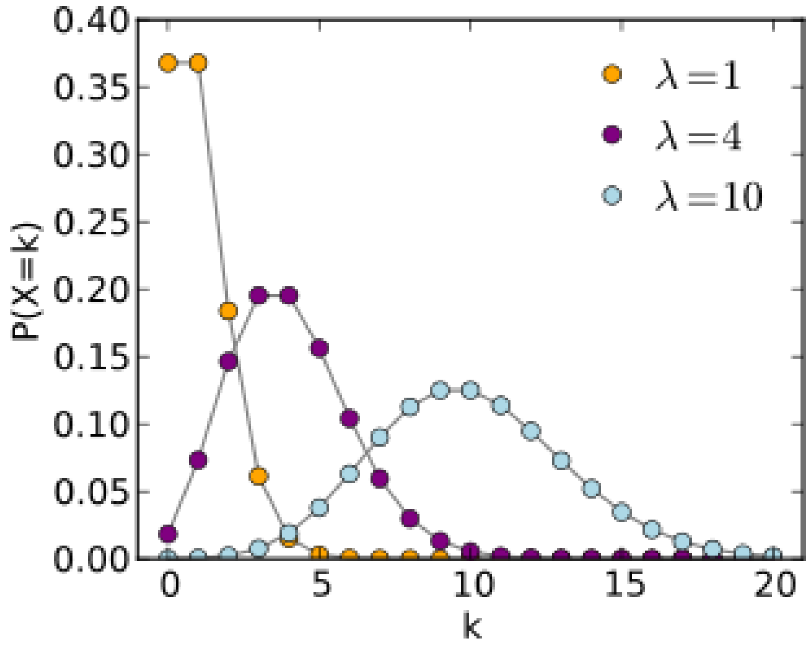
$$\lambda = e^{\beta_0 + \beta_1 \cdot x_1}$$

Family is Poisson distributions

$$Y_i \sim Poisson(\lambda))$$

R: glm_fit <- glm(y ~ x, family = Poisson(link = log))

Poisson distributions



A Poisson distribution is a probability distribution over non-negative integers

- i.e., over values 0, 1, 2, 3, …

Poisson distributions have a single parameter λ

$$X \sim Pois(\lambda)$$

$$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}\ ,\ k = 0, 1, 2, \ldots$$

- Density: dpois()
- Cumulative distribution: ppois()
- Random number: rpois()

Poisson processes

Poisson distributions models the number of outcomes that have occurred from a **Poisson process**

A **Poisson process** is a stochastic process where:

- Events (random outcome) occur at a fixed rate (λ)
- Every event is independent of the other events

Examples of Poisson processes?

- Arrival times of phone calls at a call center from 3-4am

Side note: Maximum likelihood estimate (MLE)

When building regression models, we need a way to estimate parameters

The "true" underlying model is:

$$y = \beta_0 + \beta_1 \cdot x_1 + \beta_2 \cdot x_2 + \ldots \beta_k \cdot x_k + \epsilon$$

We estimate coefficients using a data set to make predictions ŷ

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

For GLMs, the maximum likelihood estimates (MLE) is used to estimate the regression coefficients:

- MLEs find the parameters that make the data as likely as possible
  - (For linear regression with normal errors, MLE is gives the same coefficient estimates as least squares)

Example: Roy Kent saying f#ck

Ted Lasso was a Apple TV+ series that aired from July 2021 to March 2023

One of the main characters on the show was Roy Kent, who tended to say f#ck frequently

In different episodes of the show Roy was:

- A coach
- Dated Keeley Jones

Let's use Poisson regression to assess if Roy said f#ck more when he was coaching and/or when he was dating Keeley

Roy Kent

Keeley Jones

Example from season 2

Let's try it in R...