Session 4: Linear models

Overview

Multiple regression continued

- Categorical predictors and interactions
- Polynomial regression

Logistic regression

Analysis of Variance

Multiple regression

In multiple regression we try to predict a quantitative response variable y using several predictor variables $x_1, x_2, \ldots, x_k$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

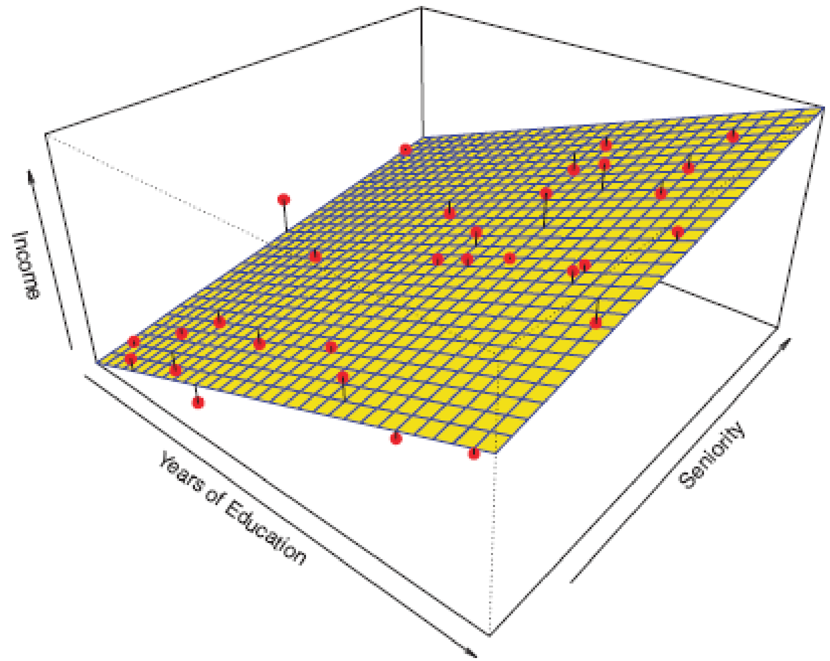
There are many uses for multiple regression models including:

- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)

Multiple regression

$$\text{salary} \;=\; \hat{\beta}_0 \;+\; \hat{\beta}_1 \cdot \mathrm{f}(\text{endowment}) \;+\; \hat{\beta}_2 \cdot \mathrm{g}(\text{enrollment})$$

Let’s explore this in R…



Categorical predictors

When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it

- e.g., we would need two dummy variables to have different intercepts for Assistant, Associate and Full Professors

$$x_{i1} = \begin{cases} 1 & \text{if Assistant Professor} \\ 0 & \text{if Full Professor} \end{cases} \qquad x_{i2} = \begin{cases} 1 & \text{if Associate Professor} \\ 0 & \text{if Full Professor} \end{cases}$$

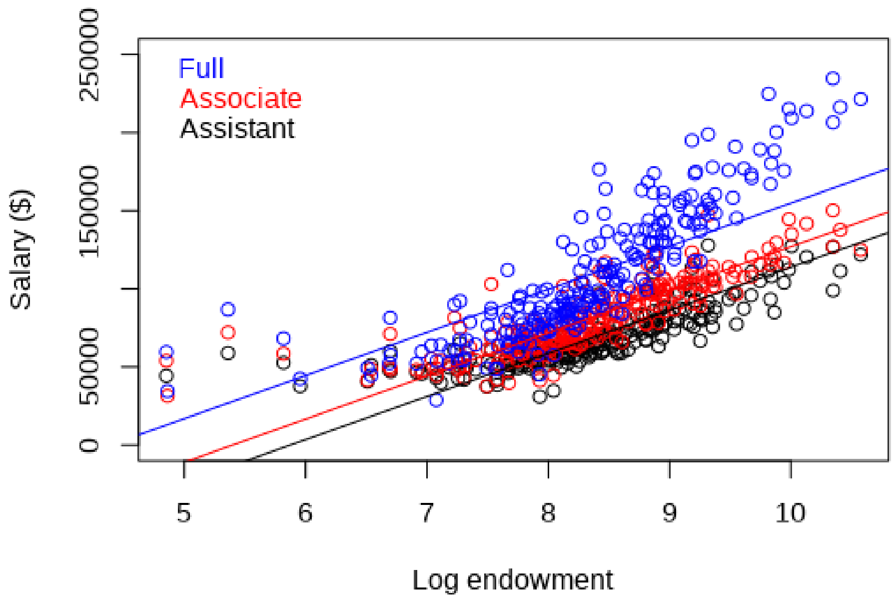
$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if Assistant Professor} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if Associate Professor} \\ \beta_0 + \epsilon_i & \text{if Full Professor} \end{cases}$$

Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it

Suppose we want to predict faculty salary as a function of endowment with separate intercepts for faculty rank




```
> summary(fit_prof_rank_offset)

Call:
lm(formula = salary_tot ~ log_endowment + rank_name, data = IPED_2)

Residuals:
   Min     1Q Median     3Q    Max 
-52464 -10844  -2703   6936  74994 

Coefficients:
                      Estimate Std. Error t value            Pr(>|t|)    
(Intercept)          -120822.1     6713.9  -18.00 <0.0000000000000002 ***
log_endowment          27569.9      791.7   34.82 <0.0000000000000002 ***
rank_nameAssociate    -27855.4     1685.5  -16.53 <0.0000000000000002 ***
rank_nameAssistant    -40973.7     1685.5  -24.31 <0.0000000000000002 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 18370 on 707 degrees of freedom
Multiple R-squared:  0.7192,	Adjusted R-squared:  0.718 
F-statistic: 603.7 on 3 and 707 DF,  p-value: < 0.00000000000000022
```

$$\hat{y}_i = \begin{cases} \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1 x_{i1}} + \boxed{\hat{\beta}_2} & \text{if assistant professor} \\ \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1 x_{i1}} + \boxed{\hat{\beta}_3} & \text{if associate professor} \\ \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1 x_{i1}} & \text{if full professor} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i3}$$

$$= -120,822 + 27,570 x_{i1} - 40,973 x_{i2} - 27,855 x_{i3}$$

Interaction terms

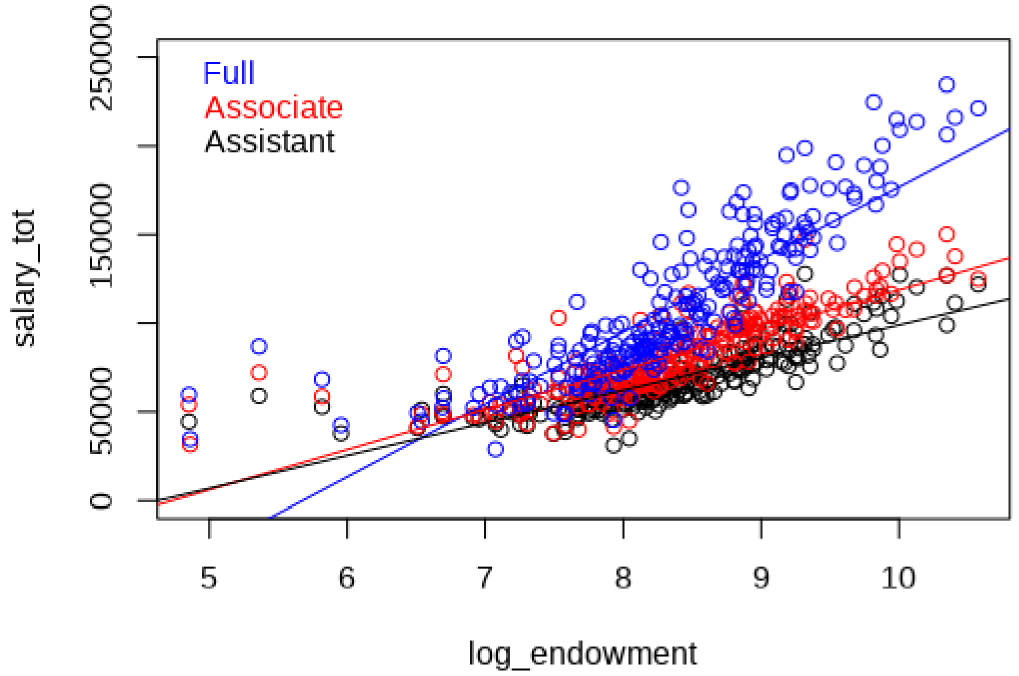
An ***interaction effect*** occurs when the response variable y is influenced by the levels of two or more predictors in a non-additive way

We can model this using an equation with an interaction term

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_3 (x_1 \cdot x_2) + \epsilon$$

An interaction term between a quantitative and categorical variable corresponds to different slopes depending for the quantitative variable depending on the value of the categorical variable

Interaction terms



If Full Professor:

$$\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment}$$

If Assistant Professor:

$$\text{salary} \approx (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{endowment}$$

Additive term if Assistant Professor

Change in slope if Assistant Professor

$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i1} \cdot x_{i2}$$

Interaction terms

$$\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment} + \beta_2 \cdot \text{assistant\_rank\_dummy} + \beta_3 \cdot (\text{assistant\_rank\_dummy} \cdot \text{endowment})$$

Let’s try it in R…

Multicollinearity

Multicollinearity occurs when our predictors (x_i's) are correlated.

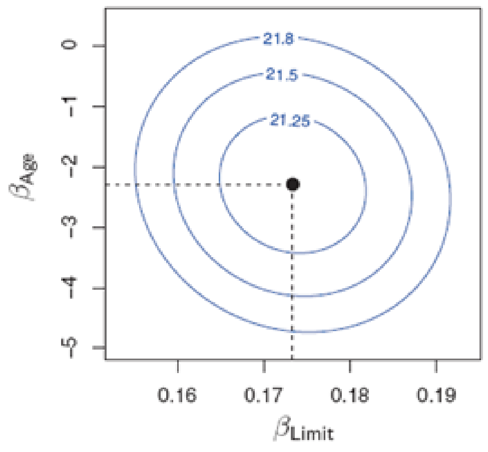
This can lead to unstable estimates of the regression coefficients

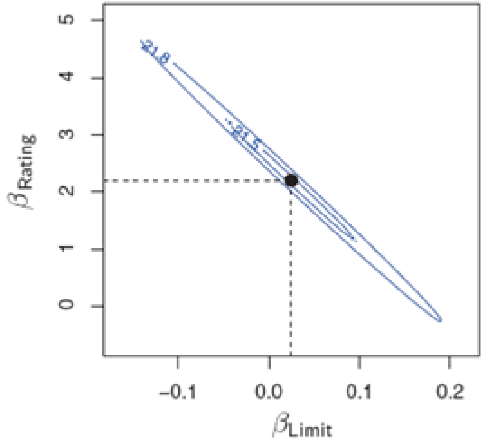
- Which will lead to large SE on the coefficients and consequently they will not appear to be statistically significant.

The **variance inflated factor** can be used to test for multicollinearity each explanatory

- Rule of thumb: VIF > 5 suspect for multicolinearity

Contours of equal SSResiduals





car::vif(lm_fit)

Non-linear relationships

Polynomial regression extends linear regression to non-linear relationships by including nonlinear transformations of predictors

$$
\begin{aligned}
\text{salary} = \;& \beta_0 + \beta_1 \cdot \text{endowment} \\
& + \beta_2 \cdot (\text{endowment})^2 + \\
& + \beta_3 \cdot (\text{endowment})^3 + \varepsilon
\end{aligned}
$$

Still a linear equation but non-linear in original predictors

Logistic regression

In **logistic regression** we try to predict whether a case belongs to one of two categories

- Does a case belong to category *a* or category *b*?
- Example: can we predict if a faculty member is an Assistant of Full professor based on the salary level?

Making predictions for a categorical variable is called **classification**

- The field of Machine Learning has developed many classification methods

In logistic regression we build a conditional probability model:

- P(Class = a | x)
- P(Assistant Professor | salary = $60,000)

Generalized linear models

Generalized linear models use a linear combinations of predictors to predict ***a function of the mean***

If Y is a binary response variable (Y = 0 or 1)

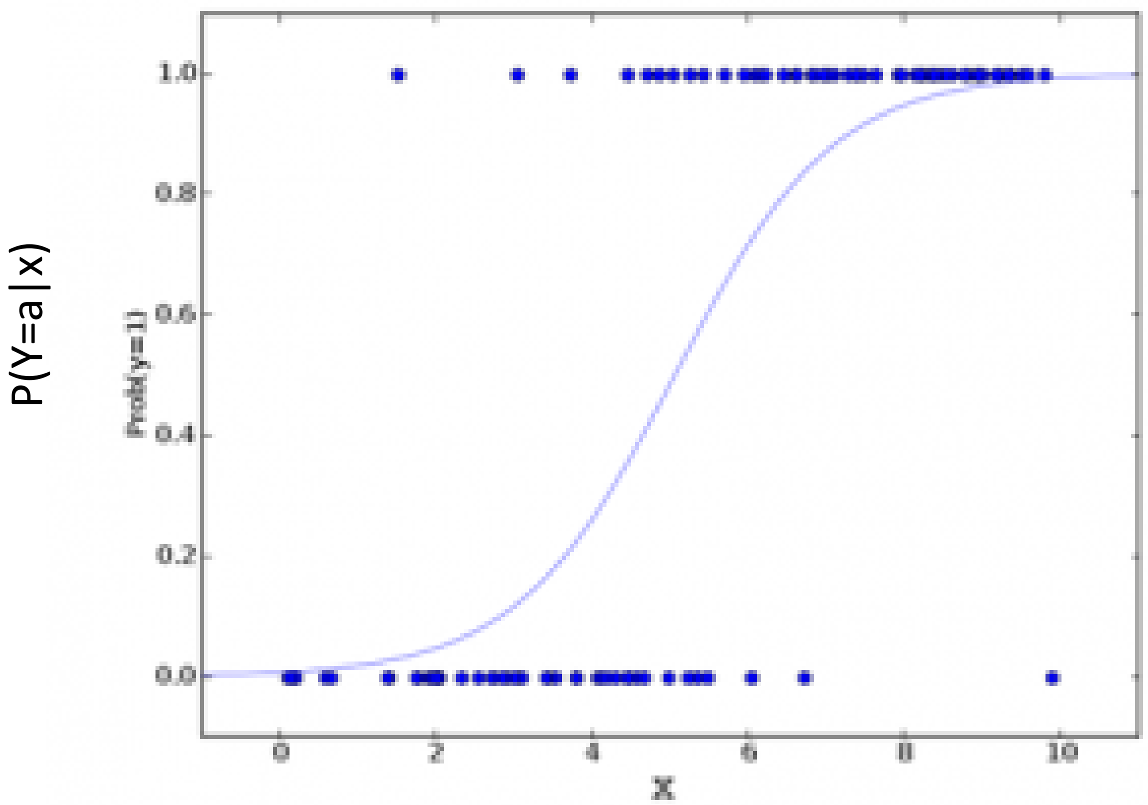
P(Y = 1|x) is the mean of Y

"Linear predictor"

$$log(\frac{P(Y{=}a|x)}{1{-}P(Y{=}a|x)}) = \beta_0 + \beta_1 \cdot x$$

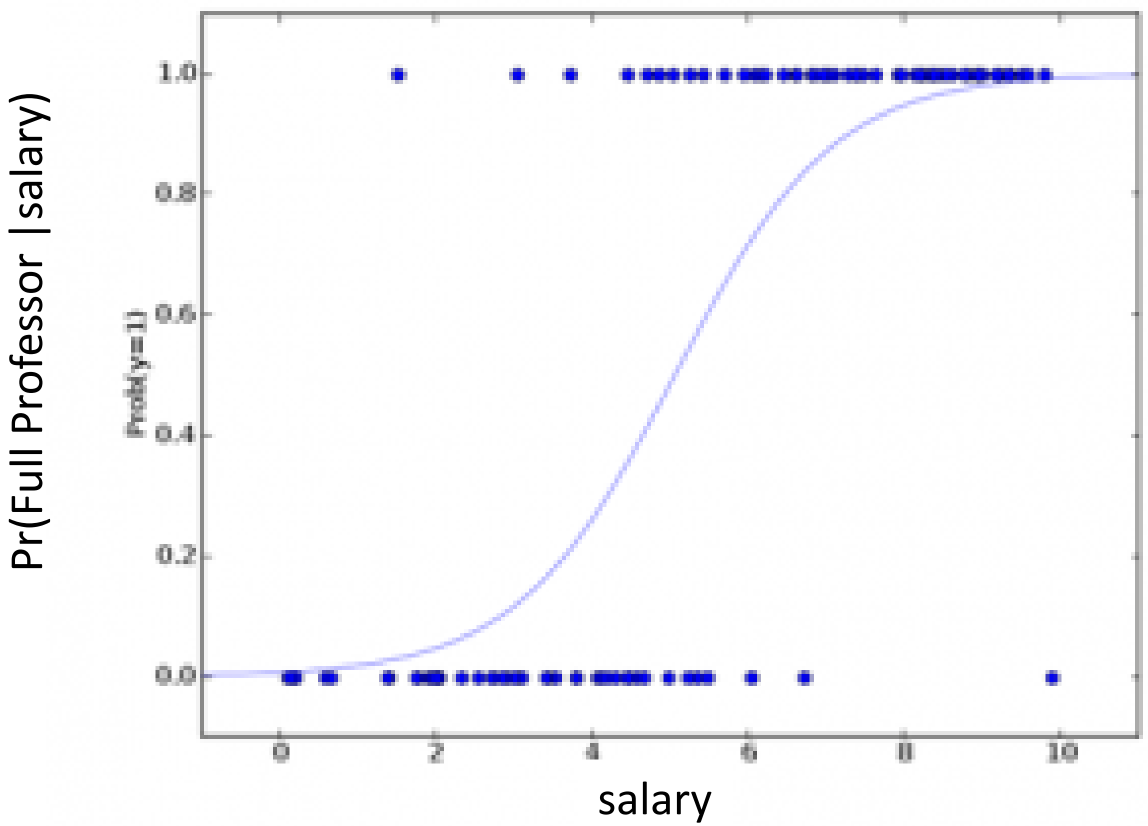
The logit function (log-odds) is a "link function" that links the mean to the linear predictor

Plotting the logistic function



$$P(Y = a|x_1) = \frac{e^{\beta_0+\beta_1 \cdot x_1}}{1+e^{\beta_0+\beta_1 \cdot x_1}}$$

Plotting the logistic function



$$P(\text{Full Professor} \mid \text{salary}) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 \cdot \text{salary}}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 \cdot \text{salary}}}$$

One-way ANOVA

A **one-way analysis of variance (ANOVA)** is a parametric hypothesis test that can be used to examine if a set of means are all the same.

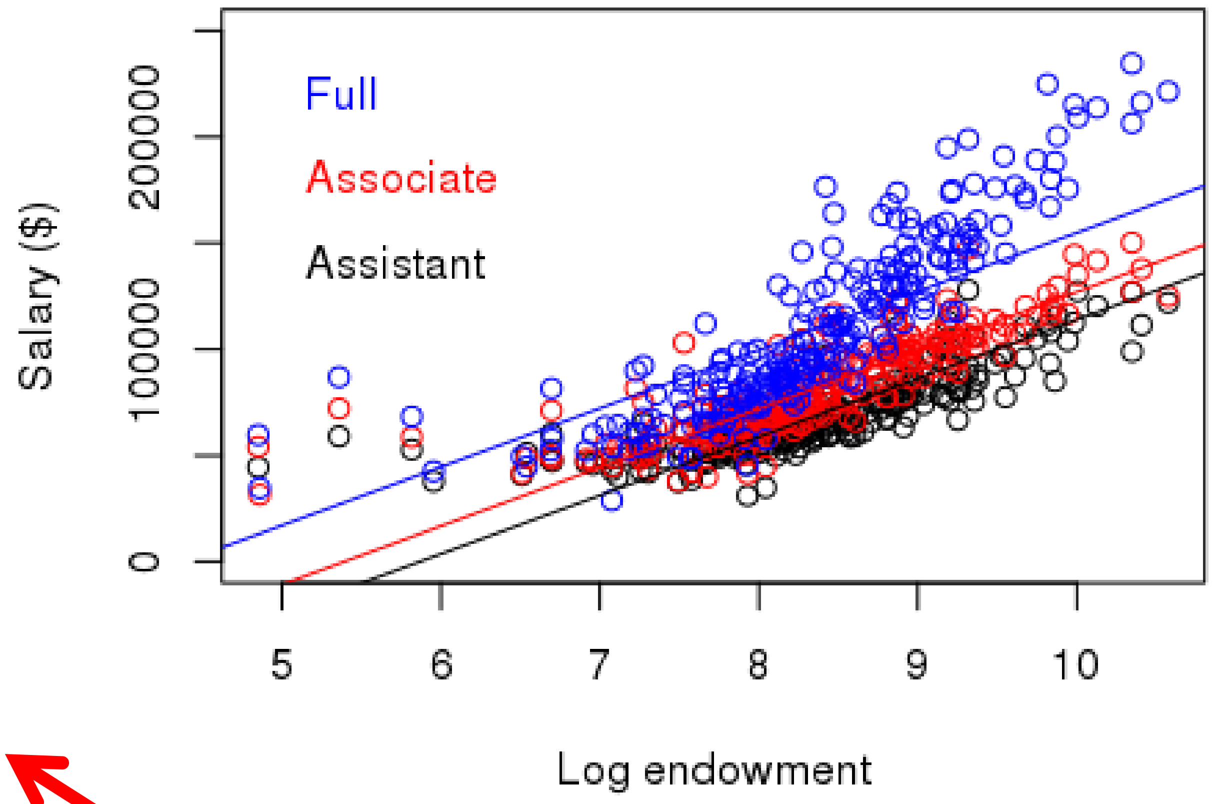
H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$

H_A: $\mu_i \neq \mu_j$ for some i, j

The statistic we use for a one-way ANOVA is the F-statistic

$$F = \frac{\text{between-group variability}}{\text{within-group variability}} = \frac{\frac{1}{K-1}\sum_{i=1}^{K} n_i(\bar{x}_i - \bar{x}_{tot})^2}{\frac{1}{N-K}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2}$$

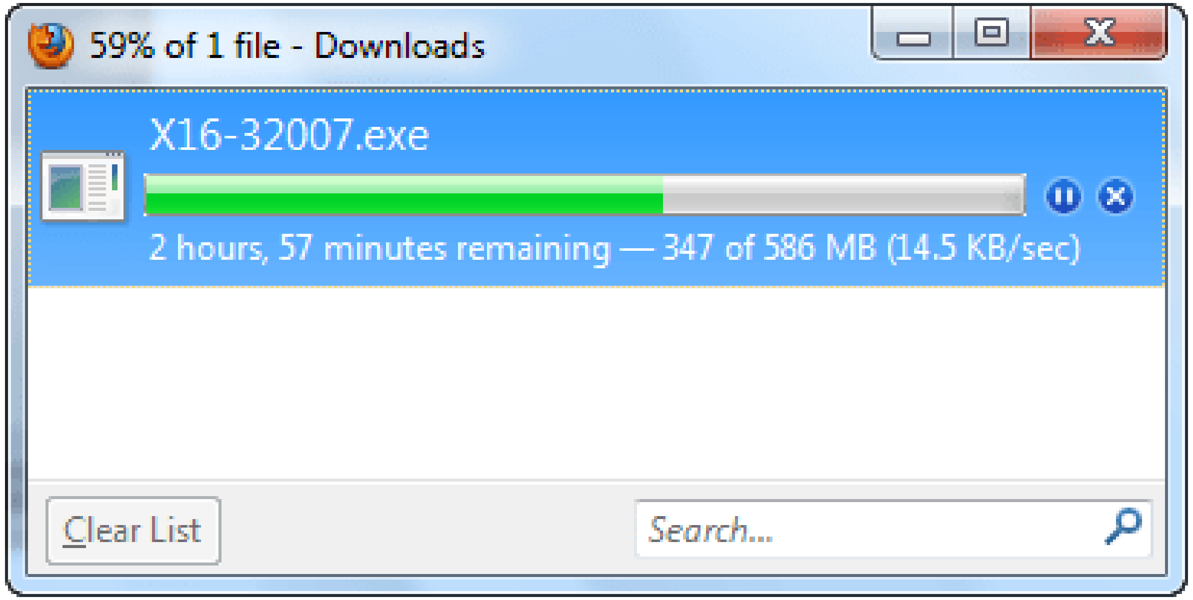
ANOVA as regression with only categorical predictors



Common slope for
log endowment

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \quad + \epsilon_i & \text{if Assistant Professor} \\ \beta_0 + \beta_2 + \quad + \epsilon_i & \text{if Associate Professor} \\ \beta_0 + \quad + \epsilon_i & \text{if Full Professor} \end{cases}$$

Let’s try it in R…

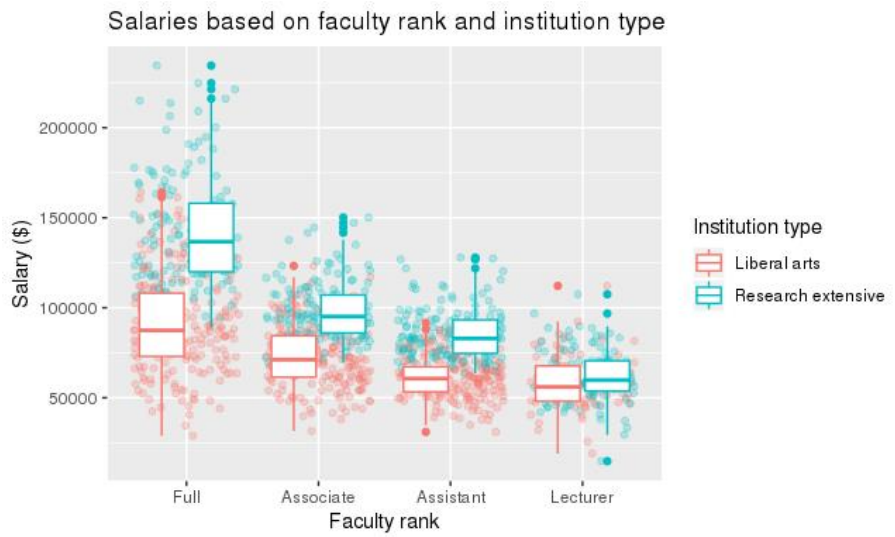


Factorial ANOVA

In a **factorial ANOVA**, we model the response variable y as a function of **more than one** categorical predictor

Example 1: Do faculty salaries depend on faculty rank, and the type of college/university

- Factors are:
    - **Rank**: Lecturer, Assistant, Associate, Full
    - **Institute**: liberal arts college, research university
    - 4 x 2 design



Factorial ANOVA

Example 2: A student at Queensland University of Technology conducted an experiment to determine what types of sandwiches ants prefer.

- Factors he looked at were:
  - **Bread**: rye, whole wheat multigrain, white
  - **Filling**: peanut better, ham and pickle, and vegemite
  - 4 x 3 design

The student creating 4 sandwiches of all combinations of bread and filling (48 sandwiches total) and randomly left pieces in front of ant nests.

He then measured how many ants were on the sandwiches 5 minutes later.

Two-way ANOVA hypotheses

Main effect for A (bread type doesn't matter or institution type doesn't matter)

H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_J = 0$

H_A: $\alpha_j \neq 0$ for some j

Main effect for B (filling doesn't matter)

H_0: $\beta_1 = \beta_2 = \ldots = \beta_K = 0$

H_A: $\beta_k \neq 0$ for some k

Interaction effect:

H_0: All $\gamma_{jk} = 0$

H_A: $\gamma_{jk} \neq 0$ for some j, k

Where:

α_j: is the "effect" for factor A at level j

β_k: is the "effect" for factor B at level k

γ_{jk} : is the interaction between level j of factor A, and level k of factor B.

Repeated measures ANOVA

In a **repeated measures ANOVA**, the same case/observational units are measured at each factor level.

Example: Do people prefer chocolate, butterscotch or caramel sauce?

Between subjects experiment: different people rate chocolate, butterscotch or caramel sauce.

- Run a between subjects ANOVA (as we have done before)

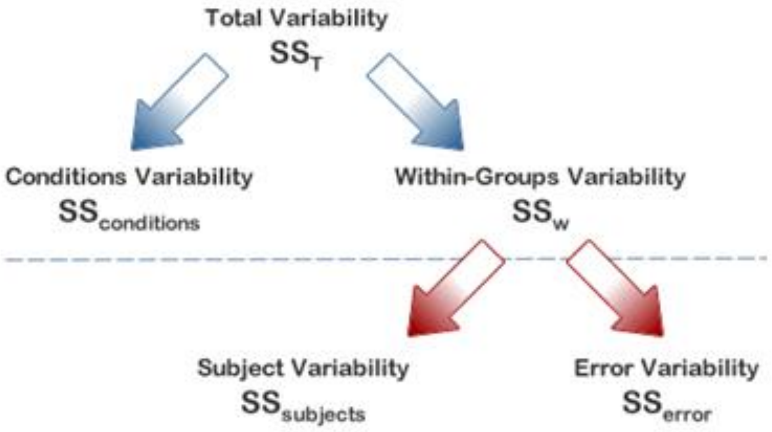
Within subjects experiment: each person in the experiment gives ratings for all three toppings.

- Run a repeated measures ANOVA

Repeated measures ANOVA

The advantages of a repeated measures ANOVA is that we can potentially reduce a lot of the variability between the cases

- This is a generalization of a paired t-test to more than two population means

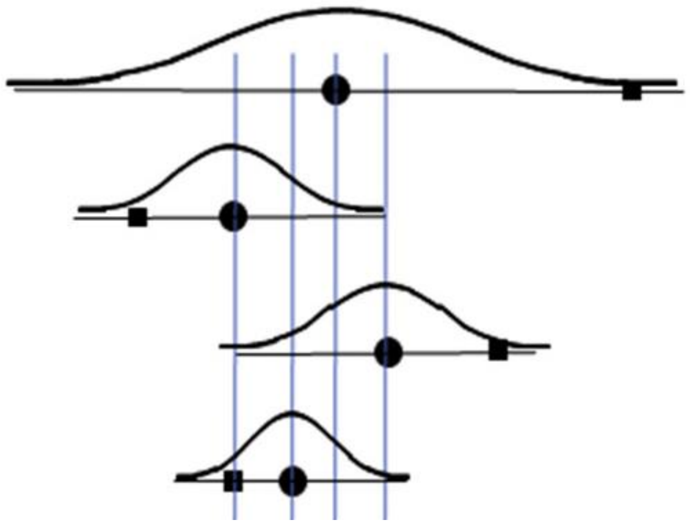


To run a repeated meseasures ANOVA, we use a factor called ID that has a unique value for each observational unit

```
aov(reaction_time ~ condition * position +  participant,
    data = popout_log_data)
```

Brief mention: random effects models

In a random effects ANOVA, factor levels are viewed as being randomly generated from an underlying distribution, rather than having a fixed number of levels.



For example, we could view participants in an experiment as being a random sample from participants in a population.

- We then just estimate a mean and standard deviation for the underlying population, rather a separately ID for each participant.
  - This leads to few parameters and hence more degrees of freedom.

You can run mixed effects models in R using the lme4 package