Cointegration Pairs Trading Strategy On Derivatives [1]

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November 27, 2013

[1] Research supported in part by grants from GRF-RGC-HKSAR.

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Introduction

Arbitrage

Arbitrage: free lunch; earning extra profit without taking additional risk. The no arbitrage assumption serves as the building block of modern finance. It is the cornerstone of the celebrated Black-Scholes models for option pricing.

Statistical Arbitrage: An attempt to profit from the pricing efficiencies that are identified through the use of mathematical models. Statistical arbitrage attempts to profit from the likelihood that prices will trend together toward a historical norm. Unlike arbitrage, statistical arbitrage is not risk-free.

Style Investing

Barberis and Shleifer (2003) discussed style investing.

Style: Assets with similar characteristics.

Switchers
–Allocate funds at the level of a style.
–The amount allocated in each style depends on the past performance of that style relative to other styles.

Switchers cause co-movements for the stocks in the same style.

Arbitrage Opportunity in Short-run

- Wilson and Marashdeh (2007) argued that co-movements between stock prices imply market efficiency in long-run equilibrium.
  –Co-movements cause arbitrage opportunity in short-run.
  –Inefficiency in the short-run is eliminated by arbitrage activities.
  –Resulting efficiency in the long-run equilibrium.
- It is reasonable to think that stock has co-integration property in the long-run.
- We may take advantage of such a property to find statistical arbitrage opportunities in the short-run.

Introduction

Some financial instruments move more or less in sync with each other, because they are driven by similar fundamental (e.g. economic) factors. For example,

- Stock prices of Coca Cola and Pepsi.
- Currency Pairs AUD/USD vs NZD/USD .

However, they do not move EXACTLY the same because of their individual technical factors, which can be assumed to be noises on top of the common movement.

Statistical Approach: Cointegration

– Assume that the underlying are stochastic processes sharing the same stochastic drift.

– Filter out the co-movement of pairs of market instruments by identifying possible stationary series which is a linear combination of two non-stationary series (e.g. prices of two stocks).

Cointegration

Definition

Two non-stationary time series $\{X_t\}$ and $\{Y_t\}$ are cointegrated if some linear combination $aX_t + bY_t$, with a and b being constants, is a stationary series.

There are two popular approaches to establish the cointegration relationship:

- Engle-Granger methodology (1987)
- Johansen methodology (1988)

Cointegration Pairs Trading Strategy on Stocks

Cointegration Pairs Trading Strategy on Stocks

- The notion of cointegration has been widely used in finance and econometrics, in particular in constructing statistical arbitrage strategy in the stock market.
- Profitability has been reported using the cointegration strategy on stocks trading, see Chan (2010).

Example: (Chan, 2010)

Let X_t and Y_t be two stock prices at time t. Assume that $a\log X_t + b\log Y_t$ is stationary, i.e. $\log X_t$ and $\log Y_t$ are cointegrated.

By Taylor expansion,

$$
\begin{aligned}
a\log X_t + b\log Y_t &\approx a(\log X_{t_0} + \frac{X_t - X_{t_0}}{X_{t_0}}) + b(\log Y_{t_0} + \frac{Y_t - Y_{t_0}}{Y_{t_0}}) \\
&= \frac{a}{X_{t_0}}X_t + \frac{b}{Y_{t_0}}Y_t + a(\log X_{t_0} - 1) + b(\log Y_{t_0} - 1).
\end{aligned}
$$

Because $a(\log X_{t_0} - 1) + b(\log Y_{t_0} - 1)$ is a constant, the stationarity of $a\log X_t + b\log Y_t$ implies that $\frac{a}{X_{t_0}}X_t + \frac{b}{Y_{t_0}}Y_t$ is approximately stationary, i.e. $\frac{a}{X_{t_0}}X_t + \frac{b}{Y_{t_0}}Y_t$ should exhibit mean-reverting property.

We can initiate a position with $c\frac{a}{X_{t_0}}$ shares of X and $c\frac{b}{Y_{t_0}}$ shares of Y for any given value c, where c can be considered as the starting initial capital.

Implementation of Trading Strategy on Stocks

- Investigate possible cointegrated series from pairs of log(stock prices) [or log(exchange rates)] based on the Johansen test (typically 5% of pairs).
- For each cointegration pairs, check if the current level of the stationary series (i.e. $a \log X_t + b \log Y_t$)) is too low/high against its historical mean (e.g. Z-score $= -2, +2$).
- Enter the trade (i.e. buy/sell $\frac{a}{X_t}$ of X and $\frac{b}{Y_t}$ shares of Y at time t_0 and expect the stationary series $a \log X_t + b \log Y_t$) to mean-revert back to it historical average level.
- For a portfolio with 46 stocks, there are $C_2^{46} = 575$ possible combination of stock pairs and E(Number of cointegrated series at t_0) $\sim 575x5\% \sim 29$ pairs.
- Works well for stock prices in the same sector, harder to interpret for stocks from different sectors.
- Not much juice on simple financial instruments, as other statistical techniques based on mean-reversion assumption (e.g. PCA) is able to detect similar trades.

- Would like to extend the cointegration strategy on derivatives. For example, extend from Currency pairs of AUD/USD vs NZD/USD to Option on AUD/USD vs option on NZD/USD.

Cointegration Pairs Trading Strategy on Derivatives

Implied volatility is the assessment of the future variation in an underlying asset.

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Even though

$$
\begin{array}{ccc}
S_X & \xleftrightarrow{\textit{not cointegrated}} & S_Y \\
\downarrow & & \downarrow \\
\sigma_X^2 & \xleftrightarrow{\textit{cointegrated}} & \sigma_Y^2
\end{array}
$$

Implied volatility is the assessment of the future variation in an underlying asset.

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$$\begin{array}{ccc} S_X & \xleftrightarrow{\textit{not cointegrated}} & S_Y \\ \downarrow & & \downarrow \\ \sigma_X^2 & \xleftrightarrow{\textit{cointegrated}} & \sigma_Y^2 \end{array}$$

because they may be affected by the same exogenous event so that people have the same perspective to the variations of the underlying assets.

Similar with the cointegration pairs trading strategy on stocks , a divergence from the mean level of the cointegration pairs can be captured to make a profit.

How to trade volatility?

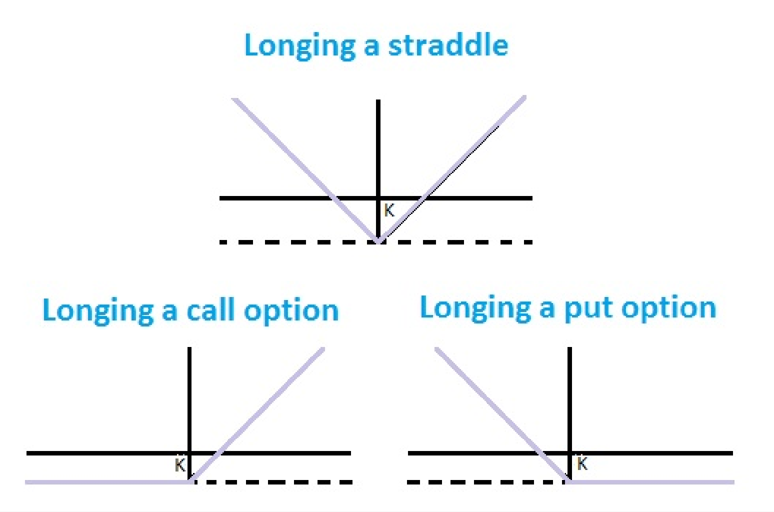
How to trade volatility?

Straddle!!

Why Straddle?

What is Straddle?

Definition: A long (short) straddle is long (short) a call option and a put option at the same strike price and expiration date.



What is Straddle?

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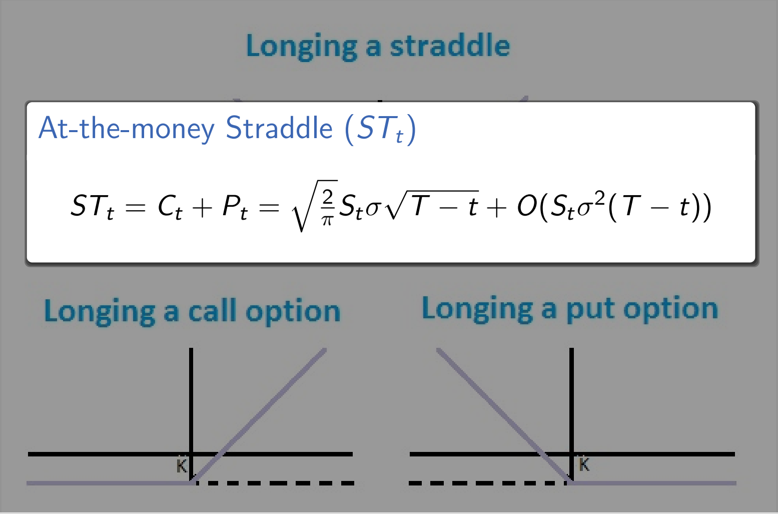

At-the-money Straddle (ST_t)

$$ST_t = C_t + P_t = \sqrt{\frac{2}{\pi}} S_t \sigma \sqrt{T-t} + O(S_t \sigma^2 (T-t))$$

How the cointegration strategy works?

Suppose the trading signal is

$$TS_t = a\sigma_t^X - b\sigma_t^Y \sim I(0), \text{ where } a, b > 0,$$

the z-score of the trading signal is

$$Z_t = \frac{TS_t - \mu}{\sigma},$$

where μ are σ are the mean and the standard deviation of the trading signal.

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where μ are σ are the mean and the standard deviation of the trading signal.

$TS_t << \mu \Rightarrow \sigma_t^X$ too low comparing to σ_t^Y, $\Rightarrow$ long ST_t^X and short ST_t^Y.
$TS_t >> \mu \Rightarrow \sigma_t^X$ too high comparing to σ_t^Y, $\Rightarrow$ long ST_t^Y and short ST_t^X.

How does the cointegration strategy work?

Setting up the Trading Portfolio

For $t \in (t_0, T)$,the value of the straddles portfolio Π_t becomes

$$\Pi_t = A \times (C^X_{S_{t_0}} + P^X_{S_{t_0}}) - B \times (C^Y_{S_{t_0}} + P^Y_{S_{t_0}}),$$

where $C^X_{S_{t_0}}$, $P^X_{S_{t_0}}$, $C^Y_{S_{t_0}}$, $P^Y_{S_{t_0}}$ at time t_0 are at-the-money options.

Based on the Taylor expansion, $\triangle\Pi_t$ can be approximated by

$$\begin{aligned}\triangle\Pi_t = \frac{\partial\Pi_t}{\partial t} \triangle t + \frac{\partial\Pi_t}{\partial S^X_t} \triangle S^X_t + \frac{\partial\Pi_t}{\partial S^Y_t} \triangle S^Y_t + \frac{1}{2}\frac{\partial^2\Pi_t}{\partial (S^X_t)^2}(\triangle S^X_t)^2 \\ + \frac{1}{2}\frac{\partial^2\Pi_t}{\partial (S^Y_t)^2}(\triangle S^Y_t)^2 + \frac{\partial\Pi_t}{\partial \sigma^X_t} \triangle \sigma^X_t + \frac{\partial\Pi_t}{\partial \sigma^Y_t} \triangle \sigma^Y_t,\end{aligned}$$

How does the cointegration strategy work?

Se

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$\triangle\Pi$

$$+\frac{1}{2}\frac{\partial^2\Pi_t}{\partial(S_t^Y)^2}(\triangle S_t^Y)^2+\frac{\partial\Pi_t}{\partial\sigma_t^X}\triangle\sigma_t^X+\frac{\partial\Pi_t}{\partial\sigma_t^Y}\triangle\sigma_t^Y,$$

Vega Part : $I=\frac{\partial\Pi_t}{\partial\sigma_t^X}\triangle\sigma_t^X+\frac{\partial\Pi_t}{\partial\sigma_t^Y}\triangle\sigma_t^Y$

Delta Part : $II=\frac{\partial\Pi_t}{\partial S_t^X}\triangle S_t^X+\frac{\partial\Pi_t}{\partial S_t^Y}\triangle S_t^Y$

Gamma Part : $III=\frac{1}{2}\frac{\partial^2\Pi_t}{\partial(S_t^X)^2}(\triangle S_t^X)^2+\frac{1}{2}\frac{\partial^2\Pi_t}{\partial(S_t^Y)^2}(\triangle S_t^Y)^2$

Theta Part : $IV=\frac{\partial\Pi_t}{\partial t}\triangle t$

i.e. $\triangle\Pi_t=I+II+III+IV.$

Requirements of the trade

The trade will be dominated by vega if the trade fulfills the following requirements:

1. **Short-term period of trading;**
2. **Using long-dated option;**
3. **The mean of the trading signal is negative (positive), the position is initiated when the trading signal is too low (high). Otherwise, the position should not be initiated.**

   In the derivation below, it is shown that at time t,

$$\Theta = \frac{\partial \Pi_t}{\partial t} \approx \begin{cases} -\frac{TS_t}{2\sqrt{T-t}}, & \text{long the portfolio,} \\ \frac{TS_t}{2\sqrt{T-t}}, & \text{short the portfolio.} \end{cases}$$

   where $TS_t = Z_t\sigma + \mu$, μ and σ are the mean and standard deviation of the trading signal TS_t.

How does the cointegration strategy work?

Because the trading is in a short-term period since time t_0, the Greek Letters can be approximated. Assume that $\mu(TS) < 0$ and the portfolio of longing ST_t^X and shorting ST_T^Y is entered.

Let $A = \sqrt{\frac{\pi}{2}}\frac{a}{S_{t_0}^X}$, $B = \sqrt{\frac{\pi}{2}}\frac{b}{S_{t_0}^Y}$, where a, b are the coefficients of the cointegration.

By the approximation of $\Phi(d_1)$ and $\Phi(d_2)$:

$$\Phi(d_1) = \frac{1}{2} + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} + O(d_1^2),$$
$$\Phi(d_2) = \frac{1}{2} - \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} + O(d_2^2),$$

and the assumption that S_t is driven by the Geometric Brownian motion

$$\frac{dS_t}{S_t} = r\,dt + \sigma^r\,dW_t,$$

where σ^r is the realized volatility, I, II, III, IV can be approximated as follows.

How does the cointegration strategy work?

Because ... ek

Letters ... tfolio of

longing

Let $A =$

cointegr

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Approximation

$$\mathbb{E}(I) \approx \sqrt{T - t_0}(\mathbb{E}(TS_t) - TS_{t_0}),$$

$$\mathbb{E}(II) \approx \frac{\Delta t \sqrt{T - t_0} r_{t_0} TS_{t_0}}{2},$$

$$\mathbb{E}(III) \approx \frac{\Delta t}{2\sqrt{T - t_0}}\left(\frac{a\mathbb{E}(\sigma_X^r)^2}{\sigma_{t_0}^X} - \frac{b\mathbb{E}(\sigma_Y^r)^2}{\sigma_{t_0}^Y}\right),$$

$$\mathbb{E}(IV) \approx -\frac{\Delta t \times TS_{t_0}}{2\sqrt{T - t_0}},$$

where $\Delta t = t - t_0$, r_{t_0} is the risk-free rate, $(\sigma_X^r)^2$ and $(\sigma_Y^r)^2$ are the average squared annual realized volatilities.

and the ... otion

where σ^r is the realized volatility, I, II, III, IV can be approximated as follows.

How does the cointegration strategy work?

Vega Part: $I = \sqrt{T - t_0}(\mathbb{E}(TS_t) - TS_{t_0})$

This is the main profit to be captured by the trading strategy. Should the trading signal reverts back to its mean value, then $\mathbb{E}(I) > 0$.

How does the cointegration strategy work?

Proposition 1

If the trade is in a short-time period, $\mathbb{E}(II) + \mathbb{E}(IV) > 0$.

Vega Pa

This is ... ld the trading

Proposition 2

Assume that

1. Annualized volatilities (include implied volatility and realized volatility) of the underlying assets are smaller than 80% ;
2. $\frac{1}{c}\sigma^i_{t_0} < \sigma^r_i < c\sigma^i_{t_0}$, for $i = X$ and Y, $c > 1$;
3. $\sigma(TS_t) > 1$.

Then $\mathbb{E}(I) + \mathbb{E}(III) > 0$.

In conclusion

If the implied volatilities of the underlying assets are not too high and do not deviate too far from the corresponding realized volatilities, and if $\mathbb{E}(TS_t) - TS_{t_0}$ is large enough, then

$$\mathbb{E}(\Delta\Pi_t) = \mathbb{E}(I) + \mathbb{E}(II) + \mathbb{E}(III) + \mathbb{E}(IV) > 0.$$

Empirical Study with Foreign Exchange Options

Empirical Study with Foreign Exchange Options

Find out the cointegration pairs

In our analysis, FX options with 3-month expiry of 30 currency pairs (e.g. EUR/USD) for all the strikes from 2009Q4 to 2011Q4 were considered.

The Johansen test is based on the data from 2009Q4-2011Q3. The co-integration pairs identified will be used as the trading signals for 2011Q4.

Find out the cointegration pairs

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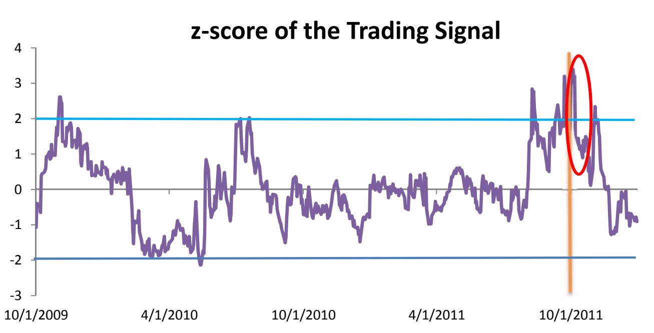
The Johansen test is based on the data from 2009Q4-2011Q3. The co-integration pairs identified will be used as the trading signals for 2011Q4.

According to the results from Johansen test, the implied volatilities of GBPNZD and GBPUSD are significantly cointegrated with the parameters $a = 0.6762$ and $b = 1.0000$, i.e. the trading signal is

$$TS = 0.6762 \times \sigma_{GBPNZD} - \sigma_{GBPUSD}.$$

Details of cointegration pairs



The z-score of the Trading Signal on Oct 3^{rd}, 2011 is 3.39 and on Oct 7^{th}, 2011, it is 1.52.

Details of the trade

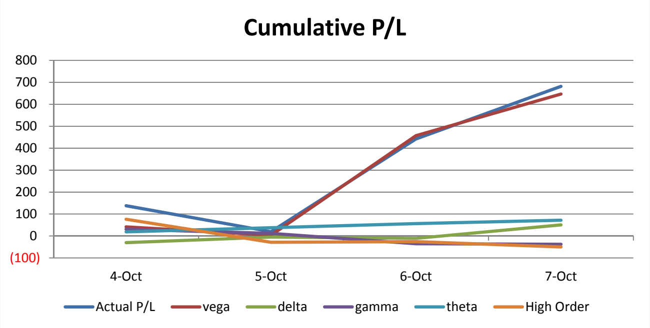


Figure: Change of P/L due to Greek Letters

The value of the portfolio is -USD3,465 on 03/10/2011, and then the value increases to -USD2,783 on 07/10/2011. Finally, we gain +USD682 by this strategy.

More examples

We identify possible cointegration in any two currency pairs, based on the data from 2009Q1-2010Q4. The cointegration pairs identified will be used as trading signals for 2011Q1. We repeated the same procedure to identify possible cointegration pairs during 2009Q2-2011Q1, 2009Q3-2011Q2 and 2009Q4-2011Q3, which would be served as the trading signals for 2011Q2, 2011Q3 and 2011Q4.

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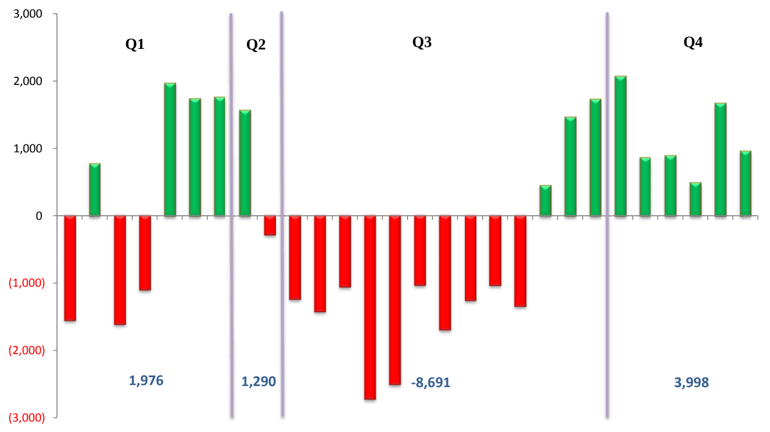
More Examples

The profit by this strategy in each trade is as follow.



Total Profit: -1,427USD

Further Trading Strategies

Strategy Performance: Straddles on HK Stocks

	Start	End	Stock 1	Stock 2	P/L	P/L due to different Greek Letters				
						Vega	Theta	Gamma	Delta	Others
1	5-Jul-10	9-Jul-10	0762	2600	1,203	2,192	155	(318)	(903)	78
2	9-Jul-10	28-Jul-10	0293	0857	3,048	1,599	564	16	853	16
3	18-Aug-10	2-Sep-10	0011	2388	2,744	2,047	578	(397)	394	122
4	4-Oct-10	29-Oct-10	0762	2600	2,419	1,192	837	(945)	1,382	47
5	17-Nov-10	8-Dec-10	0017	0239	(648)	(321)	478	(140)	(874)	209
6	17-Dec-10	28-Dec-10	0017	0239	(1,259)	(1,294)	255	20	(305)	65
7	20-Jan-11	2-Feb-11	0001	2388	1,288	(62)	315	(1,148)	2,053	129
8	21-Jan-11	26-Jan-11	1088	1898	1,722	1,715	30	88	(90)	(21)
9	11-Feb-11	18-Feb-11	0762	1898	223	(631)	101	(216)	1,014	(45)
10	16-Mar-11	18-Mar-11	0004	3988	2,887	2,827	43	(130)	141	6
11	30-Mar-11	31-Mar-11	0939	1398	241	311	14	(46)	(55)	16
				Total	13,868	$\boxed{9,575}$	3,370	(3,216)	3,610	622

Core Part of the trade
Mean Reversion of Trading Signal
=> positive Vega

Positive Theta (Carry)
Trades with positive theta (P/L over time) were chosen from our trade selection criteria

Negative Gamma
Opposite to Theta in general, a function of realized volatilities

Distribution of Delta roughly symmetric
≈0 on average

≈0 for each trade

Strategy Performance: Straddles on FX Rates

Trade	FX1	FX2	Weight	Start	End	P/L
1	CADJPY	EURNZD	0.643	3/1/11	21/1/11	(1,545)
2	GBPJPY	NZDCAD	0.587	27/1/11	21/2/11	772
3	EURJPY	EURNZD	0.725	22/2/11	4/3/11	(1,610)
4	NZDCAD	NZDJPY	2.265	24/2/11	10/3/11	(1,098)
5	AUDJPY	NZDCHF	0.530	24/2/11	14/3/11	1,964
6	GBPNZD	USDCAD	1.028	8/3/11	16/3/11	1,733
7	GBPUSD	NZDCAD	0.941	15/3/11	18/3/11	1,759
8	USDCHF	USDSEK	4.444	6/5/11	9/5/11	1,568
9	AUDCAD	NZDCHF	1.238	15/6/11	8/7/11	(278)
10	CADCHF	EURCAD	1.285	7/7/11	15/7/11	(1,232)
11	CADJPY	NZDCHF	0.395	12/7/11	21/7/11	(1,419)
12	AUDCHF	EURAUD	1.051	15/7/11	21/7/11	(1,047)
13	CADCHF	EURUSD	1.484	25/7/11	2/8/11	(2,717)
14	EURAUD	NZDCHF	0.698	27/7/11	4/8/11	(2,498)

Trade	FX1	FX2	Weight	Start	End	P/L
15	AUDCHF	AUDUSD	2.508	2/8/11	4/8/11	(1,029)
16	USDCHF	USDSEK	5.692	2/8/11	5/8/11	(1,687)
17	CADCHF	GBPCHF	1.503	5/8/11	9/8/11	(1,253)
18	GBPUSD	NZDCAD	0.420	5/8/11	11/8/11	(1,031)
19	AUDCHF	USDCHF	2.025	8/8/11	9/8/11	(1,339)
20	GBPCAD	GBPNZD	0.811	9/8/11	10/8/11	446
21	CADJPY	EURNZD	0.310	9/8/11	23/8/11	1,458
22	AUDCAD	EURNZD	0.755	2/9/11	27/9/11	1,727
23	AUDJPY	NZDJPY	1.004	8/9/11	9/9/11	2,067
24	EURAUD	NZDCAD	0.248	23/9/11	28/9/11	863
25	EURNZD	GBPJPY	4.144	3/10/11	14/10/11	886
26	GBPNZD	USDJPY	1.018	3/10/11	25/10/11	488
27	GBPUSD	NZDCAD	0.390	3/10/11	6/10/11	1,666
28	CADJPY	GBPAUD	0.575	4/10/11	17/10/11	958
					Total	(1,428)

Why doesn't work on FX Rates?

- More affected by fundamental factors (vs. technical factors for stocks)
- What happened in July – August 2011?

Risk Aversion => ↑ Realized vol => ↑ Implied vol => Cointegration Opportunities

- **Need a criterion on Implied vol based on forecasts of Realized vol**

Modeling and Forecasting Realized Volatilities

Estimation of Realized Volatility

1. Historical Estimate: Log-daily-return $r_t = \log(X_t) - \log(X_{t-1})$,
$$\hat{R}_t^2 = \sum_{i=1}^{H}(r_{t-i+1} - \bar{r}_t)^2/(H-1), \quad \text{with } \bar{r}_t = \sum_{i=1}^{H} r_{t-i+1}/H$$
2. J.P. Morgan Risk Metrics: Exponentially Weighted Moving Average of squared log-daily-return:
$$\hat{R}_t^2 = \sqrt{\frac{1-\lambda}{1-\lambda^H}\sum_{i=1}^{H}\lambda^i(r_{t-i+1} - \bar{r}_t)^2}, \quad \text{where } \lambda = 0.94.$$
3. Garman-Klass Estimates: Assume the underlying X_t follows the Geometric Brownian Motion: $dX_t = \mu X_t dt + \sigma X_t dW_t$,
$$\hat{R}_t^2 = \frac{0.17(O_t - C_{t-1})^2}{f} + \frac{0.83(u_t - d_t)^2}{(1-f)4\log 2},$$
where O_t, C_t, u_t and d_t are the Open, Close, High and Low of the underlying of the t-th day, and f is fraction of non-trading hours in a trading day.

4. High Frequency Estimate: Let $\{r_{n,t}\}$, $n = 1, \ldots, N$, $t = 1, \ldots, T$ be the log-return of an underlying asset at the nth minute of the t-th day. Assume that $r_{n,t}$ are i.i.d with mean zero and constant variance σ^2/N. The High Frequency Estimate is

$$\hat{R}_t^2 = \sum_{n=1}^{N} r_{n,t}^2 \quad \text{for } t = 1, \ldots, T.$$

Advantages of High Frequency Estimate:

a) The estimate depends on todays information ONLY $\Rightarrow$ No lagging issue as in Historical Estimate and JP Morgan Risk Metrics.

b) Unbiased estimator for σ^2. In contrast, Garman-Klass underestimates σ^2, as the high and low observed in discrete time under- and over-estimate the high and low in GBM

Forecasting Realized Volatilities R_t Andersen, Bollerslev, Diebold and Labys (2003) proposed the use of ARFIMA(1,1,0) model on the log of realized volatilities of high-frequency estimates

$$y_t = \log(\hat{R}_t)$$

$$\Phi(L)(1-L)^d y_t = \epsilon_t, \quad \epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma_\epsilon^2).$$

The model is able to

a) capture the asymmetric property of the unconditional distribution of the unconditional distribution of $\hat{R}_t$;
b) capture the long-memory property of $\hat{R}_t$;
c) provide the best 1- and 10-day ahead realized volatilities forecasts among 10+ models.

Estimation of Realized Volatility

Gamma Part: $\mathbb{E}(III) = \frac{\Delta t}{2\sqrt{T-t_0}}(\frac{a\mathbb{E}(\sigma_X^r)^2}{\sigma_{t_0}^X} - \frac{b\mathbb{E}(\sigma_Y^r)^2}{\sigma_{t_0}^Y})$, where σ_X^r and σ_Y^r are the average squared future annual realized volatilities for longing the portfolio.

Estimation of Realized Volatility

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Andersen, Bollerslev, Diebold and Labys (2003) introduced the following methods to model and forecast the realized volatility.

Modelling: High-Frequency Realized Volatility Estimation,

$$s_t^2 = \sum_{n=1}^{N} r_{n,t}^2, \text{ for } t = 1, 2, \cdots, T,$$

Forecasting: ARFIMA model:

$$\Phi(L)(1-L)^d(y_t - \mu) = \epsilon_t,$$

where $y_t = \log s_t$, d is the order of integration and ϵ_t is a vector white noise process.

Additional Criterion

Implied-Realized Criterion

$$K = \frac{a\mathbb{E}(\sigma_X^r)^2}{\sigma_{t_0}^X} \Big/ \frac{b\mathbb{E}(\sigma_Y^r)^2}{\sigma_{t_0}^Y} \quad \begin{cases} \geq d, & \text{if} \quad TS < 0, \\ \leq u, & \text{if} \quad TS > 0, \end{cases}$$

Here, d set as 0.7 and u set as 1.3.

Additional Criterion

Implied-Realized Criterion

$$K = \frac{a\mathbb{E}(\sigma_X^r)^2}{\sigma_{t_0}^X} \Big/ \frac{b\mathbb{E}(\sigma_Y^r)^2}{\sigma_{t_0}^Y} \quad \begin{cases} \geq d, & \text{if} \quad TS < 0, \\ \leq u, & \text{if} \quad TS > 0, \end{cases}$$

Here, d set as 0.7 and u set as 1.3.

Gamma-Vega Criterion

$$K = \left|\frac{Vega}{Gamma}\right| = \frac{|\sqrt{T - t_0}(\mathbb{E}(TS_t) - TS_{t_0})|}{\left|\frac{\triangle t}{2\sqrt{T - t_0}}\left(\frac{a\mathbb{E}(\sigma_X^r)^2}{\sigma_{t_0}^X} - \frac{b\mathbb{E}(\sigma_Y^r)^2}{\sigma_{t_0}^Y}\right)\right|} \geq I,$$

Here, I set as 11.

Performance of the Original Strategy

Trade	FX1	FX2	Weight	Start	End	P/L
1	CADJPY	EURNZD	0.643	3/1/11	21/1/11	(1,545)
2	GBPJPY	NZDCAD	0.587	27/1/11	21/2/11	772
3	EURJPY	EURNZD	0.725	22/2/11	4/3/11	(1,610)
4	NZDCAD	NZDJPY	2.265	24/2/11	10/3/11	(1,098)
5	AUDJPY	NZDCHF	0.530	24/2/11	14/3/11	1,964
6	GBPNZD	USDCAD	1.028	8/3/11	16/3/11	1,733
7	GBPUSD	NZDCAD	0.941	15/3/11	18/3/11	1,759
8	USDCHF	USDSEK	4.444	6/5/11	9/5/11	1,568
9	AUDCAD	NZDCHF	1.238	15/6/11	8/7/11	(278)
10	CADCHF	EURCAD	1.285	7/7/11	15/7/11	(1,232)
11	CADJPY	NZDCHF	0.395	12/7/11	21/7/11	(1,419)
12	AUDCHF	EURAUD	1.051	15/7/11	21/7/11	(1,047)
13	CADCHF	EURUSD	1.484	25/7/11	2/8/11	(2,717)
14	EURAUD	NZDCHF	0.698	27/7/11	4/8/11	(2,498)
15	AUDCHF	AUDUSD	2.508	2/8/11	4/8/11	(1,029)
16	USDCHF	USDSEK	5.692	2/8/11	5/8/11	(1,687)
17	CADCHF	GBPCHF	1.503	5/8/11	9/8/11	(1,253)
18	GBPUSD	NZDCAD	0.420	5/8/11	11/8/11	(1,031)
19	AUDCHF	USDCHF	2.025	8/8/11	9/8/11	(1,339)
20	GBPCAD	GBPNZD	0.811	9/8/11	10/8/11	446
21	CADJPY	EURNZD	0.310	9/8/11	23/8/11	1,458
22	AUDCAD	EURNZD	0.755	2/9/11	27/9/11	1,727
23	AUDJPY	NZDJPY	1.004	8/9/11	9/9/11	2,067
24	EURAUD	NZDCAD	0.248	23/9/11	28/9/11	863
25	EURNZD	GBPJPY	4.144	3/10/11	14/10/11	886
26	GBPNZD	USDJPY	1.018	3/10/11	25/10/11	488
27	GBPUSD	NZDCAD	0.390	3/10/11	6/10/11	1,666
28	CADJPY	GBPAUD	0.575	4/10/11	17/10/11	958
					Total	(1,428)

Period with High Realized Volatilities (July – August 2011)

Performance of Trading Strategy 2

Trade	FX1	FX2	Weight	Start	End	P/L
1	CHFJPY	EURNZD	0.906	14/1/11	8/2/11	248
2	NZDCAD	USDJPY	0.837	17/3/11	21/3/11	457
3	NZDCAD	USDCHF	0.796	6/5/11	31/5/11	3,415
4	AUDCAD	NZDCHF	1.238	15/6/11	8/7/11	(278)
5	AUDNZD	USDCHF	0.952	3/8/11	5/8/11	(1,885)
6	EURGBP	GBPNZD	1.186	9/8/11	10/8/11	792
7	AUDCAD	EURNZD	0.755	2/9/11	27/9/11	1,727
8	AUDJPY	NZDJPY	1.004	8/9/11	9/9/11	2,067
9	AUDNZD	EURNZD	0.817	3/10/11	12/10/11	1,143
					Total	7,686

- Most of the trades in July-August were filtered out

Performance of Trading Strategy 3

Trade	FX1	FX2	Weight	Start	End	P/L
1	CADJPY	EURNZD	0.643	3/1/11	28/1/11	(1,545)
2	GBPJPY	NZDCAD	0.587	27/1/11	21/2/11	772
3	EURJPY	EURNZD	0.725	22/2/11	14/3/11	(1,610)
4	NZDCHF	NZDJPY	1.819	24/2/11	16/3/11	1,372
5	USDCAD	USDJPY	0.832	14/3/11	8/4/11	(1,121)
6	GBPUSD	NZDCAD	0.941	15/3/11	18/3/11	1,759
7	NZDCAD	USDCHF	0.796	6/5/11	31/5/11	3,415
8	AUDCAD	NZDCHF	1.238	15/6/11	8/7/11	(278)
9	AUDNZD	USDCHF	0.952	3/8/11	8/8/11	(1,885)
10	GBPCAD	GBPNZD	0.811	9/8/11	10/8/11	446
11	AUDCAD	EURNZD	0.755	2/9/11	27/9/11	1,727
12	AUDJPY	NZDJPY	1.004	8/9/11	9/9/11	2,067
13	GBPNZD	USDJPY	1.018	3/10/11	25/10/11	488
14	EURNZD	GBPCHF	2.171	3/10/11	12/10/11	1,098
15	CADJPY	GBPAUD	0.575	4/10/11	17/10/11	958
					Total	7,663

Summary of Trading Strategies

	Original	Criteria 1	Criteria 2
# of Trades Implemented	28	12	15
Total Profit	(1,428)	7,686	7,663
# of Positive Trades	14	10	11
% of Positive Trades	50%	83%	73%
P/L Summary			
Mean	(51)	641	511
S.D.	1,501	1,508	1,535
Max	2,067	3,415	3,415
Min	(2,717)	(1,885)	(1,885)
Sharpe Ratio (Annualized)	-0.27	1.47	1.44

Conclusion and Further Discussion

Conclusion

- Cointegration strategy has been applied into the derivatives.
- If the trading signal can show mean-reverting in a short-term and the realized volatility is not too high and deviates too far from the implied volatility, the portfolio makes profit.
- The additional criteria, Implied-Realized Criterion and Gamma-Vega Criterion, are effective.

Further Discussion

- The number of traded underlying assets can be more than two.
- The dynamic hedging can be imposed in the strategy to minimize the impact of delta. This method may weaken the "short-term period trading" restriction, but costly.
- The method for modeling and forecasting realized volatility can be revised and the notion of fractional cointegration may be pursued for the further study.

Thank You

Engle-Granger's Methodology

Consider a p-dimensional non-stationary $I(1)$ time series $\{\mathbf{x_t}\}$. $\mathbf{x_t}$ can be partitioned into $(x_{1t}, \mathbf{x}'_{\mathbf{2t}})'$, where x_{1t} is a scaler and $\mathbf{x_{2t}}$ is an $(p-1) \times 1$ vector.

Through the ordinary least squares (OLS) method, we have

$$x_{1t} = \hat{\beta}\mathbf{x}'_{\mathbf{2t}} + \mathbf{\hat{u}_t}.$$

If $\mathbf{\hat{u}_t} = x_{1t} - \hat{\beta}\mathbf{x}'_{\mathbf{2t}} \sim I(0)$, it means that there exist $\hat{\theta} = (1, -\hat{\beta}')'$ such that $\hat{\theta}\mathbf{x_t} \sim I(0)$, i.e. $\mathbf{x_t}$ is cointegration.

Johansen's Methodology

Consider a p-dimensional non-stationary $I(1)$ time series $\{X_t\}$, which follows a VAR(k) process:

$$X_t = \Phi_1 X_{t-1} + \Phi_2 X_{t-2} + \cdots + \Phi_k X_{t-k} + \varepsilon_t, t = \ldots, -1, 0, 1, \ldots,$$

where $\Phi_1, \Phi_2, \ldots, \Phi_k$ are $p \times p$ matrices, and ε_t is Gaussian random vector with mean 0 and covariance matrix Ω.

Note that the above equation can be rewritten as a Vector Error Correction Model (VECM):

$$\Delta X_t = \Gamma X_{t-1} + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-k+1} + \varepsilon_t,$$

where $\Gamma = \sum_{i=1}^{k} \Phi_i - I, \Gamma_l = -\sum_{j=l+1}^{k} \Phi_j, l = 1, \ldots, k-1$. Hence, Γ_l, $l = 1, \ldots, k-1$ are unrestricted.

Johansen's Methodology

If $\Gamma = \alpha_{p\times r}\beta'_{p\times r}$, where $r < p$, then $\beta' X_t$ is stationary, where α is the adjustment coefficients and β is the cointegration vector.

Trace test:

$$L_{trace} = -N \sum_{i=r+1}^{p} \log(1 - \hat{\lambda}_i),$$
$$H_0 : K_c = r \quad vs \quad H_1 : K_c = p.$$

Maximum eigenvalue test:

$$L_{eig} = -N \log(1 - \hat{\lambda}_{r+1}),$$
$$H_0 : K_c = r \quad vs \quad H_1 : K_c = r + 1.$$

Here N is the sample size, $\hat{\lambda}_i$ is the i-th largest canonical correlation and K_c is the number of cointegrating vector.

Appendix A

First of all,

$$C_t = S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
$$P_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1),$$

where $d_1 = \frac{\log(\frac{S_t}{K})+(r-q+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$.

By the Taylor expansion,

$$\Phi(d_1) = \Phi(0) + \phi(0)d_1 + \phi'(0)d_1^2 + O(d_1^2) = \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} + O(d_1^2),$$
$$\Phi(d_2) = \Phi(0) + \phi(0)d_2 + \phi'(0)d_2^2 + O(d_2^2) = \frac{1}{2} + \frac{d_2}{\sqrt{2\pi}} + O(d_2^2),$$

Then,

$$C_t = \frac{S_t e^{-q(T-t)}}{\sqrt{2\pi}}\sigma\sqrt{T-t} + O(S_t\sigma^2(T-t)),$$
$$P_t = \frac{S_t e^{-q(T-t)}}{\sqrt{2\pi}}\sigma\sqrt{T-t} + O(S_t\sigma^2(T-t)).$$

Hence,

$$ST_t = C_t + P_t = \sqrt{\frac{2}{\pi}}S_t\sigma\sqrt{T-t} + O(S_t\sigma^2(T-t))$$

Appendix B

Proposition

If the trade is in a short-time period, $\mathbb{E}(II) + \mathbb{E}(IV) > 0$.

Proof.

In section 3.3.1, we know that theta (IV) is a positive net carry in the trade. Compared the expected value of theta (IV) with that of delta (II),

$$\left|\frac{\mathbb{E}(IV)}{\mathbb{E}(II)}\right| = \frac{1}{(T-t_0)r_{t_0}}.$$

Because $T - t_0$ is smaller than one (year) and r_{t_0} is very small (0%-5% in most countries), $|\mathbb{E}(II)| << |\mathbb{E}(IV)|$. Hence,

$$\mathbb{E}(II) + \mathbb{E}(IV) \geq |\mathbb{E}(IV)| - |\mathbb{E}(II)| > 0,$$

i.e. the loss due to delta is likely to be compensated by the positive carry from IV. □

Appendix C

Proposition

Assume that

1. *Annualized volatilities (include implied volatility and realized volatility) of the underlying assets are smaller than 80% ;*
2. $\frac{1}{c}\sigma^i_{t_0} < \sigma^r_i < c\sigma^i_{t_0}$, *for* $i = x, y$, $c > 1$;
3. $\sigma(TS_t) > 1$.

Then $\mathbb{E}(I) + \mathbb{E}(III) > 0$.

Proof

Under assumptions (1) and (2), one can show that

$$|\mathbb{E}(III)| = |\frac{\Delta t}{2\sqrt{T-t_0}}(\frac{a\mathbb{E}(\sigma^r_X)^2}{\sigma^X_{t_0}} - \frac{b\mathbb{E}(\sigma^r_Y)^2}{\sigma^Y_{t_0}})| \leq \frac{0.4c^2\triangle t}{\sqrt{T-t_0}}.$$

cont.

Recall that the trade requires using long dated options and to trade in a short-term period. To strike a balance between the trading requirements and options liquidity, the maturity of the options should be at least three months and the trade should not last over one month. i.e.

$$\frac{T-t_0}{\Delta t} \geq 3.$$

Vega is the main profit in the trade. Compared the expected value of gamma with that of vega,

$$\left|\frac{\mathbb{E}(I)}{\mathbb{E}(III)}\right| = \left|\frac{\sqrt{T-t_0}(\mathbb{E}(TS_t)-TS_{t_0})}{\frac{\Delta t}{2\sqrt{T-t_0}}\left(\frac{a\mathbb{E}(\sigma_X^r)^2}{\sigma_{t_0}^X}-\frac{b\mathbb{E}(\sigma_Y^r)^2}{\sigma_{t_0}^Y}\right)}\right| \geq \frac{7.5}{c^2}(\mathbb{E}(TS_t) - TS_{t_0}).$$

cont.

The trade is initiated when TS_{t_0} is too negative comparing to the mean level $(\mathbb{E}(TS_t))$, and c is close to 1. Hence,

$$\frac{7.5}{c^2}(\mathbb{E}(TS_t) - TS_{t_0}) > \frac{15}{c^2}\sigma(TS_t) >> 1,$$

and therefore

$$\mathbb{E}(I) + \mathbb{E}(III) \geq |\mathbb{E}(I)| - |\mathbb{E}(III)| > 0.$$