Draft version December 14, 2020
Typeset using LaTeX default style in AASTeX63

The Utility of the Beta Function Formalism in a Quintessence Cosmology

Emery Gunselman[1]

[1] The University of Arizona

(Dated: December 14, 2020)

ABSTRACT

This paper establishes the utility of the beta function formalism within the framework of a quintessence cosmology. In this instance, the beta function is defined as the derivative of the scalar field with respect to the natural log of the scale factor. The specific quintessence cosmology that is invoked as an example is an inverse power law dark energy potential, with the power set to negative three. Various cosmological relations are written using the beta function and the evolution of these parameters is then examined. The parameter space is then constrained through the use of fundamental constants.

1. INTRODUCTION

Quintessence is a cosmology in which the dark energy equation of state is dynamic. It involves a minimally coupled homogeneous scalar field which evolves in a specified potential. The utility of the beta function is revealed through its ability to define various forms of evolutionary equations as functions of the scale factor which can more easily be measured compared to the scalar field. The versatility of the beta function is demonstrated through its ability to encompass the behavior of a range of models. For instance, quintessence matches the behavior of a ΛCDM cosmology when the power in the potential is zero. In the following sections, the dark energy potential's relation with the beta function is implemented followed by various evolutionary equations expressed with regard to the beta function. Further observational constraints are then put in place to demonstrate how certain cosmological parameters and fundamental constants may evolve in a quintessence framework.

2. THE POTENTIAL

The dark energy potential is given by an inverse power potential,

$$V(\phi) = M^{4+p}\phi^{-p} \tag{1}$$

where M has the units of the reduced Planck mass M_p and the potential $V(\phi)$ has units of M_p^4. ϕ is the scalar field which has units of reduced Planck masses as well. Invoking the Friedmann equation, the potential can be written as,

$$V(\phi) = 3\Omega_\phi(\frac{H(\phi)}{\kappa})^2 \tag{2}$$

where $H(\phi)$ is the Hubble parameter, Ω_ϕ is the ratio of the dark energy density to the critical density, and $\kappa = \frac{1}{M_p}$ to maintain consistent units.

3. THE BETA FUNCTION FORMALISM

The significance of the beta function formalism lies in the fact that it allows the evolution of the scalar field to be written in the same form as a renormalization group equation (2). Moreover, the beta function is extremely useful in expressing the evolution of fundamental constants and cosmological parameters in terms of the scale factor, a, which is observable unlike the scalar field, ϕ. The beta function is defined as the derivative of the scalar field, ϕ with respect to the natural logarithm of the scale factor of the universe a,

$$\beta(\phi) \equiv \frac{\kappa d\phi}{d\ln(a)} = \kappa\phi' \tag{3}$$

where the prime on ϕ denotes the derivative with respect to $\ln(a)$. Since ϕ has units of mass the product $\kappa\phi$ is dimensionless and ϕ is specified in units of reduced Planck masses. The reduced Planck mass is used to simplify several cosmological equations. Natural units are used where G, c and $\hbar$ are set to one and κ is consistently used throughout the paper in order to maintain proper units. Although the versatility for the beta function in various models is noteworthy, this paper will focus on a specific dark energy potential, $V(\phi)$, in order to calculate various cosmological parameters using the beta function. In this instance, the beta function is determined by the potential (3) (2),

$$V_m(\phi) = V_0 \exp\{-\kappa \int \beta(\phi) d\phi\} \tag{4}$$

where $V_m(\phi)$ is the model potential given by (1). The beta function for an inverse power potential given by (1) is (5),

$$\beta(\phi) = \frac{p}{\kappa\phi} = \frac{\kappa d\phi}{d\ln(a)} \tag{5}$$

as seen through (4). Now, integrating (5) obtains the scalar field as a function of the scale factor,

$$\kappa\phi = \sqrt{2p\ln(a) + (\kappa\phi_0)^2} \tag{6}$$

which is useful for expressing other quantities involving the scalar field in terms of the scale factor. ϕ_0 is the current value of the scalar field.

4. QUINTESSENCE

The standard quintessence dark energy density, ρ_ϕ, and pressure, P_ϕ equations are

$$\rho_\phi \equiv \frac{\dot{\phi}^2}{2} + V(\phi), \qquad P_\phi \equiv \frac{\dot{\phi}^2}{2} - V(\phi) \tag{7}$$

and the dark energy equation of state $w(\phi)$ is

$$w(\phi) = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)}. \tag{8}$$

Combining the density and pressure equations in (7) gives

$$P_\phi + \rho_\phi = \dot{\phi}^2 \tag{9}$$

which with the knowledge of (8) results in,

$$\frac{P_\phi + \rho_\phi}{\rho_\phi} = w + 1 = \frac{\dot{\phi}^2}{\rho_\phi}. \tag{10}$$

From the definition of the Hubble parameter, $H = \frac{\dot{a}}{a}$,

$$\dot{\phi} = \frac{d\phi}{da} a \frac{da}{dt}\frac{1}{a} = \frac{d\phi}{d\ln(a)} H = \phi' H. \tag{11}$$

Using the following relation,

$$\rho_\phi = \frac{3\Omega_\phi H^2}{\kappa^2} \tag{12}$$

along with (12), a new relationship for $(w+1)$ can be established.

$$\frac{\dot{\phi}^2}{\rho_\phi} = \frac{(\phi'\kappa)^2 H^2}{3\Omega_\phi H^2} = \frac{(\kappa\phi')^2}{3\Omega_\phi} = (w+1) \tag{13}$$

Rearranging (13),

$$\kappa\phi' = \sqrt{3\Omega_\phi(w+1)}. \tag{14}$$

Utilizing the definition of beta, (3), and (5) with (14) gives,

$$\frac{p}{\kappa\phi} = \sqrt{3\Omega_\phi(w+1)}. \tag{15}$$

Then using the current value of $/omega$ results in,

$$\kappa\phi_0 = \frac{p}{\sqrt{3\Omega_{\phi_0}(w_0+1)}} \tag{16}$$

(16) and (6) give the evolution of ϕ as a function of the scale factor a, as (16) determines the initial value of the scalar field which can then be plugged into (6) to find the evolution of the scalar field as a function of the scale factor. The evolution of the scalar field is demonstrated in Figure 1 for $p = -3$ and various values of w.

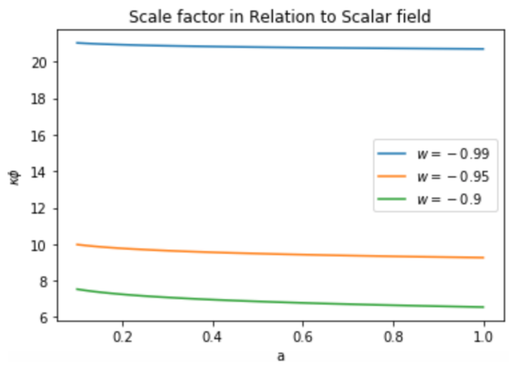

Figure 1. The evolution of the scalar field for $p = -3$

5. PARAMETER RELATIONSHIPS

This section's goal is to establish the connection between various cosmological parameters in quintessence necessary for the analysis preformed. The relations between the scalar ϕ, the Hubble parameter H, the dark energy equation of state w and the value of M are given. Combining (1) and (2) results in,

$$M^{4+p}\phi^{-p} = 3\Omega_\phi(\frac{H}{\kappa})^2. \tag{17}$$

The boundary conditions are provided by setting the values of H, Ω and w to their current values. Solving for M gives,

$$M = [3\Omega_{\phi_0}(\frac{H_0}{\kappa})^2\phi_0^p]^{\frac{1}{4+p}}. \tag{18}$$

The terms on the right are all set values so as a result, M is a set value as well.

6. THE DARK ENERGY POTENTIALS

With the implementation of the beta function, a customary superpotential is used, defined as,

$$W(\phi) = -2H(\phi) = -2\frac{\dot{a}}{a}. \tag{19}$$

Without considering matter, the potential is given as (3),

$$V(\kappa\phi) = \frac{3}{4}W_0^2\exp\{-\int_{\kappa\phi_0}^{\kappa\phi}\beta(x)dx\}(1-\frac{\beta^2(\kappa\phi)}{6}) \tag{20}$$

while the superpotential is (3),

$$W(\kappa\phi) = W_0\exp\{-\frac{1}{2}\int_{\kappa\phi_0}^{\kappa\phi}\beta(x)dx\} \tag{21}$$

Where W_0 is the current value of W coinciding with the current value of the Hubble parameter and (21) is the beta potential which is the model potential multiplied by $(1-\frac{\beta^2(\kappa\phi)}{6})$. The difference in the model and beta potential stems from the following,

$$\rho_\phi \equiv \frac{\dot{\phi}^2}{2} + V(\phi) \tag{22}$$

Rearranging,

$$V(\phi) = \rho_\phi - \frac{\dot{\phi}^2}{2} \tag{23}$$

Factoring out ρ_ϕ,

$$V(\phi) = \rho_\phi(1 - \frac{\dot{\phi}^2}{2\rho_\phi}) \tag{24}$$

Now using (11) and (13) we obtain,

$$V(\phi) = \rho_\phi(1 - \frac{(\phi' H)^2\kappa^2}{6\Omega_\phi H^2}) \tag{25}$$

Using (3), (12), and simplifying, results in,

$$V_\beta(\phi) = \frac{3\Omega_\phi H^2}{\kappa^2}(1 - \frac{\beta^2}{6\Omega_\phi}) \tag{26}$$

The beta potential is generally a good approximation of the model potential when $\frac{\beta^2(\kappa\phi)}{6} << 1$. The error is less than 0.25% for all of the w_0 values and less than 0.1% for w_0 of -0.97 and -0.99 which is much better in comparison to current observational constraints.

7. A UNIVERSE WITH MATTER

In order to make the calculations relevant to observations, the matter density of the Universe must be included in the equations. With regards to the beta function, the matter density of the Universe is given by,

$$\rho_m(\kappa\phi) = \rho_{m0}\exp(-3\int_{\kappa\phi_0}^{\kappa\phi} \frac{d\kappa\phi}{\beta(\kappa\phi)}) \tag{27}$$

where the current matter density is ρ_{m0}. Plugging in the definition of the beta function the matter density becomes,

$$\rho_m(a) = \rho_{m0}\exp(-3\int_1^a dln(a)) = \rho_{m0}a^{-3} \tag{28}$$

Introducing matter to the superpotential results in a differential equation (3),

$$\frac{WW_{,\phi}}{\kappa^3} + \frac{\beta W^2}{2\kappa^2} = -2\frac{\rho_m}{\beta} \tag{29}$$

where $W_{,\phi}$ is the derivative of W with respect to ϕ. To solve this differential equation, the integrating factors are necessary. The respective integrating factors for the power and inverse power law are $(\kappa\phi)^{-p}$ and $(\kappa\phi)^p$. Using the inverse power law integrating factor (5) gives the superpotential as,

$$W_i(a) = -\{-\frac{4\rho_{m_0}}{3}(\frac{2p}{3})^{-\frac{p}{2}}\exp(\frac{3\phi_0^2}{2p})(\phi(a))^{-p}[\Gamma(1+\frac{p}{2}, 3\ln(a) + \frac{3\phi_0^2}{2p}) - \Gamma(1+\frac{p}{2}, \frac{3\phi_0^2}{2p})] + W_0^2(\frac{\phi_0}{\phi(a)})^p\}^{1/2} \tag{30}$$

where Γ is the incomplete Gamma function.

8. THE DARK ENERGY DENSITY

In a universe with zero curvature the ratio of the dark energy density to the critical density is,

$$\Omega_\phi = \frac{3(H/\kappa)^2 - \rho_{m_0}a^{-3}}{3(H/\kappa)^2} \tag{31}$$

An important check on the Hubble parameter relation found is if the late time acceleration of the Universe occurs at the correct time. Figure 3 demonstrates this behavior correctly as the acceleration initiates around a scale factor of 0.6 which is consistent with observations.

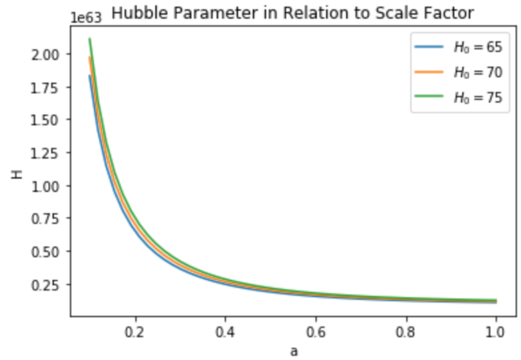

Figure 2.

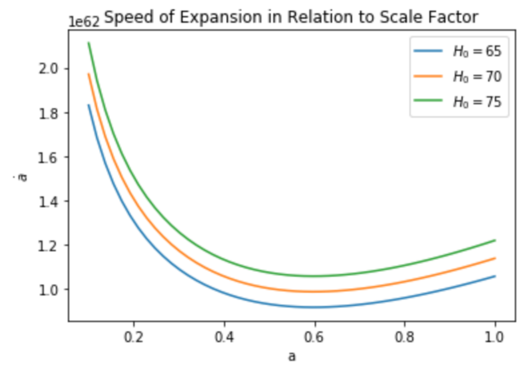

Figure 3.

9. THE DARK ENERGY EQUATION OF STATE

The dark energy equation of state, is an important cosmological parameter whose value discriminates between static and dynamic cosmologies (5). Plugging (4) and (10) into (7) gives the equation of state in terms of the beta function as,

$$1 + w(\phi) = \frac{\beta^2}{3}\frac{1}{(1 - \frac{4\rho_{m_0} a^{-3}}{3(W/\kappa)^2})} = \frac{\beta^2}{3}\frac{1}{(1 - \Omega_m)} = \frac{\beta^2(\phi)}{3\Omega_\phi} \tag{32}$$

10. THE EVOLUTION OF THE FUNDAMENTAL CONSTANTS α AND μ

Now, we transition from the evolution of cosmological parameters as a result of the rolling scalar field to the evolution of select fundamental constants. The dimensionless fundamental constants examined are the fine structure constant α and the proton to electron mass ratio μ. The relationship between the variation of μ or α and ϕ is given by,

$$\frac{\Delta x}{x} = \zeta_x \kappa(\phi - \phi_0) = \zeta_\mu \int_1^a \beta(a\prime) d\ln(a\prime) \tag{33}$$

where x is either μ or α and ζ_x is the dimensionless coupling constant for the interaction.

REFERENCES

[1]Bagdonaite, J.,Dapra, Jansen, P., Bethlem, H.L., Ubachs, W., Henkel, C., & Menten, K.M. 2013, Phys. Rev. Letters, 111, 231101

[2]Binetruy, P., Kiritisis, E. Mabillard, J., Pieroni, M. and Rosset, C. 2015 JCAP, 1504, no. 04, 033

[3]Cicciarella, F. and Pieroni, M. 2017 JCAP, 1708, no. 8, 010 arXiv:1611.10074v1 [gr-qc]

[4]Kanekar, N., Ubachs, W., Menten, K.M., Bagdonaite, J., Brunthaler, A., Henkel, Muller, C.S., Bethlem, H.L. and Dapra, M. 2015, MNRAS 448, L104.

[5]Thompson, R. 2018, MNRAS, 477, 4104