1 Optimal Quadratic Estimators

Given a model covariance matrix $C = S + N$ including both the signal and noise covariance, we can form the optimal quadratic estimator for some paramters θ by:

$$
\begin{aligned}
F_{mn} &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{C}^{-1}\mathbf{C}_{,m}\mathbf{C}^{-1}\mathbf{C}_{,n}\right] \\
q_n &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{C}^{-1}\mathbf{C}_{,n}\mathbf{C}^{-1}\mathbf{N}\right] \\
f_n &= \frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{C}_{,n}\mathbf{C}^{-1}\mathbf{y}. \\
\hat{\theta}_m &= \sum_N \left(F^{-1}\right)_{mn}(q_n - f_n)
\end{aligned} \tag{1}
$$

Here, $\mathbf{y}$ is the full data covariance matrix (think of this as making a single long vector from the two measured timeseries $C(t_i)$ and $L(t_i)$). $\mathbf{C}$ is the full covariance of this vector, and the $\mathbf{C}_{,n}$ notation is meant to indicate the derivative of the covariance matrix with respect to the n^{th} parameter we are trying to measure.

2 The Problem

We observe two light curves for quasars – $C(t)$, the continuum, and $L(t)$, the line emission.

We believe that $C(t)$ is that it is a stationary random process best modeled as a damped random walk parameterized by σ and μ, such that:

$$\mathrm{cov}\left(C(t_i), C(t_j)\right) = \xi_{CC} = \sigma^2 e^{-\frac{|t_i - t_j|}{\mu}} \tag{2}$$

We allow that the light curves are observed at irregular times, t_i, but insist that C and L are always observed simultaneously.

The idea for $L(t)$ is that line emission lags the continuum by some time that depends (possibly straightforwardly) on the BH mass, and since the broad-line-emitting region (BLR) is larger than the continuum-emitting region (CR), the BLR emission is also less synchronized. We represent this lag and desynchronicity by convolution with a transfer function $\psi(t)$ such that:

$$L(t) = \int_{-\infty}^{\infty} C(t')\psi(t' - t)\, dt' \tag{3}$$

If, for example, the BLR was an exact copy of the CR light curve, but with some lag τ, then we would have $\psi(t) = \delta(t - \tau)$. We will parameterize ψ, until we have a good reason not to, as a set of binned amplitudes, with bin width w:

$$\psi(t) = \sum_m \psi_m \Pi\left(t - t_m | w\right) \tag{4}$$

where $\Pi(t|w)$ is a top hat function of unit area and width w.

3 Covariance Matrices

In order to make use of equation 3.1, we need to calculate the full data covariance matrix. For our purposes, the observations are independent and the errors are known, so it is the signal covariance matrix between observation times $S_{ij} = S(t_i - t_j)$ that needs to be calculated. Fortunately, this can be done analytically for both timeseries.

$\mathrm{cov}(C(t_i), C(t_j))$ is known by hypothesis, and fortunately the DRW correlation function is analytically tractable, so the BLR-CR and BLR-BLR correlations can, with some effort, be written down.

3.1 BLR covariance

To write the BLR covariance:

$$C_{LL}(t) = \int_{-\infty}^{\infty} L(t_i)L(t_i - (t_i - t_j))dt_i \tag{5}$$

$$= \int_{-\infty}^{\infty} dt_i \left[\int_{-\infty}^{\infty} dt'\; C(t')\psi(t' - t_i)\right]\left[\int_{-\infty}^{\infty} dt''\; C(t'')\psi(t'' - t_i + t)\right] \tag{6}$$

For the above and what follows, we will write the unprimed t for $t_i - t_j$. Taking the Fourier Transform (FT) of this expression with respect to t causes $\psi \mapsto \hat{\psi} e^{-i\omega(t''-t_i)}$. The additional phase factor turns the other integrals into either FTs or inverse FTs, and we are left with:

$$= |\hat{\psi}|^2 |\hat{C}|^2 \tag{7}$$

Inverse-transforming this expression, we have that the line emission covariance is simply the convolution of the CR correlation function with a new quantity, ξ_ψ:

$$\xi_\psi = \int_{-\infty}^{\infty} dt' \, \psi(t')\psi(t'-t) \tag{8}$$

$$C_{LL}(t) = \xi_{LL} = \int_{\infty}^{\infty} dt' \xi_{CC}(t')\xi_\psi(t'-t) \tag{9}$$

Next, we hope to write ξ_{LL} in terms of our chosen parameterization, given by equation 4. Doing this requires evaluating the following integrals:

$$\xi_\psi^{mn}(t) = \psi_m \psi_n \int_{-\infty}^{\infty} dt' \, \Pi(t' - t_m|w)\Pi(t' - t - t_n|w) \tag{10}$$

$$C_{LL}^{mn} = \int_{-\infty}^{\infty} dt' \, \xi_{CC}(t')\xi_\psi^{mn}(t'-t) \tag{11}$$

The integral in equation 10 is straightforward – the convolution of identical unit-area top hat functions of width w (and height w^{-1}) is an isosceles triangle centered at the origin of height w^{-1} and base $2w$, which I'll denote as $T(x|w)$:

$$T(x|w) = \begin{cases} \frac{1}{w} - \frac{|x|}{w^2} & \text{for } |x| \leq w \\ 0 & \text{else} \end{cases} \tag{12}$$

Now the autocorrelation of the binned transfer function with itself is:

$$\xi_\psi^{mn}(t) = \psi_m \psi_n \, T(t - (t_m - t_n)|w) \tag{13}$$

We can now perform the integral in equation 11.

$$C_{LL} = \sum_{m,n} \psi_m \psi_n \int_{-\infty}^{\infty} dt' \; \sigma^2 e^{-\frac{|t'|}{\mu}} T((t'-t) - (t_m - t_n)|w) \tag{14}$$

First, make the integration variable swap $t' \mapsto z = (t'-t) - (t_m - t_n)$, and note that the new limits of integration are $-w \leq z \leq w$. This can be done analytically; there are four cases.

$$C_{LL} = \sum_{m,n} \psi_m \psi_n e^{-\frac{|t+(t_m-t_n)}{\mu}} \times \begin{cases} \frac{\mu^2}{w^2} e^{-\frac{w+y}{\mu}} \left(e^{\frac{w}{\mu}} - 1\right)^2 & \text{for } y > w \\ \frac{\mu^2}{w^2} e^{-\frac{y-w}{\mu}} \left(e^{\frac{w}{\mu}} - 1\right)^2 & \text{for } y < -w \\ \frac{\mu}{w^2} e^{-\frac{w+y}{\mu}} \left(-\mu - \mu e^{2\frac{y}{\mu}} + 2\mu e^{\frac{w+2y}{\mu}} - 2w e^{\frac{w+y}{\mu}} - 2y e^{\frac{w+y}{\mu}}\right) & \text{for } -w < y \leq 0 \\ \frac{\mu}{w^2} e^{-\frac{y+w}{\mu}} \left(\mu - 2\mu e^{\frac{w}{\mu}} + \mu e^{2\frac{y}{\mu}} + 2w e^{\frac{y+w}{\mu}} - 2y e^{\frac{y+w}{\mu}}\right) & \text{for } 0 < y \leq w \end{cases} \tag{15}$$

This is one part of the covariance matrix which appears (along with its ψ-derivatives) in the estimator in equation .

3.2 CR-BLR covariance

To write the cross-covariance:

$$\text{cov}(C(t_i), L(t_j)) = \int_{-\infty}^{\infty} dt' C(t_i) \int_{-\infty}^{\infty} C(t'')\psi(t'' - t_i - t) \tag{16}$$

As before, taking the Fourier transform with respect to t produces $\psi \mapsto \hat{\psi} e^{-i\omega(t''-t_j)}$. This converts the remaining integrals into Fourier and inverse Fourier transforms; the covariance matrix in frequency space is now:

$$\hat{C}_{CL}(\omega) = |\hat{C}|^2 \hat{\psi} \tag{17}$$

Performing the inverse transform gives us the expression for the time-domain covariance:

$$C_{CL}(t) = \int_{-\infty}^{\infty} dt' \; \xi_{CC}(t')\psi(t'-t) \tag{18}$$

Re-writing ψ in its parameterized form, we have:

$$C_{CL}(t) = \sum_m \psi_m \int_{-\infty}^{\infty} dt' \; \xi_{CC}(t')\Pi(t'-t-t_m|w) \tag{19}$$

Substituting in the expression for ξ_{CC} yields:

$$C_{CL}(t) = \sum_m \sigma^2 \psi_m \int_{-\infty}^{\infty} dt' e^{-\frac{|t'|}{\mu}} \Pi(t'-t-t_m|w) \tag{20}$$

The convolution with the top-hat filter can be done analytically, yielding an expression for C_{CL}:

$$C_{CL} = \sum_m \psi_m \sigma^2 \times \begin{cases} \frac{\mu}{w} e^{-\frac{w+y}{\mu}} \left(e^{2\frac{w}{\mu}} - 1\right) & \text{for } y > w \\ -\frac{\mu}{w} e^{-\frac{w+y}{\mu}} \left(1 + e^{2\frac{y}{w}} - 2e^{\frac{y+w}{\mu}}\right) & \text{for } |y| \leq w \\ 2\frac{\mu}{w} e^{\frac{y}{\mu}} \sinh\left(\frac{w}{\mu}\right) & \text{for } y < w \end{cases} \tag{21}$$

Note that, when I actually implementated, I found that this expression was just about exactly a factor of two too large; for $\psi_m = \delta_{m,0}$, the zero-lag covariance should be equal to C_{CC}, which is σ^2.

3.3 Covariance Matrix Summary

The full signal covariance matrix S can be written:

$$S = \begin{pmatrix} C_{CC} & C_{CL} \\ C_{CL} & C_{LL} \end{pmatrix} \tag{22}$$

where the expressions for the covariances matrices are, for the sake of neatness and completeness:

$$C_{CC,ij} = \sigma^2 e^{-\frac{|t_i - t_j|}{\mu}} \tag{23}$$

$$C_{CL} = \sum_m \psi_m \sigma^2 \times \begin{cases} \frac{\mu}{w} e^{-\frac{w+t_i-t_j-t_m}{\mu}} \left(e^{2\frac{w}{\mu}} - 1\right) & \text{for } t_i - t_j - t_m > w \\ -\frac{\mu}{w} e^{-\frac{w+t_i-t_j-t_m}{\mu}} \left(1 + e^{2\frac{t_i-t_j-t_m}{w}} - 2e^{\frac{t_i-t_j-t_m+w}{\mu}}\right) & \text{for } |t_i - t_j - t_m| \leq w \\ 2\frac{\mu}{w} e^{\frac{t_i-t_j-t_m}{\mu}} \sinh\left(\frac{w}{\mu}\right) & \text{for } t_i - t_j - t_m < w \end{cases} \tag{24}$$

$$C_{LL} = \sum_{m,n} \psi_m \psi_n e^{-\frac{|t+(t_m-t_n)}{\mu}} \times \tag{25}$$

$$\begin{cases} \frac{\mu^2}{w^2} e^{-\frac{w+(t_i-t_j)+(t_m-t_n)}{\mu}} \left(e^{\frac{w}{\mu}} - 1\right)^2 \\ \frac{\mu^2}{w^2} e^{-\frac{(t_i-t_j)+(t_m-t_n)-w}{\mu}} \left(e^{\frac{w}{\mu}} - 1\right)^2 \\ \frac{\mu}{w^2} e^{-\frac{w+(t_i-t_j)+(t_m-t_n)}{\mu}} \left(-\mu - \mu e^{2\frac{(t_i-t_j)+(t_m-t_n)}{\mu}} + 2\mu e^{\frac{w+2(t_i-t_j)+(t_m-t_n)}{\mu}} - 2we^{\frac{w+(t_i-t_j)+(t_m-t_n)}{\mu}} - 2\left[(t_i - t_j) + (t_m - \right.\right. \\ \frac{\mu}{w^2} e^{-\frac{(t_i-t_j)+(t_m-t_n)+w}{\mu}} \left(\mu - 2\mu e^{\frac{w}{\mu}} + \mu e^{2\frac{(t_i-t_j)+(t_m-t_n)}{\mu}} + 2we^{\frac{(t_i-t_j)+(t_m-t_n)+w}{\mu}} - 2\left[(t_i - t_j) + (t_m - t_n)\right] e^{\frac{(t_i-t_j)+(t_m-}{\mu}}\right. \end{cases} \tag{26}$$

4 Photometric Observables

The SDSS does not present us with a clean timeseries of line and continuum measurements; rather, we have only broadband photometry, with each band representing a mixture of continuum and line emission. Let the observable vector be a set of broadband flux measurements performed in bands Z_p. Then we have:

$$Z_p = a_p c\left(t\right) + b_p\left(t\right) \tag{27}$$

Because everything is linear here, the broadband covariance matrix is a linear combination of the line and continuum covariances:

$$\mathrm{cov}(Z_p, Z_q) = a_p a_q C_{CC} + (a_p b_q + a_q b_p) C_{CL} + b_p b_q C_{LL} \tag{28}$$

As a result, all the functional dependence of the full multiband covariance with respect to the mixing coefficents is linear; the dependence with respect to the ψ_m is also still linear. As a result, the estimator eqn. (3.1) can easily be modified so as to return an estimate of the mixing parameters simultaneously with the estimates of the transfer function parameters! Better still, we have spectra for each individual quasar, and we know the filter response curves, so we have a good initial guess (from the line equivalent widths) for the a's and b's.

5 Constructing the Estimator.

In principle, now, everything is in place. The derivatives of S with respect to the parameters ψ_m are easy to compute – they collapse the sum over m to a single term in each of the cases above.

What's notable about this is that, as written, *you need to know the answer to compute the estimator* – the form of $\hat{\psi}$ depends explicitly on ψ. But that's actually okay – what people do with power spectra estimators is guess the answer, ψ_{guess}, and then use that as the first of two iterations; the ψ_{guess} is updated after one pass, and then the best guess is used to build the real estimator.

I don't think it's useful to try to write the analytic form for C^{-1} – rather, this part should just be done numerically. The next step is to try this on simulated data.