

Added Variable Plot Importance and Joint Added Variable Plot Importance Measures  
for Random Forests

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I want to thank a few people.



# Preface

This is an example of a thesis setup to use the reed thesis document class (for LaTeX) and the R bookdown package, in general.





# Table of Contents

<b>Introduction</b>	<b>1</b>
<b>Chapter 1: Introduction to Trees, Random Forests, and the Bootstrap</b>	<b>3</b>
1.1 CART	4
1.2 Random Forests	6
1.2.1 How Bagged Forests Work	6
1.2.2 The Random Forest Algorithm	8
1.2.3 Variable Importance Measures	8
1.2.4 Issues with Random Forests	9
1.3 Focus of this Thesis	10
<b>Chapter 2: Variable Importance and Inference for Random Forests</b>	<b>13</b>
2.1 Introduction	13
2.2 Theoretical Analysis of MDA Variable Importance Measures	13
2.2.1 Maximal Subtrees and Theoretical Results	14
2.2.2 Extension to Forest Ensembles	16
2.3 Theoretical Analysis of MDI Variable Importance	18
2.4 Dealing with Bias in Random Forest Variable Importance	20
2.4.1 Bias in Random Forest Variable Importance Measures	20
2.4.2 Conditional Variable Importance	21
2.4.3 Infforest Variable Importance	21
<b>Chapter 3: Added Variable Plot Importance</b>	<b>25</b>
3.1 Introduction	25
3.2 Added Variable Plots in Linear Regression	26
3.3 Added Variable Plots in Random Forests	26
3.4 Added Variable Plot Importance	29
3.5 Extensions of Added Variable Plot Importance	31
<b>Chapter 4: Simulation and Results</b>	<b>33</b>
4.1 Introduction	33
4.2 Simulation Design	33
4.3 Simulation Results	35
<b>Chapter 5: Joint Added Variable Plot Importance</b>	<b>41</b>

5.1	Introduction . . . . .	41
5.2	Partial F-Test . . . . .	41
5.3	Joint Added Variable Plots . . . . .	42
5.4	Joint Added Variable Plot Importance . . . . .	43
5.5	Permutation Tests Using Joint Added Variable Plot Importance . . .	44
5.6	Simulation and Results of Joint Added Variable Plot Importance . . .	45
5.6.1	Simulation Design . . . . .	45
5.7	Simulation Results . . . . .	46
<b>Conclusion . . . . .</b>		<b>51</b>
<b>Appendix A: The First Appendix . . . . .</b>		<b>53</b>
<b>References . . . . .</b>		<b>55</b>

# List of Tables

4.1	Covariance Matrix for Correlated Predictors . . . . .	34
4.2	Informative and Uninformative Variables For Each Scenario . . . . .	35
4.3	Which Scenario and Set of Predictors Each Simulation Corresponds to	35
4.4	Simulation Results for Added Variable Plot Importances for Sampling with Replacement . . . . .	36
4.5	Simulation Results for MDA Variable Importances for Sampling with Replacement . . . . .	36
4.6	Simulation Results for Added Variable Plot Importances for Sampling without Replacement . . . . .	37
4.7	Simulation Results for MDA Variable Importances for Sampling without Replacement . . . . .	37
4.8	Added Variable Importance P-values for Sampling with Replacement	38
4.9	Added Variable Importance P-values for Sampling without Replacement	38
5.1	Partitions for Simulation Runs 1 and 2, respectively . . . . .	45
5.2	JAVPI values from first simulation run . . . . .	46
5.3	MDA values from simulation . . . . .	47



## List of Figures



# Abstract

The preface pretty much says it all.

Second paragraph of abstract starts here.





# Dedication

You can have a dedication here if you wish.



# Introduction



# Chapter 1

## Introduction to Trees, Random Forests, and the Bootstrap

This thesis is about utilizing the predictive capabilities of random forests for statistical inference via variable importance measures. The random forest is a statistical learning algorithm developed by Leo Breiman and his collaborators in the early 2000's that leverages bagging and CART (Classification and Regression Trees) methodology to produce quite good predictions of an underlying classification or regression surface. Random forests often outperform classical linear models, are simple to set-up and train, and come with few assumptions about the underlying data generating process. For this thesis, we are interested in the use of random forest variable importance measures for statistical inference. More specifically, random forest variable importance measures are often unstable with respect to correlated predictors, so our focus is on developing variable importance measures that can produce stable results even when there are correlated predictors in the dataset.

Machine learning approaches to regression can often produce models with good predictive accuracy compared to parametric modelling approaches. However, machine learning algorithms often lack in the interpretability and inferential capabilities of more traditional statistical modelling. The challenge of developing descriptive and inferential tools for machine learning algorithms is in finding a balance between complexity and interpretability. While we may want a variable importance measure that fully utilizes the predictive capabilities of the algorithm, we also want the variable importance measure to be simple enough for statistical practitioners to use and interpret.

Throughout this thesis we will be primarily interested in the regression settings for random forests. While random forests are capable of handling both regression and classification responses, our discussion will be simpler if we focus just on the regression setting. Also, we will be assuming that for the data at hand, there exists an underlying regression function  $Y = f(X) + \varepsilon$  where  $f(X)$  is an arbitrary, not necessarily linear function, and  $\varepsilon$  is generally assumed to be Gaussian error, but may have some other error structure.

## 1.1 CART

We begin our discussion of CART (Classification and Regression Trees) by considering the following problem. Suppose we have data generated from the piecewise function  $Y = f(X) + \varepsilon$  where [at this point insert a very non-linear piecewise function.] with Gaussian error  $\varepsilon$ . One approach to fitting a model to this data would be to fit a linear regression model. However, the particular form of the data is not well suited for linear regression due to the non-linearity of the data. Another approach we might try as follows. We might try to find a method that can fit a regression line to each piecewise component of the data. One such method is to fit CART trees. We fit a CART tree to the data and perform cost-complexity pruning to output the model shown in figure [figure not yet made]. In this example, we produced a regression surface which was able to take into account the piecewise, non-linear nature of the data and produce a reasonable estimate of the underlying data generating mechanism.

The CART methodology allows us to fit a model that can take into account non-linear regression or classification surfaces. The basic idea of CART is that if we can split the predictors into roughly homogenous partitions in terms of the response  $Y$ , then we can fit a simple model to predict the response of each partition. CART trees were first introduced by Breiman, Friedman, Olshen, & Stone (1984) and are a flexible method, capable of handling classification and regression settings.

More formally, suppose we have a training data set  $Z = \{Z_1, \dots, Z_n\}$ , where  $Z_i = (X_i, Y_i)$  is an  $p + 1$  dimensional vector in  $\mathbb{R}^{p+1}$ . Here we have  $X_i$  is a  $p$ -dimensional predictor variable and  $Y_i$  is the response. In particular, we can consider  $Z$  to be a  $n \times (p + 1)$  array where the rows are observations and the columns are the response and predictor variables. CART works by partitioning the data through binary recursive splits via optimizing some loss function. Trees in CART are called trees because partitioning the data through binary recursive splits forms a tree-like structure. We adopt notation evocative of this tree structure. Any subset of the training data  $Z$  is called a node while the entire data set  $Z$  is called the root node. Nodes are of two kinds: they are either terminal (sometimes called leaves) or not terminal. Non-terminal nodes are nodes that are split on in the tree growing process, while terminal nodes are nodes which are not split on. Generally, a node is a terminal node if some stopping rule is reached. Note that terminal nodes form a partition of the training data  $Z$ .

We aim to grow a tree with roughly homogenous terminal nodes in terms of  $Y$ . As Breiman et al. (1984) note, there are several factors that we need to consider:

1. How to select splits of the data.
2. When to stop splitting the data.
3. How to assign classes or values to terminal nodes.

CART address point 1 decreasing the nodal impurity of a node  $t$ . Nodal impurity is defined through some nodal impurity measure  $i(s, t)$ , usually the GINI index in the classification setting and the residual sum of squares in the regression setting.

We present the regression setting and then the classification setting. If we have a continuous response  $Y$ , then we could try to predict  $Y$  by partitioning the data using the decision tree structure and predicting that points that fall within a particular partition will take on the average response on that partition. To choose the best binary split on the data, we need to search across the splitting variables and splitting points for the split which maximizes the reduction in the RSS between the parent node and daughter nodes. In particular, we want to find the split which maximizes  $RSS_l(j, s) + RSS_r(j, s)$  where  $j$  is the proposed splitting variable and  $s$  is the proposed splitting point, and  $RSS_l$  is the RSS of the proposed left node and  $RSS_r$  is the RSS of the proposed right node. The assignment of response within a node is given by the average of the response. We present the following algorithm.

---

**Algorithm 1** Construction of Regression tree

---

- 1: **while** minimum node size not reached **do**
  - 2:     **for** each node  $t$  **do**
  - 3:         **for**  $j = 1, \dots, p$  and  $s = 1, \dots, n$  **do**
  - 4:             Compute  $RSS_l(j, s) + RSS_r(j, s)$ .
  - 5:             Pick the  $(j, s)$  which maximizes  $RSS_l(j, s) + RSS_r(j, s)$ .
  - 6:             Split the current node into left and right nodes according to  $(j, s)$ .
  - 7:             Compute left and right averages,  $\text{ave}(y_i | x_i \in t_l)$  and  $\text{ave}(y_i | x_i \in t_r)$ , respectively.
  - 8:     Output the tree  $T$ .
- 

The construction of classification trees is similar to the construction of regression trees except a different impurity measure must be used. For the node  $t$  containing  $N_t$  observations, let

$$\hat{p}_{tk} = \frac{1}{N_t} \sum_{x_i \in t} \mathbb{I}(y_i = k)$$

where  $k = 0$  or  $k = 1$ . Then  $\hat{p}_{tk}$  measures the proportion of observations of class  $k$  in node  $t$ . Using the two-class example, there are several impurity measures available in the classification setting. The most common measure is perhaps the Gini index defined by

$$2p(1 - p).$$

Other options include misclassification error

$$1 - \max(p, 1 - p),$$

and cross-entropy

$$-p \log(p) - (1 - p) \log(1 - p).$$

If  $t_L$  and  $t_R$  are left and right nodes proposed under the split, let  $\hat{p}_{tL}$  and  $\hat{p}_{tR}$  be proportion of observations falling into  $t_L$  and  $t_R$ , respectively. Denote the Gini index of the left node  $t_L$  by  $G_L$  and denote the Gini index of the right node  $t_R$  by  $G_R$ .

Then our splitting criterion is to seek the splitting variable and splitting point which minimizes

$$\hat{p}_{tL}G_L + \hat{p}_{tR}G_R.$$

The case is similar when using misclassification rate and cross-entropy as the impurity measures.

Some issues with CART trees include overfitting and variability. While there is a stopping criterion for growing the CART trees, often times naively growing a tree can result in overfitting to the data. One method of alleviating overfitting is to employ cost-complexity pruning, which searches for an optimal tree that balances fitting a good predictive tree model with overfitting to the data. We do not go into details here, but point the reader to Hastie, Tibshirani, & Friedman (2009).

A larger issue with trees is their sensitivity to small perturbations in the data. Allowing the data to vary even a bit can result in a different tree structure upon refitting. As a statistical learning method, this means that CART trees are not robust and suffer from high variance, even when constructed using cost-complexity pruning. To get around this issue of sensitivity the best solution is to employ bagging and use the random forest algorithm.

## 1.2 Random Forests

Among the limitations of CART discussed in the preceding sections, the biggest issues ARE perhaps the variability of the method and the issue of collinearity among predictors. While overfitting can be addressed using cost-complexity pruning, variability and collinearity are not fixed by pruning. Random forests deal with these two issues of CART by introducing a resampling and randomization mechanism. The natural order is to first discuss bagged forests before turning to random forests.

One method of improving CART trees is to bag them. Bagging, which stand for bootstrap aggregating, is a variation reduction technique particularly useful for improving the predictive power of weak learners. We are interested in bagging CART trees to reduce the variability of single trees under slight perturbations of the data. Generally bagged ensembles of learners produce robust predictions in comparison to running the learner once.

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### Algorithm 2 Bagged Forest algorithm

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- 1: **for**  $b = 1, \dots, B$  **do**
  - 2:     Draw a bootstrap sample  $Z_b^*$  of size  $N$  from the training data  $Z$ .
  - 3:     Grow a CART tree  $T_b$  on each bootstrap sample  $Z_b^*$ .
  - 4: Output the bagged forest ensemble  $\{T_b\}_{b=1}^B$ .
- 

### 1.2.1 How Bagged Forests Work

Bagged forests are an ensemble obtained by taking many bootstrap samples of the data and fitting a tree to each bootstrapped dataset. In this case, we grow the trees



quite deep and do not employ cost-complexity pruning. The idea behind this decision is that we want to sufficiently explore the feature space using a single bagged tree of the forest ensemble, and since we are also taking an average of the trees, we are fine with overfitting at least a little bit. As the algorithm above for bagged forests indicates, the output is an ensemble of trees  $\{T_b\}_{b=1}^B$ . Given a test data point, we form a prediction by taking the average of predictions given by the tree ensemble:

$$\hat{f}_{bf}^B(x) = \frac{1}{B} \sum_{b=1}^B T_b(x),$$

where  $\hat{f}_{bf}^B$  indicates we are taking the bagged forest estimate of the underlying regression or classification function using  $B$  many trees. Note that it is important to be consistent in the choice of splitting criterion throughout the bagging process.

While bagging is a general method, that can be applied, for example, to methods such as ordinary least squares or general linear models, it has been shown by Friedman & Hall (1999) and Chen & Hall (2003) that bagging is especially effective when used on highly non-linear models such as CART trees. Under ideal conditions, bagged estimates of non-linear estimators reduces both the bias and variance of the estimator. So bagging trees seem to be effective because of the reduction to the variance of the estimator, which produces a more robust prediction.

When we bag trees, each observation in the data is not used within each individual tree. A bootstrap replicate  $Z_b$  of the data  $Z$  will likely exclude some of the observations within the data. The observations used within the bootstrap replicate  $Z_b$  is called the in-bag data while the observations not used within the bootstrap replicate  $Z_b$  is called the out-of-bag (OOB) data and is denoted by  $\bar{Z}_b$ . The OOB data allows us to approximate the test error of the ensemble as follows. For simplicity suppose we are in the regression setting (the classification setting is similar). Then the OOB estimate of the MSE of the bagged forest is given by

$$MSE_{\text{OOB}}(T; Z) = \frac{1}{B} \sum_{b=1}^B MSE(T_b; \bar{Z}_b).$$

As the number of trees in the ensemble grows, the OOB estimate of the MSE for the bagged forest converges to the LOOCV estimate of the MSE for forest Hastie et al. (2009).

A weakness of bagged forests is collinearity between trees Hastie et al. (2009). This collinearity between trees grown from the bootstrap sample can be addressed by adding a randomization mechanism in the tree growing process. When we grow a tree from a bootstrap sample, even with the randomness induced by resampling from the data, certain features may be explored at the expense of other just as interesting features. This is a particular issue with collinear predictors. If two predictors are collinear, then the bagged forest might consistently choose one predictor over another even if the predictor not chosen leads to splits that are just as informative. This is due in part to the greedy nature of the CART algorithm when searching for optimal splits over the feature space (Cite ESL or some similar textbook for this point). The

algorithm does not take into account the second best or third best splits. Furthermore, it is not difficult to see that bagging will generally produce an ensemble of trees that are quite similar to one another, subject to some perturbations. These trees will be strongly correlated with other trees in the ensemble, so if there is a less explored part of the feature space, then the ensemble will struggle to produce good predictions over that region.

### 1.2.2 The Random Forest Algorithm

Random Forests deal with this issue of correlated trees and collinearity among predictors by choosing at random only  $m \leq p$  of the predictors to be considered as candidate splitting variables at each split in each tree in the ensemble. This randomness further reduces the variance of the bagged forest by decorrelating the trees in the ensemble. Furthermore, while individually the trees may perform worse than a single pruned tree, collectively the ensemble has a better chance of exploring the feature space fully.

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#### Algorithm 3 Random Forest algorithm

---

- 1: **for**  $b = 1, \dots, B$  **do**
  - 2:     Draw a bootstrap sample  $Z_b^*$  of size  $N$  from the training data  $Z$ .
  - 3:     Grow the CART tree  $T_b$  on  $Z_b^*$  with the following modification:
  - 4:     **while** minimum node size  $n_{\min}$  not reached across  $T_b$  **do**
  - 5:         Select  $m \leq p$  candidate splitting variables at random.
  - 6:         Pick the best splitting variable and splitting point among the  $m$  variables selected at random.
  - 7:         Split the node into two daughter nodes.
  - 8: Output the random forest ensemble  $\{T_b\}_{b=1}^B$ .
- 

To form a prediction at a test point  $x$ , we have in the regression setting

$$\hat{f}_{rf}^B(x) = \frac{1}{B} \sum_{b=1}^B T_b(x).$$

### 1.2.3 Variable Importance Measures

One of the outputs of random forests is the variable importance (VI) measure. There are two main variable importance measures in common use, with the choice of VI measure varying depending on the setting and splitting criterion chosen. The first choice is the Mean Decrease in Impurity (MDI) which is typically used in the classification setting where the GINI index or Shannon entropy is used as the splitting criterion. The second choice is Mean Decrease in Accuracy (MDA) which is typically used in the regression setting where RSS has been used as the splitting criterion. The idea of MDI is to find how much the nodal impurity  $p(t)\Delta i(s, t)$  decreases for all nodes  $t$  in which the variable of interest  $X_j$  is used and to take that average over all trees in the ensemble. Note that  $p(t)$  is the proportion of observations of a particular class in the node  $t$  and  $\Delta i(s, t)$  is the decrease in impurity over the node  $t$  when variable  $X_j$

is chosen. More important variables are those which are on average more often chose for splits and which also contribute most to reducing the nodal impurity of the trees.

---

**Algorithm 4** MDI Variable Importance
 

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- 1: Grow a random forest  $\{T_b\}_{b=1}^B$ .
  - 2: **for**  $j = 1, \dots, p$  **do**
  - 3:   **for**  $b = 1, \dots, B$  **do**
  - 4:     Compute the importance of  $X_j$  in  $T_b$  as  $VI_b(X_j) = \sum_{t \in T_b} \mathbb{I}(j_t = j)p(t)\Delta i(s, t)$  to be the sum of the decrease in impurity over nodes where variable  $X_j$  is used .
  - 5:   Compute the importance of  $X_j$  in the random forest to be  $VI(X_j) = \frac{1}{B} \sum_{b=1}^B VI_b(X_j)$ .
- 

The idea of MDA is to measure for each variable  $X_j$ , on average how much the predictive accuracy of the forest as measured using RSS suffers when the  $X_j$  component is permuted across observations within the OOB dataset  $\bar{Z}_b$  of each tree  $T_b$  in the ensemble.

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**Algorithm 5** MDA Variable Importance
 

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- 1: Grow a random forest  $\{T_b\}_{b=1}^B$ .
  - 2: **for**  $j = 1, \dots, p$  **do**
  - 3:   **for**  $b = 1, \dots, B$  **do**
  - 4:     Permute the  $X_j$  component of  $Z_b$  to obtain the dataset  $Z_b^j$ , where  $X_j$  has been permuted.
  - 5:     Compute the importance of  $X_j$  in  $T_b$  to be  $VI_b(X_j) = \frac{1}{|\bar{Z}_b|} (RSS(T_b, Z_b) - RSS(T_b, Z_b^j))$ .
  - 6:   Compute the importance of  $X_j$  in the random forest to be  $VI(X_j) = \frac{1}{B} \sum_{b=1}^B VI_b(X_j)$ .
- 

More important variables in the random forests are those for which the variable importance is large, as those are the variables for which the predictive accuracy of the random forest suffers the most when the out of bag data is permuted. Note that as variable importance measures currently defined and used, the threshold for importance of a variable is something which the researcher has to decide.

### 1.2.4 Issues with Random Forests

While random forests are good out of the box predictors, there are situations where the random forest algorithm can fail to produce good predictions. If the underlying regression function is linear, if the predictors are highly correlated, or if the data cannot be bootstrapped, then the random forest will not perform well. Relative to other methods, random forests are especially adept at handling highly non-linear functions, but can struggle with linear response compared to say linear regression.

If the predictors are highly correlated, then it has been shown by Strobl, Boulesteix, Zeileis, & Hothorn (2007) and Strobl, Boulesteix, Kneib, Augustin, & Zeileis (2008) that the trees within the forest ensemble will be biased and the variable importance measures will not be reliable due to confounding between similar looking variables. While the randomization step in the random forest algorithm can alleviate the correlation between trees in the ensemble, the collinearity between predictors can cause the predictive performance of the random forest to suffer. This is an issue with the underlying CART algorithm. Alternative algorithms from CART exist which try to alleviate issues of collinearity and correlation in predictors, but we focus primarily on the forest ensembles grown using CART in this thesis.

Finally, there are situations where the data cannot be bootstrapped. This could be due to a number of factors including that the data has a heavily-tailed distribution or if there are particularly extreme values. In this case, a different bootstrap scheme could perhaps remedy the situation where the naive bootstrap fails. One common important resampling scheme used within random forests is to use the  $m$ -out-of- $n$  bootstrap (also referred to subsampling or subagging in the literature). The  $m$ -out-of- $n$  bootstrap is a resampling scheme which resamples with or without replacement  $m \leq n$  observations from the data to form datasets with  $m$  many data points to run the bootstrap computation. Bickel et al. have shown that in important cases, subsampling can succeed where the bootstrap fails. Of course, using less of the available data is less efficient, but in the context of random forests this loss of efficiency may not be an issue. In Strobl et al. (2008), simulation results showed that subsampling instead of using the standard nonparametric bootstrap can improve the performance of random forests. The authors suggested that subsampling further reduces the variance of the ensemble by producing trees that are even more decorrelated. Their heuristic is that there is a lower probability of duplicate data points being chosen using subsampling (this probability is zero if we are subsampling without replacement), furthermore there is a lower probability of highly correlated data points being chosen as one of the  $m \leq n$  points. This certainly seems plausible, but we would like to see further simulation results or a technical result that explains why subsampling works well for forests. We would also like to note that most consistency results (Biau et al., Scornet et al., Ishwaran et al., and Wager et al.) make this subsampling assumption in their analysis of the random forest model, as subsampling constructions make the forest ensemble more amenable to mathematical analysis.

### 1.3 Focus of this Thesis

In the previous three sections we discussed the basics of trees, the bootstrap, and random forests. We would like to now discuss the direction we are going to take this thesis. We are interested in developing inferential tools using the random forest algorithm. The random forest often produces good predictions out of the box with little tuning required except growing a sufficient number of trees and choosing a good value of  $m \leq p$  to try at each split in the tree growing process. The ease of fitting random forests is one of the advantages of the random forest compared to

more complex methods like neural networks or support vector machines. Even if the random forest is easier to fit than neural nets or SVMs, the underlying random forest mechanism is quite complex. This complexity makes developing inferential tools with random forests difficult. In the next chapter we will discuss some approaches that have been developed recently by researchers. Of particular interest for us are the variable importance measures for random forests. There is the MDI variable importance and the MDA variable importance measure. While there are concerns about the bias in the MDI and MDA VI measures with random forests using CART tree's due to how CART tree's are constructed (Strobl et al., 2008), we would like to extend the inferential capabilities of random forests using CART as this is perhaps the most popular version of the algorithm in use.

The random forest VI can be viewed as a random variable with an unknown probability distribution. While determining the distribution of random forest VI's would be ideal towards developing inference for random forests, finding the distribution of random forest VI's is difficult and unclear in general. However, note that the bootstrap provides a computational method of approximating the sampling distribution of random forest VI measures. We could then proceed to obtaining estimates of the standard error and to form approximate interval estimates. We could grow a random forest once, obtain an estimate of the VI of each variable  $X_j$  and then run a bootstrap to obtain estimates of the standard error of each variable. Inference could then proceed using the VI measures along with the estimate of standard error. Note that in this scheme there is two levels of bootstrap at play. There is the bootstrap within each random forest and then the bootstrapping to obtain the standard error estimate. The estimate we produce of the VI of a variable  $X_j$  can be erratic. Running the random forest again can produce a new estimate of the VI of variable  $X_j$ . As Efron suggests in his 2014 article "Estimation and Accuracy after Model Selection," we could remedy the erratic, non-smooth nature of the VI estimate by bagging  $VI(X_j)$ . However, when we bag  $VI(X_j)$  we would require a third-level of bootstrapping to produce an estimate of the standard error of the bagged estimate of  $VI(X_j)$ . This is computationally expensive, especially considering that the computational cost of the random forest depends on the number of observations, number of tree's grown, and number of variables present. However, applying Efron's infinitesimal jackknife (IJ) technique, which we will discuss in the next chapter, we can efficiently estimate the standard error of the bagged estimate of  $VI(X_j)$  only using the two levels of bootstrapping. In this thesis we will be exploring the properties of IJ estimates of the standard error of the bagged estimate  $VI(X_j)$  utilizing different bootstrap schemes. Utilizing the IJ, we can then produce interval estimates and confidence intervals of the variable importance of variables in the random forest. This allows us to characterize the uncertainty of  $VI(X_j)$  when using variable importance measures for description, variable selection, and inference.

### ## Outline of Remaining Chapters

Chapter 2: Inference and variable importance for random forests. In this chapter we will discuss the various approaches to inference for random forests that have been developed by researchers. These include Ishwaran and Louppe's analysis of VI measures of RF. Ishwaran's analysis is of the variable importance scheme he devised.

Ishwaran's scheme is amenable to theoretical analysis and he produces asymptotic results for his scheme. Louppe analyzes the MDI variable importance in the context of a construction called totally randomized forests in the setting of finite sample spaces. Louppe investigates the effect of relevant and irrelevant variables to MDI VI in this setting and provides asymptotic results. These are two of the main papers investigating properties of VI.

The other schemes include Mentch and Hooker's prediction intervals which are produced by interpreting the random forest ensemble as a particular type of U-statistic and applying the U-statistics theory to produce prediction intervals. They then can produce hypothesis tests and confidence intervals in this context.

A second approach towards prediction intervals is Wager's application of the IJ to the random forest estimator to produce prediction intervals and confidence intervals. Wager has a couple of papers in this area which we will discuss. In this part we will also go into greater detail about the IJ, its properties, and why it works.

We will also discuss the conditional inference trees and conditional variable importance developed by Strobl, Hothorn, and their colleagues to deal with the biased manner in which the trees in random forests are constructed. In this part we will also discuss the Infforest variable importance measure developed by Aurora.

Chapter 3: In this chapter we will go into great detail about the bootstrap and why it works. We will discuss the technical details of different methods of estimating the standard error of functional statistics, and we will go into the construction of different bootstrap confidence intervals along with properties of these confidence intervals. We will also discuss different bootstrap schemes and in particular subsampling, and why subsampling works well, especially in the random forest context.

Chapter 4: We will present our different approaches to bootstrapping variable importance measures and provide simulation results. The first approach is the cases bootstrap, the second approach is the bootstrapping residuals of the random forest approach. Within the cases approach, there are two ways we will try to constrain the variability of bootstrap replicates. Adopting the notation from general linear models, we could fit a pruned tree predicting  $Y \sim X_1, \dots, X_p$ . Then we would bootstrap from the terminal nodes of this single tree to try to produce bootstrap replicates which capture the relationship between the variables. Another approach is for each predictor  $X_j$ , fit a single well pruned tree  $X_j \sim X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_p$ . We would then draw bootstrap samples from the terminal nodes of this tree which respect the partition produced by these terminal nodes. The idea is to fix the relationship between every predictor not  $X_j$  and allow the  $X_j$ 's to be exchanged across the bootstrap samples. We would produce simulation results of each approach and compare the statistical properties of these different constructions.

Chapter 5: If all goes well there might be a chapter 5 where we explore large sample properties and distribution of VI measures. This is likely going to be a difficult mathematical problem to solve in general, so a tangible course of action might be to pick a generating mechanism and easy VI measure from which we could figure out the VI distribution under some set of strong assumptions.

# Chapter 2

## Variable Importance and Inference for Random Forests

### 2.1 Introduction

In this chapter we focus on approaches of inference for random forests involving variable importance measures of random forests. Such approaches involve using variable importance measures of random forests to evaluate the relative importance of different variables in the construction of the forest. Louppe (2014) and Ishwaran (2007) focused on theoretical properties of variable importance measures while Owens (2017), Strobl et al. (2007), and Strobl et al. (2008) are centered on developing less biased variable importance measures for hypothesis testing.

### 2.2 Theoretical Analysis of MDA Variable Importance Measures

The approach to variable importances adopted in Ishwaran (2007) differs from the original variable importance measures for random forests, so we require some additional vocabulary.

Suppose  $T$  is a binary recursive tree and suppose that  $T$  has  $M$  many terminal nodes. For each point  $\mathbf{x}$  in the feature space,  $T$  maps  $\mathbf{x}$  to one of the  $M$ -many terminal nodes. In particular, if we let  $\mathcal{X}$  denote the feature space, then  $T$  is a function  $\mathcal{X} \rightarrow \{1, \dots, M\}$  defined by the equation

$$T(\mathbf{x}) = \sum_{m=1}^M m B_m(\mathbf{x}),$$

where  $B_m(\mathbf{x})$  is a 0 – 1 basis function which partition the feature space  $\mathcal{X}$ .

Let  $Z = \{(\mathbf{x}_i, Y_i) | i = 1, \dots, n\}$  denote the training data, where  $\mathbf{x}_i$  is a covariate in the feature space and  $Y_i$  is the response. We call  $T$  a binary regression tree if it is a binary recursive tree grown from  $Z$  using binary recursive splits of the form  $x_j \leq c$  and  $x_j > c$  where split values  $c$  are chosen based on the observed  $\mathbf{x}_i$  in the

training data  $Z$ . The value  $a_m$  in the terminal node is the average response of the training observations falling in the  $m$ th node. That is,

$$a_m = \frac{\sum_{i=1}^n \mathbb{I}\{T(\mathbf{x}_i) = m\} Y_i}{\sum_{i=1}^n \mathbb{I}\{T(\mathbf{x}_i) = m\}}.$$

Note that  $\mathbf{x}_i$  denotes a row of covariates for the training data  $Z$ , while  $x_j$  denotes the  $j$ th variable along the columns of the training data.

For a binary regression tree, the basis functions  $B_m(\mathbf{x})$  are product splines of the following form:

$$B_m(\mathbf{x}) = \prod_{l=1}^{L_m} [x_{l(m)} - c_{l,m}]_{s_{l,m}},$$

where  $L_m$  denotes the number of splits used to construct  $B_m(\mathbf{x})$ . For each split  $l$ , there is a splitting variable  $\mathbf{x}_{l(m)}$  which denotes the  $l(m)$ th coordinate of  $\mathbf{x}$  and a splitting value  $c_{l,m}$ . The  $s_{l,m}$  are binary  $\pm 1$  values, where for a given scalar  $x$ ,  $[x]_{+1} = \mathbb{I}(x > 0)$  and  $[x]_{-1} = \mathbb{I}(x \leq 0)$ . Note that the basis functions satisfy an orthogonality property, which gives  $B_m(\mathbf{x})B_{m'}(\mathbf{x}) = 0$  if  $m \neq m'$ . Note also that given a tree  $T$ , the predictor associated with the tree can be written as a linear combination of basis functions:

$$\hat{\mu}(\mathbf{x}) = \sum_{m=1}^M a_m B_m(\mathbf{x}).$$

We are now prepared to define Ishwaran's variable importance measure.

Informally, the *MDA* variable importance of a variable  $x_j$  is the difference between *MSE* of the tree  $T$  when  $x_j$  is randomly permuted and *MSE* of the tree  $T$  when  $x_j$  is not permuted. As such a scheme of variable importance is difficult to analyze, Ishwaran proposes a surrogate measure. For the variable  $x_j$ , we drop  $\mathbf{x}$  down the tree and follow the binary splits until either a terminal node is reached or a node with a split depending on  $x_j$  is reached. If a node with a split depending on  $x_j$  is reached, we then subsequently assign  $\mathbf{x}$  randomly to either the left or right daughter node, whenever there is a split, until we reach a terminal node. The difference in *MSE* between noising up  $x_j$  and not noising up  $x_j$  to be the variable importance of  $x_j$  in the tree  $T$ . Denote the tree that results from noising up  $x_j$  by  $T_j$ .

Such a scheme relies on the following heuristic: if we chose an adequate splitting rule to construct our tree, then we expect that variables that are split earlier in the tree are more important, since prediction will suffer the most from noising up a variable higher up in the tree than a variable close to a terminal node. This is a behavior observed in CART trees and random forests based on CART: splits closer to the root node are more influential than splits close to terminal nodes, so the MDA or MDI variable importance of variables split on close to the root node are expected to be higher than otherwise.

### 2.2.1 Maximal Subtrees and Theoretical Results

Defining a structure on binary regression trees called subtrees, we can write the predictor for the noised up tree as a deterministic component relying on terminal



nodes for no parent nodes involve a split on  $x_j$ , and a random component involving terminal nodes for which there are parent nodes involving a split on  $x_j$ . The definition of the subtree is quite intuitive. We call  $\tilde{T}_j$  a  $j$ -subtree of the tree  $T$ , if the root node of  $\tilde{T}_j$  has daughters that depend on an  $x_j$  split. A  $j$ -subtree  $\tilde{T}_j$  is a maximal  $j$ -subtree of the tree  $T$ , if there are no larger  $j$ -subtrees containing  $\tilde{T}_j$ . For a given tree  $T$  and for each variable  $x_j$ , there is a set of  $K_j$  many distinct maximal  $j$ -subtrees, which are denoted by  $\tilde{T}_{1,j}, \dots, \tilde{T}_{K_j,j}$ . Note each distinct  $\tilde{T}_{k,j}$  maximal  $j$ -subtree contains a set of distinct terminal nodes  $M_{k,j}$ . Each  $M_{k,j}$  is distinct for  $k = 1, \dots, K_j$ , since we are working with maximal  $j$ -subtrees. Define

$$M_j = \bigcup_{k=1}^{K_j} M_{k,j}$$

to be the set of terminal nodes for which there is a parent node involving a split on  $x_j$ . Ishwaran (2007) proves the following lemma about the functional form of the predictor for the tree  $T_j$ .

**Lemma 2.1.** *Let  $\hat{\mu}_j(\mathbf{x})$  denote the predictor for  $T_j$ . Then*

$$\hat{\mu}_j(\mathbf{x}) = \sum_{m \notin M_j} a_m B_m(\mathbf{x}) + \sum_{k=1}^{K_j} \tilde{a}_{k,j} \mathbb{I}\{T(\mathbf{x}) \in M_{k,j}\},$$

where  $\tilde{a}_{k,j}$  is the random terminal value assigned by  $\tilde{T}_{k,j}$  under the random left right path through  $\tilde{T}_{k,j}$ . We write  $\tilde{P}_{k,j}$  to denote the distribution of  $\tilde{a}_{k,j}$ .

For a proof, see Ishwaran (2007). It is a bit surprising that the functional form of  $\hat{\mu}_j(\mathbf{x})$  can be separated into the two components. Given this lemma and the definition of  $j$ -subtrees, we can more formally define the variable importance of  $x_j$  in the tree  $T$ .

Let  $g$  be a loss function. Often we use the squared error to evaluate loss, which corresponds to  $MSE$ , but there is no strict requirement in the definition. Denote the test data by  $(Y, \mathbf{x})$ . Then the prediction error of the predictor  $\hat{\mu}$  is given by  $\mathbb{E}(g(Y, \hat{\mu}(\mathbf{x})))$ . As a reminder, we assume that there is an underlying regression function

$$Y = \mu(\mathbf{x}) + \varepsilon,$$

where  $\varepsilon$  is independent error with zero mean and variance  $\sigma^2 > 0$ . Similarly, we can define the prediction error of the predictor  $\hat{\mu}_j$  to be  $\mathbb{E}(g(Y, \hat{\mu}_j(\mathbf{x})))$ . Set  $g(Y, \hat{\mu}(\mathbf{x})) = (Y - \hat{\mu}(\mathbf{x}))^2$  to be the  $L_2$  loss, which corresponds to  $MSE$ . Define the variable importance of the variable  $x_j$  to be

$$\Delta_j = \mathbb{E}((Y - \hat{\mu}_j(\mathbf{x}))^2) - \mathbb{E}((Y - \hat{\mu}(\mathbf{x}))^2).$$

Application of the lemma and some manipulation allows us to write

$$\Delta_j = \mathbb{E}(R_j(\mathbf{x})^2) - 2\mathbb{E}(R_j(\mathbf{x})[\mu(\mathbf{x}) - \hat{\mu}(\mathbf{x})]),$$

where

$$R_j(\mathbf{x}) = \sum_{k=1}^{K_j} \sum_{m \in M_{k,j}} (\tilde{a}_{k,j} - a_m) B_m(\mathbf{x}).$$

As noted in Ishwaran (2007), we make the assumption that the true regression function  $\mu$  is of similar form to  $T$ . That is, assume

$$\mu(\mathbf{x}) = \sum_{m=1}^M a_{m,0} B_m(\mathbf{x}),$$

where  $a_{m,0}$  are the true, but unknown, terminal values. Under this and some other large sample assumptions, Ishwaran finds that asymptotically, each maximal  $j$ -subtree will tend to contribute equally to the variable importance  $\Delta_j$ . In effect, nodes closer to the root of a maximal  $j$ -subtree will have a larger effect on  $\Delta_j$  than nodes closer to the terminal nodes.

### 2.2.2 Extension to Forest Ensembles

The framework developed in Ishwaran (2007) extends naturally to forest ensembles and his theoretical result regarding forest ensembles provides some information of the behavior of variable importance measures for random forests. First for some notation, recall that in the forest ensemble setting, we draw  $B$  many bootstrap resamples of the training data to obtain the bootstrap replicates  $Z^b = \{(\mathbf{x}_i^b, Y_i^b) | i = 1, \dots, n\}$  of the training data for  $b = 1, \dots, B$ . We then construct a binary regression tree  $T(\mathbf{x}; b)$  on each bootstrap replicate of the data and have the forest  $\hat{\mu}_F$  as the average of predictions over the trees  $T(\mathbf{x}; b)$ :

$$\hat{\mu}_F(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \hat{\mu}(\mathbf{x}; b),$$

where  $\hat{\mu}(\mathbf{x}; b)$  denotes the predictor for the tree  $T(\mathbf{x}; b)$ . Given that each  $\hat{\mu}(\mathbf{x}; b)$  is a linear combination of basis functions, we can write

$$\hat{\mu}_F(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \sum_{m=1}^{M^b} a_m^b B_m(\mathbf{x}; b).$$

Again assume that the  $\mu(\mathbf{x})$  has a similar structure to  $\mu(\mathbf{x})$ . That is,  $\mu(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \sum_{m=1}^{M^b} a_{m,0}^b B_m(\mathbf{x}; b)$ , where  $a_{m,0}$  are the true, but unknown, terminal values. We denote the noised up forest predictor for the variable  $x_j$  by  $\hat{\mu}_{F_j}(\mathbf{x})$ . As with the normal forest predictor, the noised up forest predictor is the average of the noised up predictors  $\hat{\mu}_j$  over the bootstrap resamples:

$$\hat{\mu}_j(\mathbf{x}) = \sum_{b=1}^B \hat{\mu}_j(\mathbf{x}; b).$$

As forest predictors are simply the average of individual trees, we can extend the lemma to  $\hat{\mu}_{F_j}$  with the use of  $b$ 's in appropriate places denoting the usage of the  $b$ th bootstrap resample. That is, we can write

$$\hat{\mu}_{F_j}(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \left( \sum_{m \notin M_j^b} a_m^b B_m(\mathbf{x}; b) + \sum_{k=1}^{K_j^b} \tilde{a}_{k,j}^b \mathbb{I}\{T(\mathbf{x}; b) \in M_{k,j}^b\} \right).$$

The variable importance of the variable  $x_j$  is defined to be

$$\Delta_{F_j} = \mathbb{E}((Y - \hat{\mu}_{F_j}(\mathbf{x}))^2) - \mathbb{E}((Y - \hat{\mu}(\mathbf{x}))^2).$$

Similar to the single tree case, the variable importance of the variable  $x_j$  in the forest ensemble can be written as

$$\Delta_{F_j} = \mathbb{E}(R_{F_j}(\mathbf{x})^2) - 2\mathbb{E}\left(R_{F_j}(\mathbf{x})[\mu(\mathbf{x}) - \hat{\mu}_F(\mathbf{x})]\right),$$

where

$$R_{F_j}(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \sum_{k=1}^{K_j^b} \sum_{m \in M_{k,j}^b} (\tilde{a}_{k,j}^b - a_m^b) B_m(\mathbf{x}; b).$$

We are now ready to state a result about the asymptotic form of  $\Delta_{f,j}$ :

**Theorem 2.1.** *Let  $R_{F_j,0}(\mathbf{x})$  be the function  $R_{F_j}(\mathbf{x})$ , in which each instance of  $a_m^b$  has been replaced with  $a_{m,0}^b$ . Assume that  $Y$  can be written in terms of a regression model*

$$Y = \mu(\mathbf{x}) + \varepsilon$$

and that

$$\mu(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B \sum_{m=1}^{M^b} a_{m,0}^b B_m(\mathbf{x}; b).$$

If  $a_m^b \rightarrow a_{m,0}^b$  for each  $m$  and  $b$ , then

$$\Delta_{F_j} \rightarrow \mathbb{E}\left(R_{F_j,0}(\mathbf{x})^2\right) \leq \mathbb{E}\left(\frac{1}{B} \sum_{b=1}^B \sum_{k=1}^{K_j^b} \theta_0(k, j, b)\right),$$

where

$$\theta_0(k, j, b) = \sum_{m \in M_{k,j}^b} \pi_{m,b} \tilde{P}_{k,j,0}^b \left( \tilde{a}_{k,j,0}^b - a_{m,0}^b \right)^2$$

is the node mean squared error for the  $k$ th maximal  $v$ -subtree of  $T(\mathbf{x}; b)$  and  $\pi_{m,b} = \mathbb{P}(B_m(\mathbf{x}; b) = 1 | Z)$ .

For a proof, see Ishwaran (2007). As Ishwaran (2007) notes, the bound in the above theorem becomes tighter as the trees in the forest become more and more orthogonal to each other. The above theorem is applicable if the forest predictor is consistent. This suggests that forest ensemble methods which are consistent and produce trees

that are at least approximately orthogonal to each other allow for variable importance to be characterized through node mean squared error of subtrees. Variable importance as defined above differs from the *MDA* variable importance for random forests, but the two follow a similar heuristic towards measuring variable importance via the loss of predictive accuracy when the variable  $x_j$  is perturbed. Therefore this result suggests that, perhaps, so long as the random forest estimator is consistent and the trees constructed are at least approximately orthogonal, the *MDA* variable importance can be characterized via the mean squared error of some analogous structure to maximal subtrees. This is, of course, conjectural, but we are interested whether for simpler forest algorithms, results similar to the theorem can be proven. Asymptotic results for the *MDA* and also *MDI* variable importance measures are difficult to formulate and prove for several reasons. First, the *CART* tree construction process combined with the bootstrapping and randomization procedure of the random forest is difficult to mathematically analyze. Second, it is still unknown whether random forests constructed via *CART* are consistent. However, there are other random forest variants known to be consistent, so those variants may be a natural starting point to try to prove asymptotic results about *MDA* and *MDI* variable importances. Third, the definition of *MDA* variable importance involves a permutation step that is difficult to analyze mathematically. The above results are mathematically amenable due to the noising-up procedure adopted allowing for a quite elegant analysis of the variable importance. One possible approach to dealing with this issue may be to adopt the above definition of variable importance in different random forest settings and see if similar results to Ishwaran (2007) may be derived. Another approach, which is the one more or less taken by Ishwaran (2007), is to find a mathematically tractable definition of variable importance which is approximately the *MDA* variable importance measure and proceed with analysis from there.

## 2.3 Theoretical Analysis of MDI Variable Importance

Up to now in this chapter, we have been discussing theoretical results concerning the *MDA* variable importance. Louppe (2014) has explored some of the theory for *MDI* variable importance of forest ensembles.

In order to discuss the following theoretical results regarding *MDI* variable importance, we make some modifications to the setting we are working in. Assume that we are working in a finite probability space [this assumption is necessary primarily for technical reasons] and that observations  $Y, X_1, \dots, X_p$  in the training set  $Z$  are categorical variables. Recall that if a forest ensemble is grown by choosing splits and variables maximizing the decrease in nodal impurity where  $i(s, t)$  denotes the impurity measure used, then the *MDI* variable importance of the variable  $X_j$  is defined to be

$$VI(X_j) = \frac{1}{B} \sum_{b=1}^B \sum_{t \in T_b} \mathbb{I}(j_t = j) p(t) \Delta i(s, t),$$

where  $p(t)$  is the proportion of samples which reach the node  $t$  in the tree  $T_b$ . Note that MDI variable importance is the average over the bootstrap resamples of the sum of the weighted decrease in impurity over the nodes of the tree using  $X_j$  as the splitting variable.

Rather than working with random forests, we will work with an infinite ensemble of totally randomized and fully developed trees. Totally randomized and fully developed trees are a variant of forest ensembles given by the following. A totally randomized and fully developed tree is a decision tree in which each node  $t$  is partitioned using a variable  $X_j$  picked uniformly at random among those not yet used at the parent nodes of  $t$ , where each node  $t$  is split into  $|X_j|$  sub-trees, i.e., one for each possible value of  $X_j$ , and where the recursive construction process halts only when all  $p$  variables have been used along the current branch.

Using an impurity measure called Shannon entropy, Louppe (2014) is able to show that only relevant variables (that is variables whose information content, conditional on the other variables, is useful towards explaining the response  $Y$ ) have non-zero infinite sample size variable importance. Furthermore, this means that irrelevant variables (which are variables whose information content, conditional on the other variables, is not useful towards explaining the response  $Y$ ) have a infinite sample size variable importance of zero. In fact, in this context, a variable is irrelevant if and only if it has an infinite sample size variable importance of zero. Generalizing from Shannon entropy to other impurity measures, for the GINI index (used in commonly in classification problems) and variance (used in regression via MSE), only irrelevant variables result in no decrease in nodal impurity. We would like to reiterate that these results concerning irrelevant variables are valid particularly in the setting of infinite ensembles of totally randomized and fully developed trees.

In the random forest setting, the results discussed in the previous paragraph may not be valid. In the random forest setting, a key step in the construction of trees is the randomization step in which we choose at random a subset of size  $m \leq p$  of the variables to choose the next split from. If  $m > 1$ , then it is possible for some variables to never be chosen at a node  $t$ , since there will always be other variables which have a larger decrease in nodal impurity. This results in trees in which the variables with the largest reduction in nodal impurity are centered in the trunk of the tree with variables with smaller reduction in nodal impurity are pushed to leaves, or otherwise not chosen to split on. This is an undesirable property to have since this can result in an ensemble which inadequately explores the feature space and is thus more consistently biased towards some variables over others, even if other variables can provide only slightly lower reductions in nodal impurity. As a consequence, if  $m > 1$ , then it is not the case that a variable is irrelevant if and only if it has a variable importance of zero. One implication is that in data sets where there are correlated variables, random forest variable importance measures may result in variable importances which do not reflect the actual importance of variables in explaining the response  $Y$ . In particular, it could be the case that if  $X_j$  and  $X_{j'}$  are two correlated variables, then even if both  $X_j$  and  $X_{j'}$  are important in explaining the response  $Y$ , it may be the case that  $X_j$  is consistently chosen as the splitting variable over  $X_{j'}$  if the two variables are both possible splitting variables.

## 2.4 Dealing with Bias in Random Forest Variable Importance

In the previous two sections of this chapter we discussed theoretical results pertaining to MDA and MDI variable importance measures. For MDA variable importance, a consistent ensemble with orthogonal trees have a nice asymptotic form in the framework developed in Ishwaran (2007). On the otherhand, with MDI variable importance, if the trees are grown in the random forest setting, then there are issues of the variable importance not capturing the actual importance of the variables in the training data. This could particularly arise as an issue in data sets with correlated variables where equally important variables receive different variable importances due to bias introduced by tree construction process. In using the term ‘bias,’ we would like to emphasise that this is not bias in the sense of biased estimators, but bias in the sense of which variables are chosen to be split upon in the tree growing process. Bias in the variable importance measures of random forests is concerning as we would like to be able to use random forests for inferential statistics. A couple of methods have been proposed to construct more reliable variable importance measures, which we discuss in this section.

### 2.4.1 Bias in Random Forest Variable Importance Measures

Bias in random forest variable importance measures was first substantively analyzed in Strobl et al. (2007). Strobl et al. (2007) ran a simulation study comparing MDI variable importance using the GINI index, MDA variable importance, and a conditional variable importance measure in a classification setting. The conditional variable importance measure used by Strobl et al. (2007) is a variable importance method similar to MDA variable importance, but utilizing an ensemble of conditional inference trees. We will define conditional inference trees and the conditional variable importance measures below. For now, note that Strobl et al. (2007) found that MDI variable importance using the GINI index and MDA variable importance tends to be biased from the expected variable importance value within their simulation. They also found that subsampling without replacement tends to reduce the bias in variable importance in comparison to using bootstrap resampling with replacement. In their analysis, Strobl et al. (2007) claim that the GINI index tends to favor variables with more potential cutpoints. Hence the MDI variable importance measure using the GINI index is biased towards variables with more potential cutpoints. Strobl et al. (2007) also found that in using the MDA variable importance measure, since variables that are split upon closer to the root node affect the predictive accuracy of the ensemble more than variables split upon in leaves, the MDA variable importance measure will be affected by the variable selection bias of individual trees. Furthermore, Strobl et al. (2008) find that correlated variables tend to be overselected in the tree growing process. In order to utilize random forest variable importance measures as inferential tools, several methods have been proposed to deal with the issue of variable selection bias in random forests.

### 2.4.2 Conditional Variable Importance

The MDA variable importance measure can be viewed in the context of permutation tests and hypothesis testing. In particular, we can consider that the MDA variable importance measure operates on the null hypothesis that permuting the values of the variable  $X_j$  has no effect on the predictive accuracy of the random forest predictor. This corresponds to the null hypothesis that the variable  $X_j$  is independent of the response  $Y$  and the variables  $X_1, \dots, X_p$ . If the predictive accuracy of the random forest suffers from permuting  $X_j$ , that is if the  $VI(X_j)$  is nonzero, then if also  $X_j \perp X_{-j}$ , it would follow that  $Y$  and  $X_j$  are not independent. However, it is important to note that the null hypothesis under which the MDA variable importance measure operates is:

$$H_0 : X_j \perp Y, X_{-j}.$$

If on the otherhand,  $X_j$  and  $X_{-j}$  are not independent, then  $VI(X_j)$  will be biased by the correlation between  $X_j$  and  $X_{-j}$ . It could be that  $Y$  and  $X_j$  are not independent or that  $Y$  and  $X_j$  are independent, but in either case, if the predictor variables are correlated, we see that  $VI(X_j)$  will not accurately capture the importance of  $X_j$ . Strobl et al. (2008) propose a permutation scheme corresponding to the null hypothesis

$$H_0 : (X_j \perp Y) | X_{-j},$$

that is that  $X_j$  is independent of  $Y$  conditional on the  $X_{-j}$ . Such a permutation scheme would take into account the correlation structure of the predictor variables and would allow for a less biased variable importance measure so long as the individual trees within the ensembles do not exhibit large variable selection bias.

Strobl et al. (2008) propose defining a grid within which values of  $X_j$  are permuted according to partitions of the feature space given by each tree. While Strobl et al. (2008) propose using unbiased conditional inference trees from Hothorn, Hornik, & Zeileis (2006) to determine the partition, it is possible to use partitions given by CART trees. The algorithm is presented as follows.

To determine the variables  $X_i$  to be conditioned on Strobl et al. (2008) suggest including only variables whose empirical correlation with the variable  $X_j$  suggests some reasonable threshold.

### 2.4.3 Infforest Variable Importance

While simulation results of Strobl et al. (2008) suggest that the conditional variable importance is more effective at capturing the importance of each variable  $X_j$  compared to the marginal approach followed in MDA, they note that conditional variable importance cannot completely eliminate preference for correlated variables in random forests. Owens (2017) instead suggests a partition then permute scheme which produces a sampling distribution for the variable importance of each variable  $X_j$ . Once a sampling distribution for  $VI(X_j)$  has been obtained, hypothesis testing and other sorts of statistical inference can proceed. As the Infforest variable importance is quite complicated, some explanation is needed.

**Algorithm 6** Conditional Variable Importance

- 
- 1: Grow a forest ensemble  $\{T_b\}_{b=1}^B$
  - 2: **for**  $j = 1, \dots, p$  **do**
  - 3:     **for**  $b = 1, \dots, B$  **do**
  - 4:         Compute  $RSS(T_b, Z_b)$ .
  - 5:         **for** all variables  $X_i$  to be conditioned on **do**
  - 6:             Extract cutpoints that split  $X_i$  in the tree  $T_b$  and create a grid  $\mathcal{G}$  by bisecting the sample space in each cutpoint.
  - 7:             Within the grid  $\mathcal{G}$ , permute the values of  $X_j$  and denote the permuted resample by  $Z_b^j$ .
  - 8:             Compute  $RSS(T_b, Z_b^j)$ .
  - 9:             Compute the conditional variable importance of  $X_j$  in the tree  $T_b$  to be  $VI_b(X_j) = \frac{1}{|\bar{Z}_b|} (RSS(T_b, Z_b) - RSS(T_b, Z_b^j))$ .
  - 10:     Compute the conditional variable importance of  $X_j$  in the forest ensemble to be  $VI(X_j) = \frac{1}{B} \sum_{b=1}^B VI_b(X_j)$ .
- 

After growing a forest ensemble  $\{T_b\}_{b=1}^B$ , infforest variable importance values are computed as follows. First, for each  $b = 1, \dots, B$  and each variable  $X_j$ , grow a tree  $X_j \sim X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_p$  using all  $p - 1$  predictors using the in-bag sample used to grow  $T_b$ . Denote this tree by  $T_j^{b*}$ . The rows of  $\bar{Z}_b$  are permuted within the partitions of the feature space determined by  $T_j^{b*}$ . The difference in predictive accuracy of  $T_b$  before and after the OOB sample  $\bar{Z}_b$  has been permuted is the infforest variable importance for the variable  $X_j$  in the tree  $T_b$ . Owens (2017) suggests then using the sampling distribution of  $VI(X_j)$  obtained from the infforest partition-permute scheme to test the null hypothesis that the variable importance of  $X_j$  is zero. On the other hand, if we let  $VI_b(X_j)$  denote the infforest variable importance of the variable  $X_j$  in the tree  $T_b$ , then we could define the infforest variable importance of  $X_j$  in the forest ensemble  $\{T_b\}_{b=1}^B$  to be

$$VI(X_j) = \frac{1}{B} \sum_{b=1}^B VI_b(X_j).$$

In particular, this corresponds to the mean of the sampling distribution of the infforest variable importances of  $X_j$  over the trees  $T_b$ .

Originally, Owens (2017) developed for random forests using CART trees as base learners. However, the infforest variable importance could certainly be used with random forest variants such as conditional inference forests used in Strobl et al. (2008). Another possible modification of the infforest variable importance algorithm is to only grow an auxiliary tree predicting  $X_j \sim X_{\alpha_1}, \dots, X_{\alpha_k}$ , where  $X_{\alpha_1}, \dots, X_{\alpha_k}$  are variables whose correlation with  $X_j$  exceeds some reasonable threshold. This would combine the suggestion in Strobl et al. (2008) of considering the correlation structure of the predictor variables in setting up the grid for the conditional variable importance with the infforest partition-permute scheme.

One issue with infforest variable importance is the steep computational cost of



the algorithm. The algorithm grows an auxiliary tree  $T_j^{b*}$  for each variable  $X_j$  and bootstrap resample  $b$  and then proceeds to permute the OOB sample with respect to each  $T_j^{b*}$ . Even implementing the infforest variable importance algorithm on small bootstrap resamples of around several hundred is computationally intensive.

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**Algorithm 7** Infforest Variable Importance

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- 1: Grow a forest ensemble  $\{T_b\}_{b=1}^B$
  - 2: **for**  $j = 1, \dots, p$  **do**
  - 3:   **for**  $b = 1, \dots, B$  **do**
  - 4:     Compute  $RSS(T_b, \bar{Z}_b)$ .
  - 5:     Grow a tree  $T_j^{b*}$  predicting  $X_j \sim X_{-j}$  using in-bag sample.
  - 6:     Permute rows of the OOB sample  $\bar{Z}_b$  with respect to the partitions of the feature space from  $T_j^{b*}$  to obtain the permuted OOB sample  $\bar{Z}_b^*$ .
  - 7:     Compute  $RSS(T_b, \bar{Z}_b^*)$ .
  - 8:     Compute the infforest variable importance of  $X_j$  in  $T_b$  to be  $VI_b(X_j) = \frac{1}{\bar{Z}_b} \left( RSS(T_b, \bar{Z}_b) - RSS(T_b, \bar{Z}_b^*) \right)$ .
  - 9:   Test the null hypothesis that the variable importance of  $X_j$  is zero using the distribution of values  $VI_b(X_j)$ .
-



# Chapter 3

## Added Variable Plot Importance

### 3.1 Introduction

In the previous chapter, we discussed theoretical properties and observed behavior of random forest variable importance measures. In particular, we discussed issues of bias present in the MDA variable importance measure. Strobl et al. (2008) and Owens (2017) proposed the conditional variable importance measure and the Infforest variable importance measures, respectively, as methods of accounting for bias in variable selection among correlated predictors when measuring variable importance in a forest ensemble. Conditional variable importance and Infforest variable importance aim at measuring the conditional importance of a particular variable by conditionally permuting OOB data according to some criteria measured from the data, whether that be empirical correlation in the case of conditional variable importance, or the partitions induced by a particular tree in the case of Infforest variable importance. In any case, we are primarily interested in the conditional importance of a predictor given the information provided by other predictors. While we would like to have an exact method of computing the conditional importance of a predictor in a random forest ensemble, in practice it is unclear how to construct an exact method given the complexity of the random forest ensemble. We can, however, estimate the effect of adding a predictor to the set of predictors that the forest ensemble can split on via a type of diagnostic plot called an added variable plot.

Added variable plots are a diagnostic plot which estimate the effect of adding a predictor variable to the model. In this chapter we propose a method of measuring the importance of each predictor in a random forest ensemble via a quantity we call the Added Variable Importance (AVI) that depends on the added variable plot of each predictor. The AVI attempts to estimate the effect of adding a predictor as a possible candidate in the splitting step of the random forest ensemble. To understand AVI, we regress from the random forest setting to the linear regression setting to explain how added variable plots work.

### 3.2 Added Variable Plots in Linear Regression

Suppose that we have data  $Z = \{(\mathbf{x}_i, Y_i) | i = 1, \dots, n\}$ , where  $\mathbf{x}_i$  is a covariate in the feature space, and that we fit a multiple linear regression model of the form

$$\mathbf{Y}_1 = \mathbf{X}\beta_1 + \mathbf{W}\alpha + \varepsilon_1$$

where  $\mathbf{X}$  is a  $n \times p$  matrix consisting of  $p - 1$  predictors,  $\beta = (\beta_0, \beta_1, \dots, \beta_p)^T$ ,  $\mathbf{W} = (w_1, \dots, w_n)$  is a predictor,  $\alpha$  is a scalar, and  $\text{Var}(\varepsilon) = \sigma^2 I$ .

Say we are interested in estimating the effect of the predictor  $\mathbf{W}$  in the regression model  $\hat{\mathbf{Y}}_1$ . We could first fit the models  $\hat{\mathbf{Y}}_2 = \mathbf{X}\beta_2$  and  $\hat{\mathbf{W}} = \mathbf{X}\delta$ , where  $\beta_2$  and  $\delta$  are coefficient vectors like  $\beta_1$  and then plot the residuals  $\mathbf{W} - \hat{\mathbf{W}}$  against the residuals  $\mathbf{Y} - \hat{\mathbf{Y}}_2$ . The plot we obtain by plotting  $\mathbf{W} - \hat{\mathbf{W}}$  against  $\mathbf{Y} - \hat{\mathbf{Y}}_2$  is called the added variable plot of  $\mathbf{W}$  and measures the effect of  $\mathbf{W}$  on  $\mathbf{Y}$  once we have adjusted for the effect of  $\mathbf{X}$  on  $\mathbf{W}$  and  $\mathbf{Y}$ , respectively. In particular, the residuals that we plot for the added variable plot is the portion of  $\mathbf{W}$  unexplained by  $\mathbf{X}$  on the  $x$ -axis and the portion of  $\mathbf{Y}$  unexplained by  $\hat{\mathbf{Y}}_2 = \mathbf{X}\beta_2$  on the  $y$ -axis.

In the linear regression setting the added variable plot has useful properties. In particular, let  $\alpha_{AVP}$  denote the estimate of the slope from regressing  $\mathbf{Y} - \hat{\mathbf{Y}}_2$  on  $\mathbf{W} - \hat{\mathbf{W}}$ . Then with some linear algebra it can be shown that  $\alpha_{AVP}$  is equal to the least-squares estimate of  $\alpha$  from  $\hat{\mathbf{Y}}_1 = \mathbf{X}\beta_1 + \mathbf{W}\alpha$ . For full details see Sheather (2009) and Cook & Weisberg (1982). Hence if we want to estimate the linear effect of the variable  $\mathbf{W}$  on the linear regression  $\hat{\mathbf{Y}}_1$ , we can construct and visually inspect the trend of the added variable plot of  $\mathbf{W}$ .

[Added example here]

### 3.3 Added Variable Plots in Random Forests

While added variable plots in the linear regression setting allows us to conditionally estimate the effect of the predictor on the response, the picture for more complex regression functions such as random forests is more complicated. In particular, if the relationship between the response and predictors is non-linear, or we are using a statistical learning method such as a random forest, then we do not have the nice linear algebra that undergirds the interpretation of added variable plots of linear regression models. However, we can still construct added variable plots for random forests that are similar to the added variable plots for linear regression.

Suppose we are interested in the added variable effect of variable  $X_k$ . Let  $X_{-k}$  denote the set of predictors excluding  $X_k$ . We could proceed as in the linear regression setting and fit  $\hat{\theta}_{RF}(Y|X_{-k})$  which predicts the  $Y$  given  $X_{-k}$ , and we would also fit  $\hat{\theta}_{RF}(X_k|X_{-k})$  which predicts  $X_k$  in terms of  $X_{-k}$ . We would then form the added variable plot of the predictor  $X_k$  by plotting

$$(X_k - \hat{\theta}_{RF}(X_k|X_{-k}), Y - \hat{\theta}_{RF}(Y|X_{-k})).$$

That is, we plot the part of  $X_k$  unexplained by  $X_{-k}$  against the part of the response  $Y$  unexplained by  $X_{-k}$  with respect to the random forest algorithm. We denote

this method as a residual-based added variable plot. The heuristic of this plot is similar to the added variable plot in the linear regression setting. Namely, if we want to understand the conditional relationship of the predictor  $X_k$  with respect to the response, then plotting the residuals as above provides the effect of  $X_k$  on  $Y$  once we have adjusted for the effect of  $X_{-k}$  on  $X_k$  and  $Y$ , respectively.

The output of the residual-based added variable plot for a predictor  $X_k$  would depend on the relationship between  $X_k$  and the response  $Y$ . If  $X_k$  is uninformative to the response, then we would expect that  $\hat{\theta}_{RF}(Y|X_{-k})$  would form a prediction for  $Y$  that is similar to a prediction for  $Y$  given by  $\hat{\theta}_{RF}(Y|X)$ . Hence we would expect that  $Y - \hat{\theta}_{RF}(Y|X_{-k})$  to generally be close to zero. If on the other hand,  $X_k$  is informative to the response, then we would expect that  $\hat{\theta}_{RF}(Y|X_{-k})$  would form a worse prediction for  $Y$  than that given by  $\hat{\theta}_{RF}(Y|X)$ , since we would be excluding  $X_k$  as a splitting variable. Furthermore, we would be taking into account the amount of the response explainable by  $X_{-k}$  using the random forest model, such that  $Y - \hat{\theta}_{RF}(Y|X_{-k})$  would be the amount of the response unexplained by  $X_{-k}$ . The values of  $X_k - \hat{\theta}_{RF}(X_k|X_{-k})$  will depend on the relationship between  $X_k$  and  $X_{-k}$ . In general, the importance of subtracting  $\hat{\theta}_{RF}(X_k|X_{-k})$  from  $X_k$  is that this takes into account the part of  $X_k$  that can be explained by  $X_{-k}$  using the random forest model. This can be particularly useful if  $X_k$  has a complex, dependent relationship with some of the predictors among the  $X_{-k}$ . However, if  $X_k$  and  $X_{-k}$  are independent, then  $\hat{\theta}_{RF}(X_k|X_{-k})$  will likely be a noisy prediction of  $X_k$ . Hence, if  $X_k$  is uninformative to the response, we would expect the added variable plot of  $X_k$  to be de-trended mass of point centered about the origin. While if  $X_k$  is informative, we would expect that the added variable plot of  $X_k$  to have a non-zero, possibly non-linear trend depending on the conditional relationship of  $X_k$  with the response  $Y$ .

An alternative method for added variable plots is proposed by Rendahl (2008) for black box statistical learning methods such as random forests. In particular, an added variable plot we can form is to plot  $Y - \mathbb{E}(Y|X_{-k})$  against  $\mathbb{E}(Y|X) - \mathbb{E}(Y|X_{-k})$ . That is, we can plot residuals of the model without the predictor  $X_k$  against the difference in predictions between the full model and the model with  $X_k$  removed. In the context of random forests, to obtain the added variable plot of the predictor  $X_k$ , we would plot

$$(\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k}), Y - \hat{\theta}_{RF}(Y|X_{-k})).$$

We denote this method as a model-based added variable plot.

[Add example here]

Depending on the relationship between the predictors and the response, there are several outputs we might expect from the added variable plots of the predictors in the random forest. If the predictor  $X_k$  is simply noise, i.e., if a predictor is uninformative with respect to the response, then we expect that  $\hat{\theta}_{RF}(Y|X)$  and  $\hat{\theta}_{RF}(Y|X_{-k})$  to form similar predictions of  $Y$  given  $X = x$  and  $X_{-k} = x_{-k}$ , respectively. In this scenario then, suppose for a moment that  $\hat{\theta}_{RF}(Y|X)$  and  $\hat{\theta}_{RF}(Y|X_{-k})$  are consistent estimators of  $\mathbb{E}(Y|X)$  and  $\mathbb{E}(Y|X_{-k})$ , respectively. Then by the Law of Total Expectation, we would expect that asymptotically  $\mathbb{E}_X(\hat{\theta}_{RF}(Y|X)) = Y$  and  $\mathbb{E}_{X_{-k}}(\hat{\theta}_{RF}(Y|X_{-k})) = Y$ .

Hence in the limit, by linearity of expectation,

$$\mathbb{E}(\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})) = \mathbb{E}(\hat{\theta}_{RF}(Y|X)) - \mathbb{E}(\hat{\theta}_{RF}(Y|X_{-k})) = Y - Y = 0.$$

Similarly,

$$\mathbb{E}(Y - \hat{\theta}_{RF}(Y|X_{-k})) = Y - Y = 0.$$

Hence if a predictor  $X_k$  is uninformative, provided we have grown adequately accurate forest ensembles to predict  $Y$  given  $X$  and  $Y$  given  $X_{-k}$ , respectively, then we expect that the added variable plot of  $X_k$  to either be radially centered about the origin or to otherwise be arranged in a line with a slope and  $y$ -intercept of zero.

On the other hand, suppose that the predictor  $X_k$  is informative with respect to the response. Then we would expect that when properly tuned the random forest ensemble  $\hat{\theta}_{RF}(Y|X)$  would provide reasonably accurate predictions of  $Y$ . We would also expect that the forest ensemble  $\hat{\theta}_{RF}(Y|X_{-k})$  would suffer in predictive performance due to the lack of information from  $X_k$ . Then in the limit, we would expect that

$$(\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})) \neq 0 \text{ and } Y - \hat{\theta}_{RF}(Y|X_{-k})) \neq 0$$

when we evaluate  $X = x$  and  $X_{-k} = x_{-k}$  where  $x, x_{-k}$  are training examples in the full dataset and the reduced dataset, respectively. Since the difference  $\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})$  and  $Y - \hat{\theta}_{RF}(Y|X_{-k})$  is generally non-zero when we evaluate the random forest at a particular point in the feature space and these differences are precisely what we plot on the x and y-axis of the added variable plot, we would then expect the trend of the added variable plot of  $X_k$  to be non-zero. In other words, in the case of informative predictors, we would expect there to be correlation between  $\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})$  and  $Y - \hat{\theta}_{RF}(Y|X_{-k})$ . The departure from a trend of approximately zero would depend on the relative informativeness of  $X_k$ . If  $X_k$  is strongly informative then we would expect the difference between  $Y$  and  $\hat{\theta}_{RF}(Y|X_{-k})$  and the difference between  $\hat{\theta}_{RF}(Y|X)$  and  $\hat{\theta}_{RF}(Y|X_{-k})$  to often be large. If  $X_k$  is weakly informative we would expect the two differences to often be of smaller magnitude than when  $X_k$  is strongly informative. Hence visually the trend of the added variable plot of predictors used to grow a random forest ensemble offer a method of gauging the informativeness of different predictors.

We note that in the above argument that we made an appeal to the random forest estimator  $\hat{\theta}_{RF}(Y|X)$  being a consistent estimator of  $\mathbb{E}(Y|X)$ . As discussed earlier, the random forest algorithm as introduced and implemented by Breiman and collaborators has not yet been shown to be consistent. However, empirically the random forest algorithm often offers good predictions of the response, so we expect that the added variable plot as applied to the original random forest algorithm to have the properties discussed above. We also note that there are random forest variants such as Causal Forest as introduced by Wager & Athey (2017) which have been shown to be consistent estimators of  $\mathbb{E}(Y|X)$ . Hence in such settings we expect that our discussion of added variable plots for forest ensembles to be fully valid.

One setting in which added variable plots for random forests are useful is in dealing with correlated predictors. As Strobl et al. (2008) notes, the random forest algorithm

can have difficulty determining the relative importance of correlated predictors due to masking effects. Suppose  $X_j$  and  $X_k$  are correlated predictors with  $X_j$  only weakly informative to the response. Then if both  $X_j$  and  $X_k$  are candidate splitting variables,  $X_k$  could be chosen over  $X_j$  as the splitting variable since  $X_k$  may seem to be informative due to the correlation between  $X_k$  and  $X_j$ . The better split would have been found over  $X_j$ , but the greedy nature of the CART algorithm means the algorithm does not look forward at possible splits further down the tree if  $X_j$  is chosen over  $X_k$ . Note that the tree grown using  $X_k$  at that particular node would likely have lower predictive performance than the tree grown using  $X_j$  at that node. Hence WLOG suppose the subset  $\{X_1, \dots, X_m\} \subseteq \{X_1, \dots, X_p\}$  of our predictors are correlated where  $m \leq p$ . If  $X_j \in \{X_1, \dots, X_m\}$  is not informative to the response, then we would expect  $\hat{\theta}_{RF}(Y|X)$  and  $\hat{\theta}_{RF}(Y|X_{-j})$  to provide similar predictions, while if  $X_k \in \{X_1, \dots, X_m\}$  is informative to the response, we would expect that  $\hat{\theta}_{RF}(Y|X_{-k})$  will suffer a decrease in predictive performance in comparison to  $\hat{\theta}_{RF}(Y|X)$ . Then we would expect the added variable plot for  $X_j$  to be a mass of points centered about the origin without a trend while the added variable plot for  $X_k$  should have some sort of trend whose shape depends on the informativeness of  $X_k$  and the predictive performance of  $\hat{\theta}_{RF}(Y|X)$  and  $\hat{\theta}_{RF}(Y|X_{-k})$ . This is of course despite the fact that, depending on the correlation structure and relative informativeness of  $X_k$ , that  $X_j$  may be chosen over  $X_k$  when  $X_j$  and  $X_k$  are both candidate splitting variables in the tree growing process. The full random forest model  $\hat{\theta}_{RF}(Y|X)$  may be biased in how the splitting variables are chosen due to the correlation structure of  $\{X_1, \dots, X_m\}$  as discussed in Strobl et al. (2008). However, absent the choice of  $X_k$  in the random forest model  $\hat{\theta}_{RF}(Y|X_{-k})$ , the less informative split on  $X_j$  has an increased probability of being chosen when growing the forest ensemble  $\hat{\theta}_{RF}(Y|X_{-k})$  than in the full forest ensemble  $\hat{\theta}_{RF}(Y|X)$ . Hence the random forest model  $\hat{\theta}_{RF}(Y|X_{-k})$  would have a decrease in predictive performance due to the loss of information in the predictor  $X_k$ , but also due to the increased probability of irrelevant predictors being chosen as the splitting variables due to exclusion of  $X_k$ .

### 3.4 Added Variable Plot Importance

In the previous section, we discussed the use of added variable plots for random forests including in settings where a subset of the predictors are correlated. Variable importance measures such as MDA variable importance and MDI variable importance have difficulties in dealing with correlated predictors. As discussed in the previous chapter, Strobl et al. (2008) proposed the conditional variable importance measure while Owens (2017) proposed the Infforest variable importance measure as methods that try to account for the correlation structure of the predictors when measuring the importance of predictors in the forest ensemble. In this section we propose the Added Variable Plot Importance (AVPI) measure as an alternative variable importance measure to conditional variable importance and Infforest variable importance measures when accounting for correlated predictors.

To motivate the construction of AVPI, we return briefly to added variable plots in the linear regression setting. As mentioned earlier, when we construct an added variable plot for a predictor  $\mathbf{W}$  in the linear regression model, the linear trend of the added variable plot for  $\mathbf{W}$  corresponds to the least squares estimate of the linear effect  $\alpha$  of  $\mathbf{W}$  in the full model  $\mathbf{Y}_1 = \mathbf{X}\beta_1 + \mathbf{W}\alpha + \varepsilon_1$ . Hence while we could simply visually inspect the trend of the added variable plot for  $\mathbf{W}$ , we could also run least squares regression on the added variable plot for  $\mathbf{W}$  to obtain an estimate for  $\alpha$  in the regression model  $\mathbf{Y}_1$ . The size and sign of the  $\hat{\alpha}$  from running a regression model on the added variable plot of  $\mathbf{W}$  then indicates the relative importance and effect of  $\mathbf{W}$  on the response  $\mathbf{Y}$ .

In the random forest setting, we would like to have a similar procedure to determine the importance of the variable  $X_k$  with respect to the response  $Y$  once we have accounted for the effect of the other predictors.

We have two options for plotting the added variable plot of  $X_k$  using the random forest algorithm. We have a residuals based approach where we plot

$$(X_k - \hat{\theta}_{RF}(X_k|X_{-k}), Y - \hat{\theta}_{RF}(Y|X_{-k})),$$

and we also have model based approach where we plot

$$(\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k}), Y - \hat{\theta}_{RF}(Y|X_{-k})).$$

As discussed in the previous section, depending on the relative informativeness of the predictor  $X_k$  there may or may not be a trend in the added variable plot for  $X_k$  which we could then attempt to model. Let  $W_k = Y - \hat{\theta}_{RF}(Y|X_{-k})$  and  $U_k = \hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})$ . We propose to train a bagged forest ensemble  $\hat{\theta}_{BF}(W_k|U_k)$  and compute the MDA variable importance of  $U_k$  in the bagged forest ensemble. We call the MDA variable importance of  $U_k$  in the bagged forest ensemble  $\hat{\theta}_{BF}(W_k|U_k)$  the added variable plot importance (AVPI) of  $X_k$  and denote this quantity by  $VI_{AVP}(X_k)$ . Our choice of using a bagged forest to predict  $W_k$  given  $U_k$  is motivated by the fact that unlike in the linear regression setting, there is not a clear parametric model with which to predict  $W_k$  given  $U_k$ . The bagged forest makes few assumptions on the form of the true regression function  $W_k = f(U_k) + \varepsilon$  while also providing a metric in the form of MDA variable importance to assess the importance of  $U_k$  in predicting  $W_k$  once we have grown the ensemble  $\hat{\theta}_{BF}(W_k|U_k)$ . Furthermore, the AVPI of  $X_k$  should reflect the relative importance of  $X_k$  with respect to the response  $Y$  given that the trend of the added variable plot, i.e. the degree to which  $Y - \hat{\theta}_{RF}(Y|X_{-k})$  and  $\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})$  are correlated, visually indicates the informativeness of  $X_k$ .

The purpose of the AVPI of  $X_k$  is to provide a quantitative measure of the importance of the predictor  $X_k$  with respect to  $Y$  once we have taken into account the loss in predictive performance when  $X_k$  is removed as a possible splitting variable. In particular, given that the added variable plots for random forests should reflect the informativeness of correlated predictors, the added variable plot importance for correlated predictors should be able to more accurately reflect the importance of correlated predictors with respect to the response than the MDA variable importance ran on the predictors in the full model  $\hat{\theta}_{RF}(Y|X)$ .



**Algorithm 8** Added Variable Plot Importance (AVPI)

- 
- 1: Grow the random forest ensemble  $\hat{\theta}_{RF}(Y|X)$  predicting  $Y$  using the full set of predictors.
  - 2: **for**  $k = 1, \dots, p$  **do**
  - 3:   Grow the random forest ensemble  $\hat{\theta}_{RF}(Y|X_{-k})$  predicting  $Y$  using the full set of predictors minus the predictor  $X_k$ .
  - 4:   Compute  $U_k = \hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})$  and  $W_k = Y - \hat{\theta}_{RF}(Y|X_{-k})$ .
  - 5:   Grow the bagged forest ensemble  $\hat{\theta}_{BF}(W_k|U_k)$  predicting  $W_k$  using  $U_k$ .
  - 6:   Compute the added variable plot importance of  $X_k$  to be the MDA variable importance of  $U_k$ :  $VI_{AVP}(X_k) = VI_{MDA}(U_k)$ .
- 

### 3.5 Extensions of Added Variable Plot Importance

Once we have obtained added variable plot importance values there are couple directions we can extend our framework. Once again suppose we have grown our full random forest ensemble  $\hat{\theta}_{RF}(Y|X)$  and also the random forest ensemble  $\hat{\theta}_{RF}(Y|X_{-k})$ . Let  $U_k = \hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-k})$  and  $W_k = Y - \hat{\theta}_{RF}(Y|X_{-k})$ .

Once we have computed the AVPI of  $X_k$ ,  $VI_{AVP}(X_k)$ , we can take a simulation approach to generating a sampling distribution for  $VI_{AVP}(X_k)$ . In particular, to generate a sampling distribution for the AVPI of  $X_k$ , permute  $U_k$  to obtain  $U_k^*$ . Then grow the bagged forest ensemble  $\hat{\theta}_{BF}(W_k|U_k^*)$  predicting  $W_k$  using  $U_k^*$ , and compute the MDA variable importance of  $U_k^*$  to obtain  $VI_{AVP}^*(X_k)$  as the permuted AVPI for  $X_k$ . After having permuted  $U_k$  and computed  $VI_{AVP}^*(X_k)$  for enough iterations to generate a sampling distribution for  $VI_{AVP}(X_k)$ . We can then compute a two-sided p-value using the original  $VI_{AVP}(X_k)$  as the observed test statistic. If we compute p-values for each predictor, then we could proceed to a hypothesis testing framework if we believe that the simulation to generate a sampling distribution for the AVPI of each predictor was successful. Of course depending on the number of predictors in the dataset and the relationship between the predictors, it may be necessary to control for multiple comparisons. In our opinion, as simulations in the next chapter will indicate, hypothesis testing using the AVPI is likely more sensitive to type-I errors than type-II errors, so an adjustment such as the Bonferroni correction may be appropriate.

We also note that while generating adequate sampling distributions of  $VI_{AVP}(X_k)$  for each predictor  $X_k$  is computationally expensive, each step of the process from growing the full ensemble  $\hat{\theta}_{RF}(Y|X)$  to growing each  $\hat{\theta}_{RF}(Y|X_{-k})$  to permuting  $U_k$  and growing  $\hat{\theta}_{BF}(W_k|U_k^*)$  to compute  $VI_{AVP}^*(X_k)$  can be coded to run in parallel. So performance gains in generating good sampling distributions of  $VI_{AVP}(X_k)$  for each predictor  $X_k$  are easily attainable.



# Chapter 4

## Simulation and Results

### 4.1 Introduction

In this chapter, we present simulation results of the random forest added variable plot and added variable plot importance methods. The design of our simulation follows closely to Strobl et al. (2008).

### 4.2 Simulation Design

We set up a simulation to test the added variable plot importance method on several different regression models. In particular, we were interested in how the added variable plot importance method would perform in situations where the relationship of the response to the predictors was linear, polynomial, and non-linear, and when there was and was not a correlation structure in the predictor. In addition, we were interested in comparing the effect of sampling with replacement versus sampling without replacement in the forest growing process. In the case of the independent random variables, we drew twelve predictors from a standard normal distribution. For the case of random variables with a correlation structure, we drew  $X_1, \dots, X_{12}$  from a multivariate normal distribution with mean 0 and covariance matrix  $\Sigma$  where the first four variables  $X_1, \dots, X_4$  are block-correlated with a value of 0.9, and with the other eight variables independent.

We note that our covariance is the same covariance matrix used by Strobl et al. (2008) for their simulation study on the effect of correlation on variable selection in random forests in the regression setting. We tested the AVPI on four scenario. The first was when the response was a linear combination of some of the predictors. The second scenario was when the response was a sum of polynomial terms of the predictors. The third scenario was when the response was the sum of two interaction terms. The fourth scenario was when the response was a sum of non-linear terms involving some of the predictors. In each scenario, we added gaussian noise  $N(0, 0.05)$  to the response. We ran each scenario, in effect, four times. Once with independent predictors and sampling with replacement, then with independent predictors and sampling without replacement. We then ran the simulation on correlated predictors

Table 4.1: Covariance Matrix for Correlated Predictors

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12
X1	1.0	0.9	0.9	0.9	0	0	0	0	0	0	0	0
X2	0.9	1.0	0.9	0.9	0	0	0	0	0	0	0	0
X3	0.9	0.9	1.0	0.9	0	0	0	0	0	0	0	0
X4	0.9	0.9	0.9	1.0	0	0	0	0	0	0	0	0
X5	0.0	0.0	0.0	0.0	1	0	0	0	0	0	0	0
X6	0.0	0.0	0.0	0.0	0	1	0	0	0	0	0	0
X7	0.0	0.0	0.0	0.0	0	0	1	0	0	0	0	0
X8	0.0	0.0	0.0	0.0	0	0	0	1	0	0	0	0
X9	0.0	0.0	0.0	0.0	0	0	0	0	1	0	0	0
X10	0.0	0.0	0.0	0.0	0	0	0	0	0	1	0	0
X11	0.0	0.0	0.0	0.0	0	0	0	0	0	0	1	0
X12	0.0	0.0	0.0	0.0	0	0	0	0	0	0	0	1

and sampling with replacement, then with correlated predictors and sampling without replacement. We illustrate the relationship between predictors and response for each of the four scenarios below.

Relationship Between Predictors and Response	
Scenario 1	$Y=5*X1+5*X2+2*X3-5*X5-5*X6-2*V7+err$
Scenario 2	$Y = 5*X1^4+5*X2^3+6*X3^4+5*X5^3+err$
Scenario 3	$Y=8*X1*X2+7*X5*X6+err$
Scenario 4	$Y = 3^X1+2^X2+4^X5+err$

In each scenario, variables 1 and 2 are always informative towards the predictor. In scenarios 1 and 2, variable 3 is informative to the predictor, while it is not informative for the scenario 3 and 4. In all scenarios, variable 4 is uninformative. In this way, we can see the effect of correlated predictors on AVPI. For variables 5 through 12, variable 5 is informative for each scenario, while variable 6 is informative in scenarios 1 and 2. Variable 7 is informative only in the scenario 1. Variables 8 through 12 are uninformative for each scenario. For reference, we illustrate which variables are informative and uninformative for each scenario. An  $X$  indicates that a variable is informative for that scenario while a  $-$  indicates that a variable is uninformative.

Each simulation corresponds to a particular scenario and type of predictors. We ran each scenario with first independent predictors then predictors with the correlation structure from above. In addition, we first ran each simulation using sampling with

Table 4.2: Informative and Uninformative Variables For Each Scenario

	Var1	Var2	Var3	Var4	Var5	Var6	Var7	Var8	Var9	Var10	Var11	Var12
Scenario 1	X	X	X	-	X	X	X	-	-	-	-	-
Scenario 2	X	X	X	-	X	-	-	-	-	-	-	-
Scenario 3	X	X	-	-	X	X	-	-	-	-	-	-
Scenario 4	X	X	-	-	X	-	-	-	-	-	-	-

Table 4.3: Which Scenario and Set of Predictors Each Simulation Corresponds to

Simulation 1	Scenario 1 with independent predictors
Simulation 2	Scenario 1 with correlated predictors
Simulation 3	Scenario 2 with independent predictors
Simulation 4	Scenario 2 with correlated predictors
Simulation 5	Scenario 3 with independent predictors
Simulation 6	Scenario 3 with correlated predictors
Simulation 7	Scenario 4 with independent predictors
Simulation 8	Scenario 4 with correlated predictors

replacement in the random forest ensemble, then re-ran the simulation using sampling without replacement.

For each simulated dataset, we drew 2000 entries. We then ran the added variable plot importance scheme with the permutation scheme. For each stage of AVPI, we trained forest ensembles with 1000 trees, and to generate simulated null distributions of AVPI, we ran 1000 iterations of AVPI with permuted values for the x-axis of the added variable plot.

## 4.3 Simulation Results

In the table below, we have the results of running the added variable plot importances on simulated data sets when sampling with replacement. We first note that the AVPI is quite noisy in this case. For example, in the first simulation, the relevant variables were variables 1, 2, 3, 5, 6, and 7. The AVPI reflects the importance of these variables by assigning high AVPI scores to these variables. However, irrelevant variables also had AVPI scores between 70 and 95. Simulation 2 is a similar set-up as Simulation 1, except that the predictors have the correlation structure discussed above. We see that in this case that confounding between correlated predictors is still an issue as variable 3 is assigned a AVPI score of 76 – which is lower than the scores assigned to

Table 4.4: Simulation Results for Added Variable Plot Importances for Sampling with Replacement

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1	452.93528	209.19947	184.29378	115.75481	348.0641	110.911944	200.25310	60.67242
Variable 2	436.50160	200.30775	164.79733	49.30234	355.2946	121.012598	90.09198	33.95596
Variable 3	176.15493	76.12612	141.20267	127.73681	186.0916	4.438317	68.25077	47.94023
Variable 4	69.71643	14.09013	62.70999	44.32338	190.2879	29.697785	60.86136	52.56689
Variable 5	453.05254	479.75299	160.85322	196.06473	324.4524	245.731616	92.18312	124.80432
Variable 6	437.09922	479.78513	44.56070	39.21142	325.6628	239.388844	60.91802	32.09565
Variable 7	214.78673	270.27051	70.79758	30.53213	171.8441	32.679916	41.70790	38.16311
Variable 8	94.67066	127.68758	63.23303	53.12418	182.5039	46.141387	46.65001	41.45022
Variable 9	69.99713	104.23646	56.13219	39.10399	215.7042	22.064169	53.65208	41.27242
Variable 10	70.13127	80.45368	70.77256	54.45881	152.4064	21.740682	44.84663	34.90679
Variable 11	70.51321	91.39784	65.78904	44.82417	185.0719	17.999310	60.45611	50.53217
Variable 12	68.49742	88.75896	75.87031	26.68245	200.3480	39.174289	53.36817	41.62424

Table 4.5: Simulation Results for MDA Variable Importances for Sampling with Replacement

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1	205.7751162	43.5875638	87.9694096	39.6267348	51.3775666	68.0344640	45.2615577	16.5658513
Variable 2	182.7953452	45.3203038	47.4985731	22.9426251	51.3756589	59.7224401	6.6095155	13.8717953
Variable 3	43.3705524	31.7206332	88.2067658	46.1139125	-1.8420809	19.5960032	-0.6160646	10.7902908
Variable 4	-0.8467650	22.5345613	-1.5190519	21.3672088	0.7655721	31.1151525	0.7265404	7.7755666
Variable 5	179.2816368	210.5278772	42.5696986	52.9141820	54.3017177	44.2205811	73.4474491	90.0099288
Variable 6	190.1979559	199.6886106	3.2962175	-1.8198516	50.3953957	49.1162744	-0.1722575	0.2080882
Variable 7	49.7942892	54.4672557	-1.4994542	-1.9087069	0.1191700	-2.0071842	-0.2548386	-2.8897482
Variable 8	1.8630745	-0.4211499	-1.9256818	-2.5463545	-0.2674365	0.4980884	-0.2709784	0.7483121
Variable 9	-1.7072270	-0.3489273	1.0810682	0.9995826	4.1857040	-2.7534240	0.6012231	-0.3862685
Variable 10	0.1434588	0.0824640	1.8700646	2.1839565	-0.9314400	-2.0090863	1.0238239	0.6847400
Variable 11	0.2220114	-1.1905073	-0.2861896	-1.3776237	1.6749594	-3.2776029	0.9792522	-1.9566093
Variable 12	0.2826006	1.1815968	-0.4752267	-0.8640204	1.6851063	-0.2720373	1.7977126	-2.4189332

variables 8 through 12. We also extracted the MDA variable importance values for each simulation as a comparison to the AVPI values. In particular, we note that in simulation 1, the MDA variable importance values and AVPI values more or less agree on which variables are more important to the response. Looking at the MDA variable importance values for simulation 2, we see that the full random forest model assigned MDA scores of around zero to variables 8 through 12, so we would be disinclined to believe in the AVPI scores for those variables. However, we see that there is some confounding between variables 1 through 4 due to the correlation structure between those variables. In this case, the AVPI properly deflates the importance of variable 4 in comparison to variables 1, 2, and 3, which is what we would have expected.

For simulation 3 we see that AVPI matches with the MDA variable importance in matching which variables are informative. However, with simulation 4, while variable 2 has a higher score than variable 4 with AVPI and MDA variable importance, both variables have similar scores with respect to AVPI and MDA variable importance. For simulation 5, the AVPI scores for variables 1, 2, 5, and 6 were highest although the uninformative predictors had relatively high AVPI scores. For simulation 6, which was the interaction terms with correlated predictors, the AVPI correctly downweights the importance of variables 3 and 4, while accounting for the importance of variables

Table 4.6: Simulation Results for Added Variable Plot Importances for Sampling without Replacement

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1	457.02513	212.352368	213.84990	110.83360	348.6451	110.453492	167.41957	89.34318
Variable 2	457.77972	214.188658	199.28552	69.19798	356.4324	105.020452	52.91816	48.21518
Variable 3	189.22857	98.922942	134.35239	129.33500	209.8724	17.446615	51.80081	59.05998
Variable 4	80.11575	5.013828	77.20821	50.00432	214.4154	-3.569032	50.94663	44.88170
Variable 5	453.78943	492.412633	131.13573	158.58779	320.5315	237.079037	97.21665	114.92600
Variable 6	443.55902	489.822741	84.87764	44.53580	334.8463	251.124465	36.56998	41.09561
Variable 7	201.05497	275.682102	66.14043	43.32284	259.2078	73.023361	47.50904	48.39331
Variable 8	68.14961	106.313016	66.00254	54.50657	198.7929	70.151875	47.72055	56.38571
Variable 9	78.26486	77.371760	92.99772	50.10961	235.0376	49.483426	51.44996	32.54249
Variable 10	68.90864	83.184212	80.70935	60.96954	206.1605	42.379239	49.53081	48.48468
Variable 11	88.57084	90.243103	79.04303	41.84311	176.4355	39.551375	49.41950	63.93243
Variable 12	80.23029	64.317099	76.45756	61.39489	195.9230	39.499604	62.14501	54.62036

Table 4.7: Simulation Results for MDA Variable Importances for Sampling without Replacement

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1	203.8974116	43.2217090	86.6974491	38.6782561	47.7410796	67.0884007	47.9127054	18.8238892
Variable 2	184.7896642	44.8711037	48.5145679	22.8629478	47.0771667	60.8648352	11.1373919	14.1609495
Variable 3	47.5925671	31.5300753	97.4263019	48.4838317	-2.4682257	19.4952466	-2.4124545	10.9352715
Variable 4	-1.5370464	23.4295635	-2.4456640	20.0982113	1.9451710	34.5067380	0.2163598	10.1516919
Variable 5	184.7704998	204.5983543	54.0176344	57.6470636	52.0030374	39.3199135	76.8657131	93.1538797
Variable 6	188.1526981	205.5292797	2.0321326	-0.6121418	48.3237316	43.7043053	0.3913965	1.8169532
Variable 7	51.9308522	51.0196417	-1.4260717	-2.1457808	-1.4728311	-1.0650397	1.8925123	-1.5395390
Variable 8	-0.1434761	-0.4393610	-0.1664074	-3.3893701	-0.3444081	0.2141649	-0.5496865	-0.6005596
Variable 9	-1.7221738	-0.1948834	0.8563617	0.7584246	2.9144231	-1.0565879	0.7799837	-0.7651692
Variable 10	-1.3463022	0.3026227	1.1876820	2.7230832	1.4493889	-2.0950498	0.2778606	2.8284927
Variable 11	0.1626804	-1.3699901	-0.3369013	0.6452085	1.8235057	-2.8213151	-1.2349051	-1.4908094
Variable 12	-0.5193238	2.0094494	-1.0121098	0.4916756	3.0198114	1.3935243	-0.1840246	1.7284960

1 and 2. With simulation 7 and 8, which was a highly non-linear response, there was small signal with respect to variable 2, so both AVPI and MDA importance computed low scores for variable 2. In particular, variable 2 in simulation 8 for AVPI has a particularly low score. However, for variables 1 and 5 have high AVPI scores as expected.

Overall, simulation results indicate that AVPI can determine the informativeness of predictors with fairly strong signal, although in general MDA variable importance will offer similar results without the addition of uninformative variables receiving non-zero scores. In some situations with correlated predictors, the AVPI appears to find which of the correlated predictors are informative, so long as the signal in the informative predictor is relatively strong. This indicates that AVPI may be most effective as a tool employed alongside MDA variable importance when there is some correlation structure present in the predictors. In general, MDA variable importance will be able to evaluate the importance of independent predictors, and if there is some correlation structure that leads to unstable MDA variable importance values of the correlated predictors, then AVPI can be used on just the correlated predictors to determine which of the correlated predictors are important.

One issue with the AVPI method is the relative noisiness of the method. As

Table 4.8: Added Variable Importance P-values for Sampling with Replacement

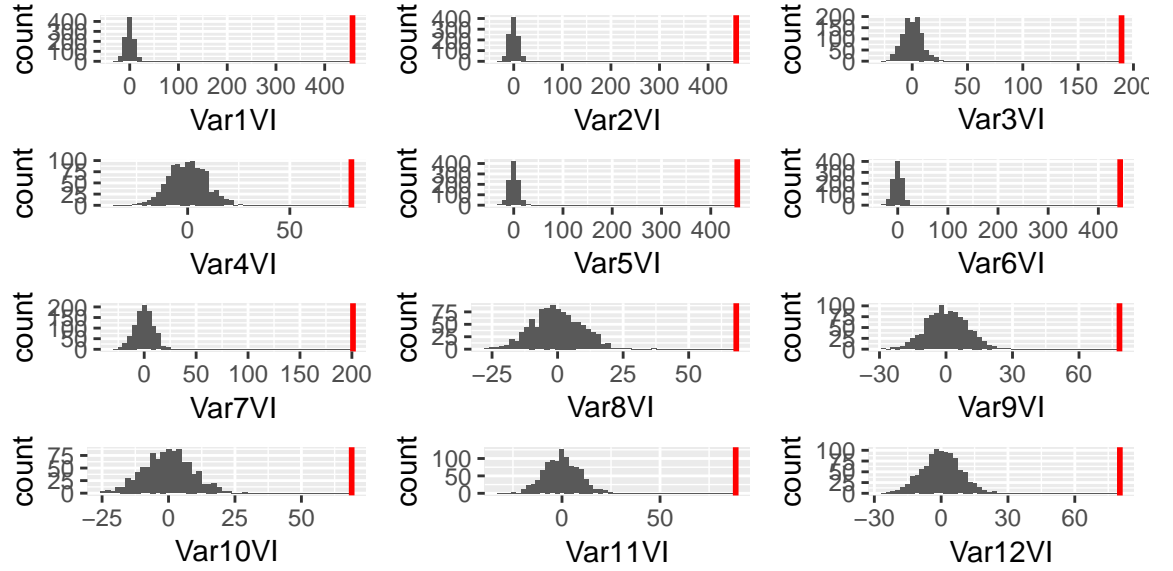
	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 2 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 3 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.6573427	0.001998	0.001998
Variable 4 P-Value	0.001998	0.1398601	0.001998	0.001998	0.001998	0.0059940	0.001998	0.001998
Variable 5 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 6 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 7 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 8 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 9 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0279720	0.001998	0.001998
Variable 10 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0219780	0.001998	0.001998
Variable 11 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0439560	0.001998	0.001998
Variable 12 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998

Table 4.9: Added Variable Importance P-values for Sampling without Replacement

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 2 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 3 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.6573427	0.001998	0.001998
Variable 4 P-Value	0.001998	0.1398601	0.001998	0.001998	0.001998	0.0059940	0.001998	0.001998
Variable 5 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 6 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 7 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 8 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998
Variable 9 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0279720	0.001998	0.001998
Variable 10 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0219780	0.001998	0.001998
Variable 11 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0439560	0.001998	0.001998
Variable 12 P-Value	0.001998	0.0019980	0.001998	0.001998	0.001998	0.0019980	0.001998	0.001998

observed in the simulation using sampling with replacement, uninformative variables can be assigned positive AVPI scores. Looking at results from re-running the simulation using sampling without replacement, we see that the same sort-of noisiness is also present in AVPI computed via random forests using sampling without replacement. Therefore the issue of noisiness for the AVPI method is not simply a matter of sampling with replacement versus sampling without replacement. The particular issue with the noisiness of the AVPI method is that it prevents us from readily employing permutation tests to determine importance of variables using AVPI as a test statistic in a hypothesis testing framework. In particular, looking at the tables of AVPI p-values for both sampling with replacement and sampling without replacement, we see that both tables are essentially identical, and that under a hypothesis testing framework, we would be committing type I errors across all simulations using a significance level of  $\alpha = 0.05$ . Furthermore, looking at the simulated null distribution of AVPI values, we see that the null distributions are generally quite similar. The simulated null distribution of AVPI values is generally symmetric with the lower end of the tails at around an importance score of 25. So any AVPI score above 30 would likely appear statistically significant using a permutation test with AVPI as the test statistic.





Thus while the AVPI may not be a stable test statistic for use in a hypothesis testing framework, our simulation results suggest that when used in conjunction with the MDA variable importance, the AVPI can be a useful tool for dealing with correlated predictors in a regression setting.



# Chapter 5

## Joint Added Variable Plot Importance

### 5.1 Introduction

In the previous chapters, we introduced the added variable plot importance (AVPI) of a predictor in the random forest ensemble. One direction we can extend our framework of added variable plot importances is to compute the joint added variable plot importance for a set of predictors. We would like to capture the joint effect of sets of predictors on the performance of the random forest ensemble in predicting the response. One scenario in which we might want to capture the joint effect of sets of predictors is if out of a set of correlated predictors, we believe that only a subset of the correlated predictors are informative to the response. In such a scenario, we would like a quantitative measure to compare the joint importance of the subset of correlated informative predictors with the subset of correlated uninformative predictors.

### 5.2 Partial F-Test

In the linear regression context, we may be interested in if a subset of predictors are informative or not towards the linear regression model. If our covariates are  $\mathbf{X}$ , then we would partition the covariates to  $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ , where  $\mathbf{X}_1$  are the first  $p - q$  predictors and  $\mathbf{X}_2$  are the last  $q$  predictors. Similarly partition the coefficients from the linear model  $\beta = (\beta_1, \beta_2)$ , where  $\beta_1$  are the coefficients for the first  $p - q$  predictors and  $\beta_2$  are the coefficients for the last  $q$  predictors. Then we would like to test whether  $\beta_2$  should be non-zero for  $\mathbf{X}_2$ . We are testing the hypotheses

$$\begin{aligned} H_0 : \mathbf{Y} &= \mathbf{X}_1\beta_1 + \varepsilon \\ H_1 : \mathbf{Y} &= \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon. \end{aligned}$$

The partial F-test allows us to test the above hypotheses. First fit the model under the null hypothesis  $H_0$  and find the residual sums of squares  $RSS_{H_0}$  and degrees of freedom  $df_{H_0}$  of the model. Next fit the model under the alternative hypothesis  $H_1$  and find

$RSS_{H_1}$  and  $df_{H_1}$ . As Weisberg (2005) notes,  $df_{H_0} > df_{H_1}$  and  $RSS_{H_0} - RSS_{H_1} > 0$ . The partial F-test statistic is

$$F = \frac{(RSS_{H_0} - RSS_{H_1}) / (df_{H_0} - df_{H_1})}{RSS_{H_1} / df_{H_1}}.$$

If  $F$  is large when compared to the  $F(df_{H_0} - df_{H_1}, df_{H_1})$  distribution, then there is evidence against the null hypothesis that the coefficients of  $\beta_2$  should be set to zero. In particular, while we cannot compute a test statistic similar to the F-test statistic for joint added variable importance, we can take the approach of comparing models fit with and without  $\mathbf{X}_2$  to determine the importance of  $\mathbf{X}_2$  in the random forest context.

### 5.3 Joint Added Variable Plots

Suppose that out of a particular set of correlated predictors, we believe that two,  $X_j$  and  $X_k$ , are particularly informative to the response while the other correlated predictors are uninformative. Denote the set of correlated predictors by  $H = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$  of which  $K = \{X_j, X_k\}$  is a subset. One approach to determining the relative importance of the variables in  $H$  is to plot added variable plots and compute added variable plot importances for each variable in  $H$ . However, if based on looking at the full random forest model variable importance, or based on some other information, we think that the variables in  $K$  are more important, we could instead train a random forest excluding the variables in  $K$ , a random forest excluding the variables in  $H \setminus K$ , and a random forest excluding the variables in  $H$ . From there we can compute and compare added variable plots and added variable plot importances for predictors in the set  $K$ , the set  $H \setminus K$ , and the set  $H$ , respectively, as we would in the single variable added variable plot scenario.

More generally, suppose we have a subset  $J = \{X_{\alpha_1}, \dots, X_{\alpha_m}\} \subseteq \{X_1, \dots, X_p\}$ , where  $m < p$ , for which we are interested in the joint added variable effect of the predictors in  $J$ . As in the case of the added variable plot of a single variable, we could grow the full random forest  $\hat{\theta}_{RF}(Y|X)$  and the reduced model  $\hat{\theta}_{RF}(Y|X_{-J})$ , where  $X_{-J}$  denotes the set of predictors not in  $J$ . The joint added variable plot for the predictors in  $J$  is then given by

$$(\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-J}), Y - \hat{\theta}_{RF}(Y|X_{-J})).$$

The rationale for forming the joint added variable plot of the predictors in  $J$  is similar to the single variable case, although care has to be taken in describing what sort-of relationship the joint added variable plot of  $J$  provides. The joint added variable plot for  $J$  captures the aggregated informativeness of the predictors in  $J$  with respect to the response. In particular, if at least one predictor in  $J$  is informative, then we would expect that the random forest ensemble  $\hat{\theta}_{RF}(Y|X_{-J})$  would have a decrease in predictive performance in comparison to the full model. If many predictors in  $J$  are informative, then we would expect that, correspondingly there would be a large decrease in predictive performance in  $\hat{\theta}_{RF}(Y|X_{-J})$ . At each node split, as the predictors in  $J$

are unavailable to be chosen as one of the  $m_{try}$  variables, there is a greater chance for an uninformative variable to be chosen as the splitting variable. On the other hand, if no variable in  $J$  is informative, then we would expect  $\hat{\theta}_{RF}(Y|X_{-J})$  to have similar predictive performance to the full model  $\hat{\theta}_{RF}(Y|X)$ . Hence for reasons similar to the single added variable plots, if at least one variable in  $J$  is an informative predictor of the response, the trend of the joint added variable plot will be strongly non-zero, while if no predictor in  $J$  is informative, we would expect the joint added variable plot to be centered about the origin and have no significant trend. If there are a mixture of informative and uninformative variables contained in  $J$ , then the joint added variable plot of  $J$  would have a non-zero trend although the magnitude and direction of the trend would depend on the composition of the variables in  $J$ .

Depending on the number of predictors in  $J$  relative to number of predictors not in  $J$ , computing the joint added variable plot for  $J$  can be quicker than computing the individual added variable plots of each variable in  $J$ . Once the variables in  $J$  are removed, there are fewer potential splitting variables for the tree growing algorithm to search through at each split on each node. We would also recommend when computing the joint added variable plot of  $J$ , to also compute the joint added variable plot of the predictors not in  $J$ , that is to also compute the random forest  $\hat{\theta}_{RF}(Y|X_J)$ . Doing so allows us to compare the predictive value of  $\hat{\theta}_{RF}(Y|X_{-J})$  and  $\hat{\theta}_{RF}(Y|X_J)$ , while allowing us to also compare the added variable effect of  $J$  versus the complement of  $J$ .

[Have examples here. One of correlated predictors, another of uncorrelated predictors]

## 5.4 Joint Added Variable Plot Importance

As with the single variable case, once we have acquired the joint added variable plot for  $J$ , we would like to have a quantitative measure of the joint importance of  $J$ . As with added variable plot importance, once we have acquired the joint added variable plot of  $J$ ,

$$(\hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-J}), Y - \hat{\theta}_{RF}(Y|X_{-J})),$$

we can try to model the trend in the joint added variable plot utilizing a bagged forest. Let  $U_J = \hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-J})$  and  $W_J = Y - \hat{\theta}_{RF}(Y|X_{-J})$ . To measure, the joint added variable importance, grow the bagged forest  $\hat{\theta}_{BF}(W_J|U_J)$  and compute the MDA variable importance of  $U_J$  in predicting  $W_J$ . We call the MDA variable importance of  $U_J$  the joint added variable importance (JAVPI) of  $J$  and denote the quantity by  $VI_{JAVP}(X_J)$ . Note that the joint added variable plot importance of  $J$  provides a quantitative measure of the relative importance of  $U_J$  in predicting the trend in the added variable plot. As mentioned previously, the trend in the added variable plot of  $J$  should reflect the aggregated informativeness of the variables in  $J$  with respect to the response, so the JAVPI of  $J$  indicates the relative aggregated importance of the variables in  $J$  as predictors of the response. In particular, higher values of JAVPI for  $J$  should indicate that the variables contained in  $J$  are more informative than lower values of JAVPI for  $J$ . That is, the value of JAVPI measures the importance of all the predictors in  $J$  to the response  $Y$  via the loss in predictive

performance in the ensemble when we remove the predictors in  $J$  as potential splitting variables in the forest growing process. As in the previous section, we also recommend that  $VI_{JAVP}(X_{-J})$ , the JAVPI of those predictors not in  $J$  be concurrently computed to provide a comparison of joint importance between sets of predictors.

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**Algorithm 9** Joint Added Variable Plot Importance (JAVPI)

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- 1: Grow the random forest ensemble  $\hat{\theta}_{RF}(Y|X)$  predicting  $Y$  using the full set of predictors.
  - 2: Grow the random forest ensemble  $\hat{\theta}_{RF}(Y|X_{-J})$  predicting  $Y$  using the full set of predictors minus the predictors in  $J$ .
  - 3: Compute  $U_J = \hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-J})$  and  $W_J = Y - \hat{\theta}_{RF}(Y|X_{-J})$ .
  - 4: Grow the bagged forest ensemble  $\hat{\theta}_{BF}(W_J|U_J)$  predicting  $W_J$  using  $U_J$ .
  - 5: Compute the added variable plot importance of  $X_J$  to be the MDA variable importance of  $U_J$ :  $VI_{JAVP}(X_J) = VI_{MDA}(U_J)$ .
- 

## 5.5 Permutation Tests Using Joint Added Variable Plot Importance

Once we have computed  $VI_{JAVP}(X_J)$ , we can again take a simulation approach to generating a sampling distribution for  $VI_{JAVP}(X_J)$ . The process is essentially analogous to computing sampling distributions of single added variable plots. To compute  $VI_{JAVP}(X_J)$ , we require the joint added variable plot  $(U_J, W_J)$ , where  $U_J = \hat{\theta}_{RF}(Y|X) - \hat{\theta}_{RF}(Y|X_{-J})$  and  $W_J = Y - \hat{\theta}_{RF}(Y|X_{-J})$ . Permute  $U_J$  to obtain  $U_J^*$  and grow the bagged forest  $\hat{\theta}_{BF}(W_J|U_J^*)$  using the permuted  $U_J^*$  as the predictor. We then use the resulting ensemble to compute  $VI_{MDA}(U_J^*) = VI_{JAVP}^*(X_J)$  as the permuted JAVPI of  $J$ . Once we have ran enough iterations of permutations of  $U_J^*$  and computation  $VI_{JAVP}^*(X_J)$ , we can compute a two-sided p-value using  $VI_{JAVP}(X_J)$  as our test statistic.

If we have a collection  $J_1, \dots, J_q$  of subsets of the predictors, we could simulate sampling distributions of  $VI_{JAVP}(X_{J_i})$  for each  $i = 1, \dots, q$  and compute two-sided p-values for each subset  $J_i$ . We could then enter into a hypothesis testing framework. We do offer some words of caution here with respect to using joint added variable plot importances in a hypothesis test framework with multiple subsets. In particular, due to sensitivity of permutation tests using JAVPI to type-I errors and due to possible overlap of predictors in the  $J_i$ , we would recommend most certainly using a multiple comparisons procedure such as the Bonferonni correction. Furthermore, we would like to emphasize the use of JAVPI and AVPI as variable selection tools and diagnostic tools for random forests, rather than tools purely of inferential statistics and hypothesis testing.

With the JAVPI, in particular, our intention in introducing the JAVPI variable importance measure was to offer a method that utilizes the random forest mechanism to measure the importance of a subset  $J$  of predictors while taking into account the

Table 5.1: Partitions for Simulation Runs 1 and 2, respectively

	Partition of Predictors	Partition of Predictors
Scenario 1	V1, V2, V3, V5, V6, V7	V1, V2, V3
Scenario 2	V1, V2, V3, V5	V1, V2
Scenario 3	V1, V2, V5, V6	V1, V2
Scenario 4	V1, V2, V5	V1, V2

effect of the predictors not in  $J$ . However, as we have noted above, if the subset  $J$  of predictors consists of a mixture of informative and uninformative predictors, then a finer analysis of the variables in  $J$  may be required if the JAVPI of  $J$  is high. Concluding that each variable in  $J$  is informative based on a high JAVPI value or low p-value from simulation would be erroneous if  $J$  is a mixture of informative and uninformative. We would also like to emphasize that it is important to check the fit and residual sum of squares of the random forest model if using the JAVPI or AVPI to see if conclusions drawn from JAVPI or AVPI are valid. Furthermore, we would like to remind the reader that inference based on residuals as we are describing with JAVPI and AVPI can be sensitive to outliers and extreme points such that statistically significant results from JAVPI or AVPI should be examined with some scrutiny.

## 5.6 Simulation and Results of Joint Added Variable Plot Importance

### 5.6.1 Simulation Design

For our simulation of JAVPI, we used our simulation set-up from chapter 4 with some changes. That is we ran the 8 simulations from chapter 4 where we have independent and correlated predictors for each scenario. Rather than test both sampling with replacement and sampling without replacement for JAVPI, we chose to run our simulation on JAVPI with just sampling without replacement. In addition, we ran two sets of simulations for JAVPI to illustrate different features of the method. The first simulation was a simulation where we partitioned out all of the informative predictors to the response, leaving no signal in the remaining data. For the second simulation, we partitioned out our predictors such that both partitions of the set of predictors contained informative predictors. In the tables below we show the partitions of our simulated data sets for both the first round and second round of simulations. All other settings for the simulations are the same as those for chapter 4: we drew 2000 samples and grew 1000 tree ensembles for both stages of JAVPI.

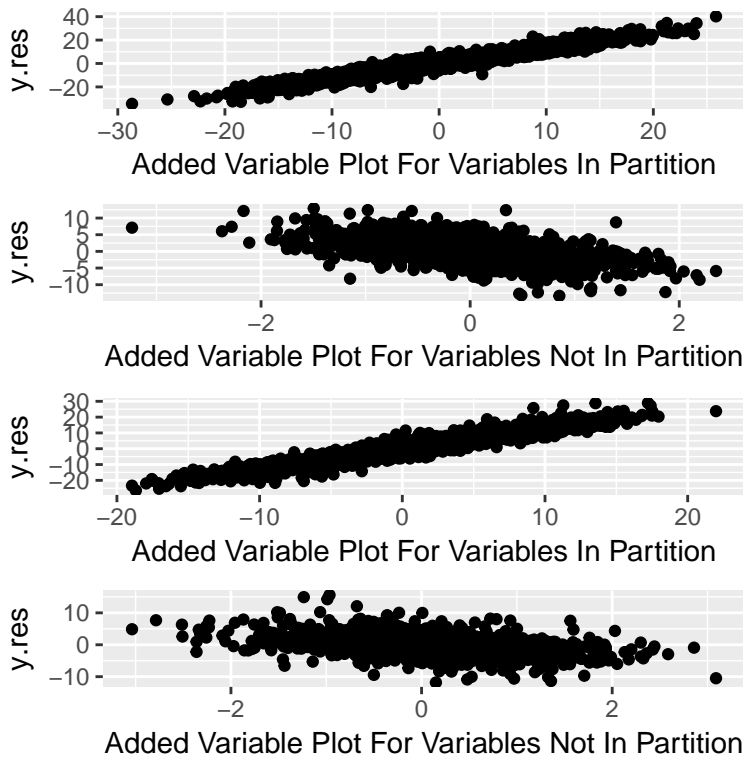
Table 5.2: JAVPI values from first simulation run

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
JAVPI of Important Variables	568.2903	569.6116	235.87012	134.14193	404.18337	300.9952	201.1710904	149.257206
JAVPI of Unimportant Variables	211.0352	111.8445	44.38363	-10.39641	49.85753	142.9379	-0.0920535	-5.136049

## 5.7 Simulation Results

Below are simulation results of JAVPI for the first simulation run. We find that, in general, JAVPI is relatively stable at distinguishing between the set of predictors containing all the signal and the set of predictors that are noise. In particular, in all cases the JAVPI values are much higher for the set of informative predictors than for the set of uninformative predictors, which is what we expected. We do note that in simulations 1, 2, and 6, the JAVPI value for the uninformative predictors is quite high. Some possible reasons for this are monte carlo variability and relatively weak signal in the predictors allowing for some masking between variables.

We examine the joint added variable plots for simulations 1, then simulation 2.

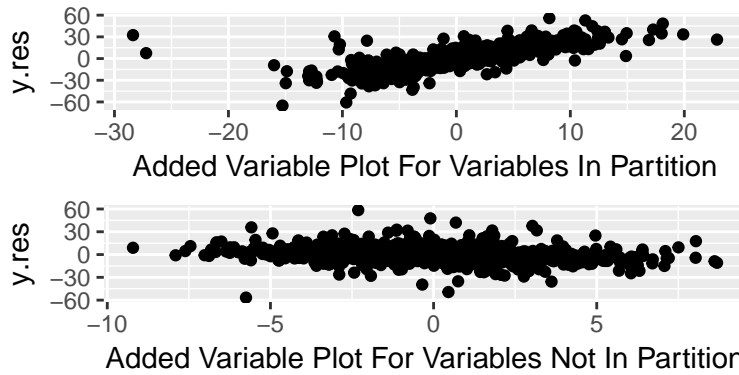


We see that there is a weak negative trend among the uninformative variables. This structure is somewhat hard to explain, although it is most likely due to noise or masking among variables. We do note that the informative predictors have a strongly positive trend in contrast to the uninformative predictors. This is in contrast to the joint added variable plots for simulations 5 below.



Table 5.3: MDA values from simulation

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
Variable 1	188.1888678	47.1926527	100.8388218	45.8739444	49.8330100	60.3014150	97.4525023	35.5213870
Variable 2	183.9331088	42.3528994	57.0499686	20.1793932	46.0267772	62.1985986	17.1483163	20.0409958
Variable 3	47.8805940	31.5347670	122.3218877	45.5134733	0.8178339	28.6188130	1.7064178	12.8387238
Variable 4	0.4348930	22.5308924	-3.6214354	21.7646631	-4.4166351	29.7790428	3.5294348	15.8408558
Variable 5	182.6877989	204.0699535	74.4365140	34.1681806	45.6762506	47.8119596	114.5667087	112.4815954
Variable 6	189.2256503	220.9777395	-0.7850783	-0.9356090	46.6726578	41.6750750	-2.5949381	0.3172609
Variable 7	50.7460484	54.7815450	-1.0197109	-0.8578919	1.6056351	-3.9313599	-1.5027717	1.5332838
Variable 8	-0.8234221	-0.4184271	-1.3429475	4.3664364	-1.1567691	-1.2540680	4.0916913	-1.2552874
Variable 9	-0.1655688	2.2750281	0.0633036	-0.8092386	-0.4823417	-2.7060695	-1.9964930	-0.2097589
Variable 10	-0.3009937	-0.0311815	0.8882865	-1.6703332	2.3257047	-0.4336373	-0.2211797	0.6841876
Variable 11	-2.0581010	-0.6666150	3.6348060	-1.7513438	-0.7229380	-2.9624221	0.4981721	0.6507235
Variable 12	0.4734702	-1.8529851	-1.0845957	-0.0398239	1.1590781	0.0759028	-0.4632472	4.1315394

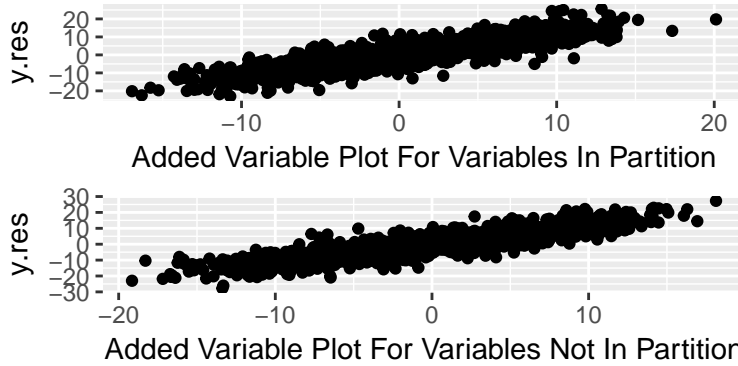


For simulation 5, the uninformative variables had no trend while the informative variables had a strong positive trend. This at least indicates that the JAVPI score captures the relative structure present in the joint added variable plot. For lower values of JAVPI, we do not expect there to be much structure in the joint added variable plot, indicating uninformative variables. Higher values of JAVPI indicate that there is structure in data. In the case of higher JAVPI scores, combining the JAVPI score with the joint added variable plot should allow us to figure out where the signal is present. For example, with simulations 1 and 2, the trend of the joint added variable plot for the informative variables is much stronger than the trend for the uninformative variables. Furthermore, the JAVPI score for the informative variables is much higher than the JAVPI scores for the uninformative variables. Some further testing, such as examining the MDA variable importance for the full set of predictors may then allow us to conclude that the set of uninformative variables are truly uninformative. If the set of uninformative variables is not too large, we could also try computing the JAVPI for each predictor in the set of uninformative variables individually. If there is some signal in those predictors, then we would expect there to be some non-zero trend in the resulting joint added variable plot, otherwise, there would be no trend in the joint added variable plot.

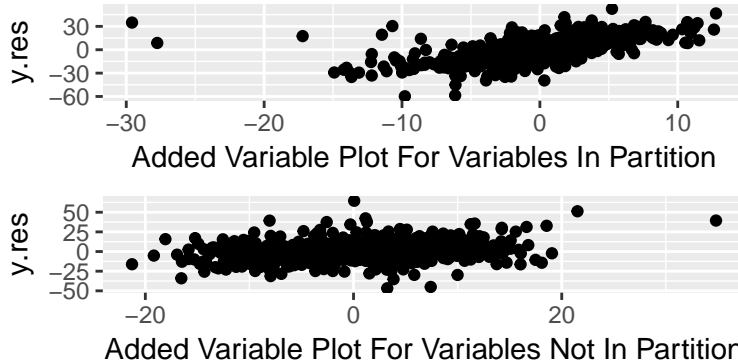
For the second simulation run, where there was signal in both partitions of the predictors, we find that, in general, the JAVPI scores for both partitions in each simulation is high, as we expected. When we include signal in both partitions, the JAVPI will generally pick up on the signal, and we can examine the joint added

variable plots for visual confirmation of what is going on. In particular, we display the joint added variable plot for simulation 1.

	Simulation 1	Simulation 2	Simulation 3	Simulation 4	Simulation 5	Simulation 6	Simulation 7	Simulation 8
JAVPI of Predictors in Partition	512.2771	419.6027	137.0338	116.9519	290.84010	193.3554	170.3404	133.4510
JAVPI of Predictors Not in Partition	492.8947	548.7029	190.8555	189.6105	67.05666	196.4716	167.9322	159.1351



For simulation 1, we found the JAVPI scores for both partitions is quite similar (around 512 versus 492), which makes sense given that we split the informative predictors evenly between both predictors. The joint added variable plot for simulation 1 is a strong positive trend as expected given the size of the JAVPI scores.



On the other hand, examining the table of JAVPI values for the second simulation run, it seems that in simulation 5, the second partition had a much lower JAVPI score than the first partition. Examining the joint added variable plots for simulation 5 immediately above, we see that while the joint added variable plot for the first partition is strongly positive, we can only describe the trend in the joint added variable plot for the second partition as being weakly positive. The JAVPI score for the second partition is high enough to suggest that there is signal in the second partition, but is much lower than the JAVPI score for the first partition. This seems very likely to be due to monte carlo variability. Choosing a different seed and re-running simulation 5 results in a appropriately high JAVPI value for the second partition, which is what we had expected.

We conclude this chapter by noting that throughout our simulations for JAVPI, the conclusions we reached with respect to independent versus correlated predictors were quite similar. Since with JAVPI, we are interested in the aggregated effect of a set of predictors, this means that we are really testing whether or not the subset of

predictors contains a discernable signal. The particular usefulness of JAVPI is not necessarily detecting where there is a signal, but detecting where there is not a signal. If a subset of predictors produce a low JAVPI score, then it is quite possible that there is very little to no signal in that subset. Hence in this manner, we could in theory test each correlated variable in a data set to try to figure out if any correlated predictors are informative to the response. Of course, such a method can be computationally intractable and impractical for data sets with many predictors, so computing the JAVPI score of different combinations of subsets could be more efficient than testing each predictor one by one. Furthermore, the JAVPI score is more computationally efficient to compute than the AVPI score since for each JAVPI score we train 5 forest ensembles, while for AVPI, if  $p$  is the number of predictors in the data set, then to compute the AVPI scores of a data set we would have to train  $2p + 1$  many forest ensembles. This suggests that for applications to permutation tests when the AVPI and JAVPI scores are not too noisy, the JAVPI score would be more efficient to compute a permutation test for. We do note that JAVPI is sensitive to the strength of the signal in the response. If the signal is weak, then the AVPI and JAVPI methods will be sensitive to noise, but if the signal in the response is strong, the AVPI and JAVPI methods will be less sensitive to noise in the data.



# Conclusion



# Appendix A

## The First Appendix





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