# CHAPTER 6 APPLICATIONS OF DEFINITE INTEGRALS

## 6.1 VOLUMES USING CROSS-SECTIONS

1. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x} - (-\sqrt{x})\right)^2}{2} = 2x$; $a = 0$, $b = 4$;
$V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$

2. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi[(2-x^2)-x^2]^2}{4} = \frac{\pi[2(1-x^2)]^2}{4} = \pi\left(1 - 2x^2 + x^4\right)$; $a = -1$, $b = 1$;
$V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi\left(1 - 2x^2 + x^4\right)dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16\pi}{15}$

3. $A(x) = (\text{edge})^2 = \left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2 = \left(2\sqrt{1-x^2}\right)^2 = 4\left(1-x^2\right)$; $a = -1$, $b = 1$;
$V = \int_a^b A(x)\,dx = \int_{-1}^1 4(1-x^2)\,dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 8\left(1 - \frac{1}{3}\right) = \frac{16}{3}$

4. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2}{2} = \frac{\left(2\sqrt{1-x^2}\right)^2}{2} = 2\left(1-x^2\right)$; $a = -1$, $b = 1$;
$V = \int_a^b A(x)\,dx = 2\int_{-1}^1 \left(1-x^2\right)dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

5. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side})\cdot(\text{side})\cdot\left(\sin\frac{\pi}{3}\right) = \frac{1}{2}\cdot\left(2\sqrt{\sin x}\right)\cdot\left(2\sqrt{\sin x}\right)\left(\sin\frac{\pi}{3}\right) = \sqrt{3}\sin x$
STEP 2) $a = 0$, $b = \pi$
STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3}\int_0^\pi \sin x\,dx = \left[-\sqrt{3}\cos x\right]_0^\pi = \sqrt{3}(1+1) = 2\sqrt{3}$
(b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4\sin x$
STEP 2) $a = 0$, $b = \pi$
STEP 3) $V = \int_a^b A(x)\,dx = \int_0^\pi 4\sin x\,dx = \left[-4\cos x\right]_0^\pi = 8$

6. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}\left(\sec^2 x + \tan^2 x - 2\sec x\tan x\right)$
$= \frac{\pi}{4}\left[\sec^2 x + \left(\sec^2 x - 1\right) - 2\frac{\sin x}{\cos^2 x}\right]$
STEP 2) $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$
STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = \frac{\pi}{4}\left[2\tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$
$= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$
(b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2\sec^2 x - 1 - 2\frac{\sin x}{\cos^2 x}\right)$
STEP 2) $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$
STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

7. (a) STEP 1) $A(x) = (\text{length})\cdot(\text{height}) = (6 - 3x)\cdot(10) = 60 - 30x$
STEP 2) $a = 0$, $b = 2$
STEP 3) $V = \int_a^b A(x)\,dx = \int_0^2 (60 - 30x)\,dx = \left[60x - 15x^2\right]_0^2 = (120 - 60) - 0 = 60$

(b) STEP 1) $A(x) = (\text{length}) \cdot (\text{height}) = (6-3x) \cdot \left(\frac{20-2(6-3x)}{2}\right) = (6-3x)(4+3x) = 24+6x-9x^2$

STEP 2) $a = 0, b = 2$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^2 (24+6x-9x^2)dx = [24x+3x^2-3x^3]_0^2 = (48+12-24)-0 = 36$

8. (a) STEP 1) $A(x) = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \left(\sqrt{x}-\frac{x}{2}\right) \cdot (6) = 6\sqrt{x}-3x$

STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^4 \left(6x^{1/2}-3x\right) dx = \left[4x^{3/2}-\frac{3}{2}x^2\right]_0^4 = (32-24)-0 = 8$

(b) STEP 1) $A(x) = \frac{1}{2} \cdot \pi\left(\frac{\text{diameter}}{2}\right)^2 = \frac{1}{2} \cdot \pi\left(\frac{\sqrt{x}-\frac{x}{2}}{2}\right)^2 = \frac{\pi}{2} \cdot \frac{x-x^{3/2}+\frac{1}{4}x^2}{4} = \frac{\pi}{8}\left(x - x^{3/2} + \frac{1}{4}x^2\right)$

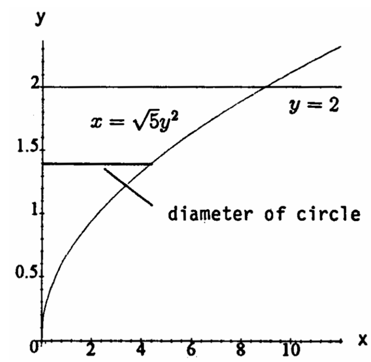
STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \frac{\pi}{8}\int_0^4 \left(x - x^{3/2} + \frac{1}{4}x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{12}x^3\right]_0^4 = \frac{\pi}{8}\left(8-\frac{64}{5}+\frac{16}{3}\right) - \frac{\pi}{8}(0) = \frac{\pi}{15}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2-0\right)^2 = \frac{5\pi}{4}y^4$;

$c = 0, d = 2; V = \int_c^d A(y)\,dy = \int_0^2 \frac{5\pi}{4}y^4\,dy$

$= \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}(2^5-0) = 8\pi$




10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2}-\left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right)$; $c = -1, d = 1$;

$V = \int_c^d A(y)\,dy = \int_{-1}^1 2(1-y^2)\,dy = 2\left[y-\frac{y^3}{3}\right]_{-1}^1 = 4\left(1-\frac{1}{3}\right) = \frac{8}{3}$

11. The slices perpendicular to the edge labeled 5 are triangles, and by similar triangles we have $\frac{b}{h} = \frac{4}{3} \Rightarrow h = \frac{3}{4}b$. The equation of the line through $(5, 0)$ and $(0, 4)$ is $y = -\frac{4}{5}x+4$, thus the length of the base $= -\frac{4}{5}x+4$ and the height $= \frac{3}{4}\left(-\frac{4}{5}x+4\right) = -\frac{3}{5}x+3$. Thus $A(x) = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \frac{1}{2}\left(-\frac{4}{5}x+4\right) \cdot \left(-\frac{3}{5}x+3\right) = \frac{6}{25}x^2 - \frac{12}{5}x + 6$ and $V = \int_a^b A(x)\,dx = \int_0^5 \left(\frac{6}{25}x^2-\frac{12}{5}x+6\right) dx = \left[\frac{2}{25}x^3-\frac{6}{5}x^2+6x\right]_0^5 = (10-30+30)-0 = 10$

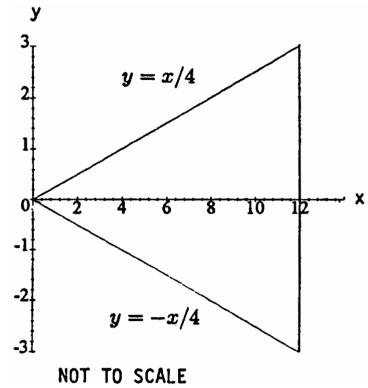
12. The slices parallel to the base are squares. The cross section of the pyramid is a triangle, and by similar triangles we have $\frac{b}{h} = \frac{3}{5} \Rightarrow b = \frac{3}{5}h$. Thus $A(y) = (\text{base})^2 = \left(\frac{3}{5}y\right)^2 = \frac{9}{25}y^2 \Rightarrow V = \int_c^d A(y)\,dy = \int_0^5 \frac{9}{25}y^2\,dy = \left[\frac{3}{25}y^3\right]_0^5 = 15-0 = 15$

13. (a) It follows from Cavalieri's Principle that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus, STEP 1) $A(x) = (\text{side length})^2 = s^2$;

STEP 2) $a = 0, b = h$; STEP 3) $V = \int_a^b A(x)\,dx = \int_0^h s^2\,dx = s^2h$

(b) From Cavalieri's Principle we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V = s^2h$

14. 1) The solid and the cone have the same altitude of 12.
    2) The cross sections of the solid are disks of diameter $x - \left(\frac{x}{2}\right) = \frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis of symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4} - \left(-\frac{x}{4}\right) = \frac{x}{2}$ (see accompanying figure).
    3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalieri's Principle we conclude that the solid and the cone have the same volume.



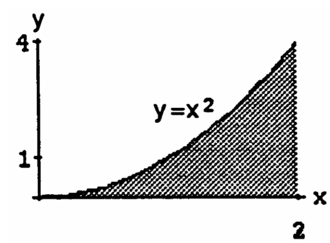

15. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$
$= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

16. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\,dy = \pi\int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi\int_0^2 \frac{9}{4}y^2\,dy = \pi\left[\frac{3}{4}y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

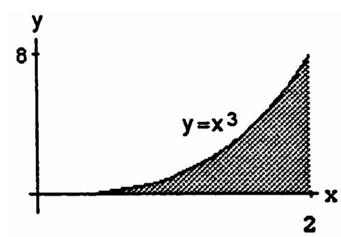
17. $R(y) = \tan\left(\frac{\pi}{4}y\right)$; $u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}dy \Rightarrow 4\,du = \pi\,dy$; $y = 0 \Rightarrow u = 0$, $y = 1 \Rightarrow u = \frac{\pi}{4}$;
$V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\,du = 4\int_0^{\pi/4}(-1 + \sec^2 u)\,du = 4[-u + \tan u]_0^{\pi/4}$
$= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

18. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration; $V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$
$= \pi\int_0^{\pi/2} (\sin x \cos x)^2\,dx = \pi\int_0^{\pi/2} \frac{(\sin 2x)^2}{4}\,dx$; $\left[u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{8} = \frac{dx}{4}; x = 0 \Rightarrow u = 0,\right.$
$\left. x = \frac{\pi}{2} \Rightarrow u = \pi\right] \rightarrow V = \pi\int_0^{\pi} \frac{1}{8}\sin^2 u\,du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

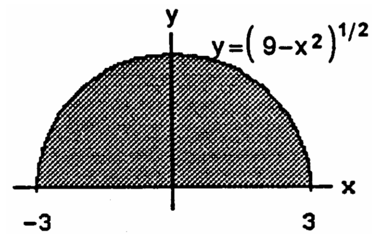
19. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 (x^2)^2\,dx$
$= \pi\int_0^2 x^4\,dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



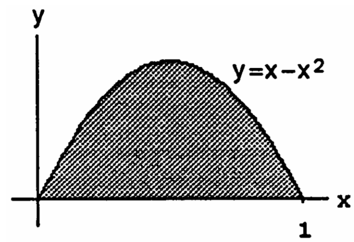

20. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 (x^3)^2\,dx$
$= \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



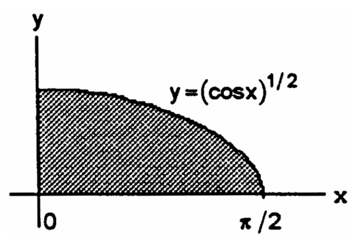
21. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^{3} \pi[R(x)]^2\,dx = \pi \int_{-3}^{3} (9 - x^2)\,dx$
$= \pi \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 2\pi\left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$



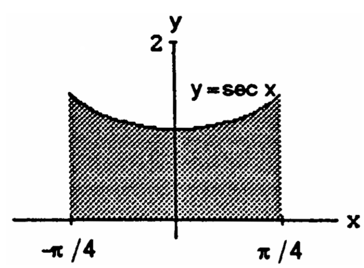

22. $R(x) = x - x^2 \Rightarrow V = \int_0^1 \pi[R(x)]^2\,dx = \pi\int_0^1 (x - x^2)^2\,dx$
$= \pi\int_0^1 (x^2 - 2x^3 + x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$
$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$



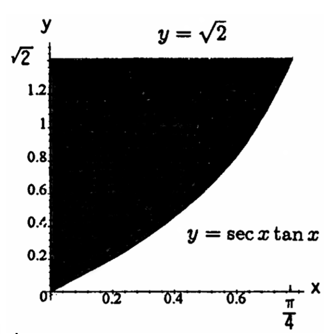

23. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx = \pi\int_0^{\pi/2} \cos x\,dx$
$= \pi\,[\sin x]_0^{\pi/2} = \pi(1 - 0) = \pi$



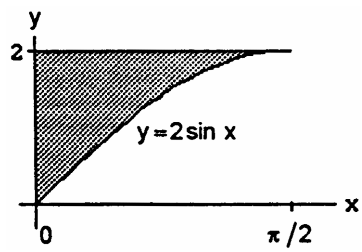

24. $R(x) = \sec x \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi[R(x)]^2\,dx = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$
$= \pi\,[\tan x]_{-\pi/4}^{\pi/4} = \pi[1 - (-1)] = 2\pi$



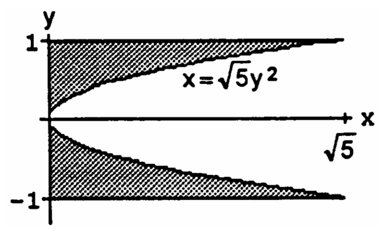

25. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi[R(x)]^2\,dx$
$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$
$= \pi\int_0^{\pi/4} \left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$
$= \pi\left(\int_0^{\pi/4} 2\,dx - 2\sqrt{2}\int_0^{\pi/4} \sec x \tan x\,dx + \int_0^{\pi/4} (\tan x)^2 \sec^2 x\,dx\right)$
$= \pi\left([2x]_0^{\pi/4} - 2\sqrt{2}\,[\sec x]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right)$
$= \pi\left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right) + \frac{1}{3}(1^3 - 0)\right] = \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$



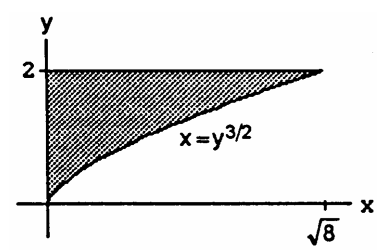
26. $R(x) = 2 - 2\sin x = 2(1 - \sin x) \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\, dx$
$= \pi \int_0^{\pi/2} 4(1 - \sin x)^2\, dx = 4\pi \int_0^{\pi/2} (1 + \sin^2 x - 2\sin x)\, dx$
$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right] dx$
$= 4\pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right)$
$= 4\pi \left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$
$= 4\pi \left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$



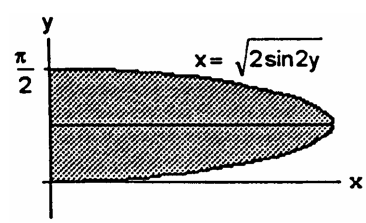

27. $R(y) = \sqrt{5}\, y^2 \Rightarrow V = \int_{-1}^{1} \pi[R(y)]^2\, dy = \pi \int_{-1}^{1} 5y^4\, dy$
$= \pi \left[y^5\right]_{-1}^{1} = \pi[1 - (-1)] = 2\pi$



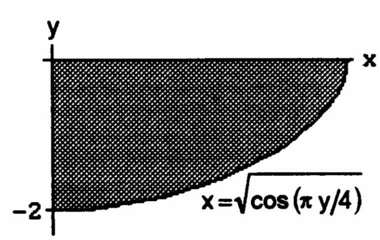

28. $R(y) = y^{3/2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\, dy = \pi \int_0^2 y^3\, dy$
$= \pi \left[\frac{y^4}{4}\right]_0^2 = 4\pi$



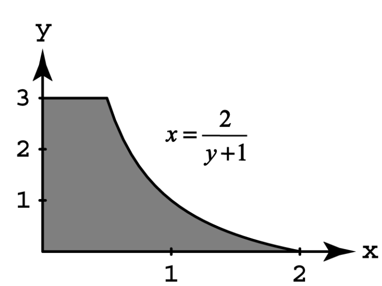

29. $R(y) = \sqrt{2\sin 2y} \Rightarrow V = \int_0^{\pi/2} \pi[R(y)]^2\, dy$
$= \pi \int_0^{\pi/2} 2\sin 2y\, dy = \pi\left[-\cos 2y\right]_0^{\pi/2}$
$= \pi[1 - (-1)] = 2\pi$



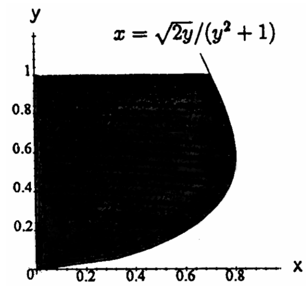

30. $R(y) = \sqrt{\cos \frac{\pi y}{4}} \Rightarrow V = \int_{-2}^{0} \pi[R(y)]^2\, dy$
$= \pi \int_{-2}^{0} \cos\left(\frac{\pi y}{4}\right) dy = 4\left[\sin \frac{\pi y}{4}\right]_{-2}^{0} = 4[0 - (-1)] = 4$




31. $R(y) = \frac{2}{y+1} \Rightarrow V = \int_0^3 \pi[R(y)]^2\, dy = 4\pi \int_0^3 \frac{1}{(y+1)^2}\, dy$
$= 4\pi \left[-\frac{1}{y+1}\right]_0^3 = 4\pi\left[-\frac{1}{4} - (-1)\right] = 3\pi$



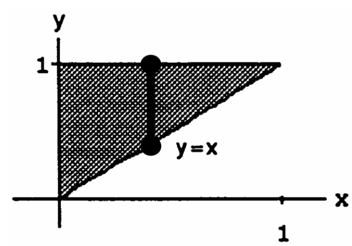
32. $R(y) = \frac{\sqrt{2y}}{y^2+1} \Rightarrow V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 2y\,(y^2+1)^{-2}\,dy$;
$[u = y^2+1 \Rightarrow du = 2y\,dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2]$
$\rightarrow V = \pi\int_1^2 u^{-2}\,du = \pi\left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2} - (-1)\right] = \frac{\pi}{2}$



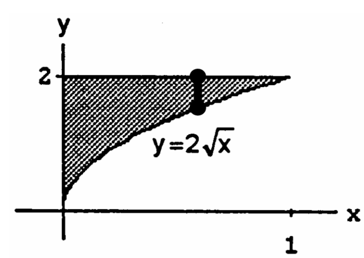

33. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)dx$
$= \int_{-\pi/2}^{\pi/2} \pi(1 - \cos x)\,dx = 2\pi\int_0^{\pi/2}(1 - \cos x)\,dx = 2\pi[x - \sin x]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$

34. For the sketch given, $c = 0$, $d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right)dy$
$= \pi\int_0^{\pi/4}(1 - \tan^2 y)\,dy = \pi\int_0^{\pi/4}(2 - \sec^2 y)\,dy = \pi[2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$

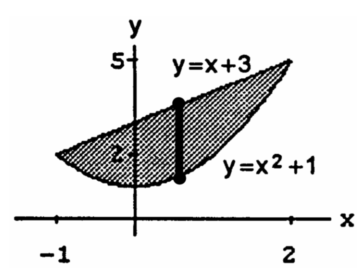
35. $r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$
$= \int_0^1 \pi\,(1 - x^2)\,dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$




36. $r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$
$= \pi\int_0^1 (4 - 4x)\,dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$




37. $r(x) = x^2 + 1$ and $R(x) = x + 3$
$\Rightarrow V = \int_{-1}^2 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$
$= \pi\int_{-1}^2 \left[(x+3)^2 - \left(x^2+1\right)^2\right]dx$
$= \pi\int_{-1}^2 \left[(x^2 + 6x + 9) - (x^4 + 2x^2 + 1)\right]dx$
$= \pi\int_{-1}^2 (-x^4 - x^2 + 6x + 8)\,dx$
$= \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^2$
$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right] = \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5\cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$



38. $r(x) = 2 - x$ and $R(x) = 4 - x^2$

$\Rightarrow V = \int_{-1}^{2} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

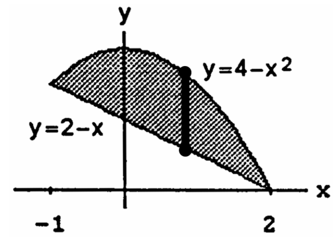
$= \pi \int_{-1}^{2} \left[(4 - x^2)^2 - (2 - x)^2\right] dx$

$= \pi \int_{-1}^{2} \left[(16 - 8x^2 + x^4) - (4 - 4x + x^2)\right] dx$

$= \pi \int_{-1}^{2} (12 + 4x - 9x^2 + x^4)\, dx$

$= \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2}$

$= \pi \left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi \left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

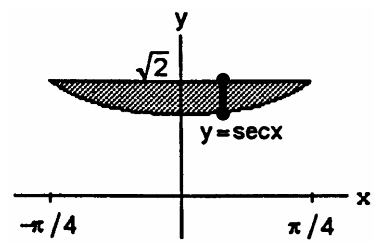



39. $r(x) = \sec x$ and $R(x) = \sqrt{2}$

$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\, dx = \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$

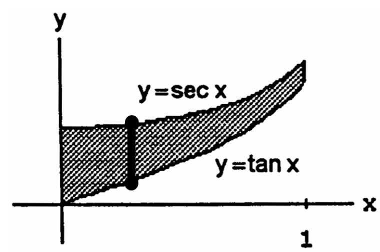
$= \pi \left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$




40. $R(x) = \sec x$ and $r(x) = \tan x$

$\Rightarrow V = \int_{0}^{1} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{0}^{1} (\sec^2 x - \tan^2 x)\, dx = \pi \int_{0}^{1} 1\, dx = \pi[x]_0^1 = \pi$

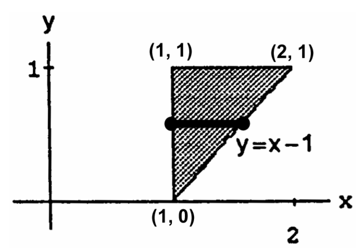



41. $r(y) = 1$ and $R(y) = 1 + y$

$\Rightarrow V = \int_{0}^{1} \pi \left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_{0}^{1} \left[(1 + y)^2 - 1\right] dy = \pi \int_{0}^{1} (1 + 2y + y^2 - 1)\, dy$

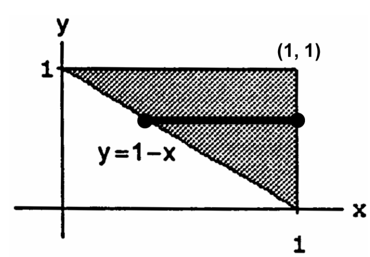
$= \pi \int_{0}^{1} (2y + y^2)\, dy = \pi \left[y^2 + \frac{y^3}{3}\right]_0^1 = \pi \left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$




42. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_{0}^{1} \pi \left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_{0}^{1} \left[1 - (1 - y)^2\right] dy = \pi \int_{0}^{1} \left[1 - (1 - 2y + y^2)\right] dy$

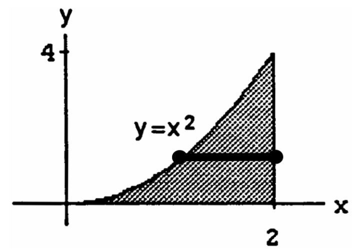
$= \pi \int_{0}^{1} (2y - y^2)\, dy = \pi \left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi \left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



43. $R(y) = 2$ and $r(y) = \sqrt{y}$

$\Rightarrow V = \int_0^4 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^4 (4 - y)\, dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$

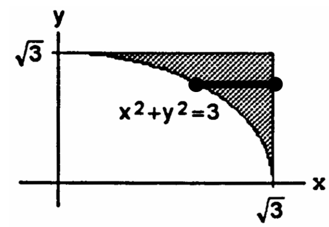



44. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$

$\Rightarrow V = \int_0^{\sqrt{3}} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^{\sqrt{3}} [3 - (3 - y^2)]\, dy = \pi \int_0^{\sqrt{3}} y^2\, dy$

$= \pi\left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$




45. $R(y) = 2$ and $r(y) = 1 + \sqrt{y}$

$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

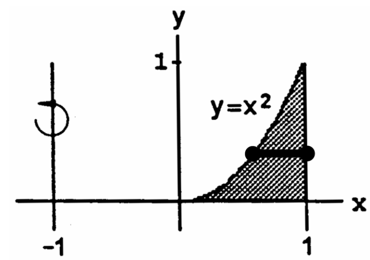
$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2\right] dy$

$= \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y\right) dy$

$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y\right) dy$

$= \pi\left[3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1$

$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18-8-3}{6}\right) = \frac{7\pi}{6}$




46. $R(y) = 2 - y^{1/3}$ and $r(y) = 1$

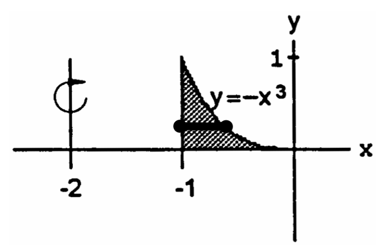
$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1\right] dy$

$= \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$

$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3}\right) dy$

$= \pi\left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1 = \pi\left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$




47. (a) $r(x) = \sqrt{x}$ and $R(x) = 2$

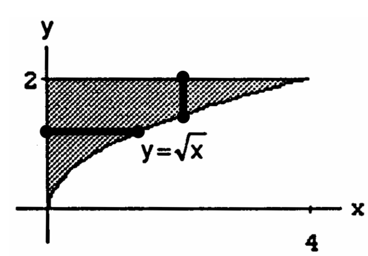
$\Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^4 (4 - x)\, dx = \pi\left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$

(b) $r(y) = 0$ and $R(y) = y^2$

$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 y^4\, dy = \pi\left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

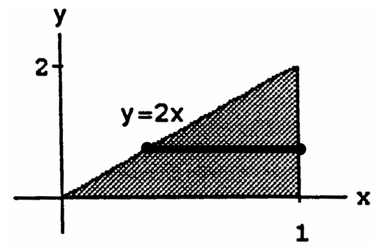



(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$

$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi\left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi\left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

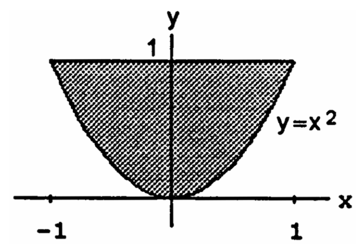
(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - (4 - y^2)^2\right] dy$
$= \pi \int_0^2 (16 - 16 + 8y^2 - y^4)\, dy = \pi \int_0^2 (8y^2 - y^4)\, dy = \pi \left[\frac{8}{3}y^3 - \frac{y^5}{5}\right]_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

48. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2}$
$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$
$= \pi \left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$




(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2}$
$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy$
$= \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi \left[3y - y^2 + \frac{y^3}{12}\right]_0^2 = \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$

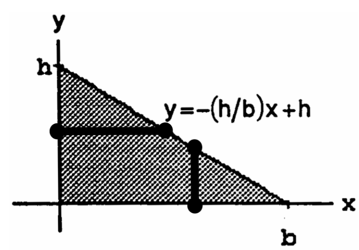
49. (a) $r(x) = 0$ and $R(x) = 1 - x^2$
$\Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \pi \int_{-1}^{1} (1 - x^2)^2\, dx = \pi \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx$
$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$
$= 2\pi\left(\frac{15-10+3}{15}\right) = \frac{16\pi}{15}$




(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[(2 - x^2)^2 - 1\right] dx$
$= \pi \int_{-1}^{1} (4 - 4x^2 + x^4 - 1)\, dx = \pi \int_{-1}^{1} (3 - 4x^2 + x^4)\, dx = \pi \left[3x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{4}{3} + \frac{1}{5}\right)$
$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[4 - (1 + x^2)^2\right] dx$
$= \pi \int_{-1}^{1} (4 - 1 - 2x^2 - x^4)\, dx = \pi \int_{-1}^{1} (3 - 2x^2 - x^4)\, dx = \pi \left[3x - \frac{2}{3}x^3 - \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{2}{3} - \frac{1}{5}\right)$
$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$

50. (a) $r(x) = 0$ and $R(x) = -\frac{h}{b}x + h$
$\Rightarrow V = \int_0^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \pi \int_0^b \left(-\frac{h}{b}x + h\right)^2 dx$
$= \pi \int_0^b \left(\frac{h^2}{b^2}x^2 - \frac{2h^2}{b}x + h^2\right) dx$
$= \pi h^2 \left[\frac{x^3}{3b^2} - \frac{x^2}{b} + x\right]_0^b = \pi h^2\left(\frac{b}{3} - b + b\right) = \frac{\pi h^2 b}{3}$




(b) $r(y) = 0$ and $R(y) = b\left(1 - \frac{y}{h}\right) \Rightarrow V = \int_0^h \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi b^2 \int_0^h \left(1 - \frac{y}{h}\right)^2 dy$
$= \pi b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy = \pi b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h = \pi b^2\left(h - h + \frac{h}{3}\right) = \frac{\pi b^2 h}{3}$

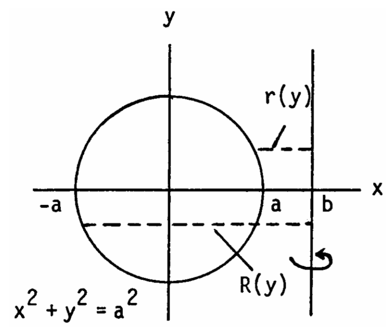
51. $R(y) = b + \sqrt{a^2 - y^2}$ and $r(y) = b - \sqrt{a^2 - y^2}$

$\Rightarrow V = \int_{-a}^{a} \pi \left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_{-a}^{a} \left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right] dy$

$= \pi \int_{-a}^{a} 4b\sqrt{a^2 - y^2}\, dy = 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy$

$= 4b\pi \cdot$ area of semicircle of radius $a = 4b\pi \cdot \frac{\pi a^2}{2} = 2a^2 b\pi^2$




52. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y dy = \pi \left[y^2\right]_0^5 = 25\pi$.

(b) $V(h) = \int A(h) dh$, so $\frac{dV}{dh} = A(h)$. Therefore $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{A(h)} \cdot \frac{dV}{dt}$.

For $h = 4$, the area is $2\pi(4) = 8\pi$, so $\frac{dh}{dt} = \frac{1}{8\pi} \cdot 3 \frac{\text{units}^3}{\text{sec}} = \frac{3}{8\pi} \cdot \frac{\text{units}^3}{\text{sec}}$.

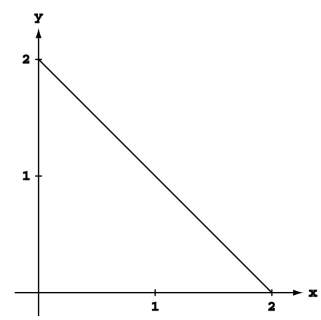
53. (a) $R(y) = \sqrt{a^2 - y^2} \Rightarrow V = \pi \int_{-a}^{h-a} (a^2 - y^2)\, dy = \pi \left[a^2 y - \frac{y^3}{3}\right]_{-a}^{h-a} = \pi \left[a^2 h - a^3 - \frac{(h-a)^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right]$

$= \pi \left[a^2 h - \frac{1}{3}\left(h^3 - 3h^2 a + 3ha^2 - a^3\right) - \frac{a^3}{3}\right] = \pi \left(a^2 h - \frac{h^3}{3} + h^2 a - ha^2\right) = \frac{\pi h^2 (3a - h)}{3}$

(b) Given $\frac{dV}{dt} = 0.2\ \text{m}^3/\text{sec}$ and $a = 5$ m, find $\left.\frac{dh}{dt}\right|_{h=4}$. From part (a), $V(h) = \frac{\pi h^2 (15 - h)}{3} = 5\pi h^2 - \frac{\pi h^3}{3}$

$\Rightarrow \frac{dV}{dh} = 10\pi h - \pi h^2 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \pi h(10 - h) \frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=4} = \frac{0.2}{4\pi(10-4)} = \frac{1}{(20\pi)(6)} = \frac{1}{120\pi}$ m/sec.

54. Suppose the solid is produced by revolving $y = 2 - x$ about the y-axis. Cast a shadow of the solid on a plane parallel to the xy-plane.
Use an approximation such as the Trapezoid Rule, to estimate $\int_a^b \pi [R(y)]^2\, dy \approx \sum_{k=1}^{n} \pi \left(\frac{d_{\hat{k}}}{2}\right)^2 \triangle y$.




55. The cross section of a solid right circular cylinder with a cone removed is a disk with radius R from which a disk of radius h has been removed. Thus its area is $A_1 = \pi R^2 - \pi h^2 = \pi\left(R^2 - h^2\right)$. The cross section of the hemisphere is a disk of radius $\sqrt{R^2 - h^2}$. Therefore its area is $A_2 = \pi \left(\sqrt{R^2 - h^2}\right)^2 = \pi\left(R^2 - h^2\right)$. We can see that $A_1 = A_2$. The altitudes of both solids are R. Applying Cavalieri's Principle we find
Volume of Hemisphere = (Volume of Cylinder) − (Volume of Cone) $= \left(\pi R^2\right) R - \frac{1}{3}\pi\left(R^2\right) R = \frac{2}{3}\pi R^3$.

56. $R(x) = \frac{x}{12}\sqrt{36 - x^2} \Rightarrow V = \int_0^6 \pi [R(x)]^2\, dx = \pi \int_0^6 \frac{x^2}{144}\left(36 - x^2\right) dx = \frac{\pi}{144} \int_0^6 \left(36x^2 - x^4\right) dx$

$= \frac{\pi}{144}\left[12x^3 - \frac{x^5}{5}\right]_0^6 = \frac{\pi}{144}\left(12 \cdot 6^3 - \frac{6^5}{5}\right) = \frac{\pi \cdot 6^3}{144}\left(12 - \frac{36}{5}\right) = \left(\frac{196\pi}{144}\right)\left(\frac{60-36}{5}\right) = \frac{36\pi}{5}\ \text{cm}^3$. The plumb bob will weigh about $W = (8.5)\left(\frac{36\pi}{5}\right) \approx 192$ gm, to the nearest gram.

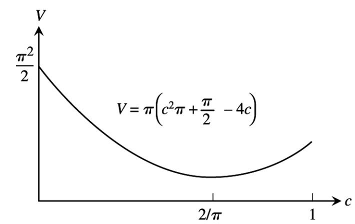
57. $R(y) = \sqrt{256 - y^2} \Rightarrow V = \int_{-16}^{-7} \pi [R(y)]^2\, dy = \pi \int_{-16}^{-7} (256 - y^2)\, dy = \pi \left[256y - \frac{y^3}{3}\right]_{-16}^{-7}$

$= \pi \left[(256)(-7) + \frac{7^3}{3} - \left((256)(-16) + \frac{16^3}{3}\right)\right] = \pi \left(\frac{7^3}{3} + 256(16 - 7) - \frac{16^3}{3}\right) = 1053\pi\ \text{cm}^3 \approx 3308\ \text{cm}^3$

58. (a) $R(x) = |c - \sin x|$, so $V = \pi\int_0^{\pi} [R(x)]^2\, dx = \pi\int_0^{\pi} (c - \sin x)^2\, dx = \pi\int_0^{\pi} (c^2 - 2c\sin x + \sin^2 x)\, dx$
$= \pi\int_0^{\pi} \left(c^2 - 2c\sin x + \frac{1-\cos 2x}{2}\right) dx = \pi\int_0^{\pi} \left(c^2 + \frac{1}{2} - 2c\sin x - \frac{\cos 2x}{2}\right) dx$
$= \pi\left[\left(c^2 + \frac{1}{2}\right)x + 2c\cos x - \frac{\sin 2x}{4}\right]_0^{\pi} = \pi\left[\left(c^2\pi + \frac{\pi}{2} - 2c - 0\right) - (0 + 2c - 0)\right] = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. Let $V(c) = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. We find the extreme values of $V(c)$: $\frac{dV}{dc} = \pi(2c\pi - 4) = 0 \Rightarrow c = \frac{2}{\pi}$ is a critical point, and $V\left(\frac{2}{\pi}\right) = \pi\left(\frac{4}{\pi} + \frac{\pi}{2} - \frac{8}{\pi}\right) = \pi\left(\frac{\pi}{2} - \frac{4}{\pi}\right) = \frac{\pi^2}{2} - 4$; Evaluate V at the endpoints: $V(0) = \frac{\pi^2}{2}$ and $V(1) = \pi\left(\frac{3}{2}\pi - 4\right) = \frac{\pi^2}{2} - (4 - \pi)\pi$. Now we see that the function's absolute minimum value is $\frac{\pi^2}{2} - 4$, taken on at the critical point $c = \frac{2}{\pi}$. (See also the accompanying graph.)

(b) From the discussion in part (a) we conclude that the function's absolute maximum value is $\frac{\pi^2}{2}$, taken on at the endpoint $c = 0$.

(c) The graph of the solid's volume as a function of c for $0 \le c \le 1$ is given at the right. As c moves away from $[0, 1]$ the volume of the solid increases without bound. If we approximate the solid as a set of solid disks, we can see that the radius of a typical disk increases without bounds as c moves away from $[0, 1]$.




59. Volume of the solid generated by rotating the region bounded by the x-axis and $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is $V = \int_a^b \pi[f(x)]^2\, dx = 4\pi$, and the volume of the solid generated by rotating the same region about the line $y = -1$ is $V = \int_a^b \pi[f(x) + 1]^2\, dx = 8\pi$. Thus $\int_a^b \pi[f(x) + 1]^2\, dx - \int_a^b \pi[f(x)]^2\, dx = 8\pi - 4\pi$
$\Rightarrow \pi\int_a^b \left([f(x)]^2 + 2f(x) + 1 - [f(x)]^2\right) dx = 4\pi \Rightarrow \int_a^b (2f(x) + 1)\, dx = 4 \Rightarrow 2\int_a^b f(x)\, dx + \int_a^b dx = 4$
$\Rightarrow \int_a^b f(x)\, dx + \frac{1}{2}(b - a) = 2 \Rightarrow \int_a^b f(x)\, dx = \frac{4 - b + a}{2}$

60. Volume of the solid generated by rotating the region bounded by the x-axis and $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is $V = \int_a^b \pi[f(x)]^2\, dx = 6\pi$, and the volume of the solid generated by rotating the same region about the line $y = -2$ is $V = \int_a^b \pi[f(x) + 2]^2\, dx = 10\pi$. Thus $\int_a^b \pi[f(x) + 2]^2\, dx - \int_a^b \pi[f(x)]^2\, dx = 10\pi - 6\pi$
$\Rightarrow \pi\int_a^b \left([f(x)]^2 + 4f(x) + 4 - [f(x)]^2\right) dx = 4\pi \Rightarrow \int_a^b (4f(x) + 4)\, dx = 4 \Rightarrow 4\int_a^b f(x)\, dx + 4\int_a^b dx = 4$
$\Rightarrow \int_a^b f(x)\, dx + (b - a) = 1 \Rightarrow \int_a^b f(x)\, dx = 1 - b + a$

## 6.2 VOLUME USING CYLINDRICAL SHELLS

1. For the sketch given, $a = 0$, $b = 2$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\left(1 + \frac{x^2}{4}\right) dx = 2\pi\int_0^2 \left(x + \frac{x^3}{4}\right) dx = 2\pi\left[\frac{x^2}{2} + \frac{x^4}{16}\right]_0^2 = 2\pi\left(\frac{4}{2} + \frac{16}{16}\right)$
$= 2\pi \cdot 3 = 6\pi$

2. For the sketch given, $a = 0$, $b = 2$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\left(2 - \frac{x^2}{4}\right) dx = 2\pi\int_0^2 \left(2x - \frac{x^3}{4}\right) dx = 2\pi\left[x^2 - \frac{x^4}{16}\right]_0^2 = 2\pi(4 - 1) = 6\pi$

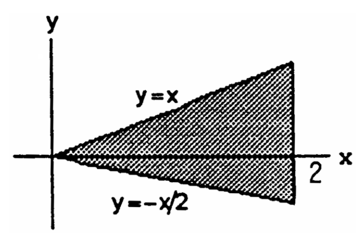
3. For the sketch given, $c = 0$, $d = \sqrt{2}$;
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^{\sqrt{2}} 2\pi y \cdot (y^2)\, dy = 2\pi\int_0^{\sqrt{2}} y^3\, dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{2}} = 2\pi$

4. For the sketch given, $c = 0$, $d = \sqrt{3}$;
$V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^{\sqrt{3}} 2\pi y \cdot [3 - (3 - y^2)]\, dy = 2\pi \int_0^{\sqrt{3}} y^3\, dy = 2\pi \left[\frac{y^4}{4}\right]_0^{\sqrt{3}} = \frac{9\pi}{2}$

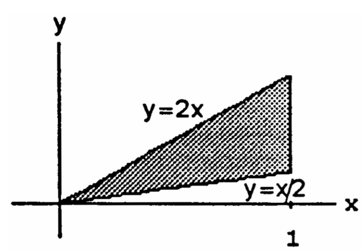
5. For the sketch given, $a = 0$, $b = \sqrt{3}$;
$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^{\sqrt{3}} 2\pi x \cdot \left(\sqrt{x^2 + 1}\right) dx$;
$\left[u = x^2 + 1 \Rightarrow du = 2x\, dx;\ x = 0 \Rightarrow u = 1,\ x = \sqrt{3} \Rightarrow u = 4\right]$
$\to V = \pi \int_1^4 u^{1/2}\, du = \pi \left[\frac{2}{3} u^{3/2}\right]_1^4 = \frac{2\pi}{3} \left(4^{3/2} - 1\right) = \left(\frac{2\pi}{3}\right)(8 - 1) = \frac{14\pi}{3}$

6. For the sketch given, $a = 0$, $b = 3$;
$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^3 2\pi x \left(\frac{9x}{\sqrt{x^3 + 9}}\right) dx$;
$[u = x^3 + 9 \Rightarrow du = 3x^2\, dx \Rightarrow 3\, du = 9x^2\, dx;\ x = 0 \Rightarrow u = 9,\ x = 3 \Rightarrow u = 36]$
$\to V = 2\pi \int_9^{36} 3u^{-1/2}\, du = 6\pi \left[2u^{1/2}\right]_9^{36} = 12\pi \left(\sqrt{36} - \sqrt{9}\right) = 36\pi$

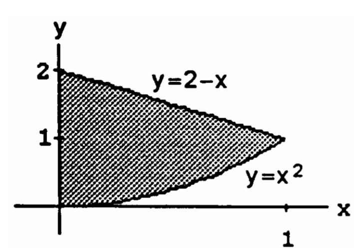
7. $a = 0$, $b = 2$;
$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi x \left[x - \left(-\frac{x}{2}\right)\right] dx$
$= \int_0^2 2\pi x^2 \cdot \frac{3}{2}\, dx = \pi \int_0^2 3x^2\, dx = \pi \left[x^3\right]_0^2 = 8\pi$



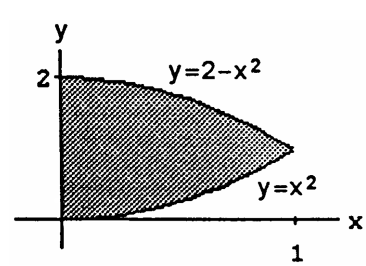

8. $a = 0$, $b = 1$;
$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^1 2\pi x \left(2x - \frac{x}{2}\right) dx$
$= \pi \int_0^1 2\left(\frac{3x^2}{2}\right) dx = \pi \int_0^1 3x^2\, dx = \pi \left[x^3\right]_0^1 = \pi$




9. $a = 0$, $b = 1$;
$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^1 2\pi x \left[(2 - x) - x^2\right] dx$
$= 2\pi \int_0^1 (2x - x^2 - x^3)\, dx = 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$
$= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi \left(\frac{12 - 4 - 3}{12}\right) = \frac{10\pi}{12} = \frac{5\pi}{6}$

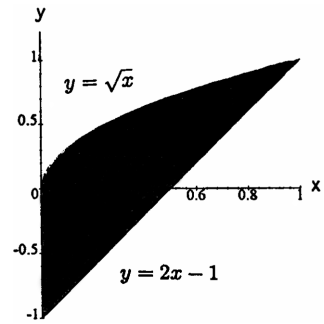



10. $a = 0$, $b = 1$;
$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right) \left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^1 2\pi x \left[(2 - x^2) - x^2\right] dx$
$= 2\pi \int_0^1 x\,(2 - 2x^2)\, dx = 4\pi \int_0^1 (x - x^3)\, dx$
$= 4\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi \left(\frac{1}{2} - \frac{1}{4}\right) = \pi$



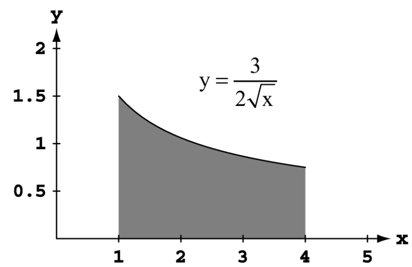
11. $a = 0, b = 1;$

$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^1 2\pi x\left[\sqrt{x} - (2x-1)\right] dx$
$= 2\pi \int_0^1 \left(x^{3/2} - 2x^2 + x\right) dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$
$= 2\pi\left(\frac{2}{5} - \frac{2}{3} + \frac{1}{2}\right) = 2\pi\left(\frac{12-20+15}{30}\right) = \frac{7\pi}{15}$




12. $a = 1, b = 4;$

$V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_1^4 2\pi x\left(\frac{3}{2}x^{-1/2}\right) dx$
$= 3\pi\int_1^4 x^{1/2}\, dx = 3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$
$= 2\pi(8-1) = 14\pi$




13. (a) $xf(x) = \begin{cases} x \cdot \frac{\sin x}{x}, & 0 < x \le \pi \\ x, & x = 0 \end{cases} \Rightarrow xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x = 0 \end{cases}$; since $\sin 0 = 0$ we have

$xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ \sin x, & x = 0 \end{cases} \Rightarrow xf(x) = \sin x,\ 0 \le x \le \pi$

(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^\pi 2\pi x \cdot f(x)\, dx$ and $x \cdot f(x) = \sin x,\ 0 \le x \le \pi$ by part (a)

$\Rightarrow V = 2\pi\int_0^\pi \sin x\, dx = 2\pi[-\cos x]_0^\pi = 2\pi(-\cos\pi + \cos 0) = 4\pi$

14. (a) $xg(x) = \begin{cases} x \cdot \frac{\tan^2 x}{x}, & 0 < x \le \frac{\pi}{4} \\ x \cdot 0, & x = 0 \end{cases} \Rightarrow xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$; since $\tan 0 = 0$ we have

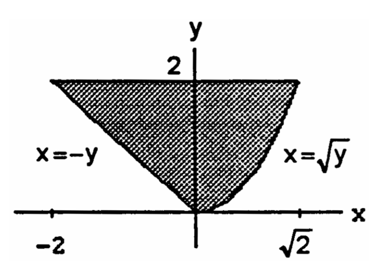
$xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ \tan^2 x, & x = 0 \end{cases} \Rightarrow xg(x) = \tan^2 x,\ 0 \le x \le \pi/4$

(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^{\pi/4} 2\pi x \cdot g(x)\, dx$ and $x \cdot g(x) = \tan^2 x,\ 0 \le x \le \pi/4$ by part (a)

$\Rightarrow V = 2\pi\int_0^{\pi/4} \tan^2 x\, dx = 2\pi\int_0^{\pi/4}(\sec^2 x - 1)\, dx = 2\pi[\tan x - x]_0^{\pi/4} = 2\pi\left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{2}$

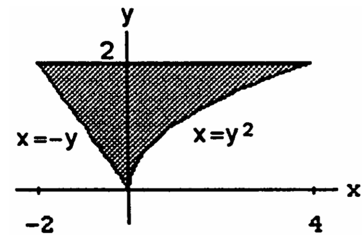
15. $c = 0, d = 2;$

$V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^2 2\pi y\left[\sqrt{y} - (-y)\right] dy$
$= 2\pi\int_0^2 \left(y^{3/2} + y^2\right) dy = 2\pi\left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2$
$= 2\pi\left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right) = 16\pi\left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right)$
$= \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$



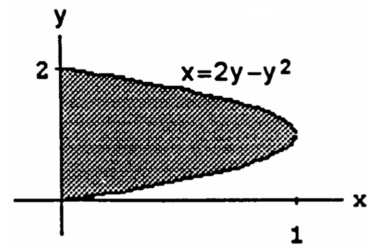
16. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\,[y^2 - (-y)]dy$$
$$= 2\pi \int_0^2 (y^3 + y^2)\,dy = 2\pi\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\frac{2}{4} + \frac{1}{3}\right)$$
$$= 16\pi\left(\frac{5}{6}\right) = \frac{40\pi}{3}$$



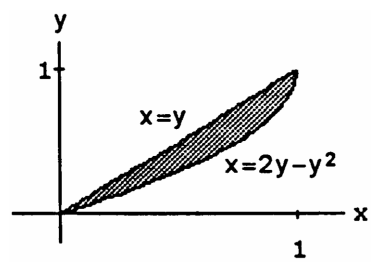

17. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\,(2y - y^2)dy$$
$$= 2\pi \int_0^2 (2y^2 - y^3)\,dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$$
$$= 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{8\pi}{3}$$



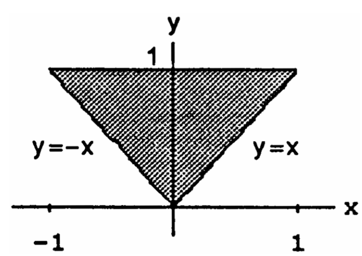

18. $c = 0, d = 1;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi y\,(2y - y^2 - y)dy$$
$$= 2\pi \int_0^1 y\,(y - y^2)\,dy = 2\pi \int_0^1 (y^2 - y^3)\,dy$$
$$= 2\pi\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$



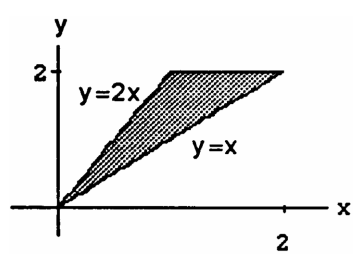

19. $c = 0, d = 1;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = 2\pi \int_0^1 y[y - (-y)]dy$$
$$= 2\pi \int_0^1 2y^2\,dy = \frac{4\pi}{3}\left[y^3\right]_0^1 = \frac{4\pi}{3}$$



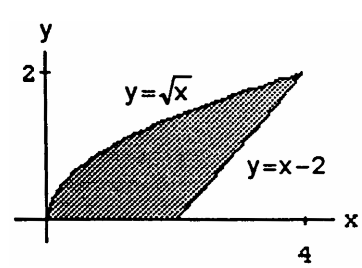

20. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi\, y\left(y - \frac{y}{2}\right)dy$$
$$= 2\pi \int_0^2 \frac{y^2}{2}dy = \frac{\pi}{3}\left[y^3\right]_0^2 = \frac{8\pi}{3}$$



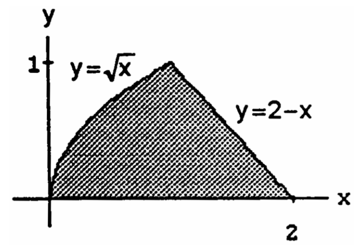

21. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\,[(2 + y) - y^2]\,dy$$
$$= 2\pi \int_0^2 (2y + y^2 - y^3)\,dy = 2\pi\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$
$$= 2\pi\left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{\pi}{6}(48 + 32 - 48) = \frac{16\pi}{3}$$



22. $c = 0, d = 1;$

$V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^1 2\pi y\left[(2-y) - y^2\right] dy$
$= 2\pi \int_0^1 \left(2y - y^2 - y^3\right) dy = 2\pi\left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$
$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}(12 - 4 - 3) = \frac{5\pi}{6}$




23. (a) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi\, x\,(3x) dx = 6\pi \int_0^2 x^2\, dx = 2\pi\left[x^3\right]_0^2 = 16\pi$

(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi\,(4-x)\,(3x) dx = 6\pi \int_0^2 \left(4x - x^2\right) dx = 6\pi\left[2x^2 - \frac{1}{3}x^3\right]_0^2 = 6\pi\left(8 - \frac{8}{3}\right) = 32\pi$

(c) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi\,(x+1)\,(3x) dx = 6\pi \int_0^2 \left(x^2 + x\right) dx = 6\pi\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^2 = 6\pi\left(\frac{8}{3} + 2\right) = 28\pi$

(d) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^6 2\pi\, y\left(2 - \frac{1}{3}y\right) dy = 2\pi \int_0^6 \left(2y - \frac{1}{3}y^2\right) dy = 2\pi\left[y^2 - \frac{1}{9}y^3\right]_0^6 = 2\pi(36 - 24) = 24\pi$

(e) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^6 2\pi\,(7-y)\left(2 - \frac{1}{3}y\right) dy = 2\pi \int_0^6 \left(14 - \frac{13}{3}y + \frac{1}{3}y^2\right) dy = 2\pi\left[14y - \frac{13}{6}y^2 + \frac{1}{9}y^3\right]_0^6$
$= 2\pi(84 - 78 + 24) = 60\pi$

(f) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^6 2\pi\,(y+2)\left(2 - \frac{1}{3}y\right) dy = 2\pi \int_0^6 \left(4 + \frac{4}{3}y - \frac{1}{3}y^2\right) dy = 2\pi\left[4y + \frac{2}{3}y^2 - \frac{1}{9}y^3\right]_0^6$
$= 2\pi(24 + 24 - 24) = 48\pi$

24. (a) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi\, x\,\left(8 - x^3\right) dx = 2\pi \int_0^2 \left(8x - x^4\right) dx = 2\pi\left[4x^2 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(16 - \frac{32}{5}\right) = \frac{96\pi}{5}$

(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi\,(3-x)\,\left(8 - x^3\right) dx = 2\pi \int_0^2 \left(24 - 8x - 3x^3 + x^4\right) dx$
$= 2\pi\left[24x - 4x^2 - \frac{3}{4}x^4 + \frac{1}{5}x^5\right]_0^2 = 2\pi\left(48 - 16 - 12 + \frac{32}{5}\right) = \frac{264\pi}{5}$

(c) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^2 2\pi\,(x+2)\,\left(8 - x^3\right) dx = 2\pi \int_0^2 \left(16 + 8x - 2x^3 - x^4\right) dx$
$= 2\pi\left[16x + 4x^2 - \frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(32 + 16 - 8 - \frac{32}{5}\right) = \frac{336\pi}{5}$

(d) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^8 2\pi\, y \cdot y^{1/3} dy = 2\pi \int_0^8 y^{4/3} dy = \frac{6\pi}{7}\left[y^{7/3}\right]_0^8 = \frac{6\pi}{7}(128) = \frac{768\pi}{7}$

(e) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^8 2\pi\,(8-y)\, y^{1/3} dy = 2\pi \int_0^8 \left(8y^{1/3} - y^{4/3}\right) dy = 2\pi\left[6y^{4/3} - \frac{3}{7}y^{7/3}\right]_0^8$
$= 2\pi\left(96 - \frac{384}{7}\right) = \frac{576\pi}{7}$

(f) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^8 2\pi\,(y+1)\, y^{1/3} dx = 2\pi \int_0^8 \left(y^{4/3} + y^{1/3}\right) dy = 2\pi\left[\frac{3}{7}y^{7/3} + \frac{3}{4}y^{4/3}\right]_0^8$
$= 2\pi\left(\frac{384}{7} + 12\right) = \frac{936\pi}{7}$

25. (a) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_{-1}^2 2\pi\,(2-x)\,\left(x + 2 - x^2\right) dx = 2\pi \int_{-1}^2 \left(4 - 3x^2 + x^3\right) dx = 2\pi\left[4x - x^3 + \frac{1}{4}x^4\right]_{-1}^2$
$= 2\pi(8 - 8 + 4) - 2\pi\left(-4 + 1 + \frac{1}{4}\right) = \frac{27\pi}{2}$

. (b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_{-1}^2 2\pi\,(x+1)\,\left(x + 2 - x^2\right) dx = 2\pi \int_{-1}^2 \left(2 + 3x - x^3\right) dx = 2\pi\left[2x + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_{-1}^2$
$= 2\pi(4 + 6 - 4) - 2\pi\left(-2 + \frac{3}{2} - \frac{1}{4}\right) = \frac{27\pi}{2}$

(c) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^1 2\pi\, y\left(\sqrt{y} - \left(-\sqrt{y}\right)\right) dy + \int_1^4 2\pi\, y\left(\sqrt{y} - (y - 2)\right) dy$
$= 4\pi \int_0^1 y^{3/2} dy + 2\pi \int_1^4 \left(y^{3/2} - y^2 + 2y\right) dy = \frac{8\pi}{5}\left[y^{5/2}\right]_0^1 + 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3 + y^2\right]_1^4$
$= \frac{8\pi}{5}(1) + 2\pi\left(\frac{64}{5} - \frac{64}{3} + 16\right) - 2\pi\left(\frac{2}{5} - \frac{1}{3} + 1\right) = \frac{72\pi}{5}$

(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^1 2\pi(4-y)\left(\sqrt{y} - \left(-\sqrt{y}\right)\right)dy + \int_1^4 2\pi(4-y)\left(\sqrt{y} - (y-2)\right)dy$
$= 4\pi\int_0^1 \left(4\sqrt{y} - y^{3/2}\right)dy + 2\pi\int_1^4 \left(y^2 - y^{3/2} - 6y + 4\sqrt{y} + 8\right)dy$
$= 4\pi\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 + 2\pi\left[\frac{1}{3}y^3 - \frac{2}{5}y^{5/2} - 3y^2 + \frac{8}{3}y^{3/2} + 8y\right]_1^4$
$= 4\pi\left(\frac{8}{3} - \frac{2}{5}\right) + 2\pi\left(\frac{64}{3} - \frac{64}{5} - 48 + \frac{64}{3} + 32\right) - 2\pi\left(\frac{1}{3} - \frac{2}{5} - 3 + \frac{8}{3} + 8\right) = \frac{108\pi}{5}$

26. (a) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx = \int_{-1}^1 2\pi(1-x)(4-3x^2-x^4)dx = 2\pi\int_{-1}^1 (x^5 - x^4 + 3x^3 - 3x^2 - 4x + 4)dx$
$= 2\pi\left[\frac{1}{6}x^6 - \frac{1}{5}x^5 + \frac{3}{4}x^4 - x^3 - 2x^2 + 4x\right]_{-1}^1 = 2\pi\left(\frac{1}{6} - \frac{1}{5} + \frac{3}{4} - 1 - 2 + 4\right) - 2\pi\left(\frac{1}{6} + \frac{1}{5} + \frac{3}{4} + 1 - 2 - 4\right) = \frac{56\pi}{5}$
(b) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^1 2\pi y\left(\sqrt[4]{y} - \left(-\sqrt[4]{y}\right)\right)dy + \int_1^4 2\pi y\left[\sqrt{\frac{4-y}{3}} - \left(-\sqrt{\frac{4-y}{3}}\right)\right]dy$
$= 4\pi\int_0^1 y^{5/4}dy + \frac{4\pi}{\sqrt{3}}\int_1^4 y\sqrt{4-y}\,dy$ [$u = 4 - y \Rightarrow y = 4 - u \Rightarrow du = -dy$; $y = 1 \Rightarrow u = 3$, $y = 4 \Rightarrow u = 0$]
$= \frac{16\pi}{9}\left[y^{9/4}\right]_0^1 - \frac{4\pi}{\sqrt{3}}\int_3^0 (4-u)\sqrt{u}\,du = \frac{16\pi}{9}(1) + \frac{4\pi}{\sqrt{3}}\int_0^3 \left(4\sqrt{u} - u^{3/2}\right)du = \frac{16\pi}{9} + \frac{4\pi}{\sqrt{3}}\left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^3$
$= \frac{16\pi}{9} + \frac{4\pi}{\sqrt{3}}\left(8\sqrt{3} - \frac{18}{5}\sqrt{3}\right) = \frac{16\pi}{9} + \frac{88\pi}{5} = \frac{872\pi}{45}$

27. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^1 2\pi y \cdot 12(y^2 - y^3)\,dy = 24\pi\int_0^1 (y^3 - y^4)\,dy = 24\pi\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$
$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{24\pi}{20} = \frac{6\pi}{5}$
(b) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^1 2\pi(1-y)\left[12(y^2 - y^3)\right]dy = 24\pi\int_0^1 (1-y)(y^2 - y^3)\,dy$
$= 24\pi\int_0^1 (y^2 - 2y^3 + y^4)\,dy = 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 24\pi\left(\frac{1}{30}\right) = \frac{4\pi}{5}$
(c) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^1 2\pi\left(\frac{8}{5} - y\right)\left[12(y^2 - y^3)\right]dy = 24\pi\int_0^1 \left(\frac{8}{5} - y\right)(y^2 - y^3)\,dy$
$= 24\pi\int_0^1 \left(\frac{8}{5}y^2 - \frac{13}{5}y^3 + y^4\right)dy = 24\pi\left[\frac{8}{15}y^3 - \frac{13}{20}y^4 + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right) = \frac{24\pi}{60}(32 - 39 + 12)$
$= \frac{24\pi}{12} = 2\pi$
(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^1 2\pi\left(y + \frac{2}{5}\right)\left[12(y^2 - y^3)\right]dy = 24\pi\int_0^1 \left(y + \frac{2}{5}\right)(y^2 - y^3)\,dy$
$= 24\pi\int_0^1 \left(y^3 - y^4 + \frac{2}{5}y^2 - \frac{2}{5}y^3\right)dy = 24\pi\int_0^1 \left(\frac{2}{5}y^2 + \frac{3}{5}y^3 - y^4\right)dy = 24\pi\left[\frac{2}{15}y^3 + \frac{3}{20}y^4 - \frac{y^5}{5}\right]_0^1$
$= 24\pi\left(\frac{2}{15} + \frac{3}{20} - \frac{1}{5}\right) = \frac{24\pi}{60}(8 + 9 - 12) = \frac{24\pi}{12} = 2\pi$

28. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^2 2\pi y\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi y\left(y^2 - \frac{y^4}{4}\right)dy = 2\pi\int_0^2 \left(y^3 - \frac{y^5}{4}\right)dy$
$= 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{2^4}{4} - \frac{2^6}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{4}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{1}{6}\right) = 32\pi\left(\frac{2}{24}\right) = \frac{8\pi}{3}$
(b) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^2 2\pi(2-y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(2-y)\left(y^2 - \frac{y^4}{4}\right)dy$
$= 2\pi\int_0^2 \left(2y^2 - \frac{y^4}{2} - y^3 + \frac{y^5}{4}\right)dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^5}{10} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{32}{10} - \frac{16}{4} + \frac{64}{24}\right) = \frac{8\pi}{5}$
(c) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^2 2\pi(5-y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(5-y)\left(y^2 - \frac{y^4}{4}\right)dy$
$= 2\pi\int_0^2 \left(5y^2 - \frac{5}{4}y^4 - y^3 + \frac{y^5}{4}\right)dy = 2\pi\left[\frac{5y^3}{3} - \frac{5y^5}{20} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{40}{3} - \frac{160}{20} - \frac{16}{4} + \frac{64}{24}\right) = 8\pi$
(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left(y^2 - \frac{y^4}{4}\right)dy$
$= 2\pi\int_0^2 \left(y^3 - \frac{y^5}{4} + \frac{5}{8}y^2 - \frac{5}{32}y^4\right)dy = 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24} + \frac{5y^3}{24} - \frac{5y^5}{160}\right]_0^2 = 2\pi\left(\frac{16}{4} - \frac{64}{24} + \frac{40}{24} - \frac{160}{160}\right) = 4\pi$

29. (a) About x-axis: $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$

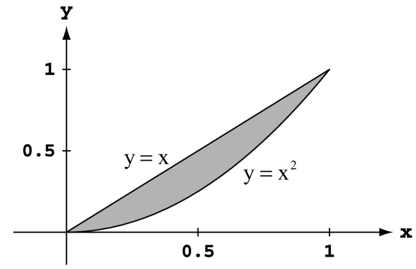
$= \int_0^1 2\pi y\left(\sqrt{y} - y\right) dy = 2\pi \int_0^1 \left(y^{3/2} - y^2\right) dy$

$= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}$

About y-axis: $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$

$= \int_0^1 2\pi x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx$

$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$




(b) About x-axis: $R(x) = x$ and $r(x) = x^2 \Rightarrow V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \int_0^1 \pi[x^2 - x^4] dx$

$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

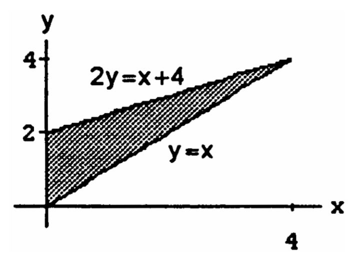
About y-axis: $R(y) = \sqrt{y}$ and $r(y) = y \Rightarrow V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right] dy = \int_0^1 \pi[y - y^2] dy$

$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

30. (a) $V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \pi \int_0^4 \left[\left(\frac{x}{2} + 2\right)^2 - x^2\right] dx$

$= \pi \int_0^4 \left(-\frac{3}{4}x^2 + 2x + 4\right) dx = \pi\left[-\frac{x^3}{4} + x^2 + 4x\right]_0^4$

$= \pi(-16 + 16 + 16) = 16\pi$




(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi x\left(\frac{x}{2} + 2 - x\right) dx$

$= \int_0^4 2\pi x\left(2 - \frac{x}{2}\right) dx = 2\pi \int_0^4 \left(2x - \frac{x^2}{2}\right) dx$

$= 2\pi\left[x^2 - \frac{x^3}{6}\right]_0^4 = 2\pi\left(16 - \frac{64}{6}\right) = \frac{32\pi}{3}$

(c) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi(4 - x)\left(\frac{x}{2} + 2 - x\right) dx = \int_0^4 2\pi(4 - x)\left(2 - \frac{x}{2}\right) dx = 2\pi \int_0^4 \left(8 - 4x + \frac{x^2}{2}\right) dx$

$= 2\pi\left[8x - 2x^2 + \frac{x^3}{6}\right]_0^4 = 2\pi\left(32 - 32 + \frac{64}{6}\right) = \frac{64\pi}{3}$

(d) $V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \pi \int_0^4 \left[(8 - x)^2 - \left(6 - \frac{x}{2}\right)^2\right] dx = \pi \int_0^4 \left[(64 - 16x + x^2) - \left(36 - 6x + \frac{x^2}{4}\right)\right] dx$

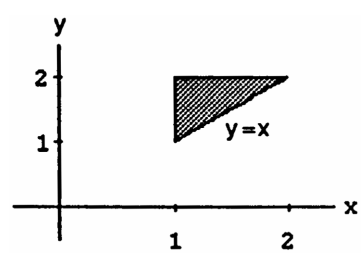
$\pi \int_0^4 \left(\frac{3}{4}x^2 - 10x + 28\right) dx = \pi\left[\frac{x^3}{4} - 5x^2 + 28x\right]_0^4 = \pi\left[16 - (5)(16) + (7)(16)\right] = \pi(3)(16) = 48\pi$

31. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_1^2 2\pi y(y - 1)\, dy$

$= 2\pi \int_1^2 (y^2 - y)\, dy = 2\pi\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$

$= 2\pi\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$

$= 2\pi\left(\frac{7}{3} - 2 + \frac{1}{2}\right) = \frac{\pi}{3}(14 - 12 + 3) = \frac{5\pi}{3}$




(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^2 2\pi x(2 - x)\, dx = 2\pi \int_1^2 (2x - x^2)\, dx = 2\pi\left[x^2 - \frac{x^3}{3}\right]_1^2$

$= 2\pi\left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 2\pi\left[\left(\frac{12-8}{3}\right) - \left(\frac{3-1}{3}\right)\right] = 2\pi\left(\frac{4}{3} - \frac{2}{3}\right) = \frac{4\pi}{3}$

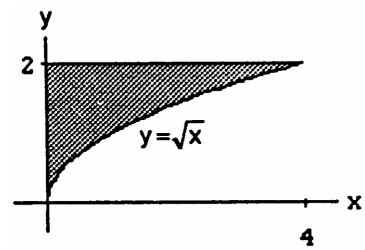
(c) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^2 2\pi\left(\frac{10}{3} - x\right)(2 - x)\, dx = 2\pi \int_1^2 \left(\frac{20}{3} - \frac{16}{3}x + x^2\right) dx$

$= 2\pi\left[\frac{20}{3}x - \frac{8}{3}x^2 + \frac{1}{3}x^3\right]_1^2 = 2\pi\left[\left(\frac{40}{3} - \frac{32}{3} + \frac{8}{3}\right) - \left(\frac{20}{3} - \frac{8}{3} + \frac{1}{3}\right)\right] = 2\pi\left(\frac{3}{3}\right) = 2\pi$

(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_1^2 2\pi(y - 1)(y - 1)\, dy = 2\pi \int_1^2 (y - 1)^2 = 2\pi\left[\frac{(y-1)^3}{3}\right]_1^2 = \frac{2\pi}{3}$

32. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y(y^2 - 0)\, dy$

$= 2\pi \int_0^2 y^3\, dy = 2\pi \left[\frac{y^4}{4}\right]_0^2 = 2\pi \left(\frac{2^4}{4}\right) = 8\pi$




(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$

$= \int_0^4 2\pi x \left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(2x - x^{3/2}\right) dx$

$= 2\pi \left[x^2 - \frac{2}{5} x^{5/2}\right]_0^4 = 2\pi \left(16 - \frac{2 \cdot 2^5}{5}\right)$

$= 2\pi \left(16 - \frac{64}{5}\right) = \frac{2\pi}{5}(80 - 64) = \frac{32\pi}{5}$

(c) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi(4 - x)\left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(8 - 4x^{1/2} - 2x + x^{3/2}\right) dx$

$= 2\pi \left[8x - \frac{8}{3} x^{3/2} - x^2 + \frac{2}{5} x^{5/2}\right]_0^4 = 2\pi \left(32 - \frac{64}{3} - 16 + \frac{64}{5}\right) = \frac{2\pi}{15}(240 - 320 + 192) = \frac{2\pi}{15}(112) = \frac{224\pi}{15}$

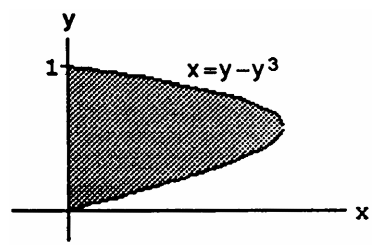
(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi(2 - y)\left(y^2\right) dy = 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi \left[\frac{2}{3} y^3 - \frac{y^4}{4}\right]_0^2$

$= 2\pi \left(\frac{16}{3} - \frac{16}{4}\right) = \frac{32\pi}{12}(4 - 3) = \frac{8\pi}{3}$

33. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi y(y - y^3)\, dy$

$= \int_0^1 2\pi \left(y^2 - y^4\right) dy = 2\pi \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{4\pi}{15}$




(b) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$

$= \int_0^1 2\pi(1 - y)\left(y - y^3\right) dy$

$= 2\pi \int_0^1 \left(y - y^2 - y^3 + y^4\right) dy = 2\pi \left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}\right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right) = \frac{2\pi}{60}(30 - 20 - 15 + 12) = \frac{7\pi}{30}$

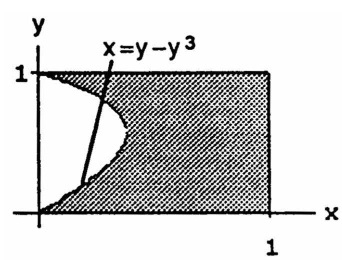
34. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$

$= \int_0^1 2\pi y \left[1 - \left(y - y^3\right)\right] dy$

$= 2\pi \int_0^1 \left(y - y^2 + y^4\right) dy = 2\pi \left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1$

$= 2\pi \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) = \frac{2\pi}{30}(15 - 10 + 6)$

$= \frac{11\pi}{15}$




(b) Use the washer method:

$V = \int_c^d \pi \left[R^2(y) - r^2(y)\right] dy = \int_0^1 \pi \left[1^2 - \left(y - y^3\right)^2\right] dy = \pi \int_0^1 \left(1 - y^2 - y^6 + 2y^4\right) dy = \pi \left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1$

$= \pi \left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right) = \frac{\pi}{105}(105 - 35 - 15 + 42) = \frac{97\pi}{105}$

(c) Use the washer method:

$V = \int_c^d \pi \left[R^2(y) - r^2(y)\right] dy = \int_0^1 \pi \left[\left[1 - \left(y - y^3\right)\right]^2 - 0\right] dy = \pi \int_0^1 \left[1 - 2\left(y - y^3\right) + \left(y - y^3\right)^2\right] dy$

$= \pi \int_0^1 \left(1 + y^2 + y^6 - 2y + 2y^3 - 2y^4\right) dy = \pi \left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1 = \pi \left(1 + \frac{1}{3} + \frac{1}{7} - 1 + \frac{1}{2} - \frac{2}{5}\right)$

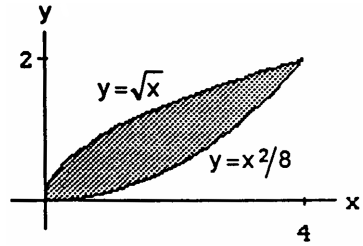
$= \frac{\pi}{210}(70 + 30 + 105 - 2 \cdot 42) = \frac{121\pi}{210}$

(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi(1 - y)\left[1 - \left(y - y^3\right)\right] dy = 2\pi \int_0^1 (1 - y)\left(1 - y + y^3\right) dy$

$= 2\pi \int_0^1 \left(1 - y + y^3 - y + y^2 - y^4\right) dy = 2\pi \int_0^1 \left(1 - 2y + y^2 + y^3 - y^4\right) dy = 2\pi \left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

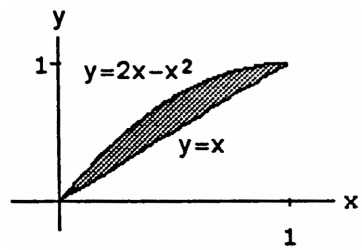
$= 2\pi \left(1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) = \frac{2\pi}{60}(20 + 15 - 12) = \frac{23\pi}{30}$

35. (a) $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^2 2\pi y \left(\sqrt{8y} - y^2\right) dy$
$= 2\pi \int_0^2 \left(2\sqrt{2}\, y^{3/2} - y^3\right) dy = 2\pi \left[\frac{4\sqrt{2}}{5}\, y^{5/2} - \frac{y^4}{4}\right]_0^2$
$= 2\pi \left(\frac{4\sqrt{2}\cdot\left(\sqrt{2}\right)^5}{5} - \frac{2^4}{4}\right) = 2\pi \left(\frac{4\cdot 2^3}{5} - \frac{4\cdot 4}{4}\right)$
$= 2\pi \cdot 4 \left(\frac{8}{5} - 1\right) = \frac{8\pi}{5}(8-5) = \frac{24\pi}{5}$




(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^4 2\pi x \left(\sqrt{x} - \frac{x^2}{8}\right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^3}{8}\right) dx = 2\pi \left[\frac{2}{5}\, x^{5/2} - \frac{x^4}{32}\right]_0^4$
$= 2\pi \left(\frac{2\cdot 2^5}{5} - \frac{4^4}{32}\right) = 2\pi \left(\frac{2^6}{5} - \frac{2^8}{32}\right) = \frac{\pi\cdot 2^7}{160}(32 - 20) = \frac{\pi\cdot 2^9\cdot 3}{160} = \frac{\pi\cdot 2^4\cdot 3}{5} = \frac{48\pi}{5}$

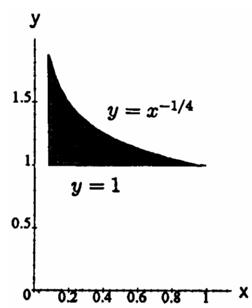
36. (a) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx$
$= \int_0^1 2\pi x \left[(2x - x^2) - x\right] dx$
$= 2\pi \int_0^1 x\,(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx$
$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$




(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^1 2\pi(1-x)\left[(2x - x^2) - x\right] dx = 2\pi \int_0^1 (1-x)(x - x^2)\, dx$
$= 2\pi \int_0^1 (x - 2x^2 + x^3)\, dx = 2\pi \left[\frac{x^2}{2} - \frac{2}{3}\, x^3 + \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{2\pi}{12}(6 - 8 + 3) = \frac{\pi}{6}$

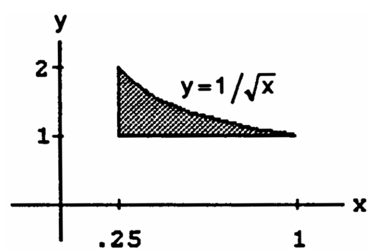
37. (a) $V = \int_a^b \pi \left[R^2(x) - r^2(x)\right] dx = \pi \int_{1/16}^1 \left(x^{-1/2} - 1\right) dx$
$= \pi \left[2x^{1/2} - x\right]_{1/16}^1 = \pi \left[(2-1) - \left(2\cdot\frac{1}{4} - \frac{1}{16}\right)\right]$
$= \pi \left(1 - \frac{7}{16}\right) = \frac{9\pi}{16}$

(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_1^2 2\pi y \left(\frac{1}{y^4} - \frac{1}{16}\right) dy$
$= 2\pi \int_1^2 \left(y^{-3} - \frac{y}{16}\right) dy = 2\pi \left[-\frac{1}{2}\, y^{-2} - \frac{y^2}{32}\right]_1^2$
$= 2\pi \left[\left(-\frac{1}{8} - \frac{1}{8}\right) - \left(-\frac{1}{2} - \frac{1}{32}\right)\right] = 2\pi \left(\frac{1}{4} + \frac{1}{32}\right)$
$= \frac{2\pi}{32}(8+1) = \frac{9\pi}{16}$




38. (a) $V = \int_c^d \pi \left[R^2(y) - r^2(y)\right] dy = \int_1^2 \pi \left(\frac{1}{y^4} - \frac{1}{16}\right) dy$
$= \pi \left[-\frac{1}{3}\, y^{-3} - \frac{y}{16}\right]_1^2 = \pi \left[\left(-\frac{1}{24} - \frac{1}{8}\right) - \left(-\frac{1}{3} - \frac{1}{16}\right)\right]$
$= \frac{\pi}{48}(-2 - 6 + 16 + 3) = \frac{11\pi}{48}$

(b) $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_{1/4}^1 2\pi x \left(\frac{1}{\sqrt{x}} - 1\right) dx$
$= 2\pi \int_{1/4}^1 \left(x^{1/2} - x\right) dx = 2\pi \left[\frac{2}{3}\, x^{3/2} - \frac{x^2}{2}\right]_{1/4}^1$
$= 2\pi \left[\left(\frac{2}{3} - \frac{1}{2}\right) - \left(\frac{2}{3}\cdot\frac{1}{8} - \frac{1}{32}\right)\right] = \pi \left(\frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{16}\right) = \frac{\pi}{48}(4\cdot 16 - 48 - 8 + 3) = \frac{11\pi}{48}$




39. (a) *Disk*: $V = V_1 - V_2$
$V_1 = \int_{a_1}^{b_1} \pi[R_1(x)]^2\, dx$ and $V_2 = \int_{a_2}^{b_2} \pi[R_2(x)]^2$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $R_2(x) = \sqrt{x}$,
$a_1 = -2,\ b_1 = 1;\ a_2 = 0,\ b_2 = 1 \ \Rightarrow$ two integrals are required