Simulating rich-get-richer effects and the effects of taxes

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***Abstract*—In this report, we examine the theoretical and practical aspects of the rich-get-richer effects in economics. We then use various Python libraries to simulate a model that approximates trends in modern economic markets, and predict that an unregulated market will lead to a concentration of wealth in few people. We then evaluate the results of these simulations to develop interesting ways to extend this model to improve the average person's wealth and run a second set of simulations.**

I. INTRODUCTION

We will first explore the model explained in by Piketty [1], and use the notation as described by Van Lear [2], to get a long-term outlook on rich-get-richer effects.

We define the capital income ratio β as

$$\beta = \frac{K}{Y} \tag{1}$$

Where Y is the national income, and K is the national capital.

The unit used for β is years of income (all measurements for wealth and income are generally in local currency). This ratio is between 5 and 6 for various West-European countries, and at 4.5 for the US. It used to be lower during the world wars and the great depression, and was higher in the 19th century.

Wealth and capital are used interchangeably here. Capital can be defined as a person's assets, with their debts subtracted. However, capital can also be represented by factors of production (of wealth).

We will come up with some more precise definitions for this. As an example, we set

$$Y = 100 \tag{2}$$

We now define ΔY as the absolute national income increase. We set the relative national income increase percentage:

$$\frac{\Delta Y}{Y} = 2\% = 0.02 \tag{3}$$

Then

$$\Delta Y = 0.02 \cdot Y = 2 \tag{4}$$

Then we define the absolute national capital increase ΔK as:

$$\Delta K = 0.1 \cdot Y = 10 \tag{5}$$

Suppose we want to leave β unchanged next year; then the ratio at which K grows is the ratio at which Y grows, or:

$$\frac{\Delta Y}{Y} = \frac{\Delta K}{K} \tag{6}$$

$$\frac{2}{100} = \frac{10}{K} \tag{7}$$

$$K = 500 \tag{8}$$

Apparently, to achieve this lack of change in β, we set K to 500. This leads to:

$$\beta = \frac{K}{Y} = \frac{500}{100} = 5 \tag{9}$$

The same then holds for $\frac{\Delta K}{\Delta Y} = 5$. When this rate increases above 5, the capital increases faster than the national income, and β goes up. When this rate is decreased, the capital increases slower than the national income, and β goes down.

Piketty estimates $\frac{\Delta K}{K}$ will go to 10% annually for most countries, and estimates that $\frac{\Delta Y}{Y}$ will slow from 3% to 1.5% annually. He projects that β will go to 7.

Now for income inequality:
If $\beta = \frac{K}{Y} = 6$ years and $\frac{\Delta K}{K} = 5\%$, then

$$K = \beta \cdot Y = 6 \cdot Y \tag{10}$$

$$\frac{\Delta K}{6 \cdot Y} = 5\% \tag{11}$$

$$\Delta K = 6 \cdot Y \cdot 0.05 = 0.3 \cdot Y \tag{12}$$

The increase in K is 30% of the national income Y.

This means that 30% of the national income consists of return on capital and investments, while 70% is from labour. Capital is more concentrated among the rich; returns on this will lead to a bigger inequality gap.

The capital return r, which is defined as the share of national income which comes from capital, is defined as

$$r = \frac{\Delta K}{Y} = 0.3 \tag{13}$$

The share of national income from labor (labor return):

$$LR = 1 - r = 0.7 \tag{14}$$

This is the same as real wages divided by the productivity of labour.

Intuitively, living in a society with increasing real wages, but even faster-increasing productivity of labour (thus leading to decreasing LR) seems nice. A person probably prefers higher wages to higher LR, as wages affect one's lifestyle, not one's share of national income. However, if LR is low, a small portion of people can come to dominate society. This will be detrimental to society in the long term.

In any case, Piketty predicts that while β will go to 7, r will fall. "As production becomes more and more capital-intensive, it gets harder and harder to find profitable uses for additional capital, or easy ways to substitute capital for labor." [1]

β will stabilise (except for during wars and such). There is no logical necessity for $\frac{\Delta K}{K}$ to be bigger than $\frac{\Delta Y}{Y}$. In fact, letting $\frac{\Delta K}{K}$ shrink until $\frac{\Delta K}{K} = \frac{\Delta Y}{Y}$ would allow for a higher level of consumption (higher national income from wages, more capital being used for consumption instead of just gathering more capital), which would be better for society, in theory. But this has not been the case in recent history.

Suppose β is stable. Wages grow as fast as productivity through technological progress. This is less than the total growth of income and GDP (due to factors such as population increase). However, if someone's income is purely from returns on capital, that person's wealth is multiplied with a factor of r each year. (r is a number such as 1.05 for a yearly return of 5%).

So after t years, a person's wealth w is:

$$w(t) = w_0 \cdot r^t \tag{15}$$

Where w_0 is the initial wealth.

After t years, the income from capital, $\frac{\Delta w(t)}{\Delta t}$, for the next year is:

$$\frac{\Delta w(t)}{\Delta t} = w_0 \cdot r^{t+1} - w_0 \cdot r^t = w_0 \cdot (r-1) \cdot r^t \tag{16}$$

So the income of such a person increases exponentially each year. Wages may also grow exponentially by a factor g (for example, a $g = 1\%$ increase each year), but this increase is never as big as the returns on capital, in practice. This is a rich-get-richer effect, and will lead to most of the capital in a society being controlled by small groups.

Thus $r \geq g$, and $\frac{\Delta K}{K} \geq \frac{\Delta Y}{Y}$. The income and wealth of the rich grow faster than the typical income from work. Thus, the rich get richer. This is not a flaw in the system, but just the ability of an economy to accept more and more capital without lowering interest rates.

It would seem like the richest people in society get their capital from investments (mostly), but this is untrue. It used to be the case, but now the top 1% and such are mostly "supermanagers" who get a lot of money for their executive positions. Only the top 0.1% or so get most of their money from investments.

This problem might be fixed by a global tax on capital (according to the authors), which we will explore in later sections.

II. The simulation model

In a group G of N people, each person i gets $w_{i,0} = 100.00$ amount of money to invest.

For every round t, we randomly divide the group G of N people into two groups. For each person i, independently draw a random number $m_{i,t}$ from distribution M (in the range [0,1]). Person i decides to invest $m_{i,t} \cdot 100\%$ of their capital, $w_{i,t}$ that round.

We then draw a random number p_t from a distribution p (in the range $[0, 0.4]$). This would be considered very volatile. However, when using a much smaller range, the effects happen too slowly, and a much larger range introduces noise and speeds up the process too much to properly observe the model's effects.

This is the return on investment. People in group 1 get $p_t \cdot 100\%$ returns on their investment, group 2 get $p_t \cdot 100\%$ loss on their investment.

For people in group 1, their capital in the next round is:

$$w_{i,t+1} = w_{i,t} \cdot m_{i,t} \cdot (1 + p_t) \tag{17}$$

For people in group 2, their capital in the next round is:

$$w_{i,t+1} = w_{i,t} \cdot m_{i,t} \cdot (1 - p_t) \tag{18}$$

In this simulation, we define $N = 10000$. Anything larger becomes really slow to simulate, but lower numbers are more susceptible to noise from the random distributions. 10000 seemed like a good compromise.

Likewise, we define the random distribution M, which is used for generating the percentages people decide to invest, to be a uniform distribution between 0 and 1 (0 and 100%). Thus, $M \sim U(0, 1)$.

Exponential numbers can generate numbers larger than one, which does not make sense for "the portion of a person's capital they decide to invest". No numbers larger than 1 will be accepted; thus, we use a truncated exponential distribution between 0 and 1. This gives us the ability to generate numbers that can be anywhere between 0 and 1, but with a high bias towards lower numbers (as in real life investments).

The random distribution p, used for generating the profit/loss percentage each person gets, is a uniformly distributed variable in the range $[0, 0.4]$. Thus $p \sim U(0, 0.4)$.

So after 1 round, half the population has experienced a loss, half has experienced a profit. After two rounds, a quarter has experienced a loss, half the population has experienced a loss and a profit, and a quarter has experienced two gains.

We see a pattern here. To see what proportion of the population has experienced some amount of losses, we can use Pascal's triangle, where an entry in row n and column k is simply $\frac{n}{k}$, and row n has $n + 1$ columns.

After four rounds, the 4th row of Pascal's triangle contains: 1 4 6 4 1 6 out of 16 people will experience two losses and two wins in round 4, 4 out of 16 will experience one loss and three gains, etc.

After n rounds $\frac{n}{\frac{n}{2}}$ people have experienced $\frac{n}{2}$ losses and $\frac{n}{2}$ wins. The number of people who experience more losses than wins is equal to the number of people who experience more gains than losses. Intuitively, one could expect that the amount of money in this market stays the same, as on average, everyone has 0 losses − gains.

However, this is not the case. The number in the market has probably decreased, as the median person has had the following multiplier applied to their capital:

$$(1+p)^{\frac{1}{2}n} \cdot (1-p)^{\frac{1}{2}n} \tag{19}$$

$$= \left(1-p^2\right)^{\frac{1}{2}n} \tag{20}$$

This number seems to decrease. Let's confirm this another way. For $p \in (0, 0.5]$:

$$\lim_{n\to\infty} (1+p)^{\frac{1}{2}n} \cdot (1-p)^{\frac{1}{2}n} = 0 \tag{21}$$

In the case that p is 0, this number stays at 1, but the chance that after n rounds, the profit/loss percentage has been precisely 0 every time is so small that this is not relevant. Thus, we can expect the amount of capital in this market to steadily decrease over time. However, we are not sure if this is realistic, and we have come up with an alternative scenario for this that we explore in Section IV.

Now we can start proving that there are rich-get-richer effects. Imagine that the profit/loss percentage p is always 50%. Thus, when a person makes a profit, they have $1.5 \cdot w$ (where w is their capital) in the next round. When they make a loss, they have $0.5 \cdot w$ capital in the next round. Then, after 1 round, people who made a profit have three times as much money as people who made a loss. So 50% of the people control 75% of the wealth.

Once again, we visualise for the n^{th} round in this simulation, that all the people in this market are represented by a number in the n^{th} row of Pascal's triangle. Here, the k^{th} number in the row divided by the sum of numbers in this row is the proportion of people who experienced k wins and $n+1-k$ losses (with $k \in [0, n]$).

The middle (median) person in this division has as many wins as losses. The people in the next group of Pascal's triangle, to the left of the median, have one more loss than wins. The people to the right have one more win and one fewer loss.

Intuition would tell us that the people to the right of the median have more money than the people to the left. However, as we just saw with the case after 1 round, this difference is not linear. The ratio of the wealth the rightmost group of people in the row of Pascal's triangle has, compared to the group next to them, is:

$$\frac{1+\mathbb{E}[p]}{1-\mathbb{E}[p]} = \frac{1.2}{0.8} = 1.5 \tag{22}$$

Thus they have 1.5 times the wealth of the group next to them, if the profit/loss ratio is always 20% and everyone invests all their money. (This is the expected value when a uniform distribution of $U(0, 0.4)$ is used). However, on average, people invest 50% of all their money each round with the $U(0, 1)$ distribution we use, making the difference between groups:

$$\frac{1+\mathbb{E}[M]\cdot\mathbb{E}[p]}{1-\mathbb{E}[M]\cdot\mathbb{E}[p]} = \frac{1+0.5\cdot 0.2}{1-0.5\cdot 0.2} = \frac{1.1}{0.9} = 1.22... \tag{23}$$

After n rounds, the rightmost (luckiest) person L has the following share of all wealth of all the people in group G:

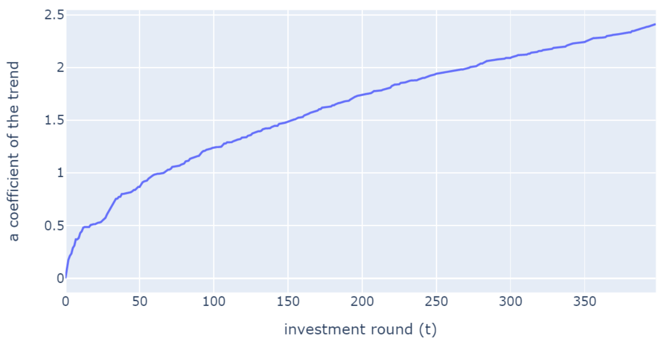

Fig. 1: a coefficient of the polynomial fit during each investment round in the simulated model.

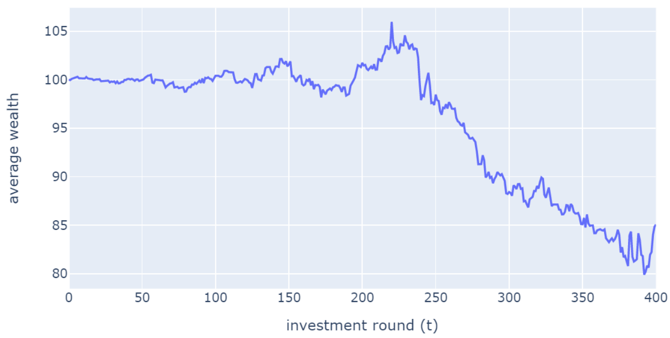

Fig. 2: Average wealth per round in the simulated model.

$$\frac{w_{L,n}}{\sum_{i\in G} w_{i,n}} = \frac{1}{\sum_{k=0}^{n} \frac{n}{k} 1.22..^{-k}} \tag{24}$$

After 1 round, this is the following portion of all wealth:

$$\frac{w_{L,1}}{\sum_{i\in G} w_{i,1}} = \frac{1}{1+1.22^{-1}} = 54\% \tag{25}$$

The rightmost (luckiest) 50% of people have the following portion of all wealth:

$$\frac{\sum_{i\in G_{\text{luckiest }50\%}} w_{i,n}}{\sum_{i\in G} w_{i,n}} = \frac{\sum_{k=0}^{n/2} \frac{n}{k} 1.22..^{-k}}{\sum_{k=0}^{n} \frac{n}{k} 1.22..^{-k}} \tag{26}$$

To illustrate how quickly this grows: after 50 rounds, this is already 80%, for $n = 100$ rounds, this is 86%, and for $n = 200$ rounds, this is 93%. Thus even without randomness, there is a rich-get-richer effect. Using random distributions instead will make the differences even more significant, as some people will be extremely lucky, while others are unlucky.

Thus, we expect to see rich-get-richer effects in the simulation. Capital will attach itself to more capital, as predicted by Piketty [1].

The source code used for generating the results can be found on GitHub [3]. The original figures can be found in the .ipynb files for these experiments.

III. Simulation Results

As expected, we saw rich-get-richer effects. However, the results were more extreme than we initially expected; the power laws (a coefficient) grew stronger over time (as seen in Figure 1), leading to a larger and larger gap between the rich and the poor. This is probably due to the combination of multiple random factors, where it is possible for a few people to get very lucky, and the inherent rich-get-richer effects discussed in Section II.

The average wealth went down over time (as seen in Figure 2), which could have been expected from the theory (discussed in Section II). However, the first time we plotted this, we were very surprised by this, as we had not finished these calculations yet.

These findings were very useful for coming up with ways to extend the model in Section IV.

IV. Extending the model

We came up with several ways to change the model to see which factors had the most considerable influence on this power law dynamic, and how we could potentially lessen the inequalities in this market.

The first idea we had is to change the distributions of the random variables. Currently, both the market volatility (the profit/loss ratio, p) and the investment rate (the % of their capital people invest per round) are modelled as uniformly distributed variables. We came up with two scenarios where we change these variables:

A. Smarter investment

The first scenario we call "smarter investment": changing the investment rate to be modelled as an exponentially distributed variable with a mean of $\frac{1}{5}$. This would mean that we would have this variable, $M \sim Exp(5)$. A mean of 0.2 was chosen as it would be similar to the previous mean, but the median would be much smaller. Because this distribution may generate variables above 1, we regenerate the number if this occurs. This should lead to even more inequality, and more outliers.

B. Extreme volatility

The second scenario is called "extreme volatility", which involves changing the market volatility to be $p \sim U(0, 1)$ instead of $U(0, 0.4)$. We expect that this might lead to less inequality over time as people make smaller, "smarter" investments.

C. Global tax

The next idea we had is to incorporate a tax, as suggested by Piketty [1]. We call this third scenario "global tax". We take a small percentage from every person's total capital each round, and redistribute this amount evenly over all participants. We do not precisely know what to expect from this, so we are very interested to see what happens. Piketty suggests an annual global wealth tax of up to 2%. As Piketty himself concedes that such a high tax would be politically impossible, we choose a compromise and use a 1% tax. The rich lose very little with a tax of only 1% per round, but the poor gain a lot from this. This prevents them from being unable to ever recover from losses, as is currently the case. We tried to model this with various percentages, and ended up using a 1% tax for the final simulations due to the fact that it is a simple number, and it works quite well for the purposes we just defined.

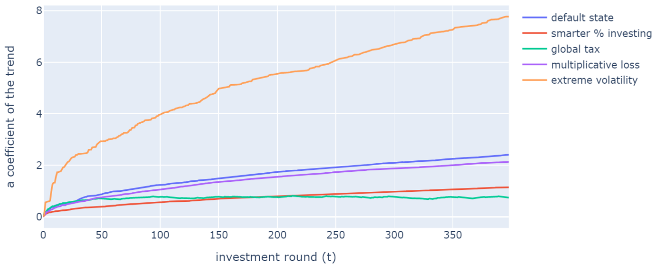

Fig. 3: a coefficient of the polynomial fit during each investment round for all the extended models.

D. Multiplicative loss

The final modification was to model loss in a different way. As described before, a person's investment is multiplied by $1+p$ when there is a profit, and by $1-p$ when there is a loss. However, this multiplication process will make the capital tend towards 0 for very large numbers when $p \in (0, 0.5]$:

$$\lim_{n\to\infty} (1+p)^{\frac{1}{2}n} \cdot (1-p)^{\frac{1}{2}n} = 0 \tag{27}$$

This leads to the average capital slowly decreasing, as seen in the simulations. We came up with a potential solution to this.

We call this final scenario "multiplicative loss". Instead of a loss taking the additive inverse of p (so $-p$ instead of p) and, adding that to 1, and multiplying that with the investment, we take the multiplicative inverse of $1+p$.
Then, a loss will lead to multiplication by $\frac{1}{1+p}$ instead of multiplying by $1-p$.

Then, multiplying a person's capital by losses and profits many times leads to:

$$\lim_{n\to\infty} (1+p)^{\frac{1}{2}n} \cdot {\frac{1}{1+p}}^{\frac{1}{2}n} = 1 \tag{28}$$

This should keep the total capital from decreasing.

We also chose to investigate this for reasons more related to economics than mathematics. The impact of a person losing half of their money is much more significant than gaining half (with $p = 50\%$, one either loses half of one's money or gets 50% extra). We could make the assumption that the impact of doubling one's money would be equal to losing half. Thus, it would be fair to say that in some theoretical markets, these have equal probabilities. Thus the probabilities of multiplying one's capital by 2 or by $\frac{1}{2}$ are equal in such a model.

V. Results of extended model simulation

Some models display interesting behaviours that deviate from the "standard model", which we will address here.

Figure 3 shows that the "smarter investing" and "extreme volatility" models are in essence very similar to the original

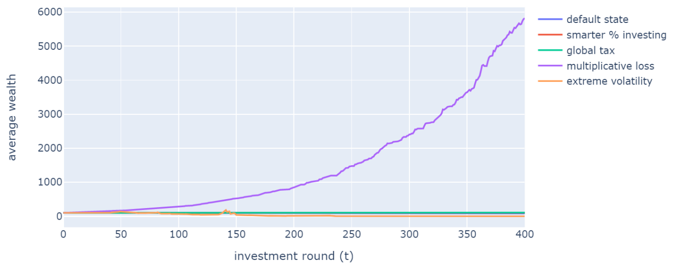

Fig. 4: Average wealth per round for all the extended models.

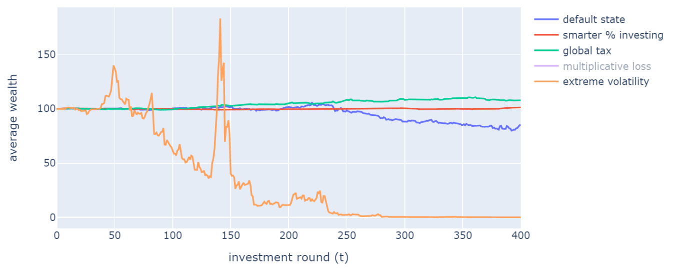

Fig. 5: Average wealth per round for all the extended models except multiplicative loss.

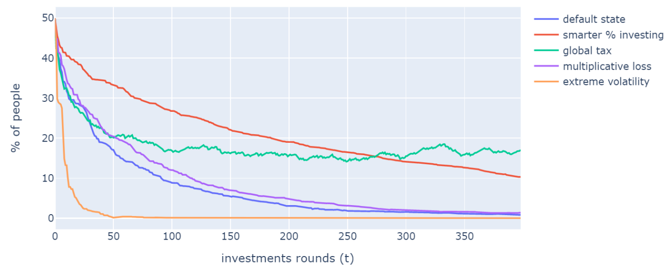

Fig. 6: Percentage of people that possess closest to half of all wealth for the extended models.

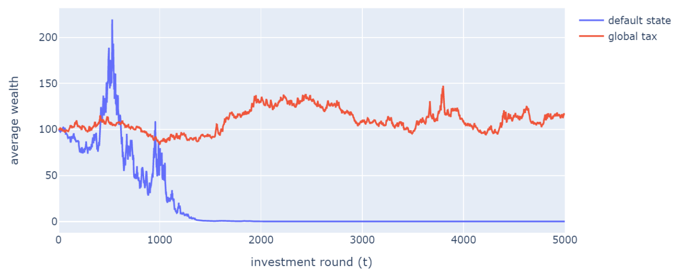

Fig. 7: Average wealth for the default scenario and the global tax scenario over a very long time.

model, except that they slow down and speed up (respectively) the process of the rich-get-richer effect. This can be explained because, at its core, the model is the same, only the variation in the random effect of the model is higher or lower, making the impact of time increments larger or smaller. Using a multiplicative loss leads to a very similar rich-get-richer effect, just slightly less pronounced. A global tax keeps the a coefficient between 0.7 and 0.8, ensuring that the distribution of wealth does not get too skewed.

Figure 4 shows that the real difference caused by the multiplicative loss scenario is that it creates a very productive market, leading to a vast increase in the average wealth. We are uncertain if this is realistic, but real economies do tend to grow over long periods of time.

Figure 5 shows that extreme volatility may lead to periods of high wealth, but can also easily lead to the market being wiped out. A global tax leads to a slight increase in average wealth, while smarter investments keep the average wealth from declining. Thus, more intelligent investing, a global tax and a multiplicative loss all lead to more positive outcomes (with more wealth and a more fair distribution), while volatility and the default scenario lead to a decline in wealth.

Figure 6 can be interpreted in the same way as Figure 3. Extreme volatility increases the concentration of wealth, while a global tax and smarter investment strategies decrease them. Multiplicative losses lead to a very slight decrease in the concentration of wealth.

Implementing a global tax leads to interesting results that we wish to explore some more. First off, the total wealth (w_{total}) no longer decreases. Instead, it slowly increases. We believe this can be explained as follows: a person losing 1% of their wealth does not cause them much harm. However, gaining $\frac{1}{N} \cdot \frac{w_{\text{total}}}{100}$ can be of tremendous help to someone who had bad luck during their earlier investments. Even after a very long run of 5000 rounds, the average wealth does not drop, while the standard model has long converged to 0 (see Figure 7). The other result is that the fit for the power law stabilises (in the case of a 1% tax) at $a = \pm 0.75\%$ (shown in Figure 3). In practice, this means that the share of people in possession of 50% of the wealth does not drop below 15% (shown in Figure 6), effectively preventing the power law from becoming too extreme. Increasing the tax makes the power law even less pronounced, while lowering it allows it to get closer to how the default model behaves. Starting at 0.003%, the a coefficient stabilises around $a = 2$, returning the model to the state of a power law.

VI. Conclusion

From the simulations we can conclude that this model has rich-get-richer effects, and power laws. Capital attaches itself to more capital, as predicted by Piketty [1] and Easley and Kleinberg [4].

The extended versions of the model mostly led to changes in how quickly the inequality in this market grows. One of the scenarios, where we added a flat tax on wealth, led to fascinating results. The stagnation of the inequality was achieved much more easily than thought, with very low tax rates. This also aligns with the conclusion from Piketty [1].

VII. Discussion

We were able to confirm the existence of the predicted power laws, and we were able to come to some unexpected conclusions on tax.

We had to make a few design decisions for the simulation. We are not entirely sure about these, but we decided to commit to them. For example, we decided not to round participants' capital to the nearest cent, because this introduced much more volatility in the model. If people lost most of their money, their capital rounded to 0, and they could never invest again. This led to uninteresting situations where only a few people have money while everyone else is bankrupt. We decided not to round any numbers, which seems unrealistic at first (real money is always rounded to cents), but it may actually still be a good approximation of the real world. After all, it is possible for stocks to be valued below 1 cent (penny stocks).

We did not end up simulating the model for a non-zero average profit/loss ratio. Normally, on a profit, one receives $x\%$ of their investment, and on a loss, they lose $x\%$ of their investment that round. It might have been interesting to do a simulation with a gain of $x + y\%$ on a profit, and a loss of $x - y\%$ on a loss, so that the average is not 0. However, due to time constraints, and the fact that we decided that other model parameters were more interesting to focus on, we discarded this idea. We do not expect that the results of this change would have been very different. We would still expect similar rich-get-richer effects, but now the total wealth would grow/decline (faster).

Different taxes produce different stabilizing points for the a coefficient. In future research, it would be interesting to plot a range of taxes against the point where a stabilises, or where the 50/50 split stabilises.

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