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Recursion relations for amplitudes in Quantum Field Theory

Bachelor Project

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Abstract

In this project we introduce the subject of on-shell recursion relations and discuss their applications. We start by introducing spinor helicity notation, QCD and little group scaling. Then we derive the on-shell recursion relations from complex deformations of the momentum, and factorization properties of the amplitudes. We consider the BCFW shift, and prove the Parke-Taylor formula with it.

We review the requirements for validity of on-shell recursion relations. Based on that we cover methods for calculating the boundary contribution for different theories, specifically focusing on subtracted recursion relations and soft shifts. We introduce a method for calculating the boundary contribution with k subtractions instead of $k+1$. We discuss when the method works, and show that it is applicable for soft shifts. Therefore new amplitudes in scalar EFTs are constructible. We show that amplitudes in $P(X)$ theories are constructible using this method, and discuss the implications of that. We argue that all scalar EFTs with any soft behavior are constructible.

Contents

1 Introduction

Quantum field theory (QFT) is the framework for some of our most successful theories of physics. It is widely used in particle physics and many other fields. Unfortunately many calculations in QFT are quite difficult. Conventional QFT also has some redundant structure. This has led people to study other ways of doing QFT calculations. The main object of interest is the S-matrix, which is the time evolution matrix from $t = -\infty$ to $t = \infty$. The non-trivial part of the S-matrix is $S = 1 + iT$, $\langle i| T |f\rangle = (2\pi)^4\delta^4(p_i - p_f)A(i \to f)$. The amplitudes A will be our main focus, they describe how a set of particles, i scatter off each other into f. The amplitude will depend on how many particles are scattering off each other (we say it is n-pt. if there are n particles involved), and which particles it is. There is much work on calculational tools in QFT and many different methods are known for calculating both tree and loop-level amplitudes as well as understanding the relations between amplitudes of different theories (see [1, 2] for a general overview of the whole topic).

The conventional method for calculating amplitudes is through Feynman diagrams. They are pictures that correspond to terms in a perturbative expansion of the amplitude. They include a set of rules for converting each diagram into a mathematical expression. Tree-level Feynman diagrams are diagrams without loops, whereas loop-level diagrams have loops. Tree-level diagrams are generally of a lower order in the perturbative expansion. In Feynman diagrams the momentum of internal particles (meaning particles not in i or f) don't have to obey $p^2 = m^2$. Particles that obey that relation are said to be on-shell, while the rest are off-shell. Unfortunately for many common processes, the number of Feynman diagrams increases very rapidly with the number of particles. Therefore, even relatively simple problems like n-pt. tree-level gluon scattering is hard to calculate for $n > 5$. Computational methods can help, but run into problems too. This forces us to search for other procedures for calculating scattering amplitudes. In this project we are specifically concerned with tree-level amplitudes, and all amplitudes will be tree-level unless otherwise stated.

Here there exists a well known method for simplifying calculations called on-shell recursion relations, discovered in [3] and explained in [4]. It is a way of calculating an amplitude in terms of lower point amplitudes. These lower point amplitudes will be on-shell, but with deformed momenta. Specifically the method involves shifting some subset of momenta into the complex plane and then taking the contour integral of the amplitude as a function of a complex variable. This integral can be calculated by considering residues of the amplitude, which will appear at points where the amplitude factorizes into lower point amplitudes. In this way the lower point amplitudes can be used to reconstruct higher point ones.

There are two reasons on-shell recursion is interesting. The first is just as a tool in the toolbox for calculating amplitudes or for proving some general results. The other reason is that on-shell recursion relations are often very general. We can sometimes calculate amplitudes purely from general properties of the S-matrix. This is because the recursion starts with spin and special relativity not a Lagrangian. All tree-level amplitudes can be found from that, if the momentum shift

(a) 3-pt. interaction of gluons *(b) 4-pt. interaction of gluons*

Figure 1: *Interaction vertices of gluons*

has good properties. By avoiding Lagrangians we avoid the extra off-shell structure they carry. Lagrangians sometimes have a unique form determined from the spin of its particles so we don't necessarily bring in less information by avoiding them.

To calculate amplitudes directly from general properties of the S-matrix we would need to prove factorization of amplitudes from that too. While this can be done (see [5] for a modern description), we will be satisfied by seeing factorization from the Feynman diagrams where it is more clear.

Beyond deriving on-shell recursion relations, we will also examine some further work on the topic. We will cover the requirements for validity of on-shell recursion. We introduce the subtracted recursion relations from [6], and consider an extension of them that, in some cases, requires fewer subtractions. We briefly review soft theorems and scalar effective field theories.

We introduce soft recursion relations [7], and by applying the previously mentioned idea show that the previously unconstructible $P(X)$ theories are constructible. We discuss this result, consider further work on the topic. We also argue that all scalar EFTs are constructible as long as they have a single soft theorem.

2 Spinor helicity notation and other background

Before introducing on-shell recursion relations we will review the theories we work in and introduce a useful notation. Most of this section has been explained in many other places. For good general introductions see [1, 8, 9], we will follow the notation and general approach of [1].

2.1 QCD and color ordering

The most important theory we will consider in this project is Quantum Chromodynamics (QCD), which is the theory of the strong force. Other theories will be introduced when relevant, in section 4.1. The strong force is mediated by gluons and acts on quarks. We focus on interactions between gluons. Gluons are massless spin 1 gauge bosons, that carry color charge. They can interact with each other through 3-pt. or 4-pt. interactions as seen in fig 1.

We will see later on that the 4-pt. interaction is unnecessary since all amplitudes can be constructed recursively, from the 3-pt. amplitudes. The Lagrangian for gluons is the Yang-Mills (YM) Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{1}$$

Where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu], \quad A_\mu = A_\mu^a T^a. \tag{2}$$

The generators T depend on the gauge group (QCD is based on $SU(3)$, but for our purposes other non-abelian groups like $SU(N)$ are allowed) and are responsible for the color. They transform in the adjoint representation of the gauge group. Color is the quantum number associated with the gauge group.

The amplitudes in YM theory will have color, and kinematic parts. We can separate these and write the full n-pt. tree-level amplitude as

$$A_n = g^{n-2} \sum_\sigma \mathrm{Tr}\big(T^{a_1} T^{\sigma(a_2} ... T^{a_n)}\big) A(1\sigma(2...n)). \tag{3}$$

Where σ's are permutations of the numbers. We can then focus on these partial (color-ordered) amplitudes, $A(1\sigma(2...n))$. We will do this in the rest of this text. This separation can be shown possible by Feynman diagrams, or alternatively by noting that any amplitude could be written as a linear combination of $\mathrm{Tr}\big(T^{a_1} T^{\sigma(a_2} ... T^{a_n)}\big)$ terms.
The color ordered amplitudes are not all independent. There are relations between the color ordered amplitudes, which are discussed in B.

2.2 Spinor helicity notation

We now introduce the spinor helicity formalism (first developed in [10], although not in its modern form). It is for our purposes just a convenient notation that simplifies various calculations of amplitudes.
Amplitudes are functions of the momenta of each particle in the interaction. In this project, we will focus on massless particles in interactions where all particles are outgoing, meaning that no particles come in from $t = -\infty$. A process with ingoing particles can by crossing symmetry (the symmetry between ingoing particles and outgoing antiparticles) be converted to one with all outgoing particles.
In many theories momentum will often appear in the form $\not{p} \equiv p_\mu \gamma^\mu$, where γ^μ is a set of matrices. This is because the Lorentz transformation properties of $\not{p}$ are necessary for equations of motion for spin-$\frac{1}{2}$ particles. The momentum $\not{p}$ can be written as a matrix with two 2x2 submatrices (In the Weyl basis of the γ-matrices)

$$\not{p} = \begin{bmatrix} 0 & p_{\alpha,\dot{\beta}} \\ p^{\dot{\alpha}\beta} & 0 \end{bmatrix}, p_{\alpha,\dot{\beta}} \equiv p_\mu(\sigma^\mu)_{\alpha,\dot{\beta}}, p^{\dot{\alpha}\beta} \equiv p_\mu(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}. \tag{4}$$

The dots on the indices indicate how the different parts transform. Note that $\det(p) = m^2 = 0$ in our case. For spin-$\frac{1}{2}$ particles in QED the Dirac equation gives us

$$-i\not{\partial}\Psi = 0, \Psi = u(p) \cdot e^{ipx} + v(p) \cdot e^{-ipx}. \tag{5}$$

This requires

$$\not{p} v_\pm(p) = 0, \overline{u}_\pm(p)\not{p} = 0. \tag{6}$$

We can choose to write that as

$$v_+(p) = \begin{bmatrix} |p]_\alpha \\ 0 \end{bmatrix}, v_-(p) = \begin{bmatrix} 0 \\ |p\rangle^{\dot{\alpha}} \end{bmatrix}, \text{etc.} \tag{7}$$

The use of these two new objects, the two component spinors $|p\rangle, |p]$, is the spinor helicity notation. Note that some of the literature uses these symbols to refer to the 4 component spinors $u_\pm, \overline{u}_\pm$, etc., while we use them for the 2 component ones. Positive helicity (meaning spin along the momentum) is associated with a square bracket, while negative helicity is associated with a normal ket. Now the Dirac eq., which for the massless case reduces to the Weyl eq. will be

$$p^{\dot{\alpha}\beta}|p]_\beta = 0, \text{etc.} \tag{8}$$

And our momenta will take the form $p_{\alpha\dot{\beta}} = -|p]_\alpha \langle p|_{\dot{\beta}}$ (The sign coming from the signature of the metric).

Let us note that one could start from just the vanishing determinant of the momentum, by using the theorem that any 2x2 matrix with vanishing determinant can be written as the product of two vectors, like

$$p_{\alpha,\dot{\beta}} = -\lambda_\alpha \tilde{\lambda}_{\dot{\beta}}. \tag{9}$$

As seen earlier the determinant of the momentum is zero (for massless particles). We can identify the λ's with our brackets, after which this gives the exact same expressions as previously. Another way to think of spinor helicity notation is that momentum transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group $SO(1,3)$. The spinors $\lambda, \tilde{\lambda}$ transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations respectively. So we have just broken the momentum down into these two simpler parts.
An extension of spinor helicity formalism with massive particles exists. It works by decomposing the momentum into two massless momenta, and using spinor helicity notation on those two. This will not be relevant for this project.

To effectively use spinor helicity notation we need some of the properties of the brackets. Firstly note that we can raise and lower indices easily with $|p\rangle^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \langle p|_{\dot{\beta}}$. We can also form products, as follows

$$\langle p_1, p_2\rangle = \langle p_1|^{\dot{\alpha}} |p_2\rangle_{\dot{\alpha}} \equiv \langle 12\rangle . \tag{10}$$

We now list a set of relations that will be used often:

$$(p_1 + p_2)^2 = 2p_1 \cdot p_2 = \langle 12\rangle [12], \tag{11}$$

$$\langle pq\rangle = -\langle qp\rangle , \langle p| q] = 0, \tag{12}$$

$$\langle 1| \gamma^\mu |2] \langle 3| \gamma_\mu |4] = 2 \langle 13\rangle [24], \tag{13}$$

$$\langle ri\rangle \langle jk\rangle + \langle rj\rangle \langle ki\rangle + \langle rk\rangle \langle ij\rangle = 0, \tag{14}$$

$$\sum_i \langle qi\rangle [ik] = 0. \tag{15}$$

The first one expresses the square of the sum of two light-like vectors. The second expresses symmetry properties of our brackets. The third is called the Fierz identity and follows from γ-matrix properties. The fourth is the Schouten identity and follows from the fact that three 2-vectors can't be linearly independent from each other. The last is an expression of momentum conservation.

We can now proceed to write other expressions in our new notation. Polarization vectors, $\epsilon^{\pm}$, are necessary to describe amplitudes with external gauge bosons. We can write the polarization vectors $\epsilon^{\pm}$ as:

$$\epsilon_{-}^{\mu} = -\frac{\langle p| \gamma^{\mu} |q]}{\sqrt{2}[qp]}, \epsilon_{+}^{\mu} = -\frac{\langle q| \gamma^{\mu} |p]}{\sqrt{2}\, \langle qp\rangle}. \tag{16}$$

Here q is the arbitrary reference momentum from gauge invariance. Other expressions can similarly be expressed in spinor helicity notation.

It is often useful to work with complex momentum $p^{\mu} \in \mathbb{C}$. Turning momentum complex highlights the analytic structure of amplitudes. With complex momentum the angle and square brackets are independent. If the momentum is real, then $[p|^{\alpha} = (|p\rangle^{\dot{\alpha}})^{*}$. With complex momentum 3-pt. amplitudes will generally be non-zero. We will use those amplitudes as the starting point in our recursion.

2.3 Little group scaling

The little group is the group of Lorentz transformations that leave the momentum of a particle invariant. It is a subgroup of the full set of Lorentz transformations. In our notation, a little group transformation can be written as

$$|p\rangle \to t\, |p\rangle\, , |p] \to t^{-1}|p]. \tag{17}$$

Clearly this will keep the momentum unchanged since $p_{\alpha\dot{\beta}} = -|p]_{\alpha}\, \langle p|_{\dot{\beta}}$. Little group scaling is important since it (together with locality) completely determines the 3-pt. gluon amplitudes. Under the transformation eq. 17, the amplitude will scale like

$$A(t\, |i\rangle\, , t^{-1}|i], ...) = t^{-2h_i} A(|i\rangle\, , |i], ...). \tag{18}$$

This can be seen by considering how external lines in Feynman diagrams would scale. This equation serves as a condition on amplitudes. At 3-pt. this condition (and locality) will completely fix all amplitudes. Consider the amplitude $A(1^{h_1}2^{h_2}3^{h_3})$. This will be a function of just angle brackets or just square brackets, since at 3-pt. all kinematical invariants are 0. We can write the amplitude generally as

$$A(1^{-}2^{-}3^{+}) = \langle 12\rangle^{a}\, \langle 23\rangle^{b}\, \langle 31\rangle^{c} \text{ or } [12]^{a}[23]^{b}[31]^{c}. \tag{19}$$

With the requirement for correct little group scaling, we get

$$2h_1 = a + c, 2h_2 = a + b, 2h_3 = b + c \text{ or } 2h_1 = -a - c, 2h_2 = -a - b, 2h_3 = -b - c. \tag{20}$$

Solving these 3 equations for the 3 unknowns we can find all 3-pt. amplitudes (up to a coupling constant). We still have an ambiguity about whether the amplitude is a function of only angle brackets or square brackets. This can be determined by dimensional analysis (using locality to determine which dimensions are permitted). We find these 3-pt. amplitudes without referring to any structure except their spin. This is connected to the fact that the Lagrangian for massless spin 1 particles, eq. 1, is unique. For a more complete treatment of little group scaling see [11].

3 On-shell recursion relations

3.1 General on-shell recursion relations

Scattering amplitudes, especially with many particles, are hard to calculate. We will now introduce on-shell recursion relations, as a tool for calculating scattering amplitudes efficiently. In this subsection we will discuss the derivation of the on-shell recursion relations, before in section 3.2 considering a specific type of recursion relation. On-shell recursion relations are both simple, effective, and general. They will generally have fewer terms than there would be Feynman diagrams. We will follow the notation and general approach of [1], as in the previous section.

In short the derivation of the recursion relations comes from shifting some subset of our momenta with a complex variable, and then finding the amplitude by integrating the amplitude as a function of the complex variable. This integration can be done through the Residue theorem, since the amplitude will factorize on its poles.
Consider the calculation of an n-pt. tree-level amplitude $A(1, 2, ..., n)$. The amplitude is a function of the momenta of each particle $p_1, ..., p_n$. The momenta satisfy $p_i^2 = 0$, and $\sum p_i = 0$, since the particles are massless and obey momentum conservation. We will now shift the momentum of some subset of our particles. We consider the simple shift

$$p_i \to \hat{p}_i = p_i + z \cdot r_i, \tag{21}$$

where z is a complex variable, and r_i is a 4-vector. The r_i are not physical quantities, they are chosen by us. More complicated shifts of the momenta are unnecessary for most purposes, see however [12] and section 3.3. The amplitude, A_n, is now a complex function $A(z)$. Importantly at $z = 0$ the amplitude is unchanged $A(0) = A_n$. Depending on the specific shift we are considering each r_i will be different. The r_i's are, however required to obey the following equations

$$r_i \cdot r_j = 0, \quad \sum r_i = 0, \quad p_i \cdot r_i = 0. \tag{22}$$

There is no implicit sum in the last equation (we are not using Einstein summation notation). These requirements are necessary for momentum conservation and for the shifted momenta to be on-shell. It will be important later to consider squares of sums of shifted momenta, since they appear in propagators. Let I be some subset of the particles. Define $P_I = \sum_{i \in I} p_i$, and likewise $\hat{P}_I$ and R_I. Then

$$\hat{P}_I^{\,2} = P_I^2 + z^2 R_I^2 + 2zP_I \cdot R_I = P_I^2 + 2zP_I \cdot R_I = \frac{-P_I^2}{z_I}(z - z_I), z_I \equiv -\frac{P_I^2}{2P_I \cdot R_I}. \tag{23}$$

At tree-level the complex amplitude function $A(z)$ is simple. It is a meromorphic function of the complex variable: all poles are simple poles, there are no branch cuts or other structure. The poles of $A(z)$ will appear at z_I, because the momenta in the propagators will take the form from eq 23.

Consider now the calculation of $A(0) = A_n$ through Cauchy's integral formula

$$\frac{1}{2\pi i}\oint_C dz\frac{A(z)}{z} = A(0) \tag{24}$$

Where C is some contour around the point zero, and not including any other poles of $\frac{A(z)}{z}$. This integral can be calculated through the Residue theorem. The amplitude can then be written as

$$A_n = A(0) = -\sum_{z_I}\text{Res}\left(\frac{A(z)}{z}, z = z_I\right) + B_0, \tag{25}$$

where B_0 is a possible residue at infinity. For a longer discussion of B_0 see section 4. In this section we will assume that $B_0 = 0$. If $A(z) \sim z^{-1}$ as $z \to \infty$, there is no residue at infinity, $B_0 = 0$. The residues at z_I are all calculable. To calculate them we use the factorization properties of amplitudes. These are statements about the properties of amplitudes when P_I^2 goes to zero, representing an internal particle going on-shell. In that limit the amplitude factorizes into lower point amplitudes. The relations say that

$$\lim_{P_I^2\to 0} P_I^2 A_n = \sum A_L A_R. \tag{26}$$

These relations apply for all sets of legs I, as long as the set I has between 2 and $n-2$ members. The left A_L and right A_R amplitudes will depend on the I. They are subamplitudes with the form $A(p_{i\in I}, -(\sum_{i\in I} p_i))$, meaning that they are amplitudes of the particles in I, and one extra particle with a momenta that serves to retain momentum conservation, the other subamplitude will depend on all the particles not in I. One can think of factorization coming from a sum of Feynman Diagrams. Some of them will give terms that have P_I^2 in the denominator (from the propagator), others will not. Those without will go to 0 in the limit, those with will clearly have two independent parts, one on the left of the propagator and one on the right. These will give the subamplitudes. More generally factorization comes from unitarity and locality. It can be argued, in general terms, through the optical theorem that amplitudes should factorize.
We will use this property in our complex case. At $z = z_I$ it is clear that $\hat{P_I}^2 = 0$, so $A(z)$ will factorize. The residue then is

$$\text{Res}\left(\frac{A(z)}{z}, z = z_I\right) = \lim_{z\to z_I}\left((z-z_I)\frac{A(z)}{z}\right) = \lim_{z\to z_I}\left(\frac{A(z)\hat{P}_I^2 z_I}{z\cdot(-P_I^2)}\right) = -A_L(z_I)\frac{1}{P_I^2}A_R(z_I). \tag{27}$$

Since taking $z \to z_I$ is equivalent to $\hat{P}_I^2 \to 0$. Applying this to our previous eq. 25, we get the full amplitude as

$$A_n = \sum_I A_L(z_I)\frac{1}{P_I^2}A_R(z_I). \tag{28}$$

This is the on-shell recursion relation. You can think of the sum over I as a sum over diagrams, see figure 2. Let us note two things. Firstly the subamplitudes are evaluated at z_I while the $\frac{1}{P_I^2}$ is

$$A = \sum_I$$

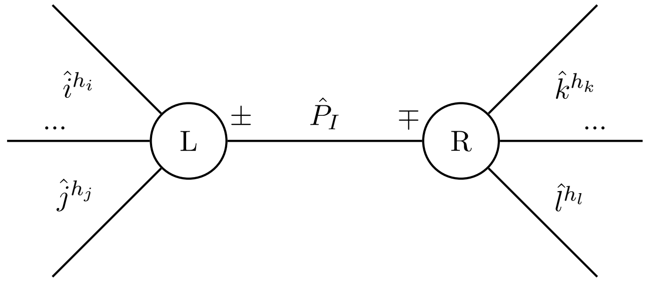

Figure 2: *Diagrammatic interpretation of the recursion relation. You sum over every possible combination of momenta I, with the constraint that $\hat{P}_I$ is shifted and you obey previous properties of the amplitude (so if it is ordered you only have to sum over ordered sets of momenta). We sum over the helicity of $\hat{P}_I$*

not. We use the notation $A(z_I) = \hat{A}$. Secondly a specific shift will give a specific recursion, but they will all have this general form. With this recursion relation, higher point amplitudes can be found from lower point ones. Since A_3 can be found directly from little group scaling (for gluons), we can then find $A_4, A_5, ...$ and all other amplitudes purely from on-shell recursion. A massive generalization of these recursion relations exists and can be gotten without many difficulties, see [13]. A loop-level generalization of on-shell recursion exists, but it is more limited in scope compared to the tree-level method [14].

3.2 BCFW recursion

Now that we have considered the general form of the recursion relations, we can consider a specific example. Let us look at the BCFW (Britto–Cachazo–Feng–Witten) recursion relation. This recursion relation was found in [3], and explained in [4]. The first paper to explain on-shell recursion generally was [4].
Here we shift

$$|i] \to |i] + z|j], |j\rangle \to |j\rangle - z\,|i\rangle\,, |i\rangle \to |i\rangle\,, |j] \to |j]. \tag{29}$$

This is equivalent to shifting two particles with the momentum $|j]\,\langle i|$. We call this a $[i, j\rangle$ shift. We will use this shift to find the tree-level MHV amplitude at n-pt. An MHV amplitude is one where two of the particles have negative helicity. An N^{k}MHV amplitude is an amplitude where $k+2$ particles have negative helicity. Our result will be the famous Parke-Taylor formula from [15]:

$$A_n = \frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,...\,\langle n1\rangle} \tag{30}$$

We want the amplitude $A(1^-, 2^-, 3^+, 4^+, ..., n^+)$. We will do a $[1, 2\rangle$ shift. Since the shifted momenta both have negative helicity the amplitude will go as z^{-1} so there is no boundary contribution. To find the MHV amplitude we first need to prove that $A(1^+, 2^+, ..., n^+) = 0$ and $A(1^-, 2^+, ..., n^+) = 0$ if $n \geq 4$. These properties are derived in appendix A.
The MHV amplitudes are tackled inductively. We assume that the $n-1$-pt. amplitude takes the Parke-Taylor form from eq. 30 (This is clearly true for the base case of $n = 3$). The n-pt.

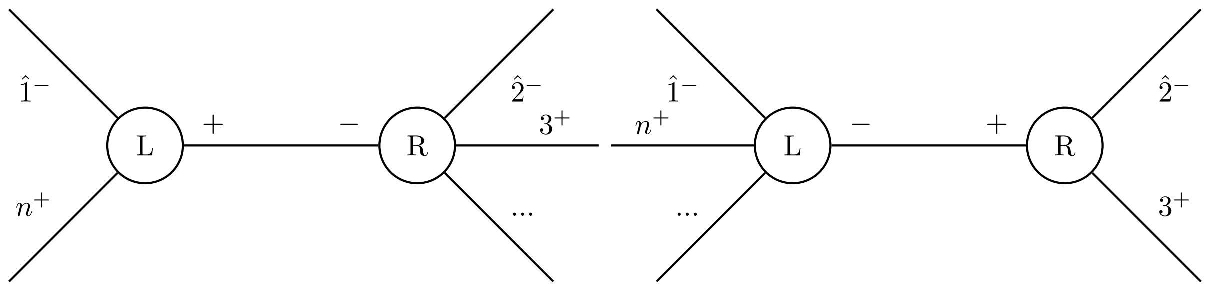

Figure 3: *Diagrams that could contribute to the Parke-Taylor amplitude.*

amplitude can be written as

$$A_n = \sum_I A_L \frac{1}{P_I^2} A_R = \sum_{k=4}^{n} \sum_{h_I=\pm} \widehat{A_{n-k+3}}(\hat{1}^-, \hat{P}_I^{h_I}, k^+, ..., n^+) \frac{1}{P_I^2} \widehat{A_{k-1}}(-\hat{P}_I^{-h_I}, \hat{2}^-, 3^+, ..., (k-1)^+). \tag{31}$$

Most of these terms end up being zero. Since 1 minus amplitudes are 0 unless they are 3-pt. we end up with two terms

$$\begin{aligned} A_n = {} & \widehat{A_3}(\hat{1}^-, -(\hat{p}_1 + p_n)^+, n^+) \frac{1}{(p_1+p_n)^2} \widehat{A_{n-1}}((\hat{p}_1 + p_n)^-, \hat{2}^-, 3^+, ..., (n-1)^+) + \\ & \widehat{A_{n-1}}(\hat{1}^-, (\hat{p}_2 + p_3)^-, 4^+, ..., n^+) \frac{1}{(p_2+p_3)^2} \widehat{A_3}(-(\hat{p}_2 + p_3)^+, \hat{2}^-, 3^+). \end{aligned} \tag{32}$$

See fig 3 for a diagrammatic view of these terms. The first term will also be zero due to a concept called special kinematics (see appendix A for a more complete description of special kinematics). Therefore we have

$$A_n = \widehat{A_{n-1}}(\hat{1}^-, (\hat{p}_2 + p_3)^-, 4^+, ..., n^+) \frac{1}{(p_2+p_3)^2} \widehat{A_3}(-(\hat{p}_2 + p_3)^+, \hat{2}^-, 3^+). \tag{33}$$

Now inserting the Parke-Taylor formula and simplifying we get

$$A_n = \frac{\left\langle 1\hat{P}_{23} \right\rangle^3}{\left\langle \hat{P}_{23} 4 \right\rangle \langle 45 \rangle ... \langle n1 \rangle} \frac{1}{[23] \langle 23 \rangle} \frac{[\hat{P}_{23} 3]^3}{[\hat{P}_{23} 2][23]} = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}. \tag{34}$$

This is the Parke-Taylor formula that we were looking for.

3.3 Requirements for validity

We consider now the requirements for the validity of on-shell recursion. As mentioned previously the recursion only works if $A(z) \sim z^{-1}$ or better as $z \to \infty$. If the amplitude does not have this behavior there will be a residue at infinity. We first consider which theories have good large z behavior. Afterwards we review methods for calculating the residue at infinity.

The large z behavior of an amplitude $A(z)$ depends on the specific shift you do, the specific particle numbers you shift and the specific theory the amplitude is calculated in. Let us first consider the large z behavior of QCD, under BCFW shifts. The amplitude will go as z^{-1} if adjacent particles with helicities $[-,-\rangle, [-,+\rangle, [+,+\rangle$ are shifted. Therefore our previous calculations were valid. A

$[+,-\rangle$ shift, will go as z^3. Nonadjacent particles give the amplitude one extra power of z^{-1} such that the behavior is z^{-2} and z^2 for the two types of shifts.
Let us now consider the methods for deriving the large z behavior of amplitudes. In [4] it was partially argued from Feynman diagrams. That method doesn't work in general cases, due to unexpected cancellations often leading to better than expected behavior. Let's consider two other methods. The first comes from [16]. The physical interpretation of the $z \to \infty$ region is that of a hard particle moving through a soft background. We can decompose our fields into a soft background, corresponding to the non shifted particles with comparably low energies, and quadratic fluctuations from the hard particles. If we decompose our field like $A_\mu = A_{soft,\mu} + a_\mu$, we can separate into a relevant Lagrangian (by expanding the terms and keeping the a_μ quadratic terms)

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}(A_{soft}+a)F^{\mu\nu}(A_{soft}+a)) \implies \mathcal{L}_{rel} = -\frac{1}{4}\,\mathrm{Tr}\big(D_{[\mu}a_{\nu]}D^{[\mu}a^{\nu]}\big) + \frac{i}{2}\,\mathrm{Tr}([a_\mu, a_\nu]F^{\mu\nu}) \tag{35}$$

Here D is the covariant derivative wrt. A_{soft}. This Lagrangian is further simplified with gauge fixing. We can on the basis of this Lagrangian analyze the large z behavior by considering how different terms will depend on z. This method gives a physical interpretation of the situation, and involves well known methods for finding the behavior. However it is built on Lagrangians, and part of the motivation for studying the on-shell recursion relations is to bypass them.
The large z behavior can also be found inductively, as done in [17]. An inductive derivation is valuable since it bypasses any Lagrangians. This method is somewhat harder to generalize to other theories and shifts.

These methods and others (like simple dimensional analysis) can be used to check whether a large group of field theories have good large z behavior. This question is addressed in different ways in [18, 19, 20]. In the last of these articles they find that (there is more to it than this, but as a summary)

$$k \leq 1 - \frac{v-3}{v-2}(m-2) - [g] - \sum_{all} \tilde{s} \tag{36}$$

where the amplitude goes as z^k, v is the number of particles that can interact at a vertex, m is the number of momenta you shift, $[g]$ is the dimension of the coupling constant and $\tilde{s}$ is the spin. They find that most useful theories are constructible. All renormalizable ones are and some non-renormalizable ones are too. Only some EFTs cannot be constructed, by this method. The intuitive answer to which theories are constructible is that recursions can't access information about interactions that are not there in the lower particle amplitudes. So it is only theories that are fixed based on some low point information that are constructible.

For most practical purposes this is sufficient, but it it physically unsatisfactory. The large z behavior depends on which legs you shift, and it depends on how many legs you shift not just on the theory. Instead we can consider calculating the residue at infinity. There have been several approaches to this some of which will be expanded on in section 4. Here we review some background.

We will focus on the idea that the boundary can be calculated by using known properties of the amplitude, specifically known points of $A(z)$. The first expression of this idea was in [21] (see also [22] which built on it). They propose the use of zeroes of the amplitudes to find the boundary (meaning z's for which $A(z) = 0$). They show that knowledge of all roots gives access to the boundary. The basic idea is to take the contour integral around each of the s zeroes $z_0^{(s)}$

$$0 = \frac{1}{2\pi i}\oint_{C_0^{(s)}} dz \frac{A(z)}{(z - z_0^{(s)})^r} = (-1)^{r-1}\sum_I \frac{\hat{A}_L \hat{A}_R}{(z_0^{(s)} - z_I)^r(-2P_I \cdot q)} + \delta_{r,1}B_0 + \sum_l B_l z_0^{(s)\ l-r+1} \tag{37}$$

Where r is the multiplicity of the zero, $C_0^{(s)}$ is a contour around the root and B_0 is the boundary (and B_l is a set of other constants). Clearly you get a whole set of these equations (from each zero). These equations can be solved to give

$$A_n = \sum_I \frac{\hat{A}_L \hat{A}_R}{P_I^2} \prod_l \left(1 - \frac{P_I^2}{P_I(z_0^{(l)})^2}\right). \tag{38}$$

They argue that some of the roots can be found by looking at factorization limits but are unsuccessful in finding a general expression for all roots. Their method can be extended to using any known points (not just roots). This method is called the subtracted recursion relation and was introduced in [6] (see also [23, 24]). For an amplitude that goes as z^k we need $k+1$ known points to fix the boundary. The relations can be derived by choosing a contour around zero, and integrating the function

$$A_n = \oint_C dz \frac{A(z) \cdot z_p^{(1)} \cdot z_p^{(2)} \cdot \ldots \cdot z_p^{(k+1)}}{z \cdot (z_p^{(1)} - z)(z_p^{(2)} - z) \cdot \ldots \cdot (z_p^{(k+1)} - z)} = \oint_C dz \frac{A(z)}{z} \prod_v \frac{z_p^{(v)}}{(z_p^{(v)} - z)}. \tag{39}$$

Where the $z_p^{(v)}$ are the point where we know the amplitude $A(z_p^{(v)})$. We recall that the formula for the residue of a pole of order n is

$$\mathrm{Res}(f(z), z = z_I) = \frac{1}{(n-1)!} \lim_{z \to z_I} \left(\frac{d^{n-1}}{dz^{n-1}} (z - z_I)^n f(z) \right). \tag{40}$$

This gives us our formula

$$A(0) = \sum_I \frac{\hat{A}_L \hat{A}_R}{P_I^2} K_{sub}(z_I) + \sum_v A(z_p^{(v)}) \cdot K_{sub\backslash v}(z_p^{(v)}), \tag{41}$$

where $K_{\mathrm{sub}}(z) = \prod_{v \in \mathrm{sub}} \frac{z_p^{(v)}}{(z_p^{(v)} - z)}$, with sub $= \{1, 2, ..., k+1\}$ and $K_{sub\backslash i}(z)$ is just $K_{sub}(z)$ without the i'th constant. This is the subtracted recursion relation. This is a generalization of the zeroes method since now all points work, not just zeroes.

The given derivation is different from that given in the cited articles. This derivation is chosen to emphasize a general pattern for extensions of on-shell recursion relations. They can generally be written as

$$A_n = \oint_C dz \frac{A(z)}{z} f(z). \tag{42}$$

Different choices of $f(z)$ can be chosen to fix bad large z behavior. Integrals like eq. 42 are equivalent to a type of nonlinear shift like $p_i \to g(z)(p_i + zq), \ldots$, where the relation between $f(z)$

and $g(z)$ will depend on the specific type of shift.

In some cases one might know $A'(z_p^{(v)})$, but not $A(z_p^{(v)})$. From the subtracted recursion relations it is not clear, whether that can be used. We will now show that it can. We fully know $A^{(k+1)}(z)$ (the $k+1$'st derivative) because we can calculate the integral (where C is a contour around w)

$$\frac{(-1)^k}{(k+1)!}A^{(k+1)}(w) = \frac{1}{2\pi i}\oint_C dz \frac{A(z)}{(w-z)^{k+2}} = \sum \text{Res}\left(\frac{A(z)}{(w-z)^{k+2}}, z = z_I\right). \tag{43}$$

Then we can integrate $A^{(k+1)}(z)$ until we get back the amplitude as

$$A(z) = A_c(z) + B_0 + B_1 \cdot z + B_2 \cdot z^2 + ..., \qquad A_c(z) = \int ... \int dz...dz A^{(k+1)}(z) \tag{44}$$

Here $A_c(z)$ is the constructible part of the amplitude. We see that the boundary term is an integration constant. This form of the amplitude can also be derived from the general analytic structure of $A(z)$, as a meromorphic function. Treating the boundary this way will be necessary for the next section, where relations between the integration constants B_i are considered.

4 Calculations of boundary contributions

4.1 EFTs and soft theorems

In this section we introduce Effective Field Theories (EFTs) and soft theorems. Soft theorems are theorems about the behavior of amplitudes when the momentum of one of the particles go to zero (see [25] for a longer review). Soft theorems generally take the form

$$\lim_{p_i \to 0} A_{n+1}(i) = \sum_j S^{(j)} A_n. \tag{45}$$

Here $S^{(j)}$ are the soft terms. $S^{(0)}$ is the leading term, and goes as $\frac{1}{p_i}$, while the sub leading term goes as $S^{(1)} \sim p_i^0$, and so on. In gauge theory these terms are universal. In general some of the terms might be zero. Soft theorems are widely useful and are often connected to asymptotic gauge symmetries. For the method in section 4.2, we only need the leading order terms. Gluons obey the soft theorem (to leading order)

$$\lim_{p_i \to 0} A(1^{h_1}, ..., i^{h_i}, ..., n^{h_n}) = S_{h_i}(i-1, i, i+1) \cdot A(1^{h_1}, ..., (i-1)^{h_{i-1}}, (i+1)^{h_{i+1}}, ..., n^{h_n}) \tag{46}$$

$$S_{h_i}(i-1, i, i+1) = \begin{cases} \frac{\langle (i-1)(i+1)\rangle}{\langle (i-1)i\rangle\langle i(i+1)\rangle} \text{ if } h_i = +1 \\ \frac{[(i-1)(i+1)]}{[(i-1)i][i(i+1)]} \text{ if } h_i = -1 \end{cases} \tag{47}$$

This theorem can be proved by analyzing Feynman diagrams with soft particles, but also with on-shell recursion relations. Recursion relations can be used to find soft theorems either through direct observation, see [26] or through a specific type of shift.

This shift was introduced in [27]. There soft theorems are derived directly from on-shell recursion methods. This method also determines the soft theorems in effective field theory extensions of

gauge theory.
Scalar field theories also have soft theorems. The theories we will consider generally obey

$$\lim_{p_i \to 0} A(i) = O(p_i^\sigma). \tag{48}$$

Here $\sigma \geq 1$ describes how fast the amplitude goes to zero. Amplitudes where a Goldstone boson (a boson resulting from spontaneous symmetry breaking) goes soft will vanish in the soft limit. This is called the Adler Zero. The soft degree σ will generally depend on the amount of derivatives per field. In some cases σ is larger than expected, we say those theories have enhanced soft behavior.

Effective Field Theories are theories that describes physics far below a specific energy scale Λ, but not at scales above that [9]. EFTs are generally nonrenormalizable. If we have a theory $\mathcal{L}_{full}$ of different particles, where some are much lighter than others we can get an effective theory valid at energies close to the mass of the light particles. This is done by integrating out the heavier particles, which creates effective higher order terms in the light particles. For example QCD has an EFT called the non-linear sigma model (NLSM) which is based on the spontaneous symmetry breaking of the approximate chiral symmetry in QCD. It is a theory of the lighter pions, without heavier particles. It is often the case that EFTs are constructed by keeping all possible terms in a Lagrangian that are consistent with some symmetry, with the higher order terms suppressed by the cutoff scale Λ.
In this project we are mainly concerned with scalar EFTs. We can write a general Lagrangian as

$$\mathcal{L} = \sum_m \sum_n \lambda_{m,n} \partial^m \phi^n, \tag{49}$$

with $\lambda_{m,n}$ being the coupling constants. Theories with several fields are not considered (although many of the results can be extended). Theories with fixed

$$\rho = \frac{m-2}{n-2} \tag{50}$$

have particularly nice properties [28]. We will focus on those. Examples are, among others, the Non-Linear Sigma Model, Dirac-Born-Infeld theory, Galileons and $P(X)$ theory. These theories can be classified by (ρ, σ), see [29, 28]. We will specifically focus on $P(X)$ theories, which are theories where

$$X = \partial\phi\partial\phi, \tag{51}$$

$$\mathcal{L} = P(X) = \sum_n c_n X^n. \tag{52}$$

These theories have $(\rho, \sigma) = (1, 1)$ (although a specific choice of c_n can give $\sigma = 2$). They obey a shift symmetry $\phi \to \phi + a$, although in applications that symmetry is often spontaneously broken. $P(X)$ theories have applications in cosmology, where they are used for some models of inflation and matter.

4.2 How do we know points on the amplitude?

There are many ways of finding known points of the amplitudes. The most systematic and clear way is through soft shifts which this subsection will introduce. The idea was developed in [7]. We consider amplitudes where the soft limit is

$$A_n \sim p^\sigma \text{ as } p \to 0 \tag{53}$$

with $\sigma \geq 1$. The method can be extended to other soft behavior too, see [23]. The EFTs with this soft behavior were previously not constructible. We shift all our momenta like

$$p_i \to p_i(1 - a_i z) \tag{54}$$

where the a_i have to be chosen to obey momentum conservation (and be non-trivial). The momentum clearly goes soft when $z = \frac{1}{a_i}$. We consider the contour integral

$$A_n = \frac{1}{2\pi i} \oint_C dz \frac{A(z)}{z \cdot \prod_i (1 - a_i z)^\sigma}. \tag{55}$$

This is just our subtracted recursion relations applied to the soft shifts. Now everything proceeds as before and we get an expression for our amplitude purely in terms of lower point amplitudes. This means that (using this method) we can construct theories that go as z^k as long as $k - n \cdot \sigma \leq -1$. With the notation introduced in the previous section for scalar EFTs we can rephrase that requirement as

$$\rho \leq \sigma \text{ and } (\rho, \sigma) \neq (1, 1) \tag{56}$$

In the next section we will show that theories with $(\rho, \sigma) = (1, 1)$ can be constructed by another procedure.

4.3 Calculating the Boundary Contribution

4.3.1 A k (not $k+1$) subtracted recursion relation?

As we have seen, the amplitude takes the form

$$A(z) = A_c(z) + B(z) = A_c(z) + B_0 + B_1 \cdot z + B_2 \cdot z^2 + ... + B_k \cdot z^k \tag{57}$$

Therefore it seems like we need $k+1$ pieces of information to fix all the constants. But that is clearly not always true. The integration constants are not independent from each other, knowing just B_0 gives all the other constants by the shift. So there exists a known transformation that takes

$$S \cdot B_0 \to B_0 + B_1 \cdot z + B_2 \cdot z^2 + ... \tag{58}$$

or just $B_0 \to (B_1, B_2, ..., B_k)$. This is the same transformation that takes $S \cdot A_n = S \cdot A_c(0) + S \cdot B_0 = A_c(z) + B(z) = A(z)$, which we know how to do by hand (just applying the shift eq. 21). If this transformation is invertible then it can be exploited to get the boundary contribution with less knowledge than previously used (k known points, instead of $k+1$). This has not been considered before to my knowledge. Whether this transformation is invertible will depend on specifics of the amplitudes and the shifts. The boundary contribution has to obey two requirements for the transformation to be invertible.

(1) The boundary can always be written as a sum of different kinematic terms

$$B_0 = \sum_N c_N T_N = \sum_N c_N s_{12}^{a_{12}} s_{13}^{a_{13}} \cdot ... \cdot s_{23}^{a_{23}} \cdot ..., \tag{59}$$

where $N = \{(a_{12}, a_{13}, ...)|\text{obeying constraints}\}$ is the set of exponents consistent with constraints on the amplitude, and c_N are coefficients for each set of exponents (in spinor helicity notation this would be $\sum c_N \langle 12\rangle^{a_{12}} \langle 13\rangle^{a_{13}} ...[12]^{b_{12}}...$). We will only keep terms independent of each other. **We require that each term in the sum is shifted under the transformation S**. That is to say that $S \cdot s_{12}^{a_{12}} s_{13}^{a_{13}} \cdot ... \cdot s_{23}^{a_{23}} \cdot ... \neq s_{12}^{a_{12}} s_{13}^{a_{13}} \cdot ... \cdot s_{23}^{a_{23}} \cdot ... \forall \{a_{12}, a_{13}, ...\} \in N$.
(2) **Every term should be identifiable solely from the z dependent parts**. As an example consider two terms like

$$\frac{(\langle 12\rangle - z\langle 32\rangle)^2}{\langle 34\rangle^2} \text{ and } \frac{(\langle 14\rangle - z\langle 34\rangle)^2 \langle 23\rangle^2}{\langle 34\rangle^4}, \tag{60}$$

(these appear under a bad BCFW shift for an MHV amplitude). If we expand each of these terms and look at the z^2 parts we get $\frac{\langle 23\rangle^2}{\langle 34\rangle^2}$ from both terms. The terms are indistinguishable at the z^2 level. If we expand to the z level we get $\frac{\langle 12\rangle\langle 23\rangle}{\langle 34\rangle^2}$ and $\frac{\langle 14\rangle\langle 23\rangle^2}{\langle 34\rangle^3}$, the terms are identifiable at the z level since we can recreate their exact form purely by the z-dependent terms. In this example the terms are identifiable, but it is possible for terms (generally terms that are only shifted once) to be unidentifiable from their z dependent parts.

If both of these conditions hold the **you only need k known points, not $k+1$ points to find the boundary**.
These conditions do not hold under generic shifts. To see concretely how the transformation is inverted if the conditions hold, consider known $B_1, B_2, ..., B_k$. We can reconstruct B_0 by considering each constant B_i, then listing all possible terms in B_0 that would give the correct B_i. Clearly some terms in B_0 would, under the shift S, give results inconsistent with our known B_i. We repeat this procedure for each i, and under our previous conditions this is enough to completely fix B_0. See appendix C for notes about concrete calculations with this method.

Even if the conditions are not true, the method still constrains the value of B_0 somewhat. It is possible that this method could be combined with other constraints on the amplitude, to get a k subtracted recursion relation for general amplitudes.

4.4 On-shell constructibility of P(X) theory

We will now show that $P(X)$ theories are constructible, by a soft recursion relation, using the previous method. As mentioned in section 4.2 theories where $(\rho, \sigma) = (1, 1)$, were not thought to be constructible. Previously only "Exceptional" theories (theories with enhanced soft behavior) were thought to be constructible. This is because the soft behavior of $P(X)$ is only strong enough to get $\frac{A(z)}{\prod_i (1-a_i z)^\sigma} \sim z^0$. This is however the exact case where the previous method is applicable.
We have to prove that $P(X)$ theories under a soft shift obey the two previously mentioned conditions. The first condition is trivially obeyed, since the soft shifts are all-line shifts. There can be no unshifted terms, if every momentum is shifted. The second condition also holds. Consider the possible terms in the boundary

$$T_N = s_{12}^{a_{12}} \cdot s_{13}^{a_{13}} \cdot ... \cdot s_{23}^{a_{23}} \cdot ... \tag{61}$$

By dimensional analysis $a_{12} + a_{13} + ...$ is determined. Therefore each term, T, will be shifted maximally ($S \cdot T_N \sim z^k \forall T_N$). We should note that $a_{ij} \geq 0$, since the boundary is a polynomial in z.
We can uniquely identify each term solely from the z^k term, since it will be $z^k T_N \cdot \prod_i a_i$ (with the product being over each momentum that appears in the term). Therefore each term is identifiable.

We have now shown that $P(X)$ theories under soft shifts obey both conditions so $P(X)$ is constructible (up to coupling constants). The constructibility has also been confirmed by hand for the lower point amplitudes, by shifting the correct amplitude and checking that no issues arise. One practical issue with k subtracted recursion relations is that the amount of terms one has to analyze rapidly increases with n. For the 6-pt. $P(X)$ amplitudes there are naively around 1000 parts (though because of the previously discussed structure one can get by with only analyzing 15 terms). Therefore, while this method can show that a theory is constructible, the actual calculation does not retain the simplicity associated with on-shell recursion.

4.4.1 Constructibility of other scalar field theories

We will now consider the constructibility of other scalar EFTs under soft recursion relations. We will argue that all scalar EFT amplitudes, that obey known soft theorems, can be calculated by soft recursion. The method is similar to that employed for the $P(X)$ example. It should be noted that this section has not been verified by calculations of specific amplitudes and should therefore be read cautiously.

Any amplitude will have the form

$$A_n = A_c(0) + B_0, S \cdot A_n = A_c(z) + B_0 + B_1 \cdot z + ... + B_k \cdot z^k \tag{62}$$

The boundary B_0 will as a scalar quantity dependent on the momenta always be expressible as

$$B_0 = \sum_N c_N T_N = \sum_N c_N s_{12}^{a_{12}} \cdot s_{13}^{a_{13}} \cdot ... \cdot s_{23}^{a_{23}} \cdot ... \tag{63}$$

We require that $\sum a_{ij}$ correspond to the correct dimension of the boundary. On-shell recursion only fixes amplitudes up to their coupling constants, which creates an ambiguity in the amplitudes. In scalar EFTs there will generally be several different coupling constants, with different dimensions. We can separate the boundary (and A_c) in terms with different dimensions, and calculate each separately. Therefore we will only consider the case with one dimension of the boundary. Under the shift, each term T_N will go as:

$$S \cdot T_N = (s_{12}^{a_{12}} \cdot (1-a_1 z)^{a_{12}} \cdot (1-a_2 z)^{a_{12}}) \cdot (s_{13}^{a_{13}} \cdot (1-a_1 z)^{a_{13}} \cdot (1-a_3 z)^{a_{13}}) \cdot ... \cdot (s_{23}^{a_{23}} \cdot (1-a_2 z)^{a_{23}} \cdot (1-a_3 z)^{a_{23}}) \cdot ... \tag{64}$$

Since the boundary under the shift will be a pure polynomial, we will require that $a_{ij} \geq 0$. We will pick a set $\{T_N\}$, where all a_{ij} follow our previous requirements and where there is no redundancy (meaning that we also throw out terms that are equivalent through relations between the Mandelstam variables). Every boundary will be expressible in terms of a linear combination of those $\{T_N\}$. The structure of the soft shift forces each higher order coefficient into a very specific form. They take the general form:

$$B_i = \sum_N \left(c_N T_N \cdot \sum_\sigma \prod_{j \in T_N}^{i} a_j \right) \tag{65}$$

Here we sum over different permutations of indices in T_N and the sum is of products of the a_i. An example will make this notation and formula more clear. Consider a term like $T_N = s_{ij} \cdot s_{kl} \cdot s_{mn}$. The terms in the coefficients coming from that will be

$$B_1 = s_{ij} \cdot s_{kl} \cdot s_{mn} \cdot (a_i + a_j + a_k + a_l + a_m + a_n) = T_N \cdot \sum_{v \in T_N} a_v \tag{66}$$

$$B_2 = s_{ij} \cdot s_{kl} \cdot s_{mn} \cdot (a_i \cdot a_j + a_i \cdot a_k + ...) = T_N \cdot \sum_{v,u \in T_N} a_v \cdot a_u \tag{67}$$

$$\vdots \tag{68}$$

$$B_k = s_{ij} \cdot s_{kl} \cdot s_{mn} \cdot a_i \cdot a_j \cdot a_k \cdot a_l \cdot a_m \cdot a_n = T_N \prod_{i \in T_N} a_i \tag{69}$$

We see here that the coefficients are not at all independent. The assumption of independence of the coefficients is the reason that previous approaches were weaker.

Like previously we have the transformation S that takes $B_0 \to (B_1, ..., B_k)$, and it is invertible. But the invertibility properties are significantly stronger here since just B_k, for example, is enough to get B_0. This suggests the possibility of a 1-subtracted recursion relation instead of a k or $k+1$ subtracted relation. All possible scalar EFT amplitudes will have at least 1 subtraction, as long as any of the momenta obey a soft theorem. We no longer require strong enough soft behavior, such that $\sigma \cdot n$ is larger than k. Let us now show how we find the boundary from one known point (specifically one soft point). We start by calculating $A_c(z)$ through the conventional method (A_c is just what you get if you go through with on-shell recursion on an amplitude with bad large z behavior). At the soft point $z = \frac{1}{a_i}$ we have

$$0 = A\left(\frac{1}{a_i}\right) = A_c\left(\frac{1}{a_i}\right) + B_0 + B_1 \cdot \frac{1}{a_i} + ... + B_k \cdot \frac{1}{a_i^k} \implies B_0 + B_1 \cdot \frac{1}{a_i} + ... + B_k \cdot \frac{1}{a_i^k} = -A_c\left(\frac{1}{a_i}\right) \tag{70}$$

The left term can be written as a linear combination of elements in the previously introduced set $\{T_N\}$:

$$B_0 + B_1 \cdot \frac{1}{a_i} + ... + B_k \cdot \frac{1}{a_i^k} = \sum_N T_N \cdot \left(c_N + c_N \frac{1}{a_i} \cdot \sum_{v \in N} a_v + c_N \frac{1}{a_i^2} \cdot \sum_{v,u \in N} a_v \cdot a_u + ... + c_N \frac{1}{a_i^k} \prod a_i \right) \tag{71}$$

The right hand side can also be written as a linear combination of elements in $\{T_N\}$ (otherwise equation 70 could not be true). Since $A_c\left(\frac{1}{a_i}\right)$ is fully known, we know all constants k_N in

$$A_c\left(\frac{1}{a_i}\right) = \sum_N k_N T_N. \tag{72}$$

Then we can find all the constants c_N (and thereby the entire boundary) by comparing

$$\left(c_N + c_N \frac{1}{a_i} \cdot \sum_{v \in N} a_v + c_N \frac{1}{a_i^2} \cdot \sum_{v,u \in N} a_v \cdot a_u + ... + c_N \frac{1}{a_i^k} \prod a_i \right) = -k_N, \tag{73}$$

and solving for c_N. With this the entire boundary contribution of an amplitude is constructed from a single subtraction.

This method is general and can be applied independent of how bad the large z behavior is. Therefore it is widely applicable to scalar EFTs, previously thought to be inconstructible. This includes scalar EFTs without enhanced soft behavior. Whereas we previously had the requirements $\rho \leq \sigma$ and $(\rho, \sigma) \neq (1,1)$, our results suggest that a single known soft limit (with any value of $\sigma \geq 1$) is sufficient. In [28] only around 5 theories obey the previous inequality, with this new method the infinitely many theories with trivial soft behavior should be constructible. Instead of a small set of exceptional theories being constructible we now see that generic theories may be.

Since on-shell recursion doesn't produce a coupling constant, this method does have ambiguities in theories with several different coupling constants of the same dimension. As mentioned earlier the calculation of the boundary can also be computationally challenging.

5 Conclusion

In this project we proved that $P(X)$ theory was constructible through soft shift on-shell recursion relations. To do this we first introduced spinor helicity notation, and reviewed some basic aspects of QFT. We then introduced on-shell recursion relations. These recursion relations are derived by deforming momenta into the complex plane, and taking the contour integral of $\frac{A(z)}{z}$. That integral is calculated by the residue theorem, and we showed that amplitudes will factorize into lower point amplitudes at the residues. The on-recursion relations show that all tree-level amplitudes can be constructed from just the lowest point amplitudes.

We considered BCFW recursion relations, which is a specific type of on-shell recursion relation. From that we proved that all plus and one minus amplitudes are zero and found that the MHV amplitude followed the Parke-Taylor formula. We discussed the requirements for validity of on-shell recursion. We focused specifically on methods to find the boundary at infinity. We introduced the $k+1$ subtracted recursion relations (and the special case of them where only roots are used). We introduced a set of scalar EFTs, and discussed their properties. We specifically focused on $P(X)$ theories. They have some relevance for cosmological models.

We showed that amplitudes where the boundary has no 0-shift terms and is identifiable can be calculated by k subtractions instead of $k+1$ subtractions. This method works by constraining the terms in the boundary based on the knowledge of the shift of the momenta. We introduced soft shifts, and proved that they only need k subtractions, not $k+1$ as previously believed. Using this we showed that $P(X)$ theories are constructible. We discussed the implications of this, and the limits of the method. It would be interesting to develop a more concrete criterion for when the method works. It would also be interesting to give concrete bounds on the information contained in higher boundary coefficients. We argue that general scalar EFTs are constructible. It would be very interesting to extend this to theories with other kinds of soft behavior. Another interesting open problem is to generalize the method.

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A All-plus and one-minus amplitudes

We now show that $A(1^+, 2^+, ..., n^+) = 0$ and $A(1^-, 2^+, ..., n^+) = 0$ if $n \geq 4$ These properties can be proven by showing that polarization vector products that appear in these amplitudes are zero. We can also show it from recursion. To prove the first statement note that the 3-pt. all plus amplitude is zero, $A(1^+, 2^+, 3^+) = 0$ by little group scaling and locality. No 3-pt. all plus amplitude can have correct little group scaling, correct dimension and be a function of all square or all angle brackets. The higher point amplitudes, $A(+++...+)$, can be calculated recursively, and are always zero since one of either the left or right subamplitudes will be zero. To prove the second statement we proceed in a similar way. This is done inductively. The 3-pt. 1 minus amplitude exists and is

$$A(1^-, 2^+, 3^+) = \frac{[23]^3}{[31][12]}. \tag{74}$$

The 4-pt. amplitude can be written as

$$A(1^-, 2^+, 3^+, 4^+) = \hat{A}(\hat{1}^-, -\hat{P}_{14}^+, 4^+) \cdot \frac{1}{P_{14}^2} \cdot \hat{A}(\hat{P}_{14}^-, \hat{2}^+, 3^+). \tag{75}$$

We now simplify that expression

$$A(\hat{1}^-, -\hat{P}_{14}^+, 4^+) = \frac{[(-\hat{P}_{14})4]^3}{[4\hat{1}][\hat{1}(-\hat{P}_{14})]}, \tag{76}$$

$$\hat{P}_{14}^2 = \langle 14\rangle [\hat{1}4] \implies [4\hat{1}] = 0, \tag{77}$$

$$\left|\hat{P}_{14}\right\rangle [4\hat{P}_{14}] = -\hat{P}_{14}|4] = -(\hat{p}_1 + p_4)|4] = -\hat{p}_1|4] = \left|\hat{1}\right\rangle [\hat{1}4] = 0 \implies [\hat{P}_{14}4] = 0, \tag{78}$$

$$\left|\hat{P}_{14}\right\rangle [\hat{1}\hat{P}_{14}] = -\hat{P}_{14}|\hat{1}] = -(\hat{p}_1 + p_4)|\hat{1}] = -\hat{p}_4|\hat{1}] = |4\rangle [\hat{1}4] = 0 \implies [\hat{1}\hat{P}_{14}] = 0. \tag{79}$$

Therefore we see that left subamplitude is 0, and $A(1^-, 2^+, 3^+, 4^+) = 0$. The $n+1$-pt. one minus amplitude will be given recursively with one of the subamplitudes always being greater than 3-pt. Therefore the $n+1$-pt. 1 minus amplitude is 0, and our second statement is proven. The calculation showing that the left subamplitude $A(\hat{1}^-, -\hat{P}_{14}^+, 4^+)$ was zero is a special cases of a general pattern. The pattern is called special kinematics, and says that an MHV amplitude with a shifted angle bracket or an anti-MHV (2 plus rest minus) amplitude with a shifted square bracket is 0.

B KK and BCJ relations

The relations between color ordered amplitudes are called the KK (Kleiss-Kuijf) and BCJ (Bern-Carrasco-Johansson) relations (discovered in [30, 31]).

$$\begin{aligned} A(1,2,...,n) = (-1)^n A(n, n-1, ..., 1), \ \sum A(1, \sigma(2,3,...,n) = 0, \\ A(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_\sigma A(1, \sigma, n), \end{aligned} \tag{80}$$

$$A(1, 2, \{\alpha\}, 3, \{\beta\}) = \sum_\sigma A(1, 2, 3, \sigma) \cdot (\text{kinematic factor}). \tag{81}$$

The first equations are the KK relations (the first σ means cyclic permutations, the second means permutations of α and β^T where the ordering in each is preserved), the second the BCJ (σ, is the combination of α and β, that retain the order in β. The kinematic factor can be found in the paper, it is a sum of products of Mandelstam invariants). With these relations you only need $(n-3)!$ color ordered amplitudes, to get the full amplitude.

One can get the KK and BCJ relations from on-shell recursion. This was done in [32]. It is done by induction, writing an amplitude in terms of lower point ones, who do obey the properties. To get the BCJ relations bonus relations are used. Bonus relations are relations between amplitudes that arise when you have better than necessary large z behavior. Then integrals like

$$0 = \oint_C dz A(z) = \sum_I \frac{z_I \hat{A}_L \hat{A}_R}{P_I^2} \tag{82}$$

or $\int zA(z)dz$, are calculable. They will give you relations between amplitudes, that can be used to get the BCJ relations.

C Simple calculations using k subtracted recursion

We will now show concretely how the method works. Let us consider a simple example to make this method clear. We will look at the amplitude $A(1^-, 2^-, 3^+, 4^+)$ under a $[3,1\rangle$ BCFW shift. This is an uninteresting amplitude, but serves to show the method. This amplitude will go as z^2, and the known part is

$$A_c(z) = \frac{(\langle 12\rangle \langle 34\rangle + \langle 14\rangle \langle 23\rangle)^3}{\langle 23\rangle \langle 34\rangle^4 (z \langle 34\rangle - \langle 14\rangle)}. \tag{83}$$

This can be found by normal on-shell recursion. Let us take B_1, B_2 as known. Obviously we would need to get them from known points in actual calculations (finding known points in BCFW is possible, but not as simple as in soft shifts), but since we are just illustrating the method we will take them as given

$$B_2 = \frac{\langle 23\rangle^2}{\langle 34\rangle^2}, B_1 = \frac{3 \cdot (\langle 12\rangle \langle 23\rangle \langle 34\rangle) + \langle 14\rangle \langle 23\rangle^2}{\langle 34\rangle^3}. \tag{84}$$

We will now use these to find B_0. We work backwards from our known constants. Let us consider all the terms in B_0 that could lead to our B_2. The value of B_2 could come from terms like (writing it in a form with a common denominator)

$$\frac{\langle 12\rangle^2 \langle 34\rangle^2}{\langle 34\rangle^4}, \frac{\langle 12\rangle \langle 14\rangle \langle 23\rangle \langle 34\rangle}{\langle 34\rangle^4}, \frac{\langle 14\rangle^2 \langle 23\rangle^2}{\langle 34\rangle^4}. \tag{85}$$

Therefore B_0 is some linear combination of those three terms

$$B_0 = a\frac{\langle 12\rangle^2 \langle 34\rangle^2}{\langle 34\rangle^4} + b\frac{\langle 12\rangle \langle 14\rangle \langle 23\rangle \langle 34\rangle}{\langle 34\rangle^4} + c\frac{\langle 14\rangle^2 \langle 23\rangle^2}{\langle 34\rangle^4} + \text{other}, \tag{86}$$

Maybe new terms will be necessary to match B_1 (the "other" in the equation), we only require that those terms not spoil the z^2 part. T. In this specific case it can easily be shown that there can't be any terms that would only contribute to B_1 (or any non-shifted terms). In more complicated examples, this is not true. We know the relation $a - b + c = 1$ from matching to the coefficient for B_2. Now we can consider how we could get a B_1 like term. Since we have two terms in B_1 we consider them separately. The first term can come from either the first or second of the previously given terms. The second term can come from either the second or the third of the previous terms. To match coefficient we get two new relations between a, b, c. If we then solve those three equations for three variables we end up getting

$$B_0 = \frac{3 \cdot \langle 12\rangle^2 \langle 34\rangle^2 + 3 \cdot \langle 12\rangle \langle 14\rangle \langle 23\rangle \langle 34\rangle + \langle 14\rangle^2 \langle 23\rangle^2}{\langle 34\rangle^4}. \tag{87}$$

This is also the true value of the boundary. Therefore we have found the boundary contribution with just 2 known constants (corresponding to 2 known points).

In the 4-pt. MHV case it can easily be proven that this method works. It can be shown that it will obey the two conditions discussed in the main text. However BCFW shifts will not obey those conditions in general. It might be interesting to see concretely how the method fails, when the conditions aren't met.

To do this we can calculate the 6 point NMHV amplitude using this method. Consider the amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$, under a $[4, 2\rangle$ shift. We will skip some intermediary calculations that are not relevant for seeing the failure. For reference the final answer should be

$$A_6 = -\frac{(-\langle 13\rangle [16] - \langle 23\rangle [26])^3}{(\langle 12\rangle [12] + \langle 16\rangle [16] + \langle 26\rangle [26]) [12][16] \langle 34\rangle \langle 45\rangle (-\langle 15\rangle [12] - \langle 56\rangle [26])} - \tag{88}$$

$$\frac{(-\langle 15\rangle [45] - \langle 16\rangle [46])^3}{(\langle 15\rangle [15] + \langle 16\rangle [16] + \langle 56\rangle [56]) [23][34] \langle 56\rangle \langle 16\rangle (-\langle 15\rangle [12] - \langle 56\rangle [26])} \tag{89}$$

Let us imagine we had knowledge of k points, which gave us knowledge of all the constants $B_1, B_2, ...$ Again it would obviously have to be very specific points for this to be true. The method also works if we just have k relations between the coefficients, but it is more cumbersome for calculations. Let us now use our method. We will skip all the intermediate calculations (these expressions are long enough that doing them by hand is hard anyway), and just state that the result using the k subtracted recursion relations would be (were keeping the z dependence, for clarity):

$$
\begin{aligned}
B(z) = &- \frac{2m_1 \langle 34\rangle^2[26]^4\left(-z\langle 14\rangle+\langle 12\rangle\right)\langle 26\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&+ \frac{6\left(-z[23]+[34]\right)\langle 15\rangle^2[25][26][45]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&- \frac{3\langle 34\rangle[26]^3\left(-z\langle 14\rangle+\langle 12\rangle\right)\langle 13\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&- \frac{3n_1\langle 34\rangle[26]^3\langle 14\rangle\langle 16\rangle\left(z\langle 34\rangle+\langle 23\rangle\right)}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&+ \frac{3\left(-z[23]+[34]\right)\langle 15\rangle^3[25]^2[45]}{[23]^3\langle 56\rangle\langle 16\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&+ \frac{3\langle 16\rangle\left(-z[23]+[34]\right)\langle 15\rangle[26]^2[45]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&- \frac{3\langle 34\rangle[26]^5\left(z\langle 34\rangle+\langle 23\rangle\right)\langle 26\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12][16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&+ \frac{6\langle 16\rangle\left(-z[23]+[34]\right)\langle 15\rangle[25][26][46]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&- \frac{3m_2\langle 34\rangle[26]^4\langle 14\rangle\left(z\langle 34\rangle+\langle 23\rangle\right)\langle 26\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&- \frac{3\langle 34\rangle[26]^4\langle 16\rangle\left(z\langle 34\rangle+\langle 23\rangle\right)\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&- \frac{3[12]\langle 34\rangle[26]^2\left(-z\langle 14\rangle+\langle 12\rangle\right)\langle 13\rangle\langle 14\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&- \frac{2n_2\langle 34\rangle^2[26]^3\left(-z\langle 14\rangle+\langle 12\rangle\right)\langle 16\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&+ \frac{3\langle 16\rangle^2\left(-z[23]+[34]\right)[26]^2[46]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&+ \frac{3\left(-z[23]+[34]\right)\langle 15\rangle^2[25]^2[46]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&- \frac{6\langle 13\rangle\left(z\langle 34\rangle+\langle 23\rangle\right)[26]^2}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)} \\
&- \frac{[12]\langle 34\rangle^2[26]^3\left(-z\langle 14\rangle+\langle 12\rangle\right)^2}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&+ \frac{3\langle 16\rangle\left(-z[23]+[34]\right)^2\langle 15\rangle[25][26]^2}{[23]^4\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&- \frac{3[12]\langle 34\rangle[26]^3\left(-z\langle 14\rangle+\langle 12\rangle\right)\langle 14\rangle\left(z\langle 34\rangle+\langle 23\rangle\right)}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&- \frac{3\langle 34\rangle[26]^4\left(-z\langle 14\rangle+\langle 12\rangle\right)\left(z\langle 34\rangle+\langle 23\rangle\right)\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
&+ \frac{\left(-z[23]+[34]\right)^2\langle 15\rangle^3[25]^3}{[23]^4\langle 56\rangle\langle 16\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&+ \frac{3\left(-z[23]+[34]\right)^2\langle 15\rangle^2[25]^2[26]}{[23]^4\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&+ \frac{\langle 16\rangle^2\left(-z[23]+[34]\right)^2[26]^3}{[23]^4\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
&- \frac{3[26]^3\left(z\langle 34\rangle+\langle 23\rangle\right)^2}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12][16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)}
\end{aligned}
\tag{90}
$$

However the real value is (this time not keeping the z dependence)

$$
\begin{aligned}
B_0 = {} & \frac{3\langle 16\rangle^2[26][46]^2}{[23]^2\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& + \frac{\langle 16\rangle^2[34]^2[26]^3}{[23]^4\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& - \frac{2\langle 34\rangle^2[26]^4\langle 12\rangle\langle 26\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{2\langle 34\rangle^2[26]^4\langle 16\rangle\langle 26\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& + \frac{6[34]\langle 15\rangle^2[25][26][45]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& + \frac{6\langle 16\rangle[34]\langle 15\rangle[25][26][46]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& - \frac{2\langle 34\rangle^2[26]^3\langle 12\rangle\langle 16\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& + \frac{3\langle 16\rangle^2[34][26]^2[46]}{[23]^3\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& + \frac{[34]^2\langle 15\rangle^3[25]^3}{[23]^4\langle 56\rangle\langle 16\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& - \frac{3\langle 34\rangle[26]^3\langle 12\rangle\langle 13\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3[12]\langle 34\rangle[26]^3\langle 12\rangle\langle 14\rangle\langle 23\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^4\langle 12\rangle\langle 23\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3[16]\langle 34\rangle[26]^2\langle 13\rangle\langle 14\rangle\langle 16\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^3\langle 13\rangle\langle 14\rangle\langle 26\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3[16]\langle 34\rangle[26]^3\langle 13\rangle\langle 16\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^4\langle 13\rangle\langle 26\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^3\langle 14\rangle\langle 16\rangle\langle 23\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^4\langle 14\rangle\langle 23\rangle\langle 26\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^4\langle 16\rangle\langle 23\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& - \frac{3\langle 34\rangle[26]^5\langle 23\rangle\langle 26\rangle\langle 46\rangle}{\left(-\langle 14\rangle[12]-\langle 46\rangle[26]\right)[12][16]\langle 45\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 14\rangle[12]+\langle 46\rangle[26]\right)^2} \\
& + \frac{3\langle 16\rangle\langle 15\rangle[25][46]^2}{[23]^2\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& + \frac{3\langle 15\rangle^2[26][45]^2}{[23]^2\langle 56\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} \\
& + \frac{3\langle 15\rangle^3[25][45]^2}{[23]^2\langle 56\rangle\langle 16\rangle\left(-\langle 15\rangle[12]-\langle 56\rangle[26]\right)\left(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56]\right)} -
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
&\frac{3[16][26]\langle 13\rangle^2}{(-\langle 14\rangle[12]-\langle 46\rangle[26])[12]\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])} \\
&-\frac{3[26]^3\langle 23\rangle^2}{(-\langle 14\rangle[12]-\langle 46\rangle[26])[12][16]\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])} \\
&-\frac{3[12]\langle 34\rangle[26]^2\langle 12\rangle\langle 13\rangle\langle 14\rangle}{(-\langle 14\rangle[12]-\langle 46\rangle[26])\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 14\rangle[12]+\langle 46\rangle[26])^2} \\
&-\frac{\langle 34\rangle^2[26]^5\langle 26\rangle^2}{(\langle 14\rangle[12]+\langle 46\rangle[26])^2(-\langle 14\rangle[12]-\langle 46\rangle[26])[12][16]\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])} \\
&-\frac{6\langle 13\rangle\langle 23\rangle[26]^2}{(-\langle 14\rangle[12]-\langle 46\rangle[26])[12]\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])} \\
&-\frac{[12]\langle 34\rangle^2[26]^3\langle 12\rangle^2}{(-\langle 14\rangle[12]-\langle 46\rangle[26])[16]\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 14\rangle[12]+\langle 46\rangle[26])^2} \\
&-\frac{[16]\langle 34\rangle^2[26]^3\langle 16\rangle^2}{(-\langle 14\rangle[12]-\langle 46\rangle[26])[12]\langle 45\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 14\rangle[12]+\langle 46\rangle[26])^2} \\
&+\frac{6\langle 15\rangle^2[25][45][46]}{[23]^2\langle 56\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])} \\
&+\frac{6\langle 16\rangle\langle 15\rangle[26][45][46]}{[23]^2\langle 56\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])} \\
&+\frac{3[34]\langle 15\rangle^3[25]^2[45]}{[23]^3\langle 56\rangle\langle 16\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])} \\
&+\frac{3[34]\langle 15\rangle^2[25]^2[46]}{[23]^3\langle 56\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])} \\
&+\frac{3[34]^2\langle 15\rangle^2[25]^2[26]}{[23]^4\langle 56\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])} \\
&+\frac{3\langle 16\rangle[34]\langle 15\rangle[26]^2[45]}{[23]^3\langle 56\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])} \\
&+\frac{3\langle 16\rangle[34]^2\langle 15\rangle[25][26]^2}{[23]^4\langle 56\rangle(-\langle 15\rangle[12]-\langle 56\rangle[26])(\langle 15\rangle[15]+\langle 16\rangle[16]+\langle 56\rangle[56])}
\end{aligned}
\tag{92}
$$

These expressions are different. We are able to find a lot of the boundary with our method. Certainly what we found is much more than nothing, and the parts we have found are "true" meaning that they do show up in the real boundary. All terms that go as z^2 where found in this example (there are 8 of them). Most of the terms that go as z^1 were found. Out of the 15 terms all were found, but 4 of them had unknown coefficients (the m_1, m_2, n_1, n_2), because they were not identifiable. The two terms with m_1, m_2 had the same leading z terms. We found none of the z^0 terms (There were 14 terms of them). This again shows why we require that all terms are shifted. We found roughly half of the terms but we are missing $14 + 4 = 18$ terms. We can see that the method fails in this case. We can also see that our method still gives some information about the boundary. It is possible that other information could be used in addition to this method to fix the boundary in general. For example by requiring the correct soft behavior of the boundary, or by other methods. We have not been able to find any concrete methods that work generally. Sometimes even/odd properties of the amplitude can be exploited. If there are relations between amplitudes of different theories, the boundaries can be expressed in terms of each other.

D AI declaration

Declaration of using generative AI tools (for students)
☒ I/we have used generative AI as an aid/tool ☐ I/we have NOT used generative AI as an aid/tool
List which GAI tools you have used and include the link to the platform (if possible): Chat GPT (https://chatgpt.com).
Describe how generative AI has been used in the exam paper: Generative AI was used to generate TIKZ code, for the figures. The output was heavily edited. GAI was also used to format the front page, and for some other help with latex commands. GAI was NOT used for any writing, or for any scientific content.