5.5 a) We disprove the statement with a counterexample:

2.5/2.5

let $A = \{1, 2\}$ and $\rho = \{(1,2)\}$. 1 pt

then ρ is not symmetric because $(1,2) \in \rho$ but $(2,1) \notin \rho$.

✓ nice

However, $\rho^2 = \emptyset$ and since $\hat{\emptyset} = \emptyset$, $\hat{\rho^2} = \rho^2$, so per definition of symmetric, the empty set (or ρ^2) is symmetric. ✓ 1 pt

This contradicts ρ^2 is symmetric $\Rightarrow \rho$ is symmetric ✓

0.5 pt

b) We disprove the statement with a counterexample.

let $A = \{1\}$ and $\rho = \emptyset$. ✓

Then ρ is symmetric and antisymmetric per definition, since $\hat{\emptyset} = \emptyset$ and $\emptyset \cap \hat{\emptyset} \subseteq id$ (since $\emptyset$ is subset of every set). ✓
However, $\rho \neq id_A$ since $id_A = \{(1,1)\}$. ✓ 2.5/2.5

c) 3/3

To prove that $\rho^2 = \mathbb{Z} \times \mathbb{Z}$ we need to show that $\mathbb{Z} \times \mathbb{Z} \subseteq \rho^2$. ($\rho^2 \subseteq \mathbb{Z} \times \mathbb{Z}$ since $\rho_1 \subseteq \mathbb{Z} \times \mathbb{Z}$ and $\rho_2 \subseteq \mathbb{Z} \times \mathbb{Z}$
✓ nice! $\Rightarrow \rho \subseteq \mathbb{Z} \times \mathbb{Z} \Rightarrow \rho^2 \subseteq \mathbb{Z} \times \mathbb{Z}$) ← very good and important argument

We show that $(a,c) \in \mathbb{Z} \times \mathbb{Z} \Rightarrow (a,c) \in \rho^2$: ✓

$(a,c) \in \rho^2$

$\Leftrightarrow \exists b((a,b) \in \rho \land (b,c) \in \rho)$ ✓ | def. of composition

$\Leftrightarrow \exists b(((a,b) \in \rho_1 \lor (a,b) \in \rho_2) \land ((b,c) \in \rho_1 \lor (b,c) \in \rho_2))$ | def. of ρ and $\cup$ ✓

$\Leftrightarrow \exists b(\underbrace{((b = a+1) \lor (b \equiv_2 a))}_{\text{condition 1}} \land \underbrace{((c = b+1) \lor (c \equiv_2 b))}_{\text{condition 2}})$ | def. of ρ_1 and ρ_2 ✓

nice usage of logic!

We see that for (a,c) to be element of ρ^2 we need to find a b that fulfills both condition 1 and 2.

condition 1: $b = a+1$ or $b \equiv_2 a$ ✓

condition 2: $c = b+1$ or $c \equiv_2 b$ ✓

We show by case distinction that we can find such a b for all $a \in \mathbb{Z}$ and $c \in \mathbb{Z}$. ✓ ✓

It must either hold that $a \equiv_2 c$ or $a \not\equiv_2 c$. ✓

Case 1 ($a \equiv_2 c$):

Then we can choose any b so that $b \equiv_2 a$.
Then $b \equiv_2 a \equiv_2 c$ and b fulfills both condition 1 and 2 ✓
since $b \equiv_2 a$ and $b \equiv_2 c$. (Since $\equiv_2$ as an equivalence relation is transitive) perfect ✓

Case 2 ($a \not\equiv_2 c$):

Then $a = (c+1) + 2m$ for some $m \in \mathbb{Z}$ per def. of $\equiv_2$. ✓
We now choose $b = c-1$, so it fulfills condition 2.
And since ✓

$$a - b = c + 1 + 2m - c + 1 = 2m + 2 = 2(m+1) \text{ for some } m \in \mathbb{Z},$$

$a \equiv_2 b$ which means b fulfills condition 1. ✓

Since we can find a b that satisfies our conditions in both cases and one of the cases has to be true for any $(a,c) \in \mathbb{Z} \times \mathbb{Z}$, ✓
$(a,c) \in \rho^2$ holds for all $(a,c) \in \mathbb{Z} \times \mathbb{Z}$. ✓

We now showed that $(a,c) \in \mathbb{Z} \times \mathbb{Z} \Rightarrow (a,c) \in \rho^2$
which implies $\mathbb{Z} \times \mathbb{Z} \subseteq \rho^2$ per def. of „$\subseteq$". ✓

Since $\rho^2 \subseteq \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \subseteq \rho^2$: $\rho^2 = \mathbb{Z} \times \mathbb{Z}$.
which is what we wanted to show.

I have nothing to say ✓

8/8