Practice Exercises Week 2

① Show that the quantors $\forall$ and $\exists$ are not minimal. i.e show that we can write all formolas using $\forall$ by only using $\exists$ and vice versa.

Solotion:

We can write $\forall x\, F$ as $\neg\neg\forall x\, F \equiv \neg\exists x \neg F$
and $\exists x\, F$ as $\neg\neg\exists x\, F \equiv \neg\forall x \neg F$

② Simplify the following formulas using the rules from Lemma 2.1 (and $T \vee A \equiv T$, $\perp \wedge A \equiv \perp, \ldots$):

a) $(B \vee A) \wedge (\neg A \vee B)$
b) $(A \vee B \vee C) \wedge (\neg A \vee B \vee C) \wedge (\neg B \vee C)$
c) $(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C)$

Do you see a pattern? Can you formulate your own rule?

Solotion:

a) $(B \vee A) \wedge (\neg A \vee B)$
$\equiv (B \vee A) \wedge (B \vee \neg A)$ (commutativity)
$\equiv B \vee (A \wedge \neg A)$ (distr. law)
$\equiv B \vee \perp$ $(A \wedge \neg A \equiv \perp)$
$\equiv B$ $(A \vee \perp \equiv A)$

b) $(A \vee B \vee C) \wedge (\neg A \vee B \vee C) \wedge (\neg B \vee C)$
$\equiv ((B \vee C) \vee A) \wedge ((B \vee C) \vee \neg A) \wedge (\neg B \vee C)$ (ass./comm.)
$\equiv ((B \vee C) \vee (A \wedge \neg A)) \wedge (\neg B \vee C)$ (distr.)
$\equiv ((B \vee C) \vee \perp) \wedge (\neg B \vee C)$ $(A \wedge \neg A \equiv \perp)$
$\equiv (B \vee C) \wedge (\neg B \vee C)$ $(A \vee \perp \equiv A)$
$\equiv C$ as above

c) use the same idea to get $A \wedge B$

The pattern is that we can reduce a formula of the form

$$(F \vee A) \wedge (F \vee \neg A) \equiv F$$

and

$$(F \wedge A) \vee (F \wedge \neg A) \equiv F$$

(3) Express the following statements in predicate logic. Assume $U = \mathbb{N}$ and use the predicates:

- $less(x, y) = 1 \iff x < y$
- $prime(x) = 1 \iff x$ is a prime
- $isFive(x) = 1 \iff x = 5$

a) "There is no largest prime" by
   i) starting your formula with $\neg \exists x$
   ii) starting your formula with $\forall x$

b) "Every prime number is equal to five"

c) "If a number is prime, then either it is five or there exists a number smaller than it"

d) "For all numbers there exists a number not equal to five smaller than it"

Solution:

a) i) $\neg \exists x (prime(x) \wedge \neg \exists y (prime(y) \wedge less(x, y)))$

ii) $\forall x (prime(x) \rightarrow \exists y (prime(y) \wedge less(x, y)))$

b) $\forall x (prime(x) \rightarrow isFive(x))$

c) $\forall x (prime(x) \rightarrow (isFive(x) \vee \exists y \, less(y, x)))$

d) $\forall x \exists y (\neg isFive(y) \wedge less(y, x))$

(4) Which of these statements are true?

a) If S is true and $S \Rightarrow T$ is true, then T is true.

b) $\models \forall x \left((P(x) \rightarrow Q(x)) \vee (Q(x) \rightarrow P(x)) \right)$

c) If $F \Rightarrow G$ is a tautology, then $F \models G$

d) $\exists x (\neg P(x)) \models \forall x \, P(x)$

e) Predicate logic is weaker than propositional logic.

Solutions:

a) YES
b) YES
c) YES
d) NO
e) NO (it lets us express more stuff