PROVING RELATION PROPERTIES

These are the best strategies to prove relation properties. About 99% of the time you will use exactly those patterns if you want to prove the property, so always try them first. If they dont work, you need to get creative!.

PROVING REFLEXIVITY

To prove that a relation on A is reflexive, we simply show that for any element a in A, the relation contains (a,a).

Your proof should start with Take any element $a \in A$ and end with $\Rightarrow (a,a) \in \rho$.

Example: Let | be the divisibility relation so $a|b$ if there exists z such that $a \cdot z = b$. We show | is reflexive on $\mathbb{Q}$.

Take any $x \in \mathbb{Q}$.

$\Rightarrow x \cdot 1 = x$ (1 neutral element)

$\Rightarrow x | x$ (def. |)

(Showing reflexivity is mostly very easy.)

PROVING SYMMETRY

To prove symmetry for a relation $\sim$ on A, take any $a, b \in A$ with $a \sim b$ and show that also $b \sim a$.

Your proof should look like this:

Take any $a, b \in A$ with $a \sim b$.

$\Rightarrow$ ...

$\Rightarrow$ ...

$\Rightarrow b \sim a$

PROVING ANTISYMMETRY

To prove antisymmetry for $\sim$ on A we take some $a, b \in A$ with $a \sim b$ and $b \sim a$ and show that $a = b$.

Example: We show that $\leq$ on $\mathbb{N}$ is antisymmetric.

Take any $a, b \in \mathbb{N}$ with $a \leq b$ and $b \leq a$.

$a \leq b \;\wedge\; b \leq a$

$\Rightarrow (a < b \vee a = b) \wedge (b < a \vee a = b)$ (def. $\leq$)

$\Rightarrow (a < b \wedge b < a) \vee (a = b)$ (distr. law)

$\Rightarrow a = b$ (since $a < b \wedge b < a$ is not possible)

(Here we assume we know that $<$ is antisymmetric ...)

PROVING TRANSITIVITY

To prove transitivity of $\sim$ on A we take any $a, b, c \in A$ with $a \sim b$ and $b \sim c$ and show that $a \sim c$.

Example: We show that $\leq$ is transitive on $\mathbb{Z}$.

Take any $a, b, c \in \mathbb{Z}$ with $a \leq b$ and $b \leq c$.

$$
\begin{aligned}
& a \leq b \wedge b \leq c \\
\Rightarrow\ & a-b \leq 0 \wedge b-c \leq 0 \\
\Rightarrow\ & (a-b)+(b-c) \leq 0 \qquad \text{(if } a, b \leq 0 \text{ then } a+b \leq 0\text{)} \\
\Rightarrow\ & a+(-b+b)-c \leq 0 \\
\Rightarrow\ & a-c \leq 0 \\
\Rightarrow\ & a \leq c
\end{aligned}
$$

(This is a little sketchy, just a short example)

DISPROVING PROPERTIES

To disprove any of the properties, just find some elements for which they don't hold and show it.

Example: We disprove that $\neq$ is transitive on $\mathbb{Z}$.

Consider 1 and 2. $1 \neq 2$ and $2 \neq 1$ but not $1 \neq 1$.

EQUIVALENCE / PARTIAL ORDERS

To prove that $\sim$ is an equivalence / p.o. relation simply show all the three properties seperately.
Start with reflexivity, since it is the easiest. Then (anti)symmetry and lastly transitivity.