Exercises Week 6

(Solutions at the back)

① POSETS

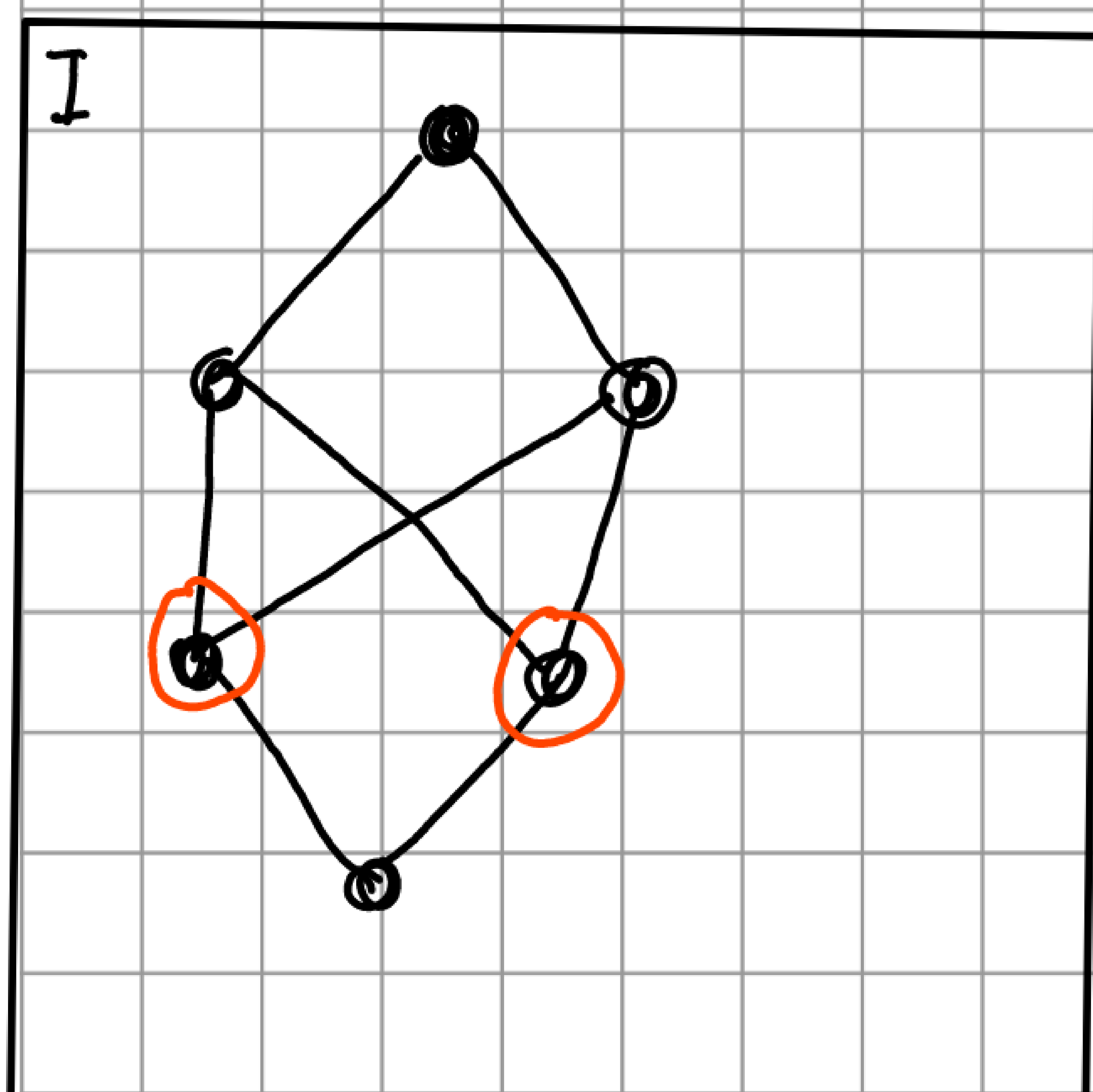
For the following posets:

a) find maxima, minima, least, greatest elements (if exist)

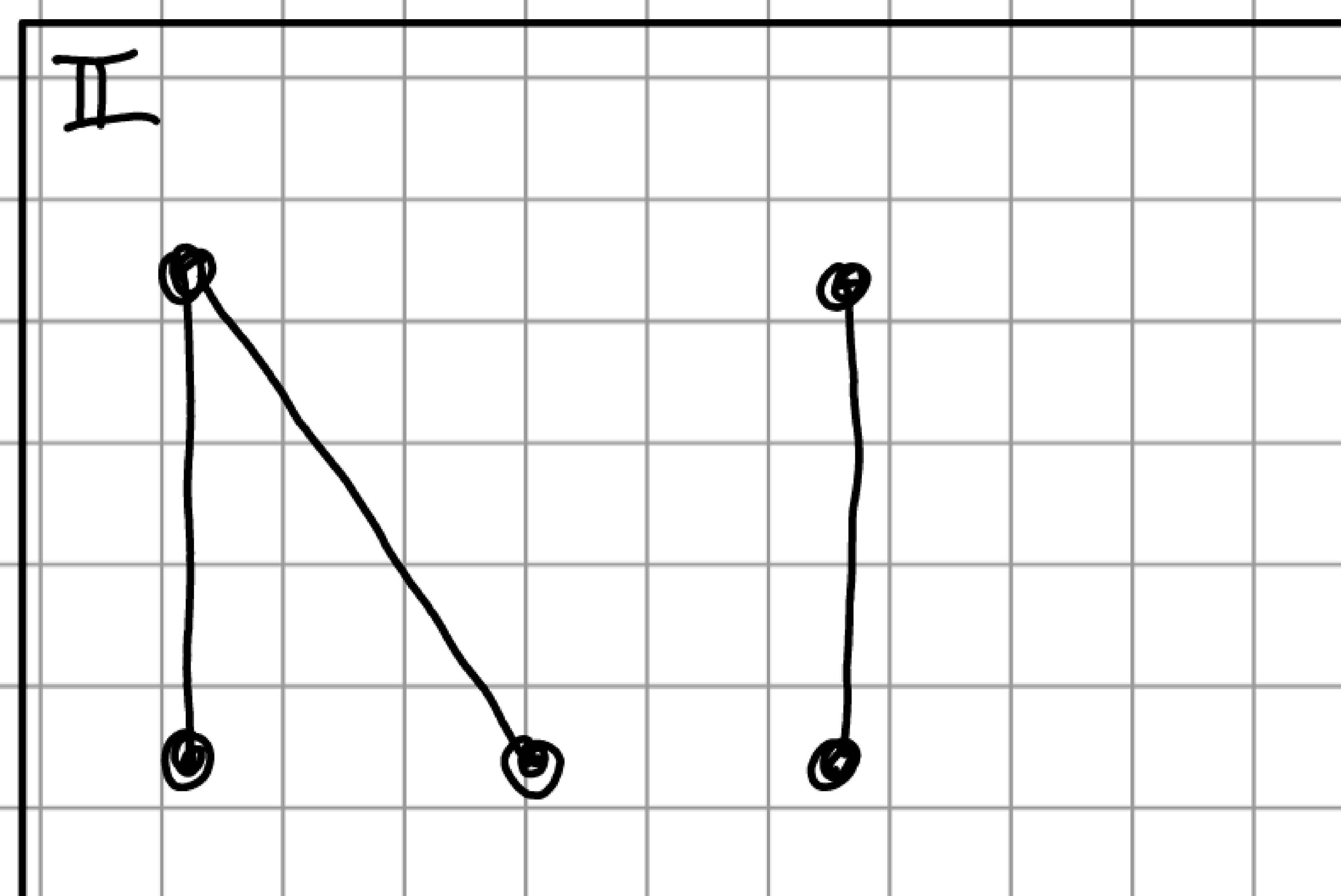
b) all lower/upper bounds, as well as the greatest lower/least upper bound (if exist) of the red set

c) decide if they are totally ordered and a lattice

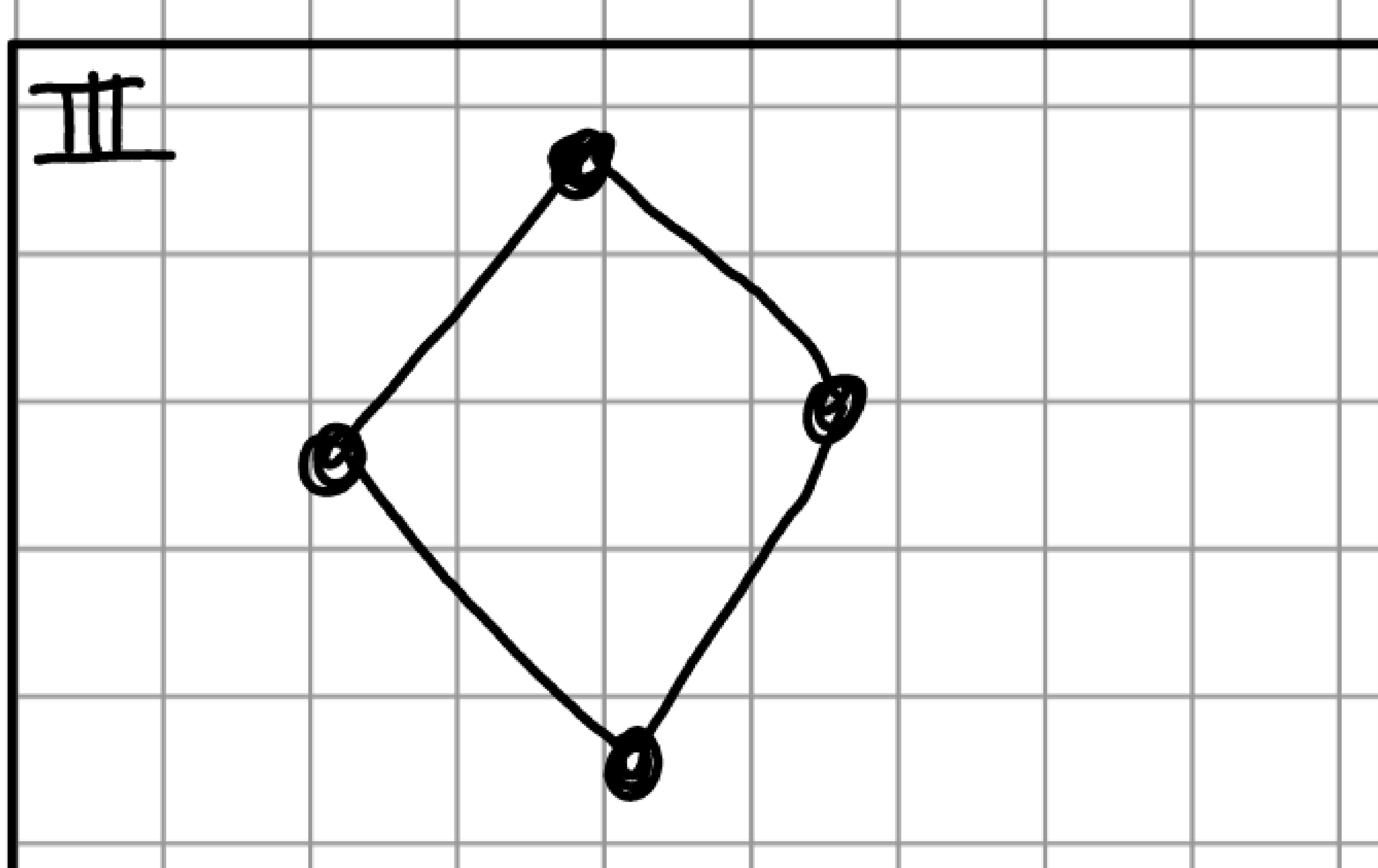
I



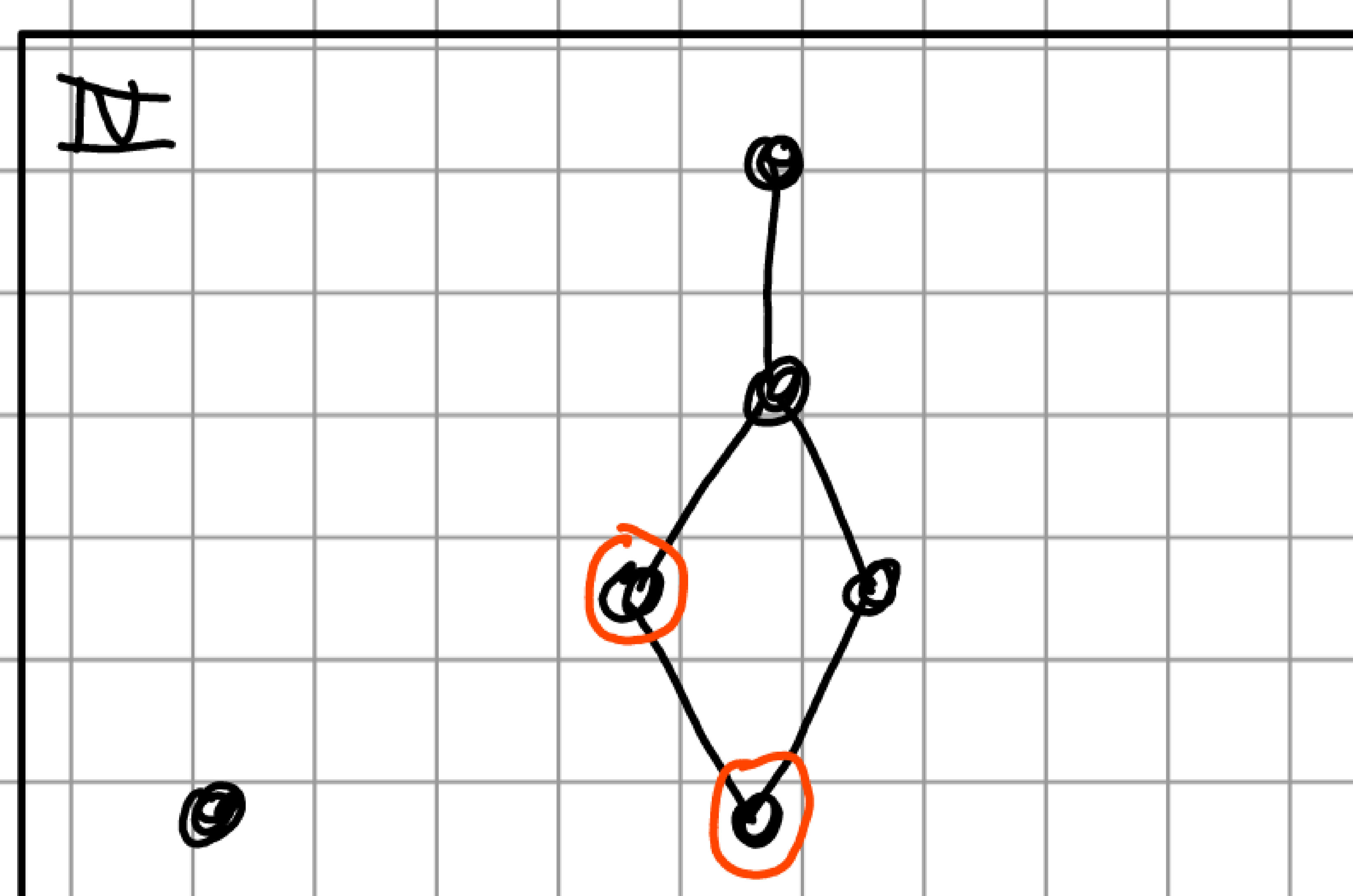
II



III



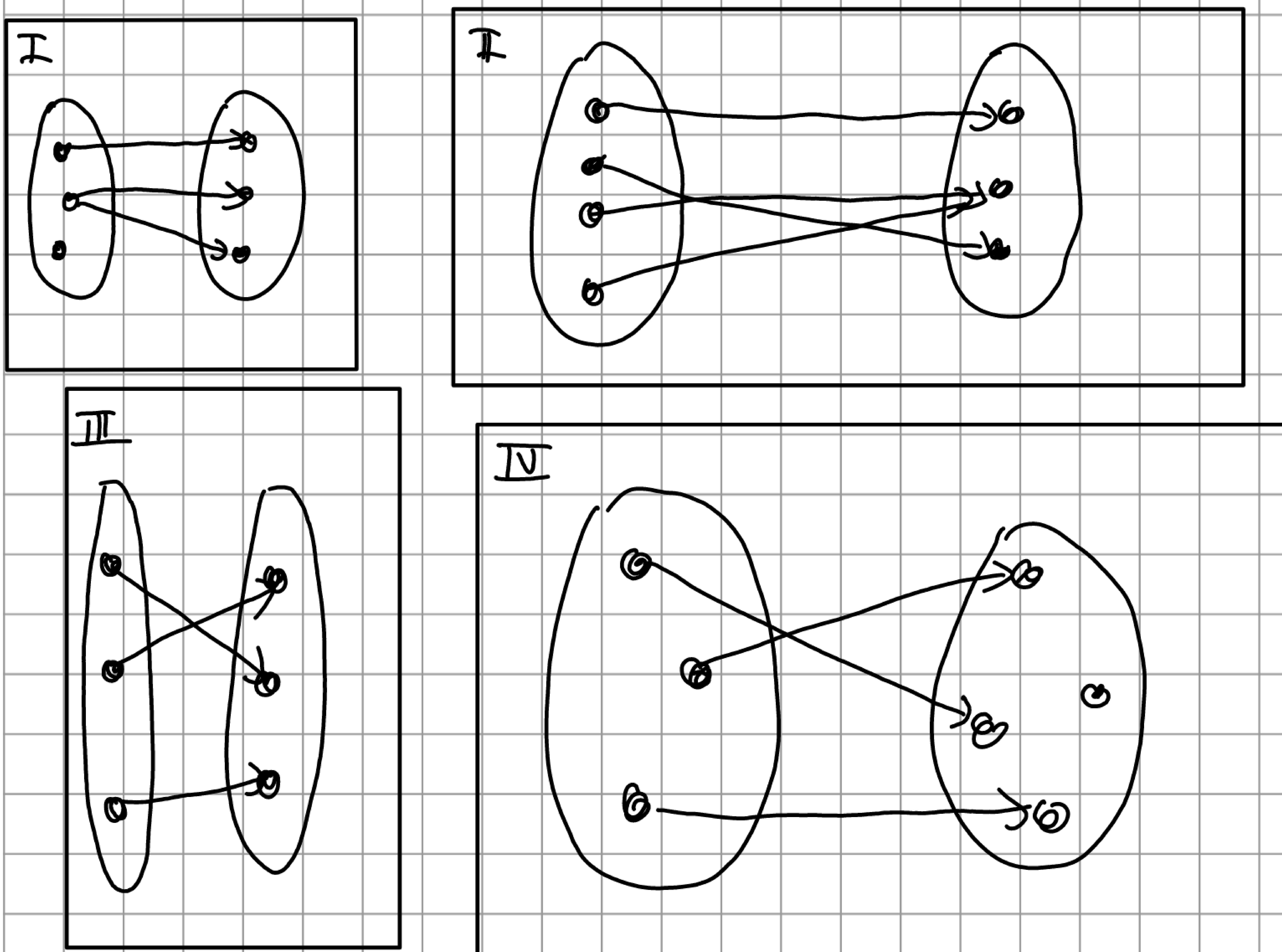
IV



② FUNCTIONS

a) Which of the following relations are functions?

b) Which are injective / surjective / bijective?



③ FUNCTIONS

a) Take some $f: A \to B$. Does f being injective imply that $\hat{f}$ is a function?

b) Show that if $f: A \to B$ and $g: B \to C$ are injective, then $g \circ f$ is also injective.

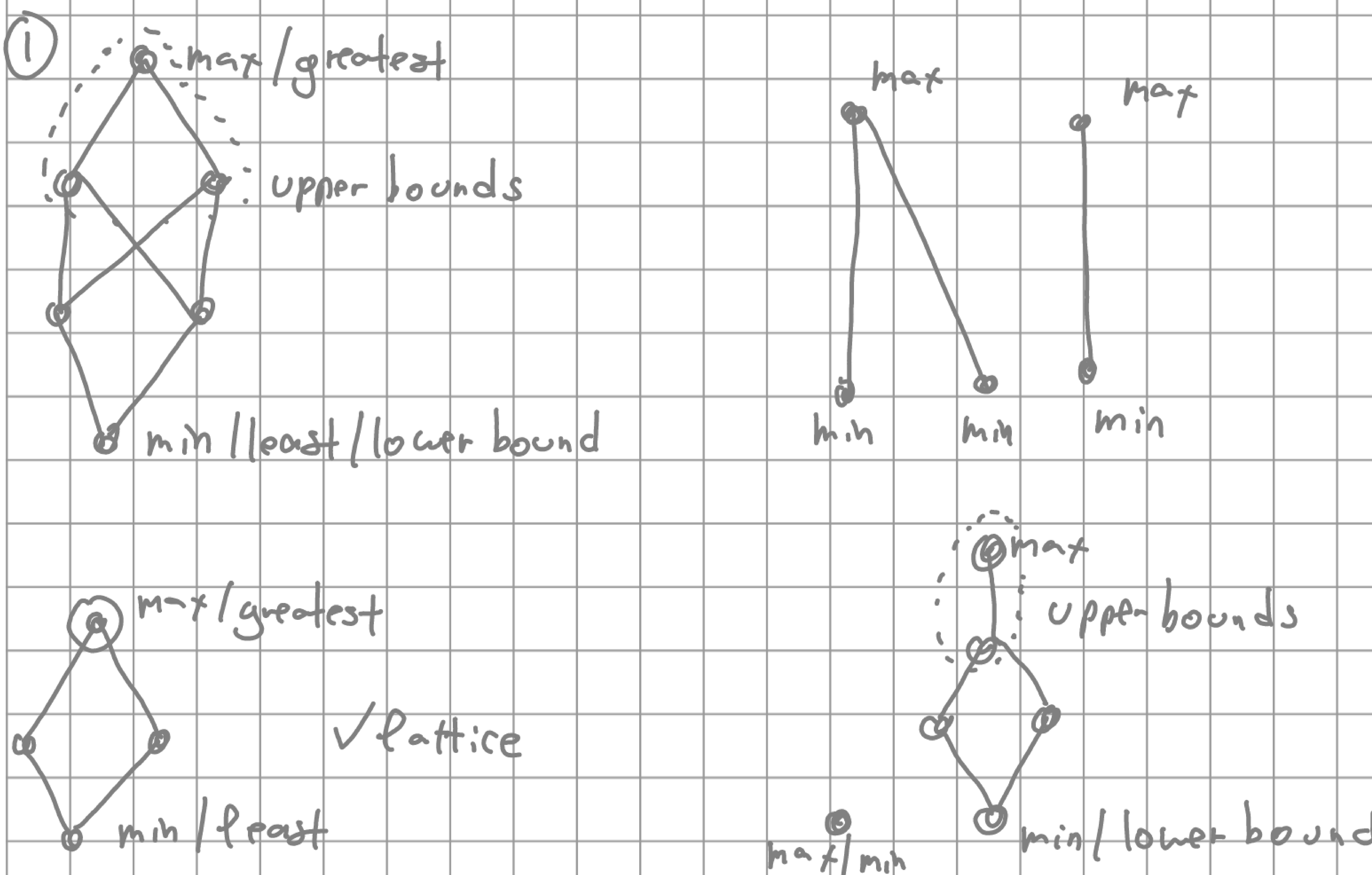
④ COUNTABILITY

a) Show that $P(\mathbb{N})$ is uncountable. (Note that $P(\mathbb{N})$ is the set of all (possibly infinite) subsets of $\mathbb{N}$.)

b) Let S be the set of all FINITE subsets of $\mathbb{N}$. Show that S is countable.

Solutions

①



② I is not a function

II is surjective

III is a bijection

IV is injective

③

a) No, take as a counterexample $A = \{1\}$, $B = \{1,2\}$ and $f = \{(1,1)\}$. Then f is injective, but $\hat{f}$ is not a function since it is not totally defined.

b) Take any $a, a' \in A$ with $(g \circ f)(a) = (g \circ f)(a') = c$ and $c \in C$.
We show that this implies $a = a'$, which shows injectivity.

$$
\begin{aligned}
& (g \circ f)(a) = (g \circ f)(a') & \\
\overset{.}{\Rightarrow}\ & g(f(a)) = g(f(a')) & (\text{def. } \circ) \\
\overset{.}{\Rightarrow}\ & f(a) = f(a') & (g \text{ injective}) \\
\overset{.}{\Rightarrow}\ & a = a' & (f \text{ injective})
\end{aligned}
$$

4

a) We show that $P(\mathbb{N})$ is uncountable by giving an injection from $\{0,1\}^\infty$ into $P(\mathbb{N})$.

Now assume that $P(\mathbb{N})$ was countable. Then we would have $P(\mathbb{N}) \hat{\le} \mathbb{N}$. Since we found an injection from $\{0,1\}^\infty$ to $P(\mathbb{N})$ this means $\{0,1\}^\infty \hat{\le} P(\mathbb{N})$ and per transitivity of $\hat{\le}$: $\{0,1\}^\infty \hat{\le} \mathbb{N}$ and therefore $\{0,1\}^\infty$ countable. This is a contradiction, since we know that $\{0,1\}^\infty$ is uncountable and therefore $P(\mathbb{N})$ is uncountable.

the magic sentence !

Propose the injection:

$$
\begin{aligned}
\varphi: \{0,1\}^\infty &\to P(\mathbb{N}) \\
b &\mapsto S
\end{aligned}
$$

where $\varphi(b) = S$ exactly if for all $x \in \mathbb{N}$: $b_x = 1 \Leftrightarrow x \in S$.
(Where $b = b_0 b_1 b_2 \ldots$ so b_x is bit x of b.)
In words: We map b to S if S contains the positions where b is one.

We show that φ is :

1) totally defined: Since $P(\mathbb{N})$ contains all infinite sets of natural numbers, we can find such an S for each b with the construction.

2) well-defined: Assume φ was not well-defined. Then there exists some $b \in \{0,1\}^\infty$ with $\varphi(b) = S$ and $\varphi(b) = T$, but $S \neq T$.
Now since $S \neq T$, there must be some $s \in S$ with $s \notin T$. (We assume without loss of generality that $s \in S$. The same works when changing S and T.)
Since $s \in S$ and $\varphi(b) = S$ we have $b_s = 1$.
But since $s \notin T$ and $\varphi(b) = T$ we have $b_s = 0$.
This is a contradiction, so $S = T$.

3) injective: Take any $b, b' \in \{0,1\}^\infty$ with $\varphi(b) = \varphi(b')$.
We show by contradiction that $b = b'$.
Assume $b \neq b'$ and name $\varphi(b) = \varphi(b') = S$.
Now take the first bit where b and b' differ.

Taking the first bit where b, b' differ is a very common tactic when proving injectivity.

Say that bit is b_x and assume w.l.o.g that $b_x = 0$ and $b'_x = 1$.
Since $b_x = 0$ and $\varphi(b) = S$ we have $x \notin S$.
Since $b'_x = 1$ and $\varphi(b') = S$ we have $x \in S$.
A contradiction.

Since we found a valid injection, this proves the injectivity of φ and so $P(\mathbb{N})$ is uncountable.

b) We show that S is countable by giving an injection from $S \to \{0,1\}^*$. This means $S \preceq \{0,1\}^*$ and we already know that $\{0,1\}^*$ is countable and so $\{0,1\}^* \preceq \mathbb{N}$. Transitivity of $\preceq$ gives $S \preceq \mathbb{N}$ and so S countable.

We choose for our injection:

$$\varphi : S \to \{0,1\}^*$$
$$A \mapsto b$$

where $\varphi(A) = b$ exactly if $|A| = \text{length}(b)$ and $b_x = 1 \Leftrightarrow x \in A$.

Note that we consider the empty bitstring λ as an element of $\{0,1\}^*$ here.

Again we show:

1) well-defined
2) totally defined
3) injective

The arguments here are very similar to the ones from the last exercise a).