RELATION EXERCISES

(Solutions at the end.)

① Which of the following are reflexive, transitive, symmetric, antisymmetric, equivalence, partial order relations? (on $\mathbb{Z}$)

a) $\leq$ d) $\subseteq$
b) $|$ ("divides") e) $\subset$
c) $\neq$ f) $>$

② Define the relation $\sim$ as: $a \sim b \Leftrightarrow \exists \lambda (\lambda \cdot a = b)$.

a) Show that $\sim$ is an equivalence relation on $\mathbb{Q} \setminus \{0\}$.
b) Why is it not on $\mathbb{Q}$?

③ This is the bonus exercise from 2024. Prove or disprove:

a) A relation ρ on A is symmetric if and only if ρ^2 is symmetric on A.
b) If ρ is a relation on A that is symmetric and antisymmetric, then it must hold that $\rho = id_A$.
c) Define ρ_1 and ρ_2 on $\mathbb{Z}$ as:

$$a \rho_1 b \Leftrightarrow b = a+1 \quad , \quad a \rho_2 b \Leftrightarrow b =_2 a$$

Then for $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$

Hint: for a) and b) think about $\emptyset$. $\emptyset$ is a relation too!

④ Show that the intersection of two antisymmetric relations on a set A is also antisymmetric.

5) Let ρ be antisymmetric. Prove that $\hat{\rho}$ is antisymmetric.

6) Let ρ and q be transitive. Disprove that $\rho \circ q$ is transitive.

Solutions

1)

	refl	sym	antisym	trans	equiv	part
$\leq$	✓	✗	✓	✓	✗	✓
$\mid$	✓	✗	✓	✓	✗	✓
$\neq$	✗	✓	✗	✗	✗	✗
$\subseteq$	✓	✗	✓	✓	✗	✓
$\subset$	✗	✗	✓	✓	✗	✗
$>$	✗	✗	✓	✓	✗	✗

2)

a) We show the three properties for $\sim$:

- reflexivity: take any $a \in \mathbb{Q}\setminus\{0\}$. Then for $\lambda = 1$ we have $1 \cdot a = a$ and therefore $a \sim a$.

- symmetry: take any $a, b \in \mathbb{Q}\setminus\{0\}$ with $a \sim b$.

$$
\begin{aligned}
& a \sim b \\
\Rightarrow\ & \exists\lambda\,(a \cdot \lambda = b) && (\text{def. } \sim) \\
\Rightarrow\ & \exists\lambda\,(a = \tfrac{1}{\lambda} \cdot b) && (\text{since } b \neq 0:\ \lambda \neq 0) \\
\Rightarrow\ & \exists\delta\,(\delta \cdot b = a) && (\delta = \tfrac{1}{\lambda}) \\
\Rightarrow\ & b \sim a && (\text{def. } \sim)
\end{aligned}
$$

- transitivity: take any $a,b,c \in \mathbb{Q}\setminus\{0\}$ with $a \sim b$ and $b \sim c$.

$$
\begin{aligned}
& a \sim b \wedge b \sim c \\
\Rightarrow\ & \exists \lambda_1 (\lambda_1 a = b) \wedge \exists \lambda_2 (\lambda_2 b = c) \\
\Rightarrow\ & \exists \lambda_1 (\lambda_1 a = b) \wedge \exists \lambda_2 \left(b = \tfrac{1}{\lambda_2} \cdot c\right) && (\lambda_2 \neq 0) \\
\Rightarrow\ & \exists \lambda_1 \exists \lambda_2 \left(\lambda_1 a = \tfrac{1}{\lambda_2} \cdot c\right) && (\text{both} = b) \\
\Rightarrow\ & \exists \lambda_1 \exists \lambda_2 (\lambda_1 \lambda_2 a = c) \\
\Rightarrow\ & \exists \delta (\delta a = c) && (\delta = \lambda_1 \lambda_2) \\
\Rightarrow\ & a \sim c && (\text{def.} \sim)
\end{aligned}
$$

(3) look at emils-site → session 5 for the solution

(4) Let ρ and σ be antisymmetric on A. Take any $a,b \in A$ with $(a,b) \in \rho \cap \sigma$ and $(b,a) \in \rho \cap \sigma$.

$$
\begin{aligned}
& (a,b) \in \rho \cap \sigma \wedge (b,a) \in \rho \cap \sigma \\
\Rightarrow\ & (a,b) \in \rho \wedge (a,b) \in \sigma \wedge (b,a) \in \rho \wedge (b,a) \in \sigma && (\text{def. } \cap) \\
\Rightarrow\ & (a,b) \in \rho \wedge (b,a) \in \rho \\
\Rightarrow\ & a = b && (\rho \text{ is antisymmetric})
\end{aligned}
$$

(5) Let ρ on A be antisymmetric. Take any $a,b \in A$ with $a \hat{\rho} b$ and $b \hat{\rho} a$.

$$
\begin{aligned}
& a \hat{\rho} b \wedge b \hat{\rho} a \\
\Rightarrow\ & b \rho a \wedge a \rho b && (\text{def. } \hat{\rho}) \\
\Rightarrow\ & a = b && (\rho \text{ is antisymmetric})
\end{aligned}
$$

(6) Consider $A = \mathbb{N}$ and the relations:

$p = \{(1,2), (3,4)\}$ and

$q = \{(2,3), (4,5)\}$.

Then both p and q are transitive. But:

$p \circ q = \{(1,3), (3,5)\}$, which is not transitive.