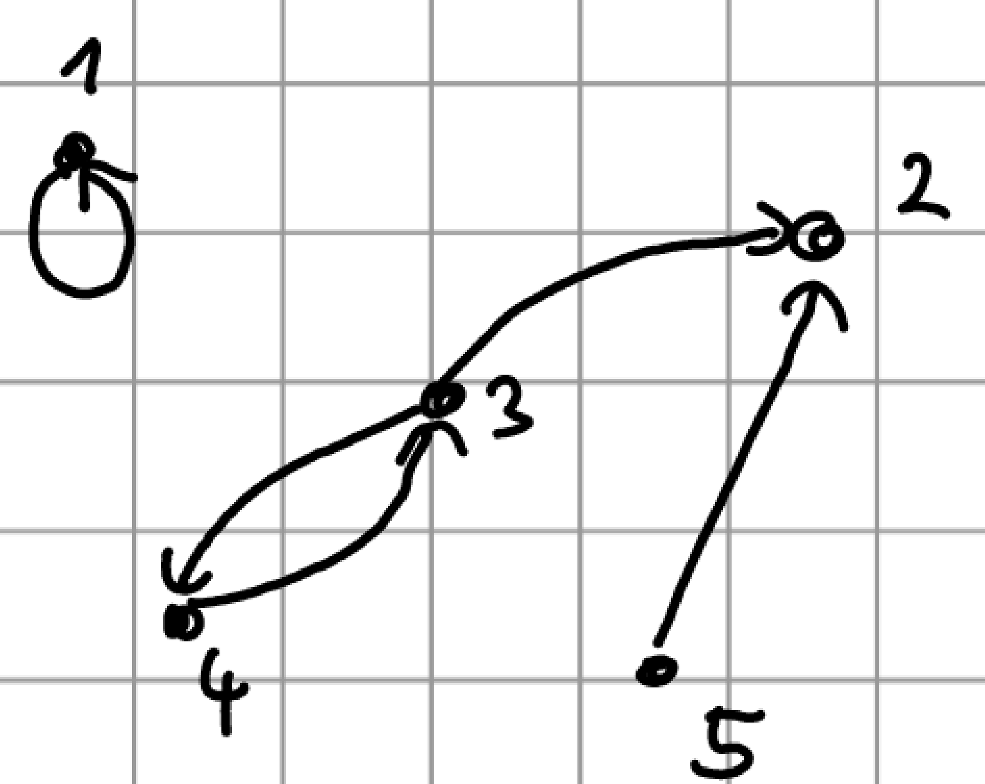
Relations

- This is not a complete summary. READ THE SCRIPT!
- Exercises are in red

REPRESENTATION

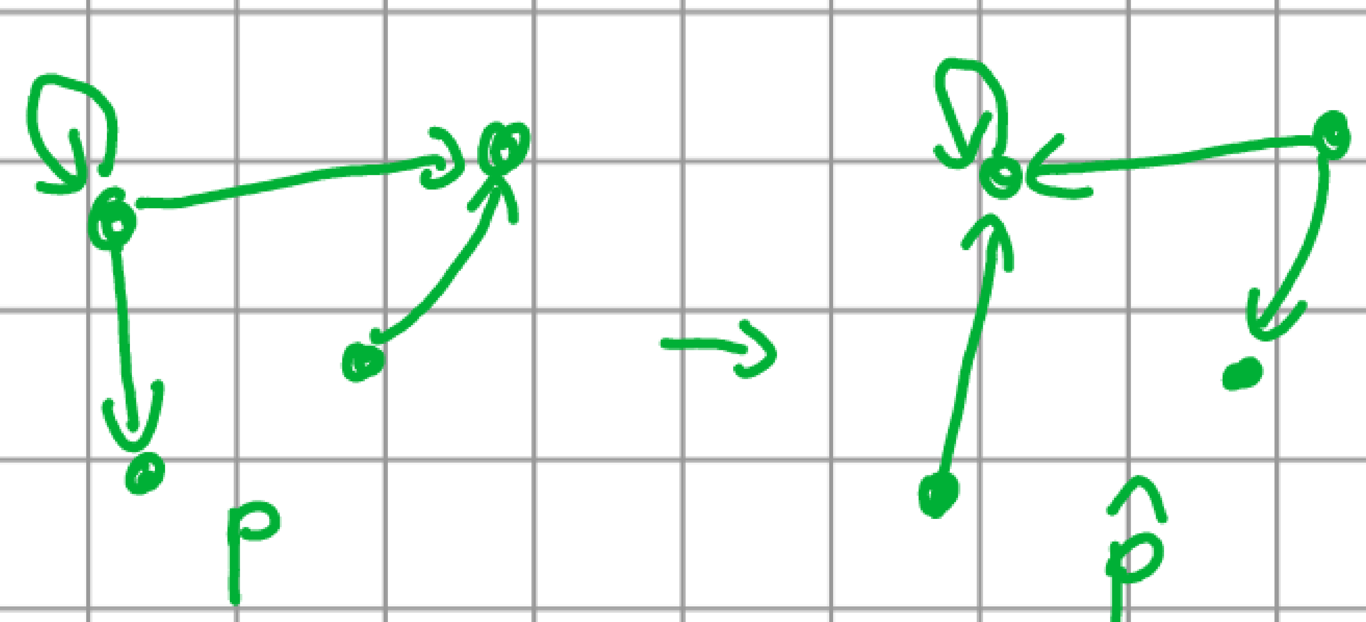
It is very useful to represent a relation $r: A \to A$ as a graph:



This would be the relation $\{(1,1),(3,2),(3,4),(4,3),(5,2)\}$.
This representation is VERY useful if you want to look for relation properties.

INVERSE

of a relation ρ is the relation $\hat{\rho}$, which contains all the tuples that ρ does, but reversed. So the inverse to the relation above would be $\{(1,1),(2,3),(4,3),(3,4),(2,5)\}$.
In graph representation, we simply turn around the arrows.



Formally $\hat{\rho} = \{(b,a) \mid (a,b) \in \rho\}$.

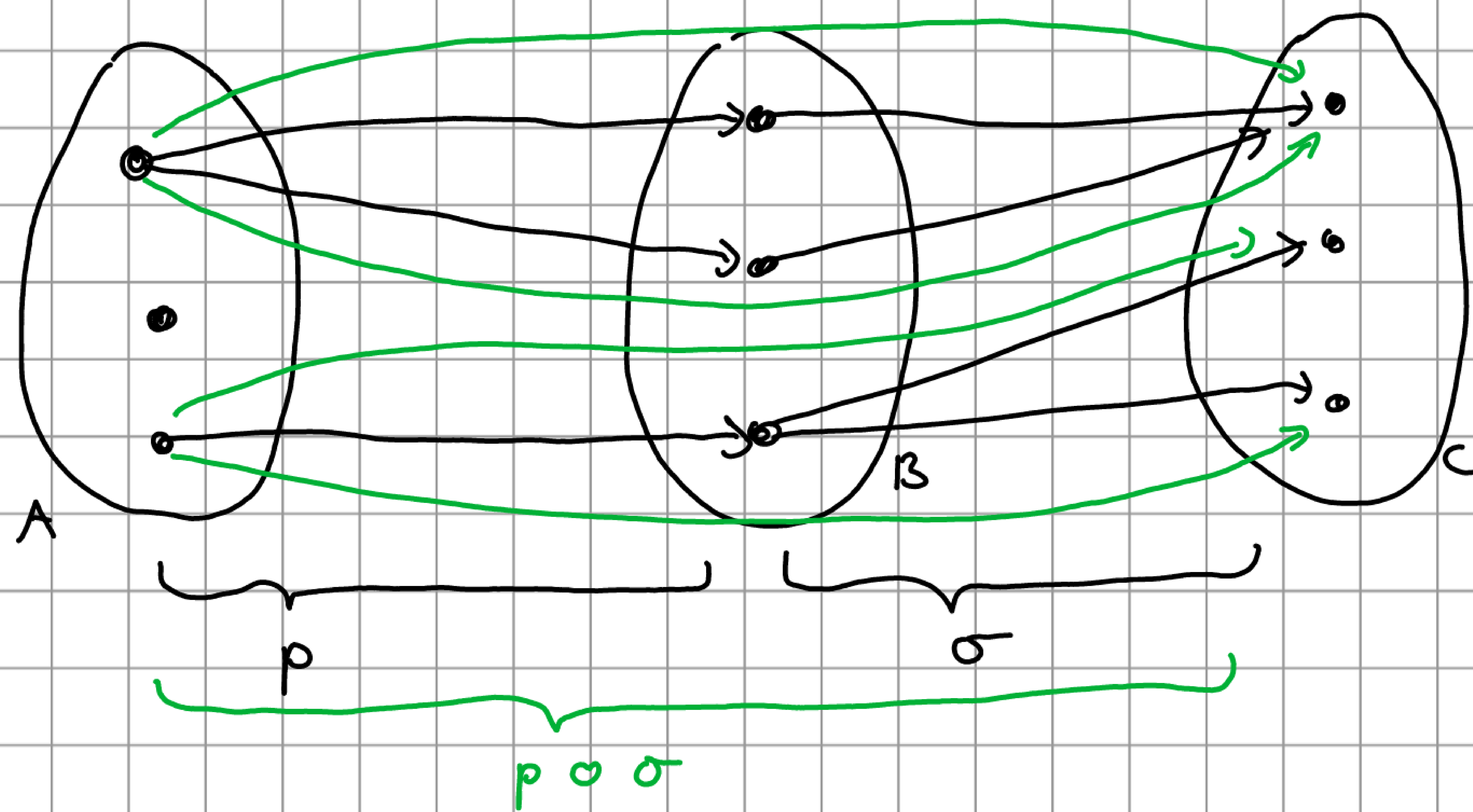
We sometimes also write ρ^{-1}.

COMPOSITION

lets us combine two relations to form a new one.
Formally:

$$\rho \circ \sigma = \{(a,c) \mid \exists b((a,b) \in \rho \wedge (b,c) \in \sigma)\}$$

Intuitively, this means that an element a is "connected" to c in the relation $\rho \circ \sigma$ if there is a middleman b who connects them.
Graphically, this becomes a lot clearer:



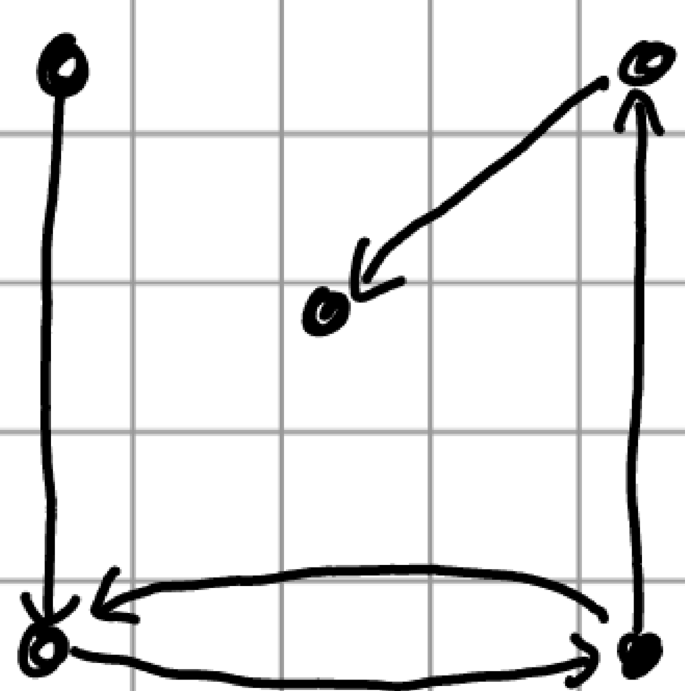
The relation $\rho \circ \sigma$ is now a relation from A to C.

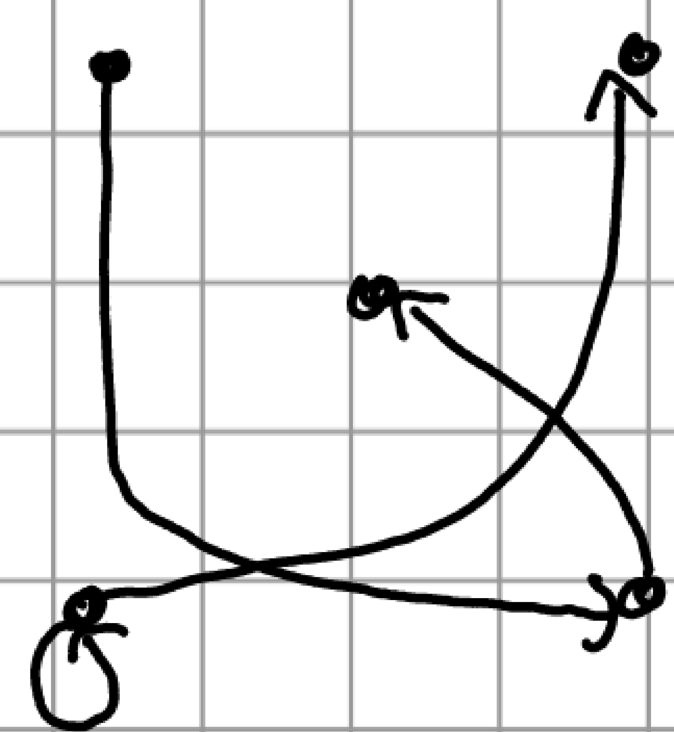
POWERS

If we combine a relation ρ with itself, we can write ρ^2 instead of $\rho \circ \rho$. Then instead of $\rho^2 \circ \rho$ we can write ρ^3 and so on.
So ρ^n is simply $\rho \circ \rho \circ \dots \circ \rho$ n-times.
Now how should we think about ρ^3 for example? For (a,c) to be in ρ^2 we needed a middleman b. Now for (a,d) to be in ρ^3 we need two middlemen.
For example, for the relation $r = \{(a,b),(b,c),(c,d)\}$ its third power r^3 would be $\{(a,d)\}$, since we can reach d from a over $a \to b \to c \to d$.

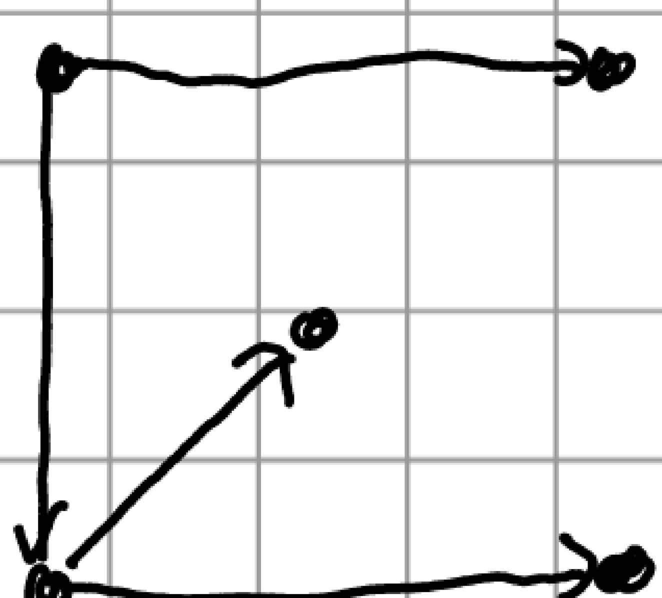
If we represent a relation as a graph, its n-th power

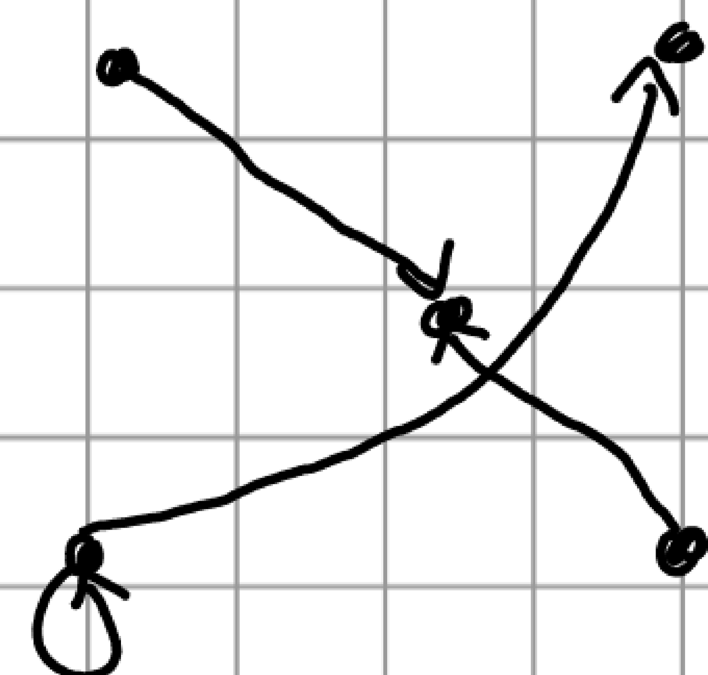
are just the walks we can take of length n.
So:

p: 

p^2: 

Look at these relations closely and try to find the path we took in p!

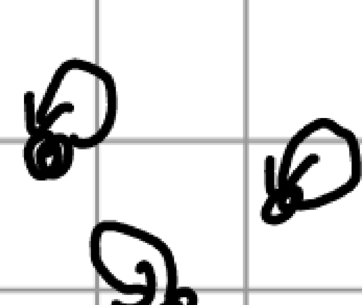
p^3: 

p^4: 

Note that relation composition is associative. This means that $p^5 = (((p \circ p) \circ p) \circ p) \circ p) = (p \circ p \circ p) \circ (p \circ p)$.
So if we wanted to find p^5 we could either find all paths of length 5 in p or find all paths by first taking an edge in p^3 and then taking an edge in p^2.
Try both methods!

RELATION PROPERTIES

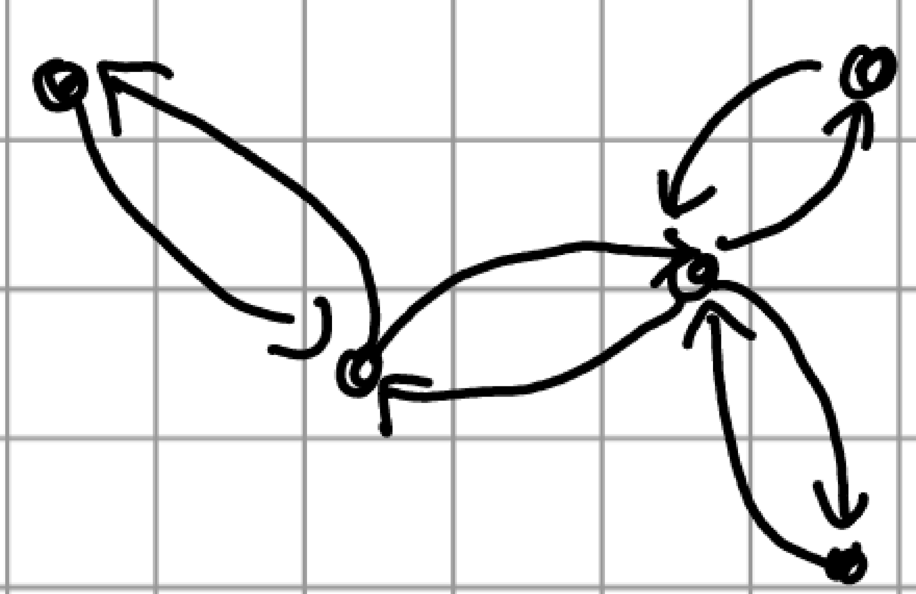
- A relation on A is REFLEXIVE if for all $a \in A$ it holds that (a,a) is in the relation.
Note that this must hold for all elements in A and not only for all elements that appear in the relation.
In the graph of a reflexive relation <u>all</u> nodes have self-loops:

 is reflexive, but  not

• A relation $\sim$ on A is SYMMETRIC if

$$a \sim b \iff b \sim a$$

Or intuitively: in a symmetric relation, if a is in relation to b, then b also has to be in relation to a. So either $(a,b) \in \sim$ and $(b,a) \in \sim$ or none of them is. The graph of a symmetric relation looks like this:

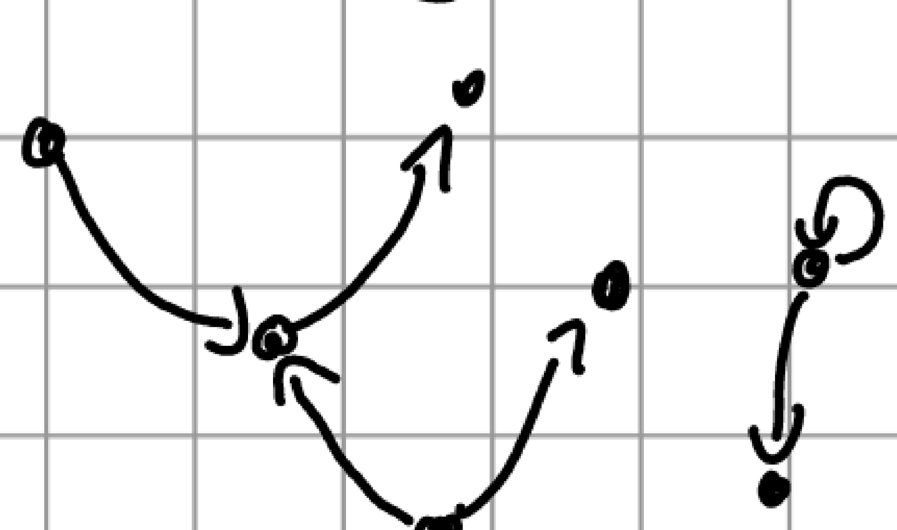


If we have an arrow from a node to another, there must also be the reversed arrow.

• A relation $\sim$ on A is ANTISYMMETRIC if

$$a \sim b \ \wedge \ b \sim a \implies a = b.$$

Or intuitively: Either a can be in relation to b or b in relation to a. Not both (a,b) and (b,a) can be in the relation. This means if $a \sim b$ and $b \sim a$ then a has to actually be equal to b.
In the graph of an antisymmetric relation there cannot be any cycles of length two:



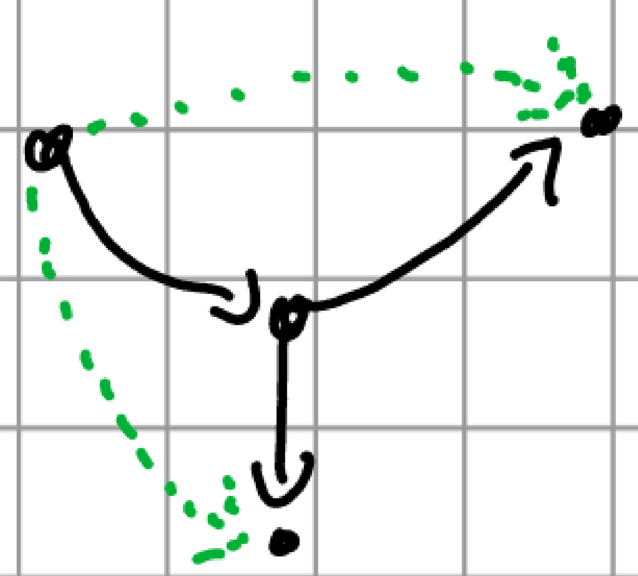
Only either a can be connected to b or b to a.

Antisymmetry is NOT the negation of symmetry!
For example, the following relation is neither symmetric nor antisymmetric:

- A relation $\sim$ on A is TRANSITIVE if

$$a \sim b \wedge b \sim c \implies a \sim c$$

Or intuitively: If we can walk from a to c in the graph of the relation through a node b, then we must also be able to walk to c from a directly.
For example:



this relation is <u>not</u> transitive. If we added the green arrows, then it would be transitive.

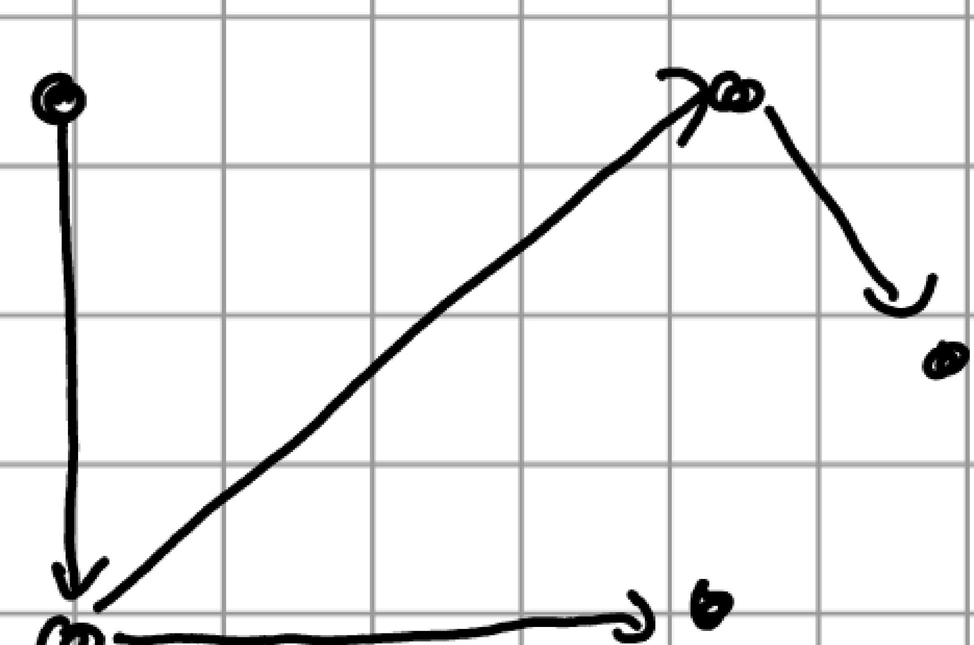
As an example: The relation $\leq$ on $\mathbb{N}$ is transitive. If $a \leq b$ and $b \leq c$ then $a \leq c$ also.

TRANSITIVE CLOSURE

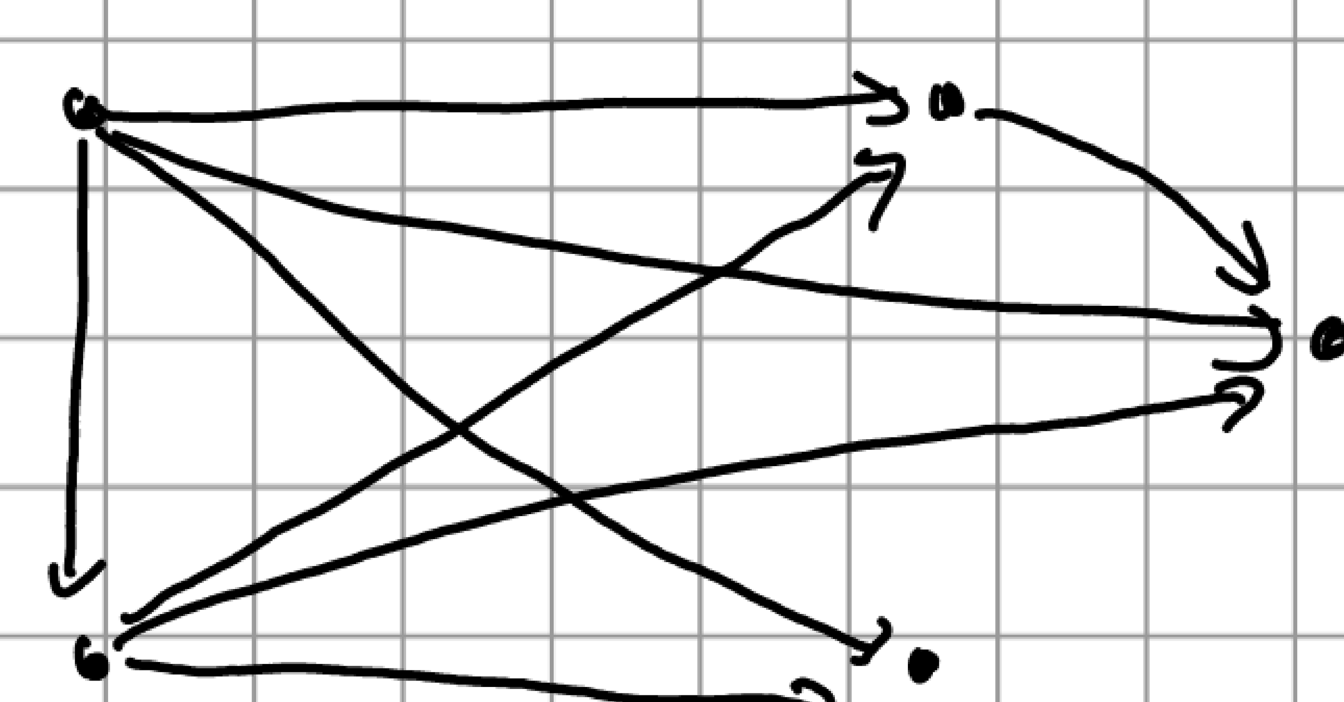
The transitive closure of p is $p^* = \bigcup_{n \in \mathbb{N}^+} p^n$.

In words: We combine all powers of p to get the transitive closure of p.
Intuitively, this means that $(a,b) \in p^*$ if we can reach b from a in some series of steps (through some number of middlemen).
If we want to get the transitive closure of p, we add all edges (a,b) to p if there is a path from a to b:

p:



p^*:



Or in other words: We "extend" ρ until it is transitive.

EQUIVALENCE RELATION

An equivalence relation satisfies three properties:

1) reflexivity
2) symmetry
3) transitivity

Intuitively, they capture our usual understanding of "equivalence". If a is equivalent to b then b is also equivalent to a and b is also equivalent to all elements that a is equivalent to ...

EQUIVALENCE CLASSES

If we have an equivalence relation $\sim$ on A then we can divide A into equivalence classes:

$$[a]_\sim = \{b \in A \mid a \sim b\}$$

For example if we choose the parity relation as $\sim$ (meaning all odd numbers are in relation to odd numbers and even numbers to even numbers) then it would divide $\mathbb{N}$ into the classes $[0]_\sim$ and $[1]_\sim$.
Note that $[0]_\sim = [2]_\sim = [4]_\sim = \dots$ It is the <u>one</u> and the same class, just multiple ways to write it.
Any element of A is exactly in one equivalence class. Why?

PARTIAL ORDER RELATIONS

A partial order relation on A is:

1) reflexive
2) antisymmetric
3) transitive

We call a a poset with the partial order $\leq$ and write $(A; \leq)$.
As the name suggest, they define an "ordering" on A.

PROVING RELATION PROPERTIES

These are the best strategies to prove relation properties. About 99% of the time you will use exactly those patterns if you want to prove the property, so always try them first. If they don't work, you need to get creative!.

PROVING REFLEXIVITY

To prove that a relation on A is reflexive, we simply show that for any element a in A, the relation contains (a,a).

Your proof should start with Take any element $a \in A$ and end with $\Rightarrow (a,a) \in \rho$.

Example: Let $|$ be the divisibility relation so $a|b$ if there exists z such that $a \cdot z = b$.
We show $|$ is reflexive on $\mathbb{Q}$.

Take any $x \in \mathbb{Q}$.

$\Rightarrow x \cdot 1 = x$ (1 neutral element)

$\Rightarrow x|x$ (def. $|$)

(Showing reflexivity is mostly very easy.)

PROVING SYMMETRY

To prove symmetry for a relation $\sim$ on A, take any $a, b \in A$ with $a \sim b$ and show that also $b \sim a$.

Your proof should look like this:

Take any $a, b \in A$ with $a \sim b$.
$\Rightarrow$ ...
$\Rightarrow$ ...
$\Rightarrow$ $b \sim a$

PROVING ANTISYMMETRY

To prove antisymmetry for $\sim$ on A we take some $a, b \in A$ with $a \sim b$ and $b \sim a$ and show that $a = b$.

Example: We show that $\leq$ on $\mathbb{N}$ is antisymmetric.

Take any $a, b \in \mathbb{N}$ with $a \leq b$ and $b \leq a$.

$a \leq b \ \wedge \ b \leq a$
$\Rightarrow (a < b \vee a = b) \wedge (b < a \vee a = b)$ (def. $\leq$)
$\Rightarrow (a < b \wedge b < a) \vee (a = b)$ (distr. law)
$\Rightarrow a = b$ (since $a < b \wedge b < a$ is not possible)

(Here we assume we know that $<$ is antisymmetric ...)

PROVING TRANSITIVITY

To prove transitivity of $\sim$ on A we take any $a, b, c \in A$ with $a \sim b$ and $b \sim c$ and show that $a \sim c$.

Example: We show that $\leq$ is transitive on $\mathbb{Z}$.

Take any $a, b, c \in \mathbb{Z}$ with $a \leq b$ and $b \leq c$.

$$
\begin{aligned}
& a \leq b \land b \leq c \\
\Rightarrow\ & a-b \leq 0 \land b-c \leq 0 \\
\Rightarrow\ & (a-b)+(b-c) \leq 0 \qquad (\text{if } a, b \leq 0 \text{ then } a+b \leq 0) \\
\Rightarrow\ & a+(-b+b)-c \leq 0 \\
\Rightarrow\ & a-c \leq 0 \\
\Rightarrow\ & a \leq c
\end{aligned}
$$

(This is a little sketchy, just a short example)

DISPROVING PROPERTIES

To disprove any of the properties, just find some elements for which they don't hold and show it.

Example: We disprove that $\neq$ is transitive on $\mathbb{Z}$.

Consider 1 and 2. $1 \neq 2$ and $2 \neq 1$ but not $1 \neq 1$.

EQUIVALENCE/PARTIAL ORDERS

To prove that $\sim$ is an equivalence / p.o. relation simply show all the three properties seperately.
Start with reflexivity, since it is the easiest. Then (anti)symmetry and lastly transitivity.

RELATION EXERCISES

(Solutions at the end.)

① Which of the following are reflexive, transitive, symmetric, antisymmetric, equivalence, partial order relations? (on $\mathbb{Z}$)

a) $\leq$ d) $\subseteq$
b) $|$ ("divides") e) $\subset$
c) $\neq$ f) $>$

② Define the relation $\sim$ as: $a \sim b \Leftrightarrow \exists \lambda (\lambda \cdot a = b)$.

a) Show that $\sim$ is an equivalence relation on $\mathbb{Q} \setminus \{0\}$.
b) Why is it not on $\mathbb{Q}$?

③ This is the bonus exercise from 2024. Prove or disprove:

a) A relation ρ on A is symmetric if and only if ρ^2 is symmetric on A.
b) If ρ is a relation on A that is symmetric and antisymmetric, then it must hold that $\rho = id_A$.
c) Define ρ_1 and ρ_2 on $\mathbb{Z}$ as:

$$a \rho_1 b \Leftrightarrow b = a+1 \quad , \quad a \rho_2 b \Leftrightarrow b =_2 a$$

Then for $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$

Hint: for a) and b) think about $\emptyset$. $\emptyset$ is a relation too!

④ Show that the intersection of two antisymmetric relations on a set A is also antisymmetric.

⑤ Let ρ be antisymmetric. Prove that $\hat{\rho}$ is antisymmetric.

⑥ Let p and q be transitive. Disprove that $p \circ q$ is transitive.

Solutions

①

	refl	sym	antisym	trans	equiv	part
$\leq$	✓	✗	✓	✓	✗	✓
$\mid$	✓	✗	✓	✓	✗	✓
$\neq$	✗	✓	✗	✗	✗	✗
$\subseteq$	✓	✗	✓	✓	✗	✓
$\subset$	✗	✗	✓	✓	✗	✗
$>$	✗	✗	✓	✓	✗	✗

②

a) We show the three properties for $\sim$:

- reflexivity: take any $a \in \mathbb{Q} \setminus \{0\}$. Then for $\lambda = 1$ we have $1 \cdot a = a$ and therefore $a \sim a$.

- symmetry: take any $a, b \in \mathbb{Q} \setminus \{0\}$ with $a \sim b$.

$$
\begin{aligned}
& a \sim b \\
\Rightarrow\ & \exists \lambda\, (a \cdot \lambda = b) && (\text{def. } \sim) \\
\Rightarrow\ & \exists \lambda\, (a = \tfrac{1}{\lambda} \cdot b) && (\text{since } b \neq 0:\ \lambda \neq 0) \\
\Rightarrow\ & \exists \delta\, (\delta \cdot b = a) && (\delta = \tfrac{1}{\lambda}) \\
\Rightarrow\ & b \sim a && (\text{def. } \sim)
\end{aligned}
$$

- transitivity: take any $a, b, c \in \mathbb{Q} \setminus \{0\}$ with $a \sim b$ and $b \sim c$.

$$
\begin{aligned}
& a \sim b \wedge b \sim c \\
\Rightarrow\; & \exists \lambda_1 (\lambda_1 a = b) \wedge \exists \lambda_2 (\lambda_2 b = c) \\
\Rightarrow\; & \exists \lambda_1 (\lambda_1 a = b) \wedge \exists \lambda_2 \left(b = \tfrac{1}{\lambda_2} \cdot c\right) && (\lambda_2 \neq 0) \\
\Rightarrow\; & \exists \lambda_1 \exists \lambda_2 \left(\lambda_1 a = \tfrac{1}{\lambda_2} \cdot c\right) && (\text{both} = b) \\
\Rightarrow\; & \exists \lambda_1 \exists \lambda_2 (\lambda_1 \lambda_2 a = c) \\
\Rightarrow\; & \exists \delta (\delta a = c) && (\delta = \lambda_1 \lambda_2) \\
\Rightarrow\; & a \sim c && (\text{def. } \sim)
\end{aligned}
$$

(3) look at emils-site → session 5 for the solution

(4) Let ρ and σ be antisymmetric on A. Take any $a, b \in A$ with $(a,b) \in \rho \cap \sigma$ and $(b,a) \in \rho \cap \sigma$.

$$
\begin{aligned}
& (a,b) \in \rho \cap \sigma \wedge (b,a) \in \rho \cap \sigma \\
\Rightarrow\; & (a,b) \in \rho \wedge (a,b) \in \sigma \wedge (b,a) \in \rho \wedge (b,a) \in \sigma && (\text{def. } \cap) \\
\Rightarrow\; & (a,b) \in \rho \wedge (b,a) \in \rho \\
\Rightarrow\; & a = b && (\rho \text{ is antisymmetric})
\end{aligned}
$$

(5) Let ρ on A be antisymmetric. Take any $a, b \in A$ with $a \hat{\rho} b$ and $b \hat{\rho} a$.

$$
\begin{aligned}
& a \hat{\rho} b \wedge b \hat{\rho} a \\
\Rightarrow\; & b \rho a \wedge a \rho b && (\text{def. } \hat{\rho}) \\
\Rightarrow\; & a = b && (\rho \text{ is antisymmetric})
\end{aligned}
$$

(6) Consider $A = \mathbb{N}$ and the relations:

$p = \{(1,2), (3,4)\}$ and

$q = \{(2,3), (4,5)\}$.

Then both p and q are transitive. But:

$p \circ q = \{(1,3), (3,5)\}$, which is not transitive.