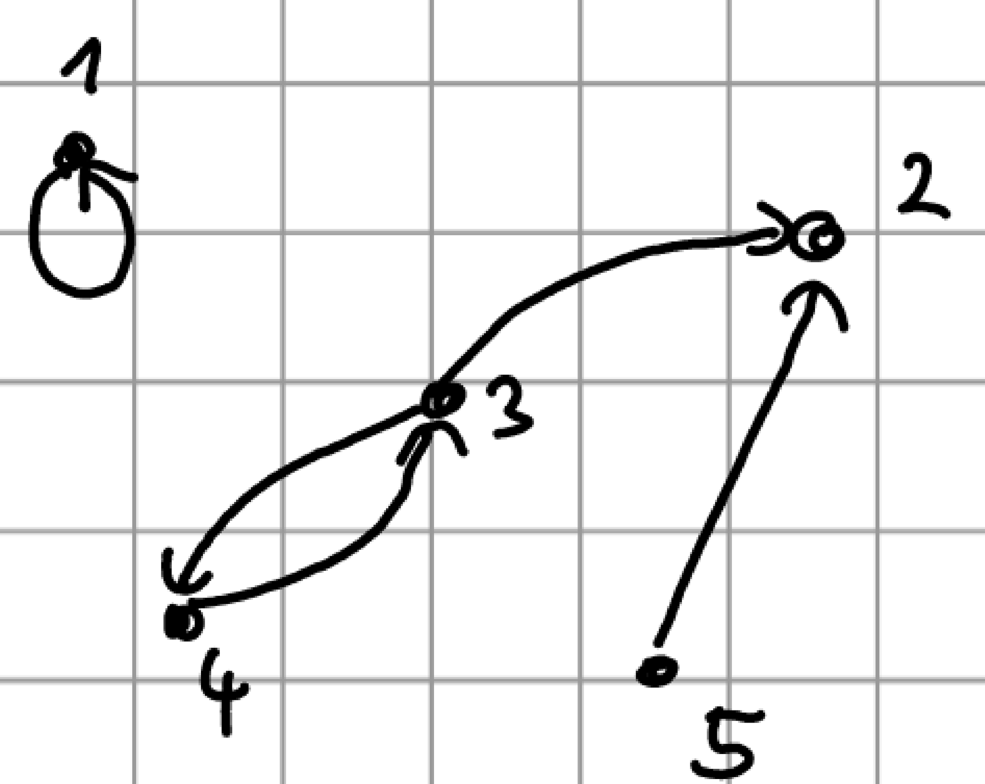
Relations

- This is not a complete summary. READ THE SCRIPT!
- Exercises are in red

REPRESENTATION

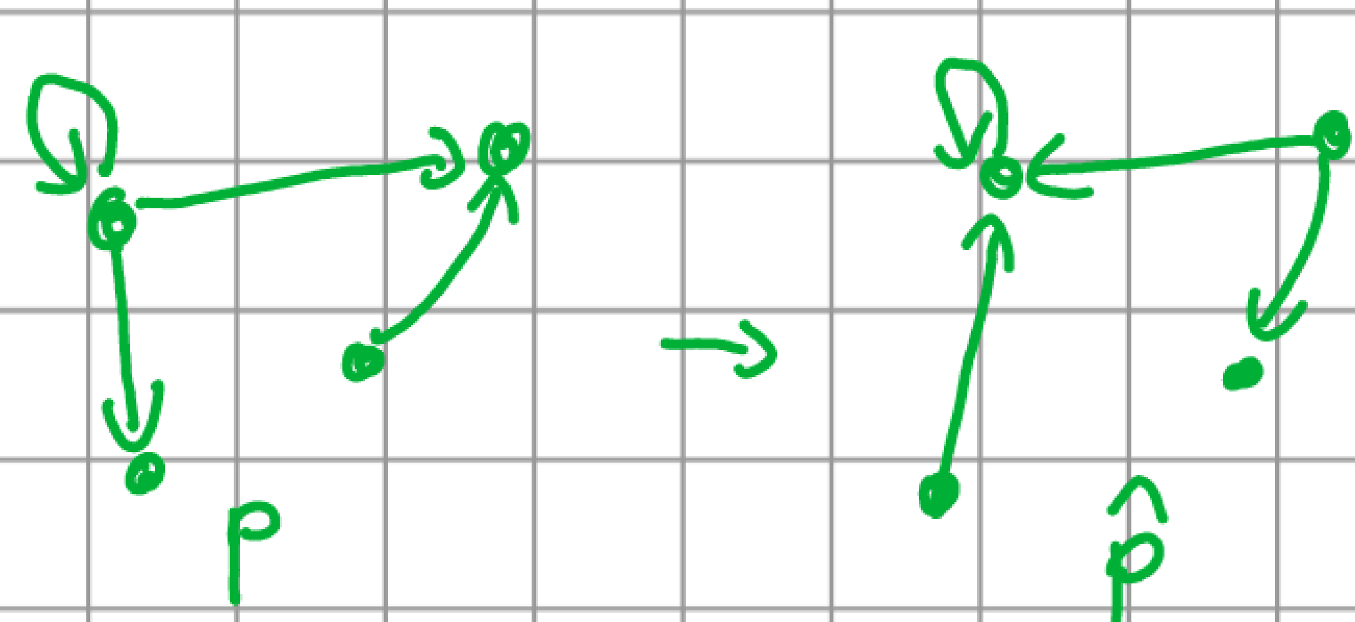
It is very useful to represent a relation $r: A \to A$ as a graph:



This would be the relation $\{(1,1),(3,2),(3,4),(4,3),(5,2)\}$.
This representation is VERY useful if you want to look for relation properties.

INVERSE

of a relation p is the relation $\hat{p}$, which contains all the tuples that p does, but reversed. So the inverse to the relation above would be $\{(1,1),(2,3),(4,3),(3,4),(2,5)\}$.
In graph representation, we simply turn around the arrows.



Formally $\hat{p} = \{(b,a) \mid (a,b) \in p\}$.

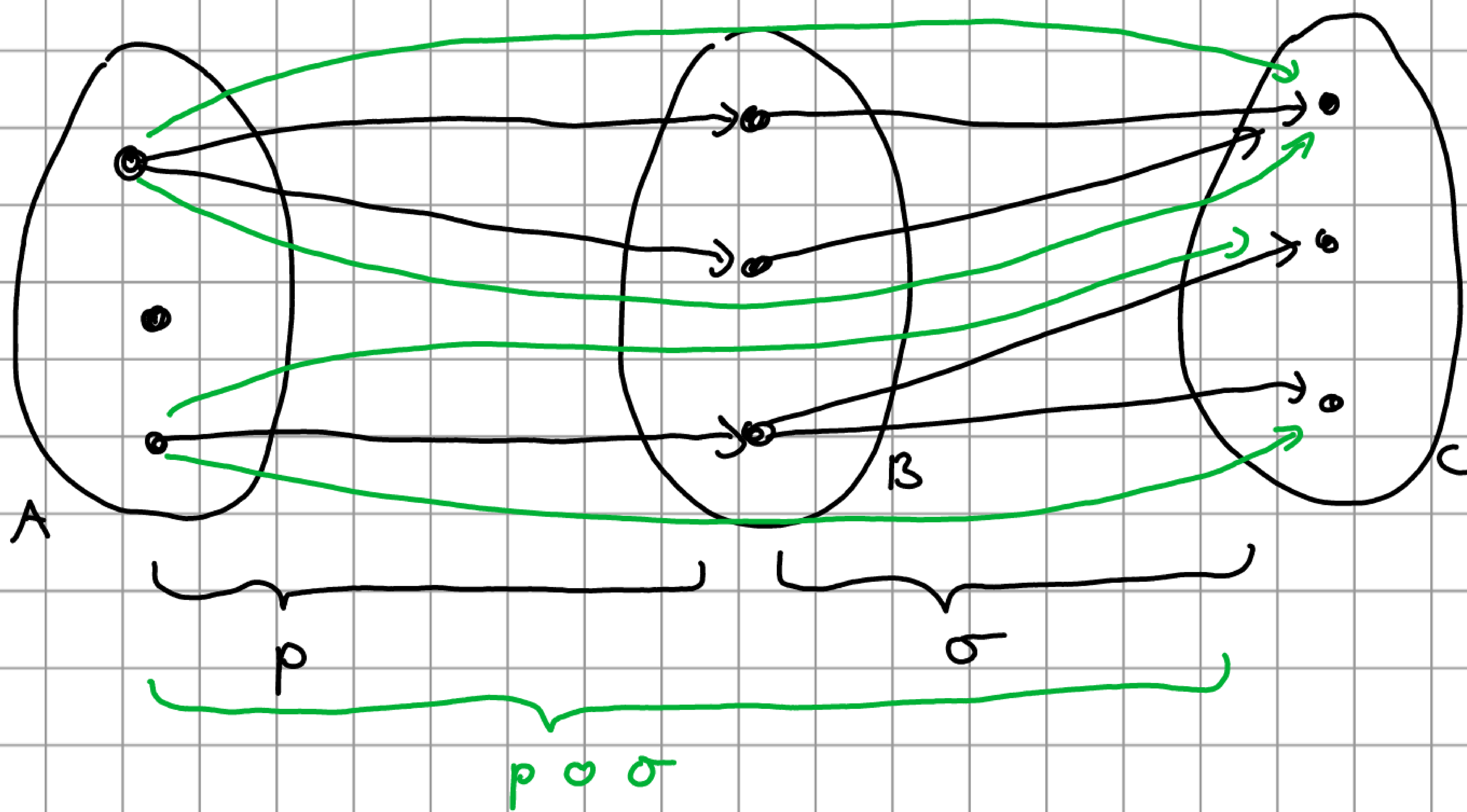
We sometimes also write p^{-1}.

COMPOSITION

lets us combine two relations to form a new one.
Formally:

$$p \circ \sigma = \{(a,c) \mid \exists b((a,b) \in p \wedge (b,c) \in \sigma)\}$$

Intuitively, this means that an element a is "connected" to c in the relation $\rho \circ \sigma$ if there is a middleman b who connects them.
Graphically, this becomes a lot clearer:



The relation $\rho \circ \sigma$ is now a relation from A to C.

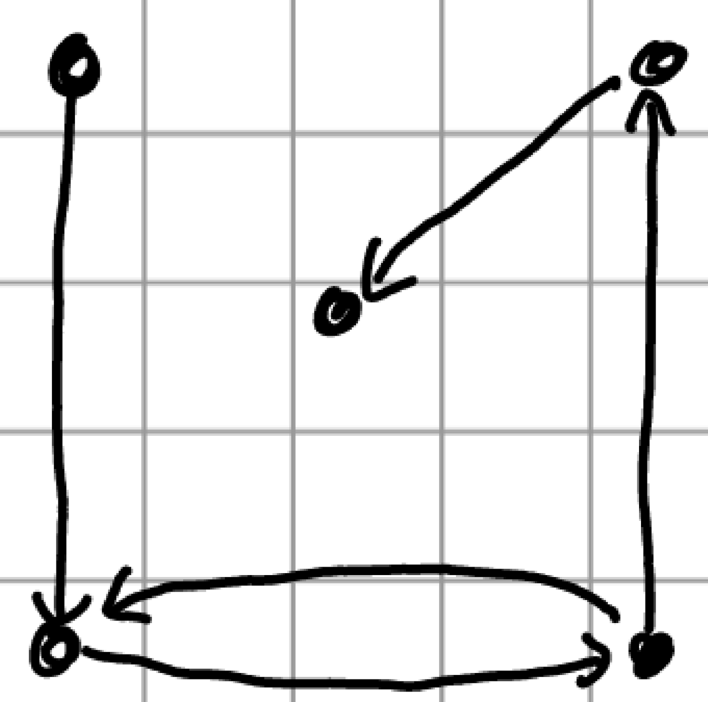
POWERS

If we combine a relation ρ with itself, we can write ρ^2 instead of $\rho \circ \rho$. Then instead of $\rho^2 \circ \rho$ we can write ρ^3 and so on.
So ρ^n is simply $\rho \circ \rho \circ \dots \circ \rho$ n-times.
Now how should we think about ρ^3 for example? For (a,c) to be in ρ^2 we needed a middleman b. Now for (a,d) to be in ρ^3 we need two middlemen.
For example, for the relation $r = \{(a,b),(b,c),(c,d)\}$ its third power r^3 would be $\{(a,d)\}$, since we can reach d from a over $a \to b \to c \to d$.

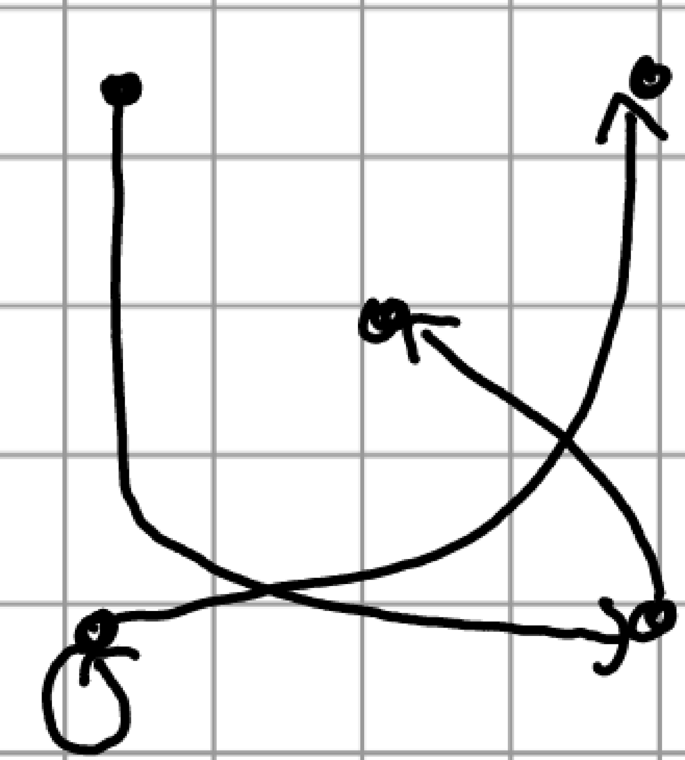
If we represent a relation as a graph, its n-th power are just the walks we can take of length n.

So:

p:

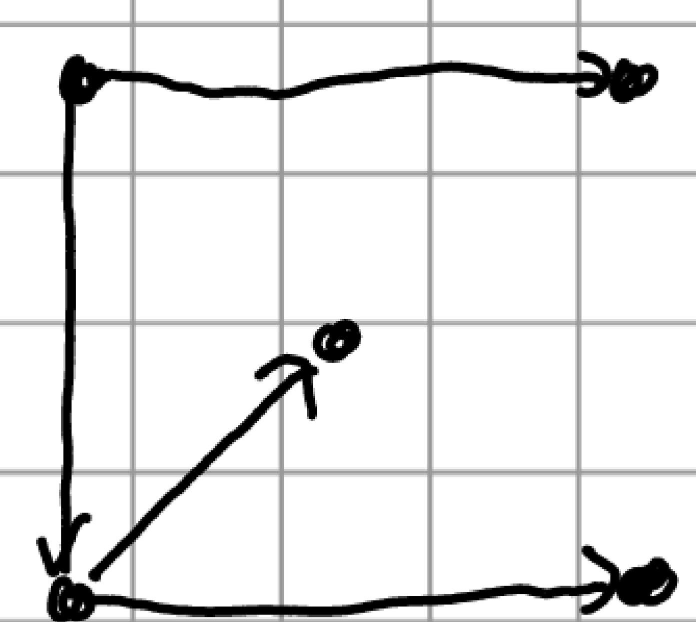


p^2:

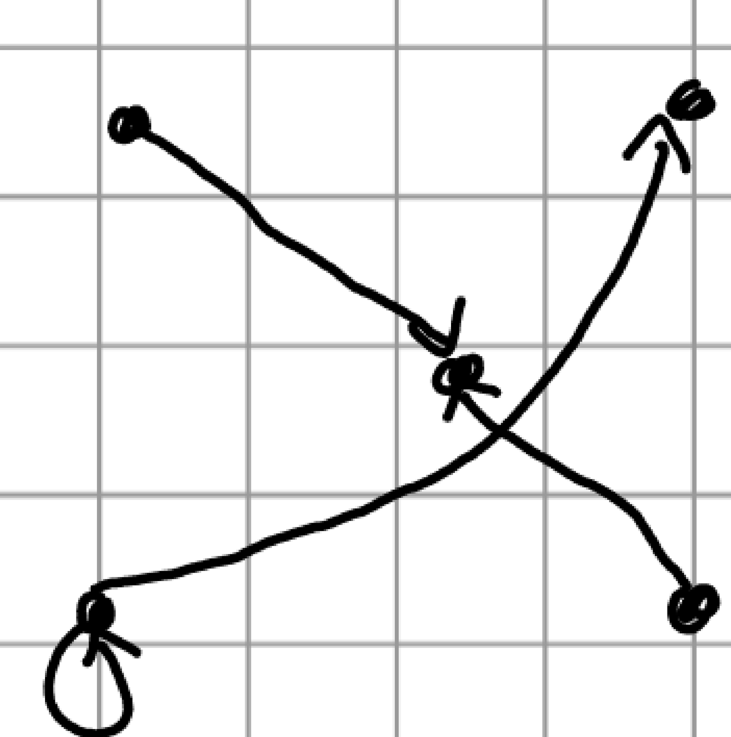


Look at these relations closely and try to find the path we took in p!

p^3:



p^4:



Note that relation composition is associative. This means that $p^5 = ((((p \circ p) \circ p) \circ p) \circ p) = (p \circ p \circ p) \circ (p \circ p)$.
So if we wanted to find p^5 we could either find all paths of length 5 in p or find all paths by first taking an edge in p^3 and then taking an edge in p^2.
Try both methods!

RELATION PROPERTIES

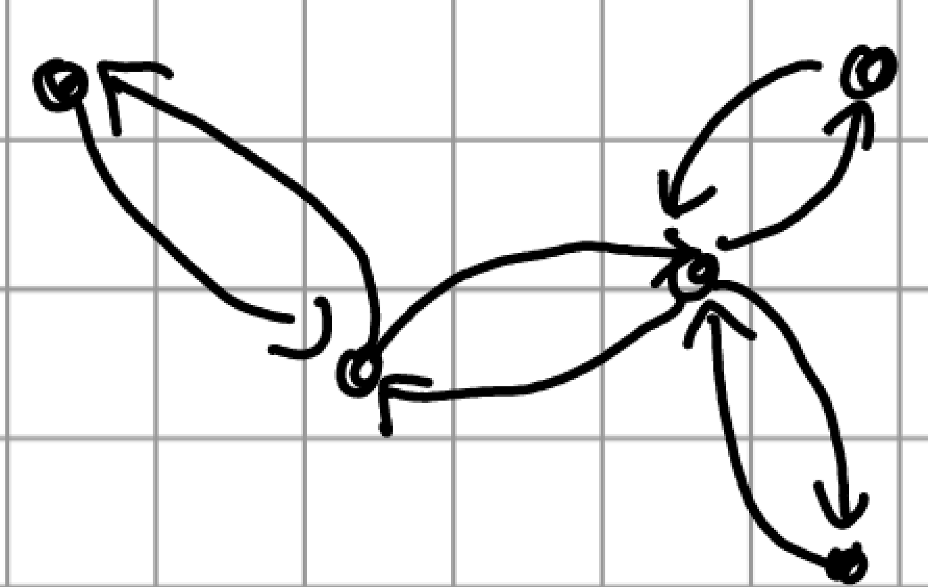
- A relation on A is REFLEXIVE if for all $a \in A$ it holds that (a,a) is in the relation.
Note that this must hold for all elements in A and not only for all elements that appear in the relation.
In the graph of a reflexive relation <u>all</u> nodes have self-loops:

 is reflexive, but  not

- A relation $\sim$ on A is SYMMETRIC if

$$a \sim b \iff b \sim a$$

Or intuitively: in a symmetric relation, if a is in relation to b, then b also has to be in relation to a. So either $(a,b) \in \sim$ and $(b,a) \in \sim$ or none of them is. The graph of a symmetric relation looks like this:

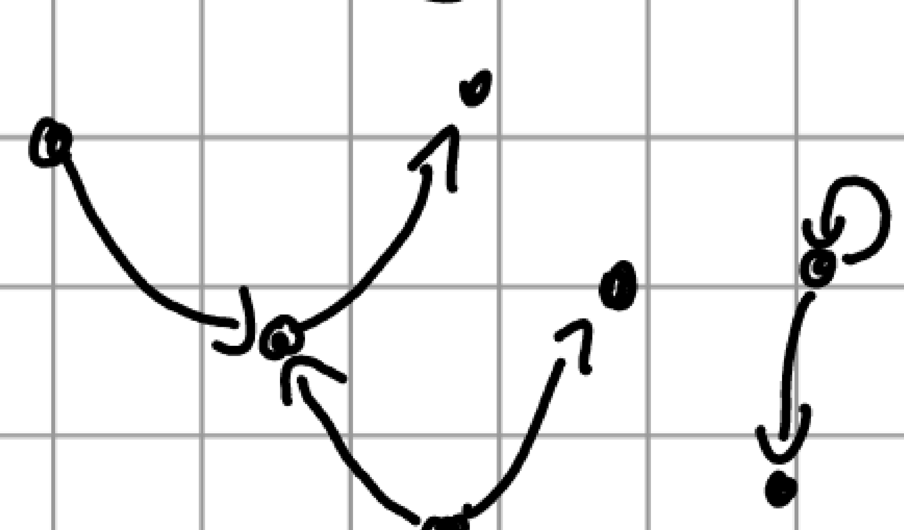


If we have an arrow from a node to another, there must also be the reversed arrow.

- A relation $\sim$ on A is ANTISYMMETRIC if

$$a \sim b \land b \sim a \Rightarrow a = b.$$

Or intuitively: Either a can be in relation to b or b in relation to a. Not both (a,b) and (b,a) can be in the relation. This means if $a \sim b$ and $b \sim a$ then a has to actually be equal to b.
In the graph of an antisymmetric relation there cannot be any cycles of length two:



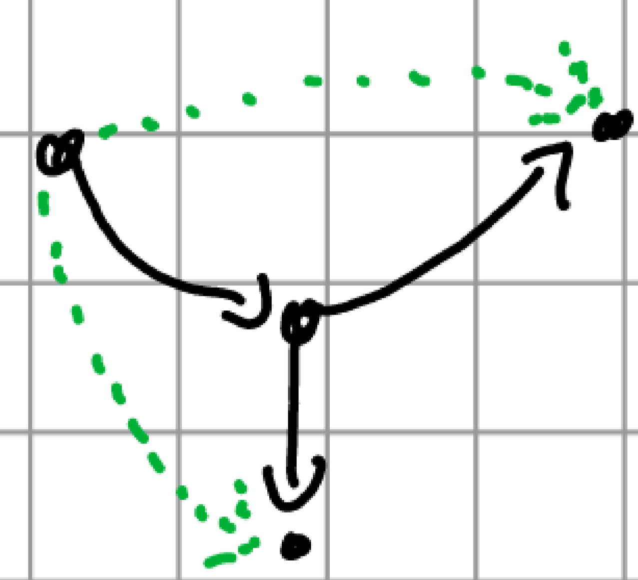
Only either a can be connected to b or b to a.

Antisymmetry is NOT the negation of symmetry!
For example, the following relation is neither symmetric nor antisymmetric:

- A relation $\sim$ on A is TRANSITIVE if

$$a \sim b \wedge b \sim c \implies a \sim c$$

Or intuitively: If we can walk from a to c in the graph of the relation through a node b, then we must also be able to walk to c from a directly.
For example:



this relation is <u>not</u> transitive. If we added the green arrows, then it would be transitive.

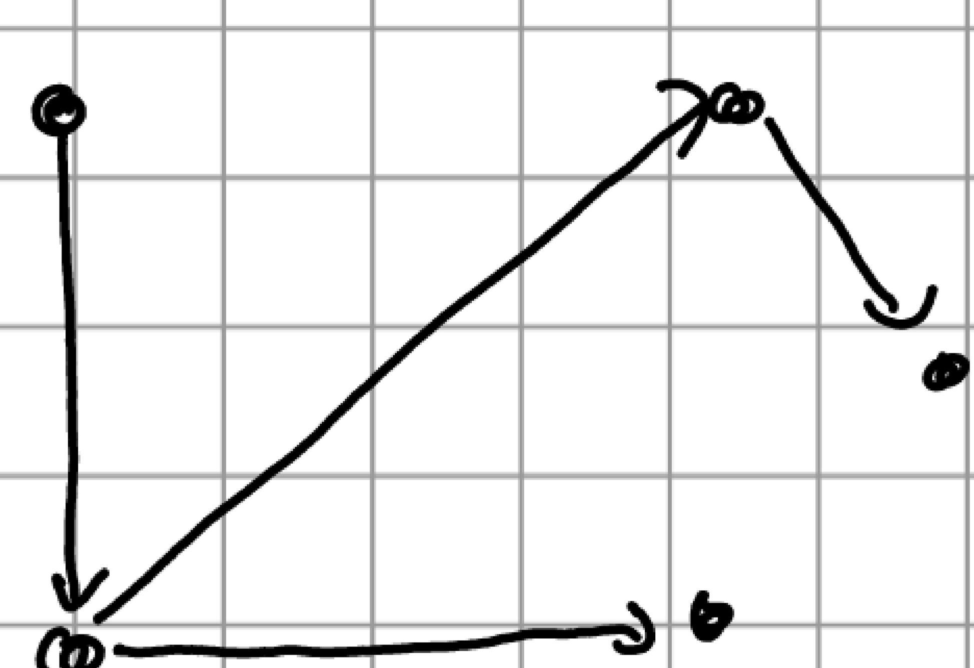
As an example: The relation $\leq$ on $\mathbb{N}$ is transitive.
If $a \leq b$ and $b \leq c$ then $a \leq c$ also.

TRANSITIVE CLOSURE

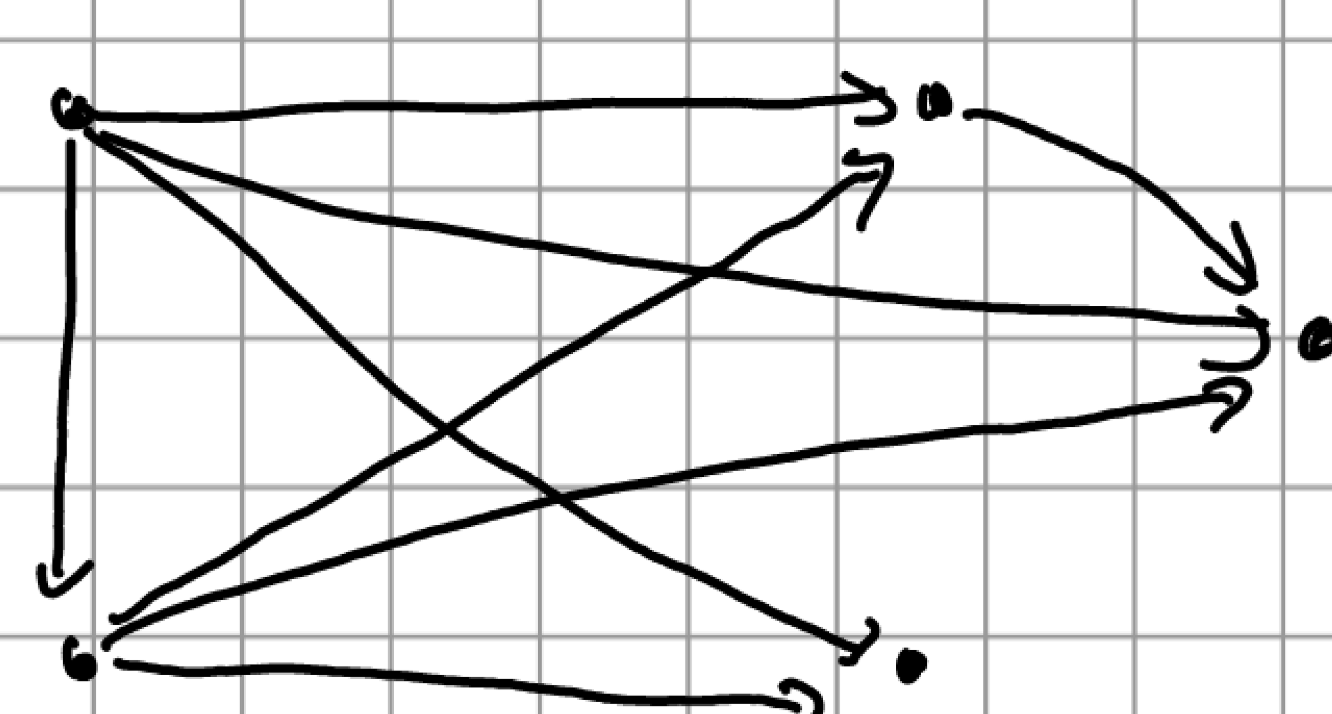
The transitive closure of ρ is $\rho^* = \bigcup_{n \in \mathbb{N}^+} \rho^n$.

In words: We combine all powers of ρ to get the transitive closure of ρ.
Intuitively, this means that $(a, b) \in \rho^*$ if we can reach b from a in some series of steps (through some number of middlemen).
If we want to get the transitive closure of ρ, we add all edges (a, b) to ρ if there is a path from a to b:

ρ:



ρ^*:



Or in other words: We "extend" ρ until it is transitive.

EQUIVALENCE RELATION

An equivalence relation satisfies three properties:

1) reflexivity
2) symmetry
3) transitivity

Intuitively, they capture our usual understanding of "equivalence". If a is equivalent to b then b is also equivalent to a and b is also equivalent to all elements that a is equivalent to ...

EQUIVALENCE CLASSES

If we have an equivalence relation $\sim$ on A then we can divide A into equivalence classes:

$$[a]_\sim = \{ b \in A \mid a \sim b \}$$

For example if we choose the parity relation as $\sim$ (meaning all odd numbers are in relation to odd numbers and even numbers to even numbers) then it would divide $\mathbb{N}$ into the classes $[0]_\sim$ and $[1]_\sim$.
Note that $[0]_\sim = [2]_\sim = [4]_\sim = \ldots$ It is the <u>one</u> and the same class, just multiple ways to write it.
Any element of A is exactly in one equivalence class. Why?

PARTIAL ORDER RELATIONS

A partial order relation on A is:

1) reflexive
2) antisymmetric
3) transitive

We call a a poset with the partial order $\leq$ and write $(A; \leq)$.
As the name suggest, they define an "ordering" on A.