⑤ The direct product of $H \times K$ is the group:
$\langle H \times K; \triangledown \rangle$ with $(h_1, k_1) \triangledown (h_2, k_2) = (h_1 * h_2, k_1 * k_2)$ per def. 5.9./ lemma 5.4.

To prove that it is isomorphic to G, we need to find an isomorphism from G to $\langle H \times K; \triangledown \rangle$.
Let's look at the map ψ:

$$\begin{aligned} \psi : G &\to \langle H \times K; \triangledown \rangle \\ g &\mapsto (h, k) \end{aligned}$$

with $\psi(g) = (h, k)$ where $h \in H$ and $k \in K$ such that $h * k = g$.

ψ is a funtion since per (i) we know that such h, k exist for all $g \in G$ and we quickly prove that it is well-defined:

We know that the only representation of $e \in G$ is $e * e$, since if $h_e * k_e = e$ with $h_e \in H$, then k_e is the inverse of h_e and therefore $k_e \in H$. The only $k_e \in K$ which fulfills this is e per (ii).
Now assume $\psi(g) = (h_1, g_1) = (h_2, g_2)$.
Then the unique inverse of g is $\hat{g} = \hat{h}_2 * \hat{g}_2$ since:

$g * \hat{g}$
$= h_2 * g_2 * \hat{h}_2 * \hat{g}_2$ | $g = h_2 * g_2$ since $\psi(g) = (h_2, g_2)$
$= (h_2 + \hat{h}_2) * (g_2 * \hat{g}_2)$ | G is commutative group
$= e * e = e$ | G3 and e is neutral element

Then however:

$g * \hat{g} = e$
$\Rightarrow h_1 * k_1 * \hat{h}_2 * \hat{g}_2 = e$ | $g = h_1 * k_1$ cause $\psi(g) = (h_1, g_1)$
$\Rightarrow (h_1 * \hat{h}_2) * (k_1 * \hat{k}_2) = e$ | G is commutative group
$\Rightarrow h_1 * \hat{h}_2 = e$ and $h_1 * \hat{h}_2 = e$ | since $(h_1 * \hat{h}_2) \in H$ and $(h_1 * \hat{k}_2) \in K$ we know from above they have to be e.
$\Rightarrow h_1 = \hat{\hat{h}}_2$ and $h_1 = \hat{\hat{k}}_2$ | def. of inverse
$\Rightarrow h_1 = h_2$ and $h_1 = h_2$ | $\hat{\hat{a}} = a$

Which shows that ψ is well-defined.

We now need to prove that ψ is a homomorphism, so that it satisfies $\psi(g * g') = \psi(g) \triangledown \psi(g')$ for all $g, g' \in G$.

So let g, g' be arbitrary ($\in$ of G), then:

$$
\begin{aligned}
& \psi(g * g') \\
&= \psi(h * k * h' * k') && \text{with } h*k = g,\ h'*k' = g' \text{ which exist per (i)} \\
&= \psi((h * h') * (k * k')) && \mid * \text{ is commutative / associative} \\
&= (h * h',\ k * k') && \mid \text{def. of } \psi \\
&= (h, k) \triangledown (h', k') && \mid \text{def. of } \triangledown \\
&= \psi(h * k) \triangledown \psi(h' * k') && \mid \text{def. of } \psi \\
&= \psi(g) \triangledown \psi(g') && \mid \text{def. of } g, g' \text{ from above}
\end{aligned}
$$

Which is exactly the homomorphism-property.

It is left to show that ψ is a bijection, so injective, surjective:

- injective: for arbitrary $g, g' \in G$ with $g = h * k$, $g' = h' * k'$:

$$
\begin{aligned}
& g \neq g' && \text{(assume)} \\
\Rightarrow\ & h * k \neq h' * k' && \mid \text{def } g, g' \\
\Rightarrow\ & h \neq h' \lor k \neq k' && \mid \text{otherwise } h = h' \land k = k' \Rightarrow h * k = h' * k' \mid \text{de morgan} \\
\Rightarrow\ & (h, k) \neq (h', k') && \mid \text{def. of tupel equality} \\
\Rightarrow\ & \psi(g) \neq \psi(g') && \mid \text{def. of } \psi
\end{aligned}
$$

which is the definition of injective

- surjective: For arbitrary $(h, k) \in \langle H \times K, \triangledown \rangle$ there must exist $g \in G$ such that $h * k = g$ per (i). Consequently $\psi(g) = (h, g)$ per def. of ψ.

Since we showed that ψ is a homomorphism and bijective, it is an isomorphism and therefore G is isomorphic to $H \times K$.

b) $\langle \mathbb{Z}_{15}^*, \odot \rangle = \{1, 2, 4, 7, 8, 11, 13, 14\}$

- $A = \{1, 2, 4, 8\}$ is a subgroup of it, since $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 1$, meaning it is generated by 2 and therefore a group per def. 5.14.

- $B = \{1, 11\}$ is another subgroup since $11 \odot_{15} 11 = 1$ meaning it is closed.

- We calculate all products between elements of A and B:

$\odot_{15}$	1	2	4	8
1	1	2	4	8
11	11	7	14	3

and see that those are exactly $\langle \mathbb{Z}_{15}^*, \odot \rangle$. And since the groups only share 1 (the neutral element), they fulfill (i) and (ii) and so $A \times B \simeq \langle \mathbb{Z}_{15}^*, \odot \rangle$.

- And since both groups are cyclic, they are also isomorph to $\mathbb{Z}_4$ and $\mathbb{Z}_2$ per theorem 5.7.

- Now we can construct an isomorphism ψ from $A \times B$ to $\mathbb{Z}_4 \times \mathbb{Z}_2$ with $\psi((x,y)) = (a, b)$ s.t. $f(x) = a$ and $g(y) = b$ if f is the isomorphism from $A \to \mathbb{Z}_4$ and g from $B \to \mathbb{Z}_2$.
Since g, f are isomorphisms, ψ is well defined and bijective and since the operations on $A \times B$ are defined component wise ψ is also a homomorphism.

- This shows that $\mathbb{Z}_4 \times \mathbb{Z}_2 \simeq A \times B \simeq \langle \mathbb{Z}_{15}^*, \odot \rangle$

  meaning $\mathbb{Z}_4 \times \mathbb{Z}_2 \simeq \langle \mathbb{Z}_{15}^*, \odot \rangle$