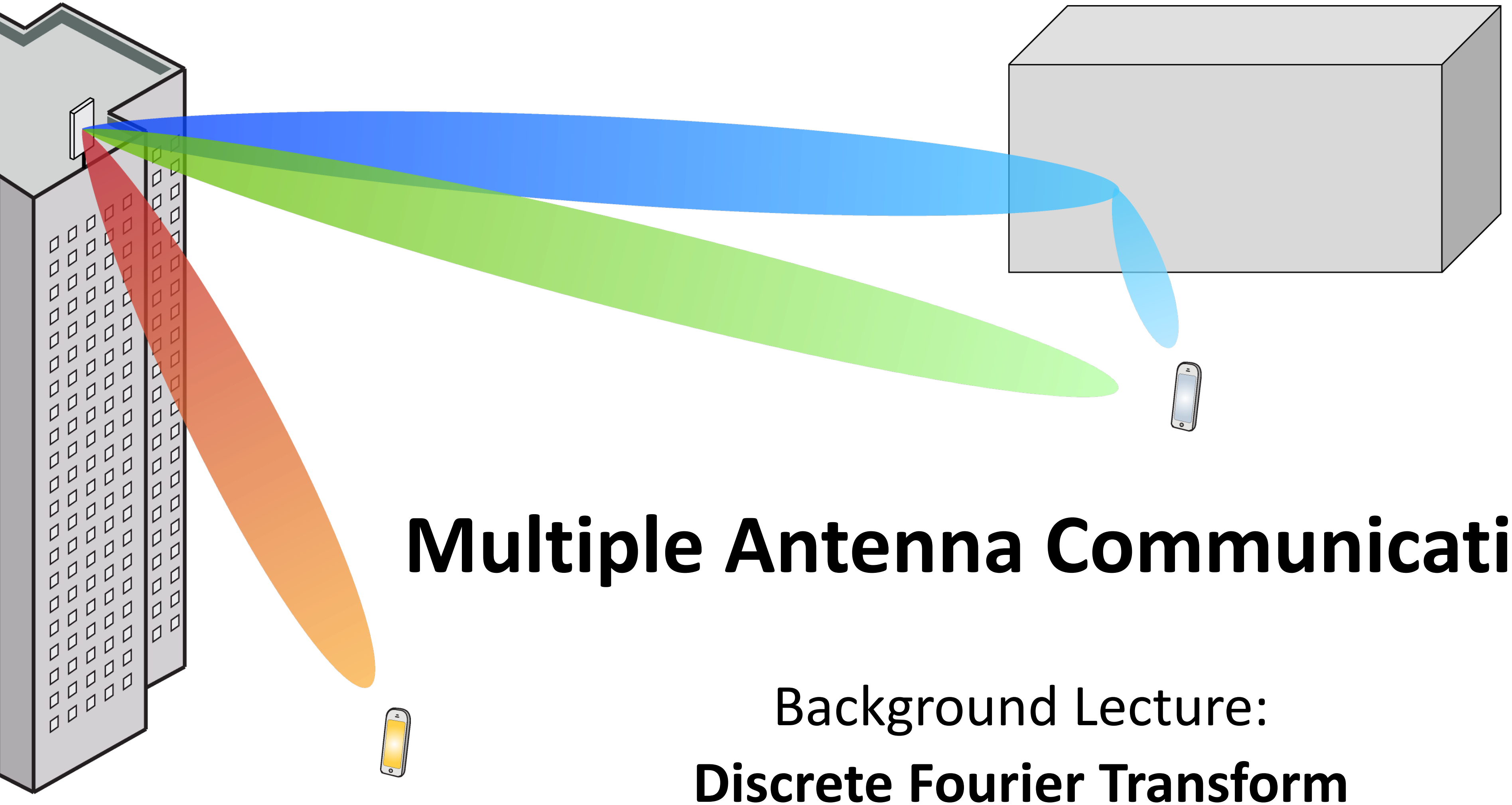


Multiple Antenna Communications

Background Lecture:

Discrete Fourier Transform

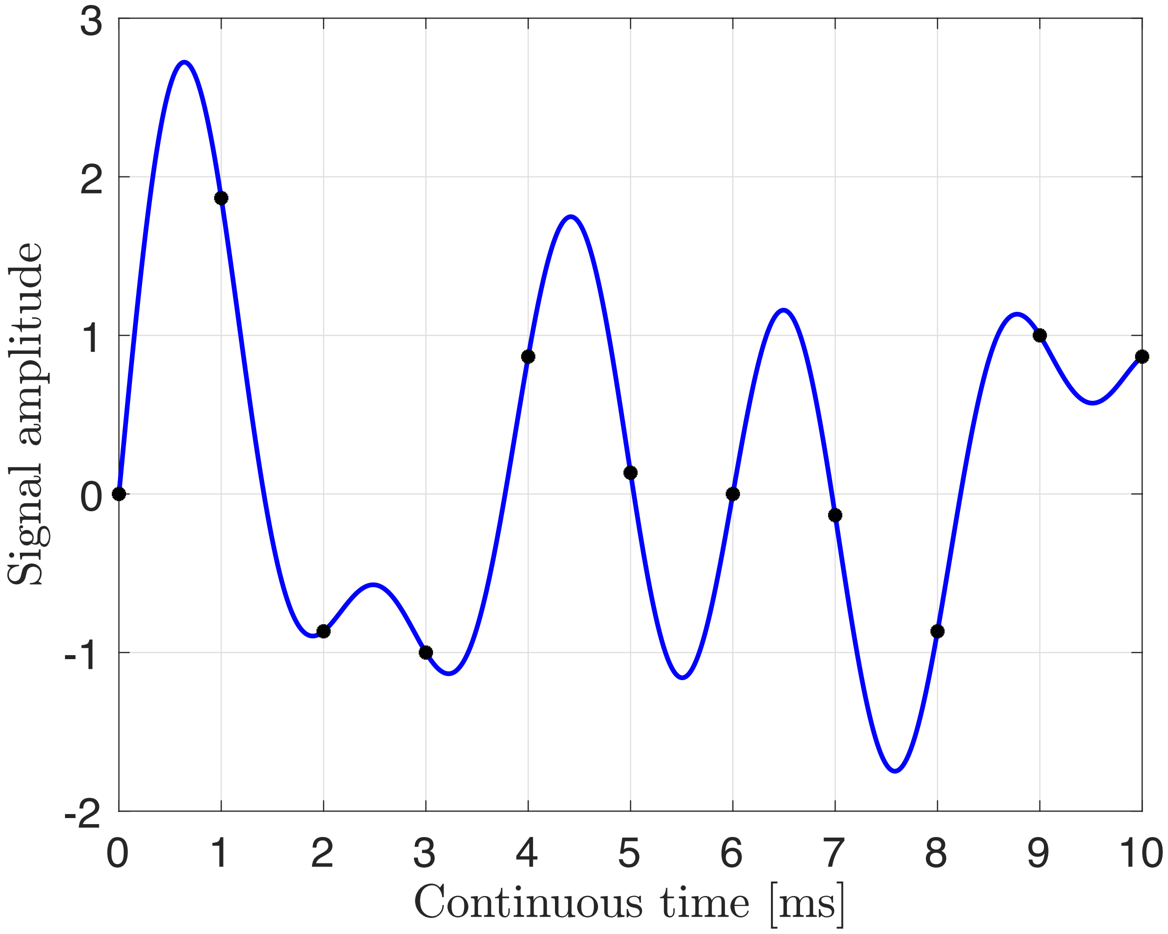
Emil Björnson

Outline

- Sampled signals are vectors
- Discrete Fourier transform
- Analysis of discrete-time filtering

Sampling of Continuous Signal

- Take discrete samples of continuous-time signal
  - Here: Sample rate 1 kHz
  - Store result in a vector $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1.9 \\ -0.9 \\ -1 \\ 0.9 \\ 0.1 \\ 0 \\ -0.1 \\ -0.9 \\ 1 \\ 0.9 \end{bmatrix}$



Sampling theorem

Complete representation if

Sample rate $>$ 2 $\cdot$ Highest frequency

Discrete Fourier Transform (DFT)

- S-length discrete-time sequence: $\chi[0], \dots, \chi[S-1]$

Discrete Fourier transform

$\bar{\chi}[0], \dots, \bar{\chi}[S-1]$ sequence describing frequency-domain content:

$$\bar{\chi}[\nu] = \frac{1}{\sqrt{S}} \sum_{s=0}^{S-1} \chi[s] e^{-j2\pi s\nu/S}, \qquad \nu = 0, \dots, S-1$$

Other definitions

Some omit $1/\sqrt{S}$, but important to get

$$\sum_{s=0}^{S-1} |\chi[s]|^2 = \sum_{\nu=0}^{S-1} |\bar{\chi}[\nu]|^2$$

- Matrix representation:

$$\underbrace{\begin{bmatrix} \bar{\chi}[0] \\ \vdots \\ \bar{\chi}[S-1] \end{bmatrix}}_{\bar{\boldsymbol{x}}} = \boldsymbol{F}_S \underbrace{\begin{bmatrix} \chi[0] \\ \vdots \\ \chi[S-1] \end{bmatrix}}_{\boldsymbol{x}}$$

DFT matrix: $\upsilon_S = e^{-j2\pi/S} = \cos(2\pi/S) - j\sin(2\pi/S)$

$$\boldsymbol{F}_S = \frac{1}{\sqrt{S}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \upsilon_S & \upsilon_S^2 & \dots & \upsilon_S^{S-1} \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & \upsilon_S^{S-1} & \upsilon_S^{2(S-1)} & \dots & \upsilon_S^{(S-1)(S-1)} \end{bmatrix}$$

Inverse Discrete Fourier Transform

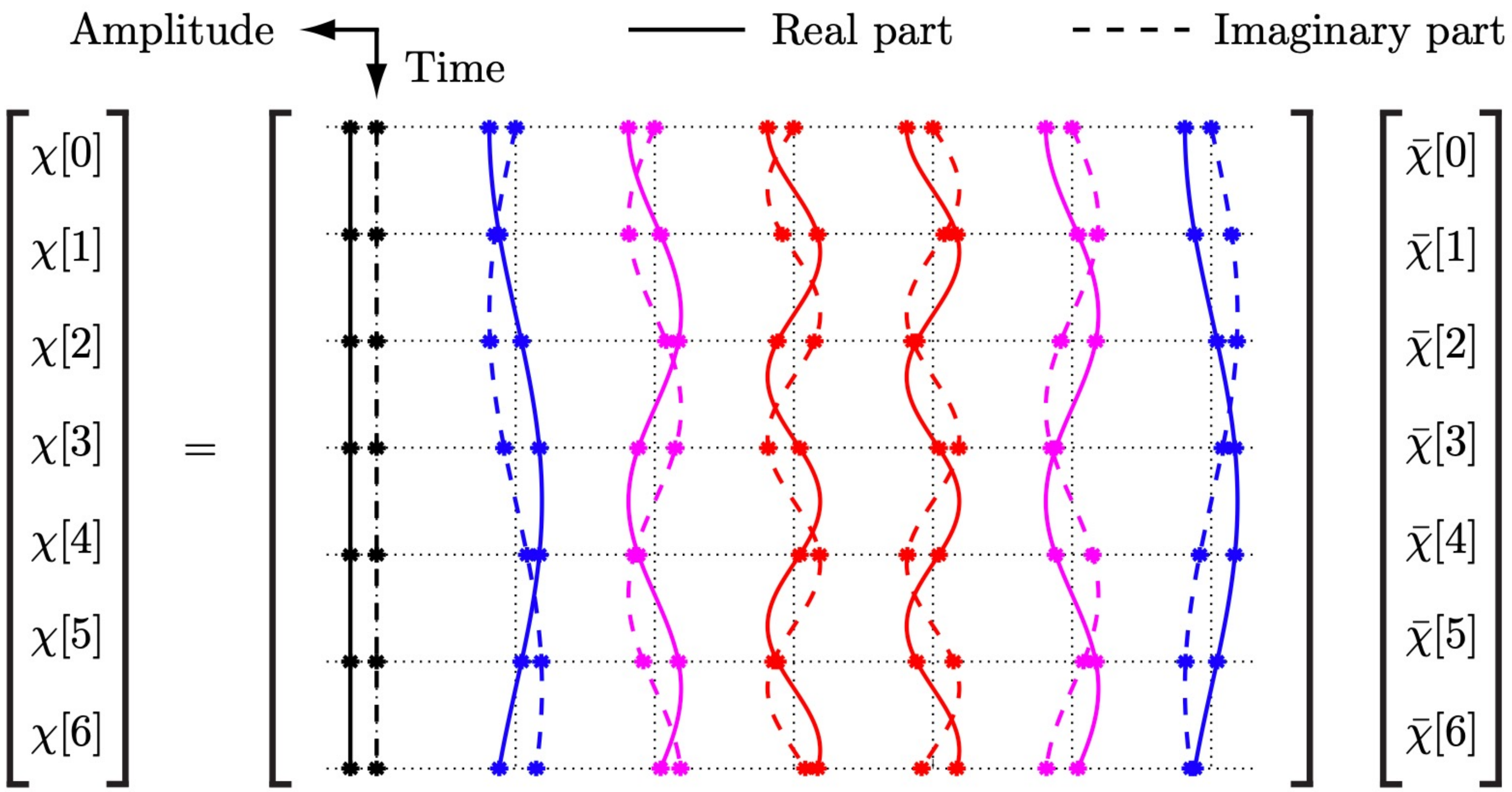
Unitary matrix

- Matrix notation: $\boldsymbol{\chi} = \boldsymbol{F}_S^{-1}\,\overline{\boldsymbol{\chi}} = \boldsymbol{F}_S^{H}\,\overline{\boldsymbol{\chi}}$

Columns of $\boldsymbol{F}_S^H$

Basis vectors in $\mathbb{C}^S$

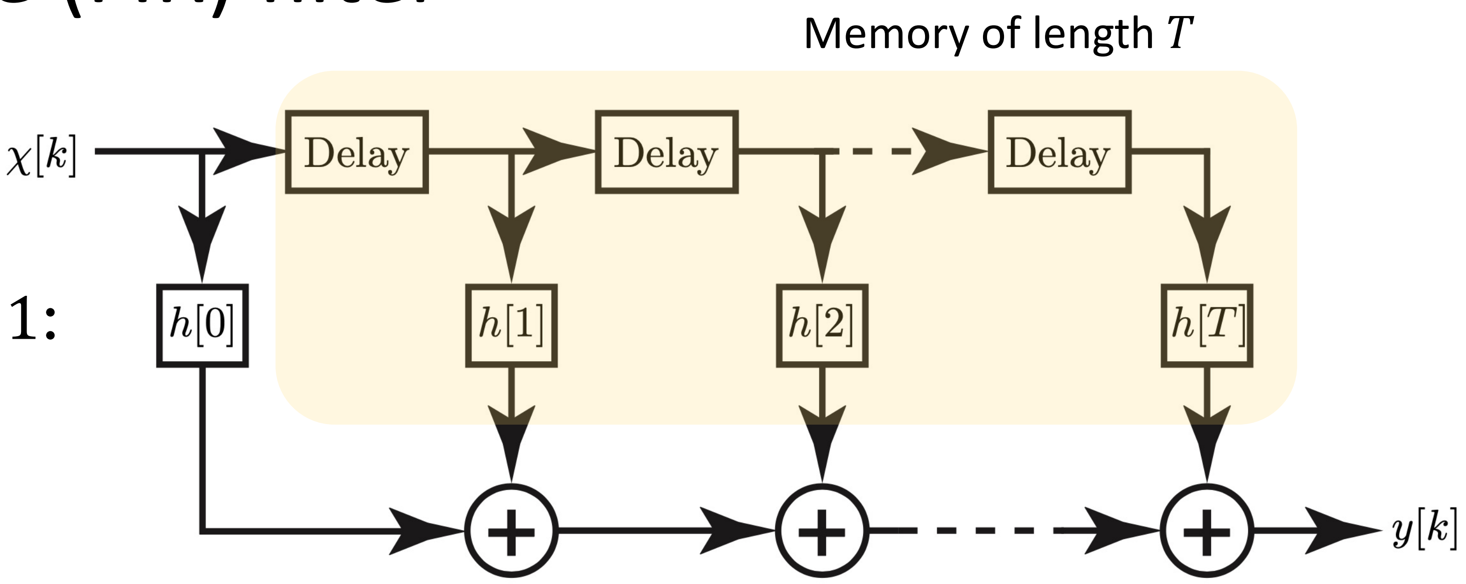
Represent frequencies



Finite impulse response (FIR) filter

- Input: $\chi[k]$, k discrete time

- Impulse response, length $T+1$:
  $h[0], h[1], \dots, h[T]$

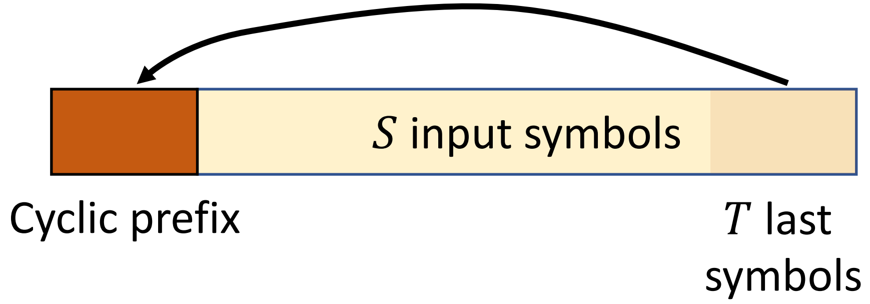


- Output signal:

$$y[k] = (h * \chi)[k] = \sum_{\ell=0}^{T} h[\ell]x[k-\ell]$$

How do we start using this filter?

Initiate with cyclic prefix



- Input $\chi[0], \ldots, \chi[S-1]$
  - Prefix: $\chi[-1] = \chi[S-1], \ldots, \chi[-T] = \chi[S-T]$

- Output signal:

Cyclic convolution

$$y[k] = \sum_{\ell=0}^{T} h[\ell]x[(k-\ell)_{\mathrm{mod}\,S}] = (h \circledast \chi)[k], \qquad k = 0, \ldots, S-1$$

Relation in DFT domain:

$$\bar{y}[\nu] = \bar{h}[\nu]\bar{\chi}[\nu], \qquad \nu = 0, \ldots, S-1$$

where $\bar{h}[\nu] = \sum_{s=0}^{T} h[s]e^{-j2\pi s\nu/S}$

Matrix representation of FIR filtering

$$\underbrace{\begin{bmatrix} y[0] \\ \vdots \\ y[S-1] \end{bmatrix}}_{=\boldsymbol{y}} = \underbrace{\begin{bmatrix} h[0] & 0 & \dots & \dots & \dots & 0 & h[3] & h[2] & h[1] \\ h[1] & h[0] & 0 & \dots & \dots & \dots & 0 & h[3] & h[2] \\ h[2] & h[1] & h[0] & 0 & \dots & \dots & \dots & 0 & h[3] \\ h[3] & h[2] & h[1] & h[0] & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & h[3] & h[2] & h[1] & h[0] & 0 & \ddots & \vdots \\ \vdots & \dots & 0 & h[3] & h[2] & h[1] & h[0] & 0 & \vdots \\ \vdots & \dots & \dots & 0 & h[3] & h[2] & h[1] & h[0] & 0 \\ 0 & \dots & \dots & \dots & 0 & h[3] & h[2] & h[1] & h[0] \end{bmatrix}}_{=\boldsymbol{C}_h} \underbrace{\begin{bmatrix} \chi[0] \\ \vdots \\ \chi[S-1] \end{bmatrix}}_{=\boldsymbol{\chi}}$$

Using DFT

$$\boldsymbol{\chi} = \boldsymbol{F}_S^H \, \bar{\boldsymbol{\chi}}$$
$$\boldsymbol{y} = \boldsymbol{F}_S^H \, \bar{\boldsymbol{y}}$$

$$\underbrace{\begin{bmatrix} \bar{y}[0] \\ \vdots \\ \bar{y}[S-1] \end{bmatrix}}_{=\bar{\boldsymbol{y}}} = \underbrace{\begin{bmatrix} \bar{h}[0] & 0 & \dots & 0 \\ 0 & \bar{h}[1] & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{h}[S-1] \end{bmatrix}}_{=\boldsymbol{D}_{\bar{h}}} \underbrace{\begin{bmatrix} \bar{\chi}[0] \\ \vdots \\ \bar{\chi}[S-1] \end{bmatrix}}_{=\bar{\boldsymbol{\chi}}}$$

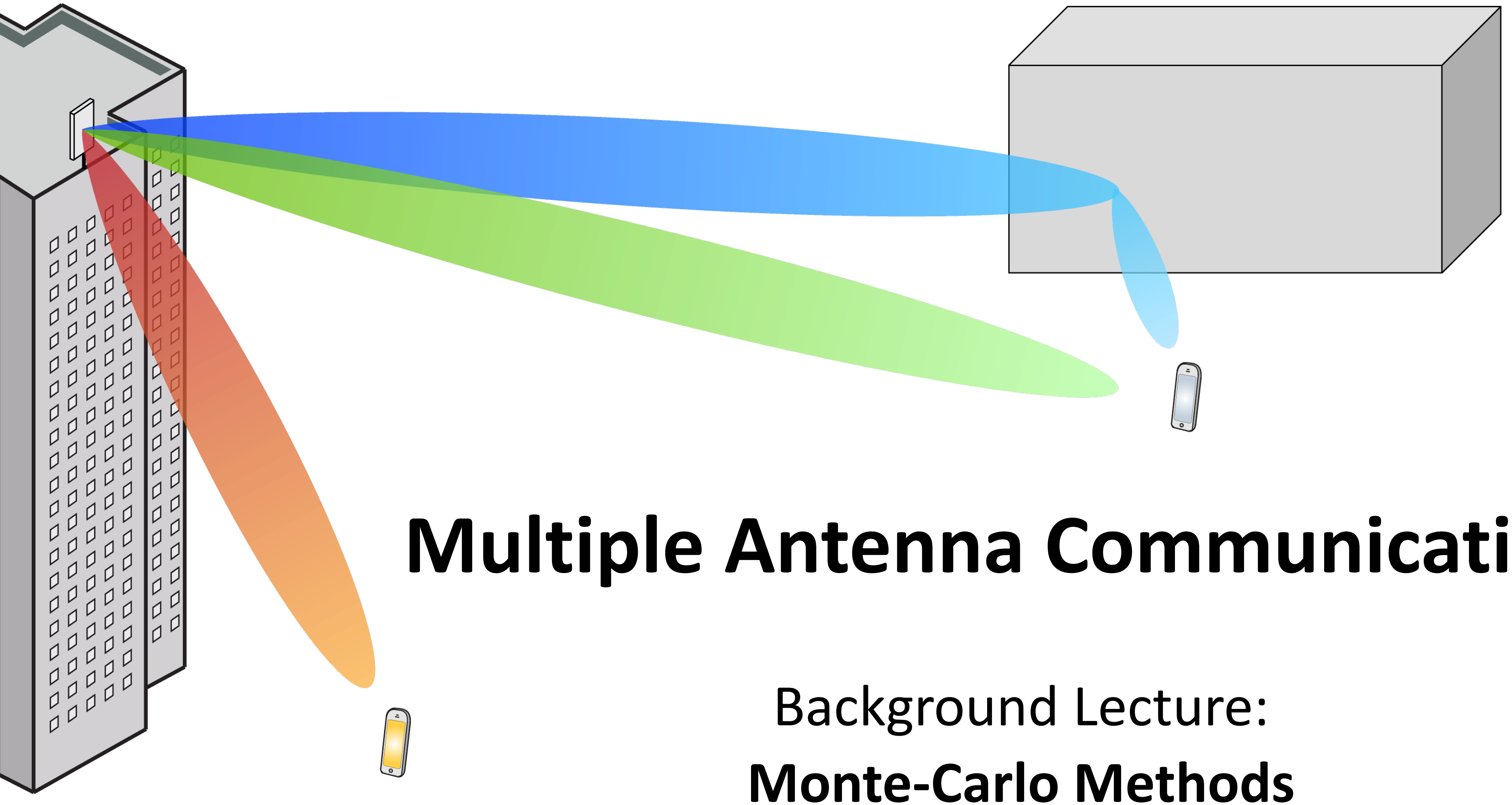
Circulant matrix

$$\boldsymbol{C}_h = \boldsymbol{F}_S^H \boldsymbol{D}_h \boldsymbol{F}_S$$

Summary

- Discrete Fourier transform
  - Natural basis for vectors with samples
  - Linear combination of sampled complex exponentials

- One application:
  - Study output of FIR filter
  - No inter-symbol interference in frequency domain



Multiple Antenna Communications

Background Lecture:

Monte-Carlo Methods

Emil Björnson