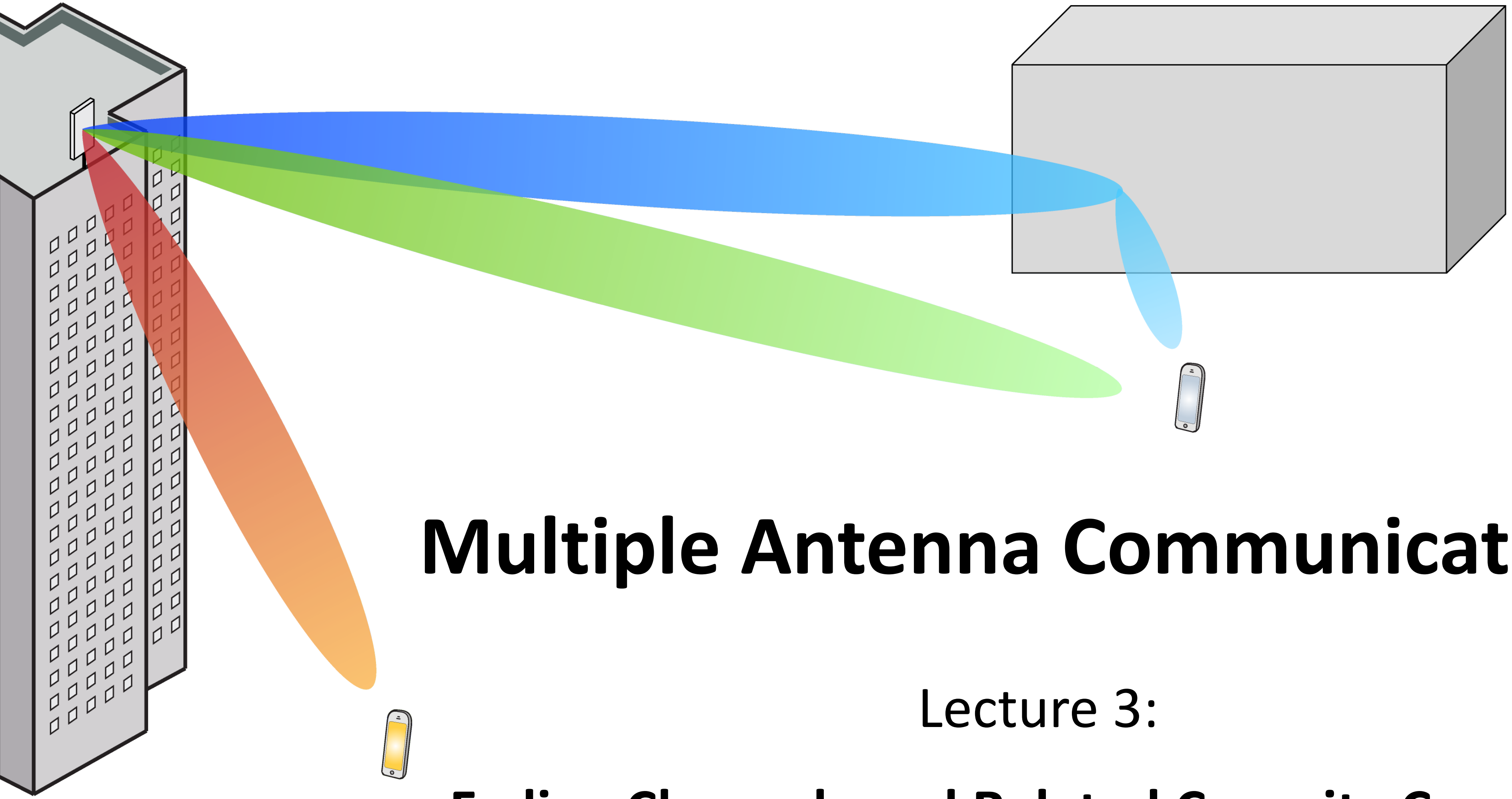


Multiple Antenna Communications

Lecture 3:

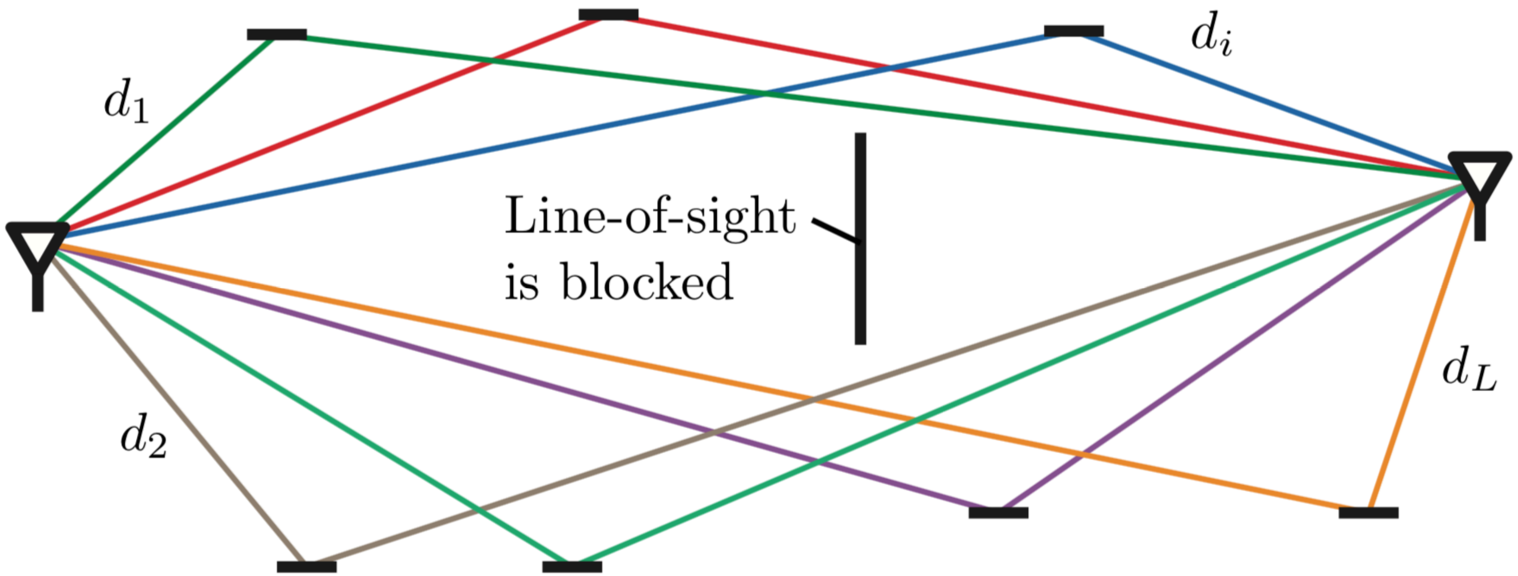
Fading Channels and Related Capacity Concepts

Emil Björnson

Outline

- Multipath propagation and Rayleigh fading
- Slow fading
    - Outage probability
    - Outage capacity
    - Spatial diversity
- Fast fading
    - Ergodic capacity
    - Channel hardening

Multipath propagation



Non-line-of-sight channel:
- Scattering

- Channel with L propagation paths:

$$g = \sum_{i=1}^{L} \alpha_i e^{-j2\pi \frac{d_i - d}{\lambda}}$$

Reference distance

Channel attenuation

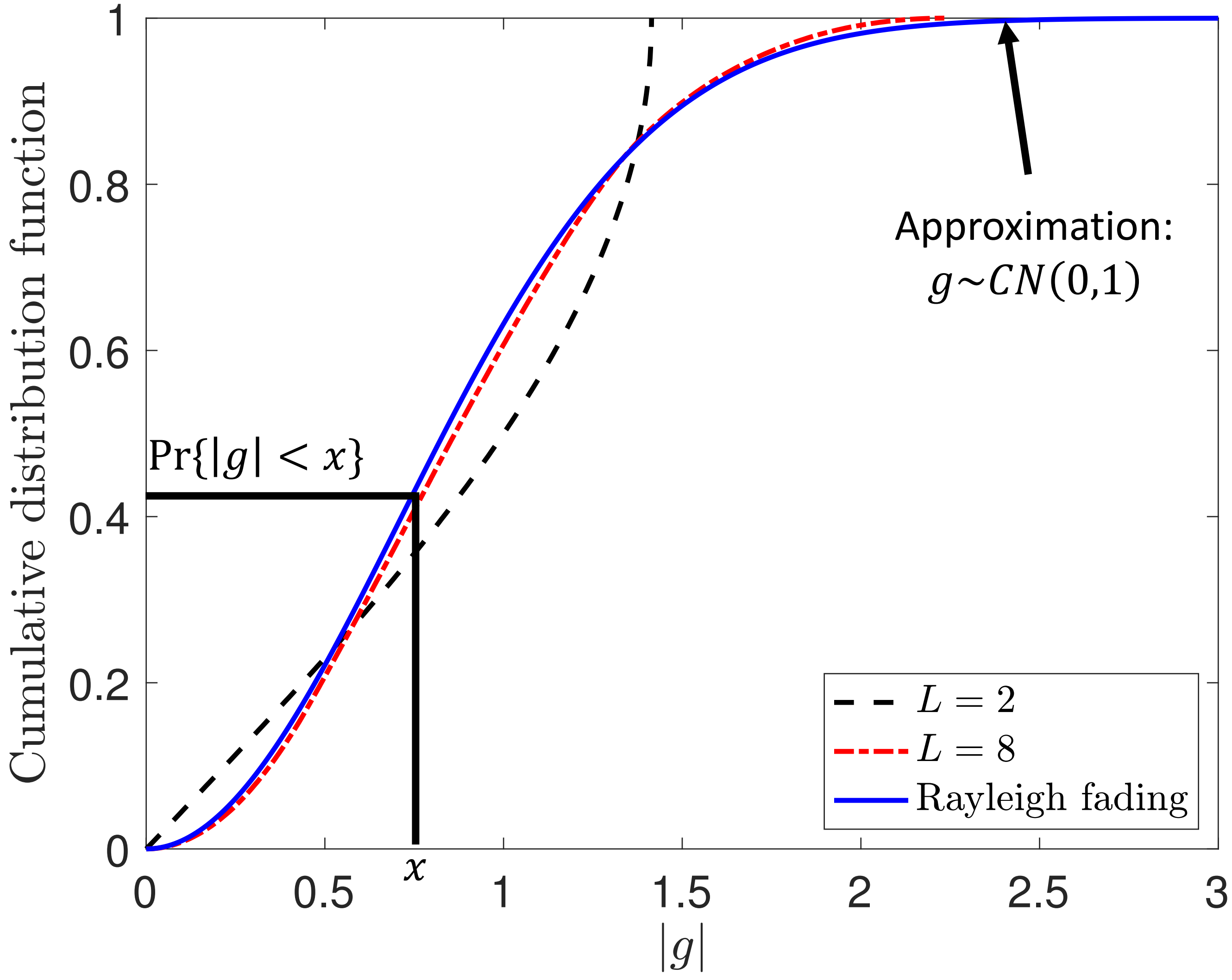
Wavelength

Multipath fading

- Example:
  - $\alpha_i^2 = \frac{1}{L}$
  - $\theta_i = 2\pi \frac{d_i - d}{\lambda} \sim U(0, 2\pi)$

- Channel magnitude:

$$|g| = \left| \sum_{i=1}^{L} \sqrt{\frac{1}{L}} e^{-j\theta_i} \right|$$



Rich scattering: Rayleigh fading

Central limit theorem

Let $X_1, \ldots, X_L$ be a sequence of L real-valued independent and identically distributed random variables with zero mean and variance σ^2. As $L \to \infty$,

$$\frac{1}{\sqrt{L\sigma^2}} \sum_{i=1}^{L} X_i$$

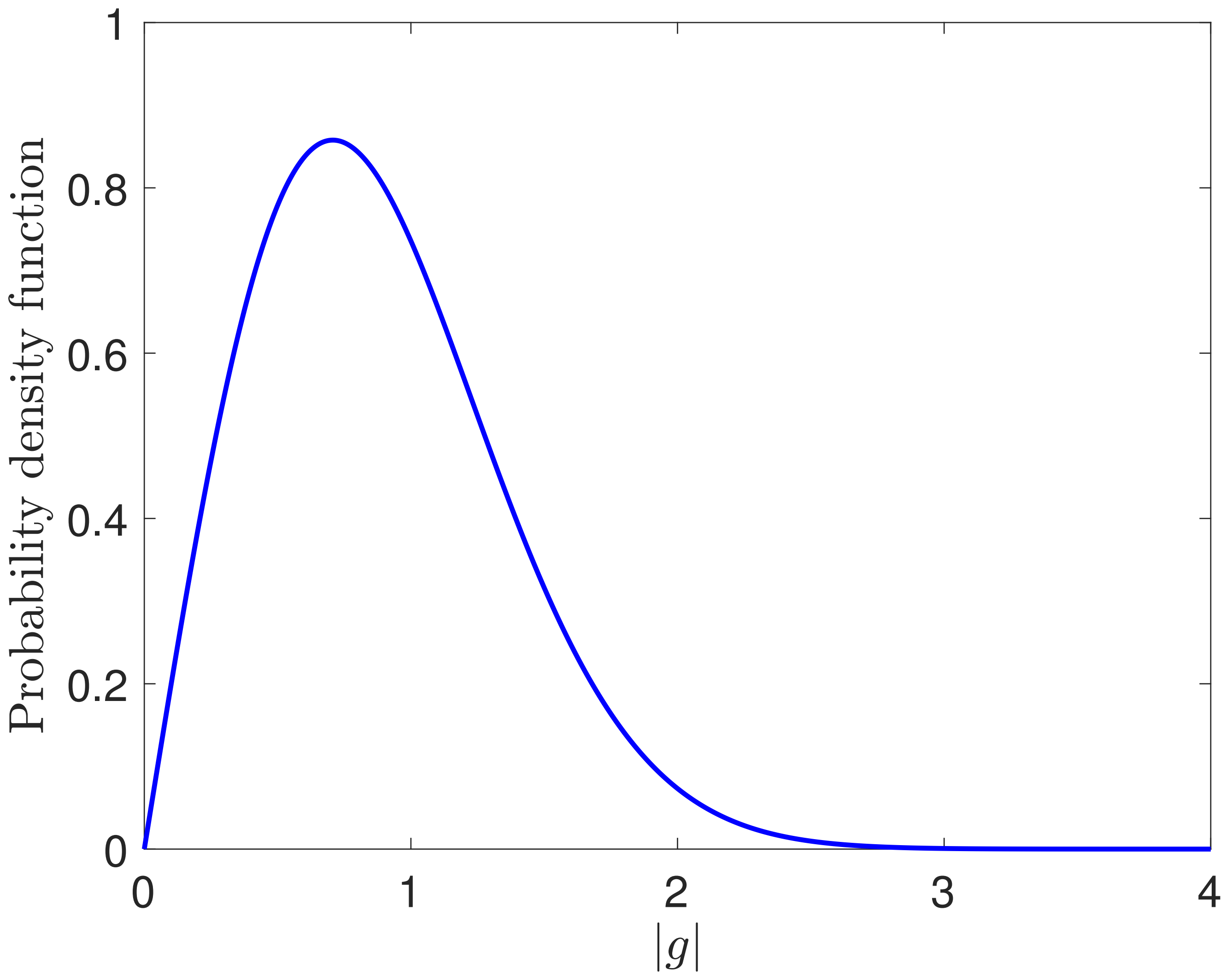
converges to a standard Gaussian distribution $N(0,1)$.

- Rich multipath propagation: $X_i = \alpha_i e^{-j\theta_i} = \alpha_i \cos(\theta_i) - j\,\alpha_i \sin(\theta_i)$
  - Very large number paths: Gaussian distribution
  - Channel response: $g \sim CN(0, \beta)$
  - Called *Rayleigh fading* since $|g| \sim$ Rayleigh$(\sqrt{\beta/2})$

Rayleigh fading, $|g| \sim \text{Rayleigh}(1/\sqrt{2})$

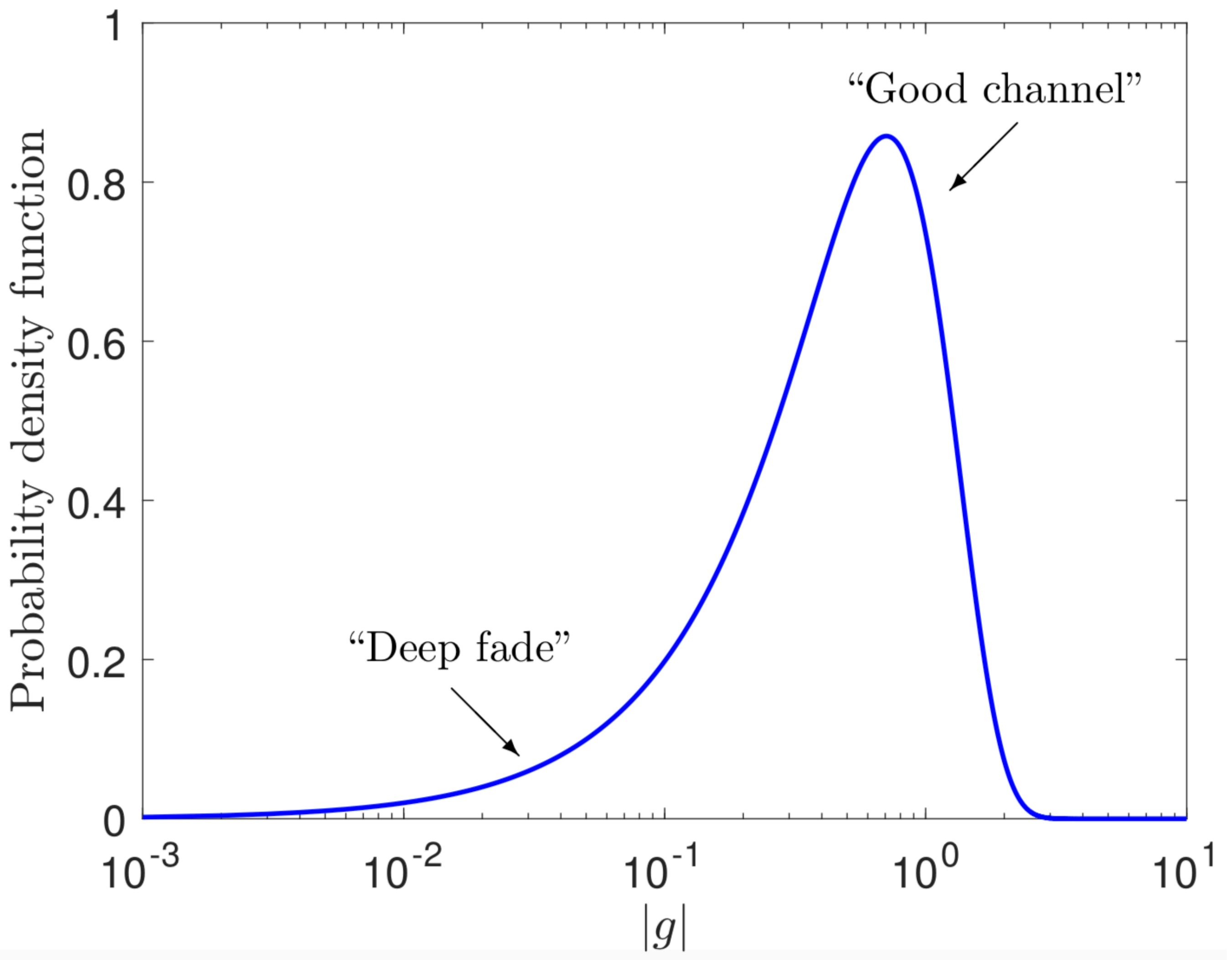
- Amplitude of channel changes over time

- In this case: $g \sim CN(0,1)$



Rayleigh fading, zooming in on tail

- Risk of very small channel gain



Two issues:

- Variations in channel quality
- Unpredictable

Capacity of fading channel

- AWGN channel with a random channel response $g[l]$:

$$y[l] = g[l] \cdot x[l] + n[l]$$

  - $x[l] \sim CN(0, q)$, energy per sample: $q = P/B$
  - $n[l] \sim CN(0, N_0)$

- Two categories:
  - Slow fading: $g[l]$ takes one realization during communication
  - Fast fading: $g[l]$ takes "all" realizations during communication

Reality might be somewhere in between

Slow fading

- Received signal

$$y[l] = g \cdot x[l] + n[l]$$

  - Fixed channel $g[l] = g$ for the entire transmission
  - Assumption: Receiver knows g, but not the transmitter

- Capacity for a realization g:

q/N_0

$$C_g = \log_2(1 + |g|^2 \mathrm{SNR})$$

Transmitter does not know C_g
Cannot encode the data to achieve it!

Opportunistic transmission

- Suppose transmitter encode using the rate R bit/symbol

- Two possible events:
  - If $R \leq C_g$: Successful transmission
  - If $R > C_g$: Large error probability

System is in outage if $R > C_g$

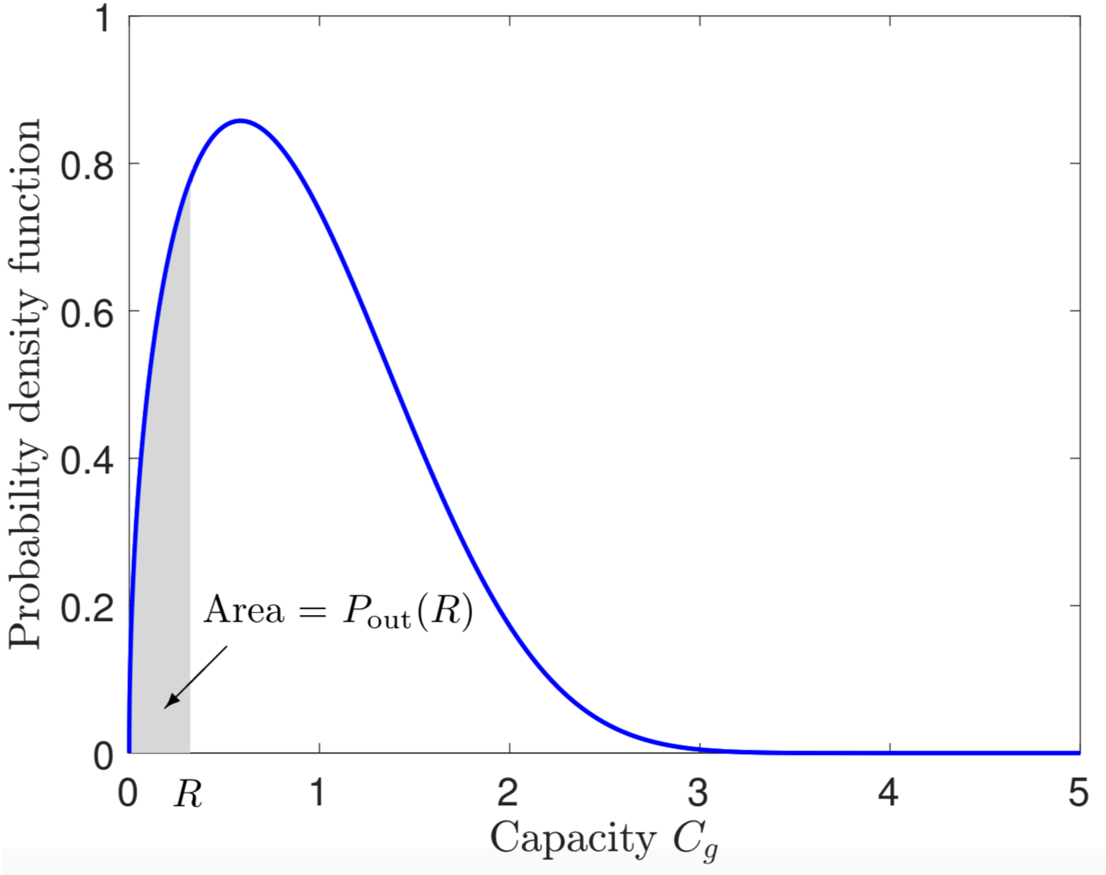
- Outage probability for rate R

$$p_{out}(R) = \Pr\left\{C_g < R\right\} = \Pr\{\log_2(1 + |g|^2 \mathrm{SNR}) < R\}$$

High SNR:
$e^{-x} \approx 1 - x$

Outage probability with $g \sim CN(0,1)$

- Outage probability for rate R: $p_{out}(R) = \Pr\{C_g < R\} = 1 - e^{-\frac{2^R - 1}{\text{SNR}}} \approx \frac{2^R - 1}{\text{SNR}}$



Outage probability decays with $\text{SNR} = q/N_0$ as SNR^{-1}

Outage capacity

- Difference from deterministic AWGN channel
  - Only $R = 0$ can guarantee zero error probability
    - Capacity is zero

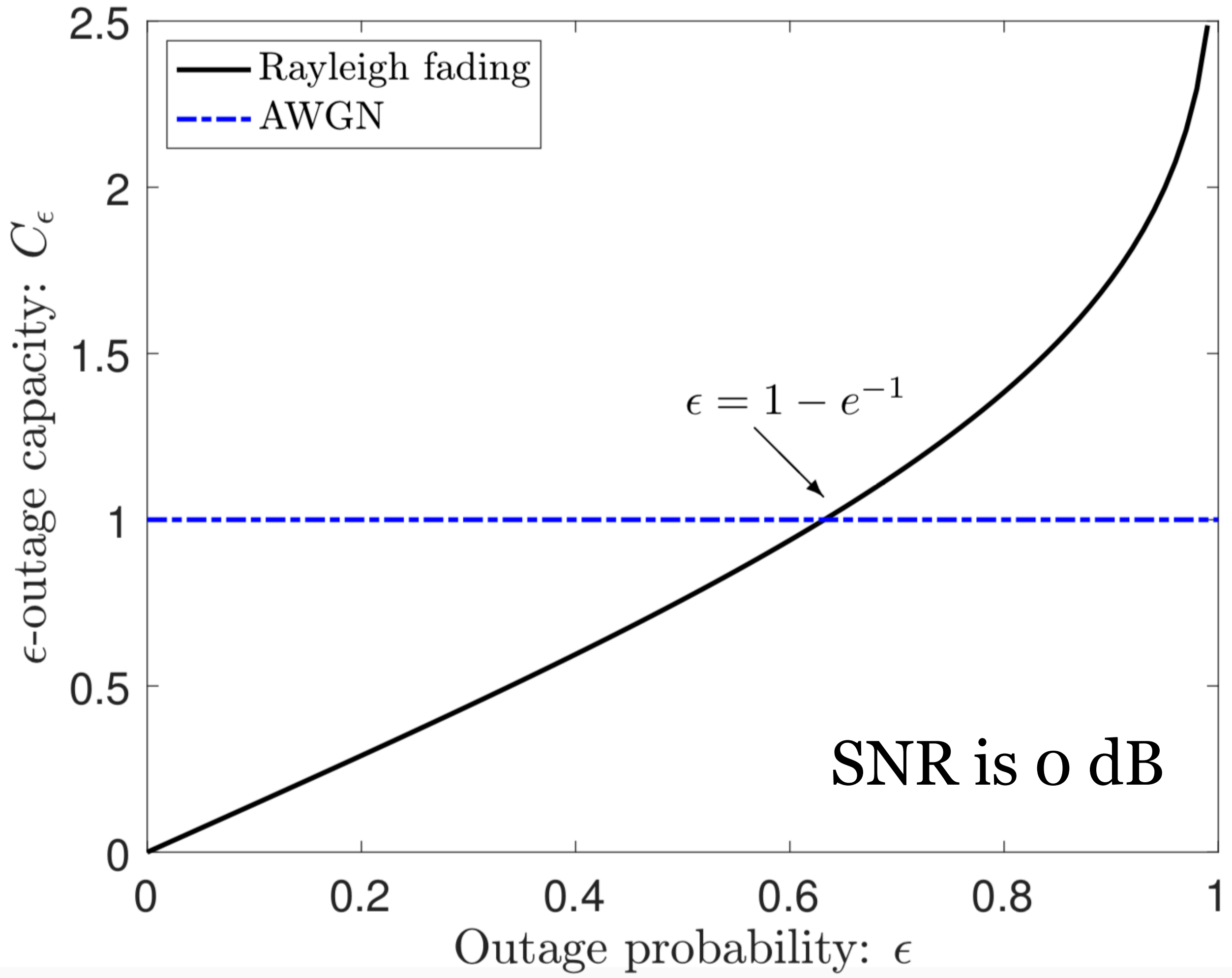
- ϵ-Outage capacity C_ϵ:
  - Largest rate R such that $p_{out}(R) \leq \epsilon$

Interpretation:
With probability $1 - \epsilon$, we can communicate at C_ϵ with is zero error probability

Outage capacity with $g \sim CN(0,1)$

$$1 - e^{-\frac{2^R - 1}{\mathrm{SNR}}} \leq \epsilon \qquad \Longrightarrow$$

$$\mathrm{C}_\epsilon = \log_2\Big(1 + \mathrm{SNR}\,\underbrace{\ln\big((1-\epsilon)^{-1}\big)}_{\text{Difference from AWGN channel}}\Big)$$



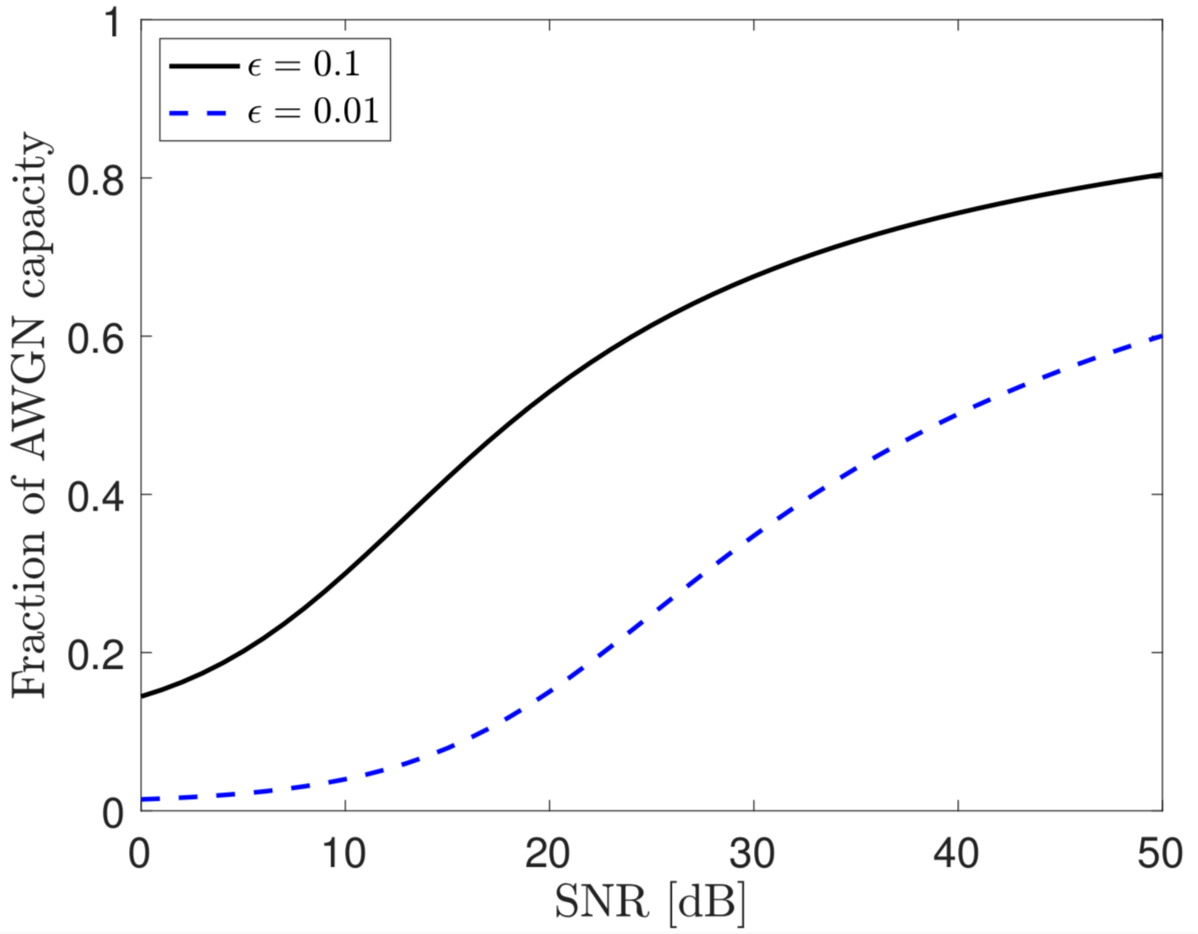
- Low ϵ: Better with AWGN channel
- High ϵ: Better with fading channel

Outage capacity with small outage probability

- Fraction of AWGN capacity:

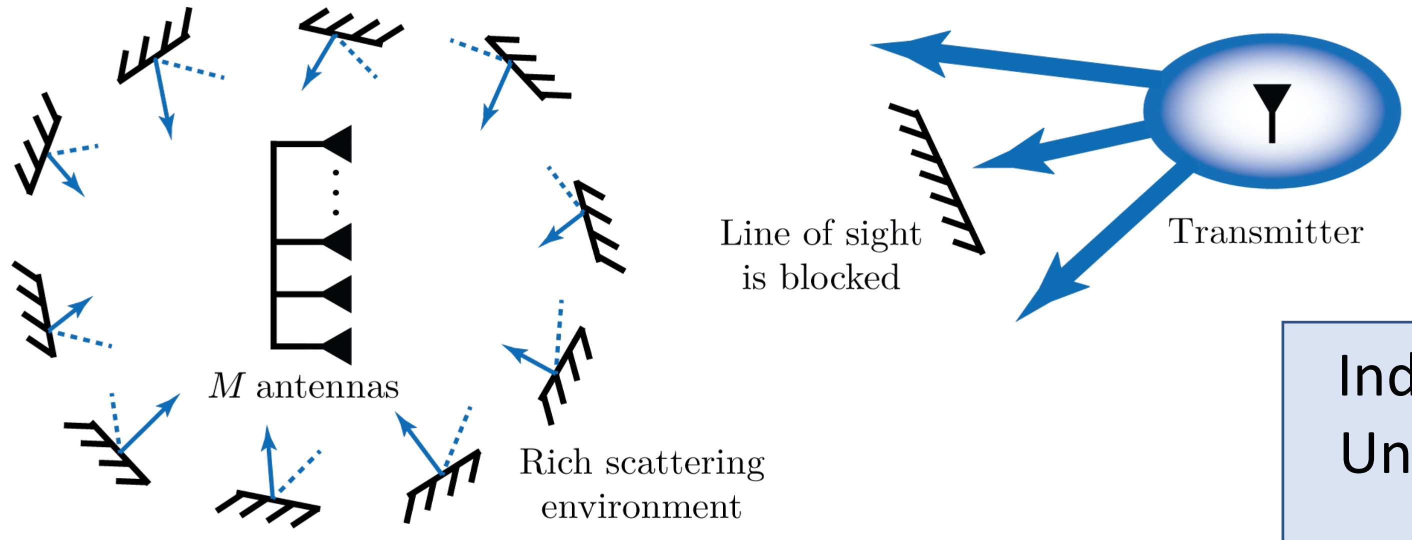
$$\frac{\log_2\left(1 + \mathrm{SNR}\ln\left((1-\epsilon)^{-1}\right)\right)}{\log_2(1+\mathrm{SNR})}$$

  - Much lower capacity than with AWGN channel



Can we improve the situation?

Fading multiple antenna channels



Independent fading:
Uniform linear array
with $\Delta = \lambda/2$

- Independent and identically distributed Rayleigh fading
  - Channel vector: $\boldsymbol{g} \sim CN(\boldsymbol{0}, \beta \boldsymbol{I}_M)$
  - Distribution of $\|\boldsymbol{g}\|^2$: $$f_{\|\boldsymbol{g}\|^2}(x) = \frac{x^{M-1} e^{-\frac{x}{\beta}}}{(M-1)!\beta^M} \leq \frac{x^{M-1}}{(M-1)!\beta^M}$$

M receive antennas and i.i.d. Rayleigh fading

High SNR

- Outage probability

$$p_{out}(R) = \Pr\{\log_2(1 + \|\boldsymbol{g}\|^2 \text{SNR}) < R\} = \int_0^{\frac{2^R - 1}{\text{SNR}}} f_{\|\boldsymbol{g}\|^2}(x)\, dx \leq \left(\frac{2^R - 1}{\text{SNR}}\right)^M \frac{1}{M!}$$

Spatial diversity gain

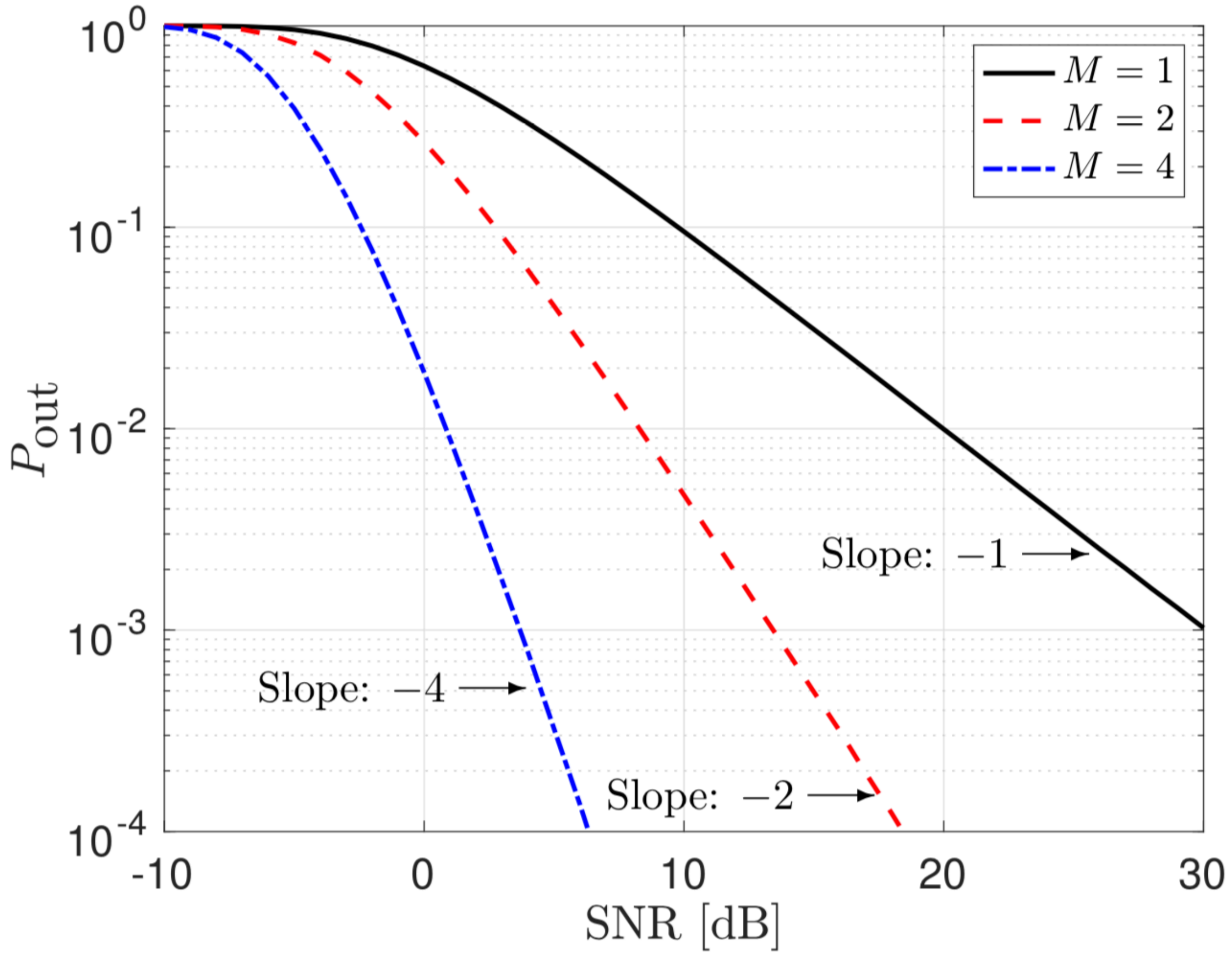
$p_{out}(R)$ proportional to SNR^{-M}

M is the diversity order

Outage probability with M receive antennas

- Outage probability decays as
  $$\mathrm{SNR}^{-M}$$

  - Makes a huge difference!



Multiple receive antennas gives:
- Beamforming gain
- Diversity gain

Fast fading

- Received signal

$$y[l] = g[l] \cdot x[l] + n[l]$$

- Block fading
    - One realization of channel $g[l]$ per l (or a finite-sized block of symbols)
    - New independent realization every time (ergodic process)

Opportunistic transmission

- Suppose transmitter encode using the rate R bit/symbol
  - There are L fading realization: $g[1], \dots, g[L]$

- Reliable communication if

$$\sum_{l=1}^{L} \log_2(1 + \mathrm{SNR}\,|g[l]|^2) > LR$$

  - Many fading realizations:

$$R < \frac{1}{L}\sum_{l=1}^{L} \log_2(1 + \mathrm{SNR}\,|g[l]|^2) \to \mathbb{E}\{\log_2(1 + |g|^2 \mathrm{SNR})\}$$

As $L \to \infty$

Mean value with respect to channel fading

Ergodic capacity

- This is called ergodic capacity:

$$\mathbb{E}\{\log_2(1 + |g|^2 \mathrm{SNR})\}$$

  - Deterministic: Transmitter knows it even if g is unknown
  - There are no outage issues!

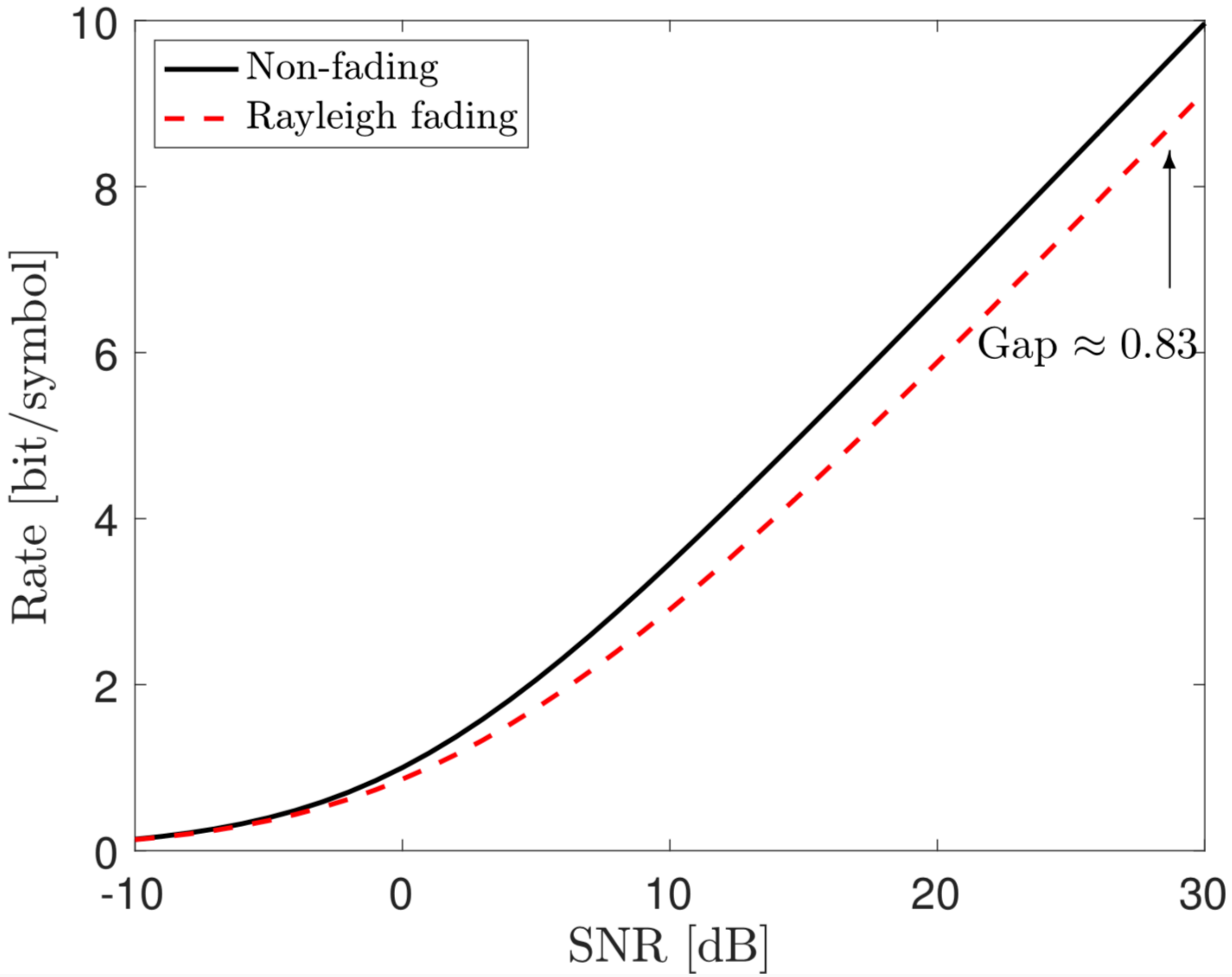
- Extension to SIMO case with channel:

$$\mathbb{E}\{\log_2(1 + \|\boldsymbol{g}\|^2 \mathrm{SNR})\}$$

Comparison with AWGN channel

- AWGN channel:
$$\log_2(1 + \mathrm{SNR})$$

- Rayleigh fading, $g \sim CN(0,1)$:
$$\mathbb{E}\{\log_2(1 + |g|^2 \mathrm{SNR})\}$$



Low SNR: Little difference
High SNR: Ergodic capacity is lower

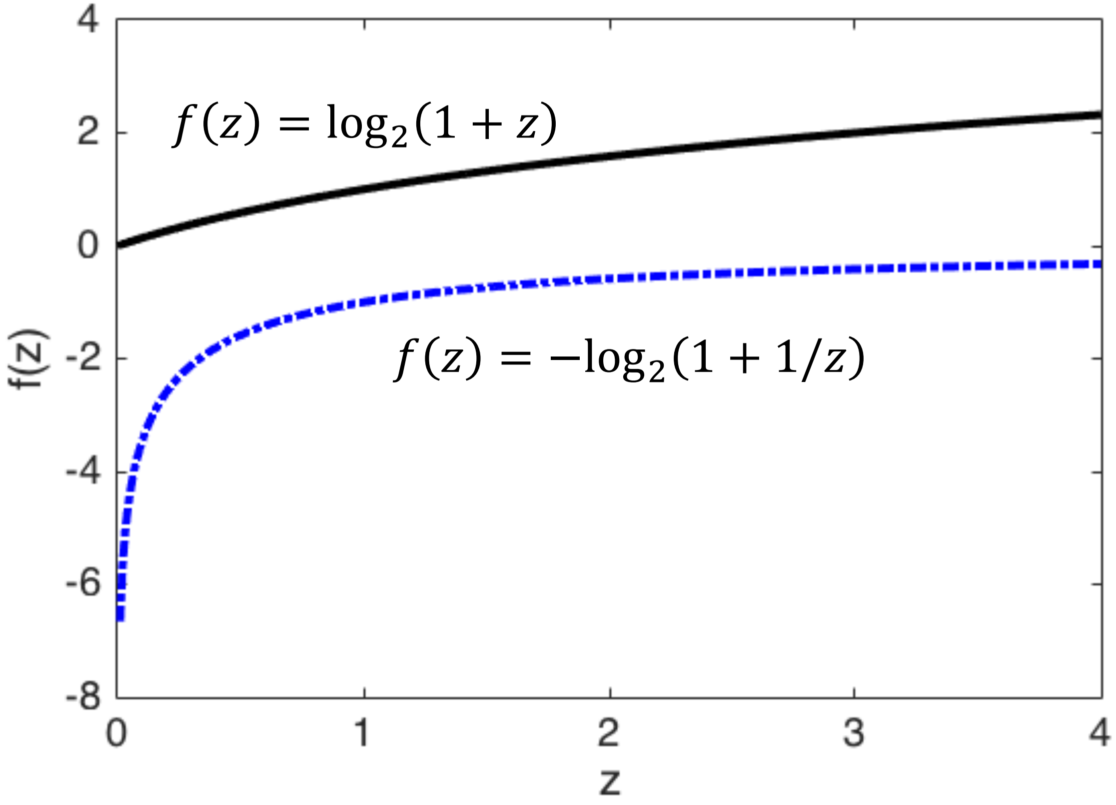
Jensen's inequality and concave functions

- For any random variable z and concave function $f(\cdot)$,

$$\mathbb{E}\{f(z)\} \leq f(\mathbb{E}\{z\})$$

A function is concave if

- Any line between two points on the curve is below the curve
- Second derivative is negative



Ergodic capacity with SIMO channel

- Can be used to prove

$$\log_2\left(1+\frac{\mathrm{SNR}}{\mathbb{E}\{\|\boldsymbol{g}\|^{-2}\}}\right) \leq \mathbb{E}\{\log_2(1+\|\boldsymbol{g}\|^2\mathrm{SNR})\} \leq \log_2(1+\mathbb{E}\{\|\boldsymbol{g}\|^2\}\mathrm{SNR})$$

Jensen's inequality with $f(z) = -\log_2(1+1/z)$

Jensen's inequality with $f(z) = \log_2(1+z)$

- Rayleigh fading with $\boldsymbol{g}$ having i.i.d. $CN(0,1)$ elements:

$$\log_2(1+(M-1)\mathrm{SNR}) \leq \mathbb{E}\{\log_2(1+\|\boldsymbol{g}\|^2\mathrm{SNR})\} \leq \log_2(1+M\,\mathrm{SNR})$$

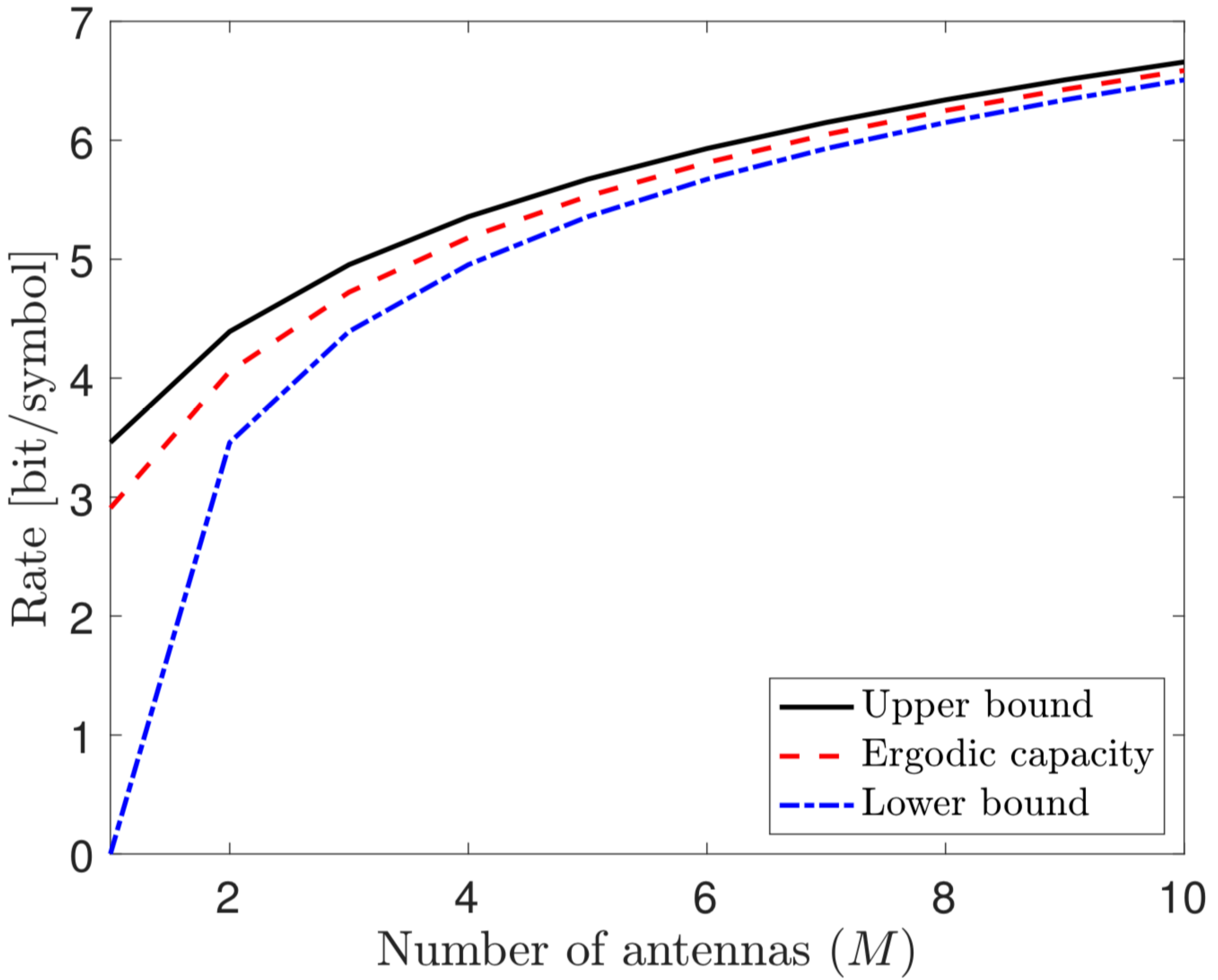
Line-of-sight channel with $\|\boldsymbol{g}\|^2 = M-1$

Line-of-sight channel with $\|\boldsymbol{g}\|^2 = M$

Comparison

- Small M
  - Large loss from channel fading
- Larger M
  - Small loss

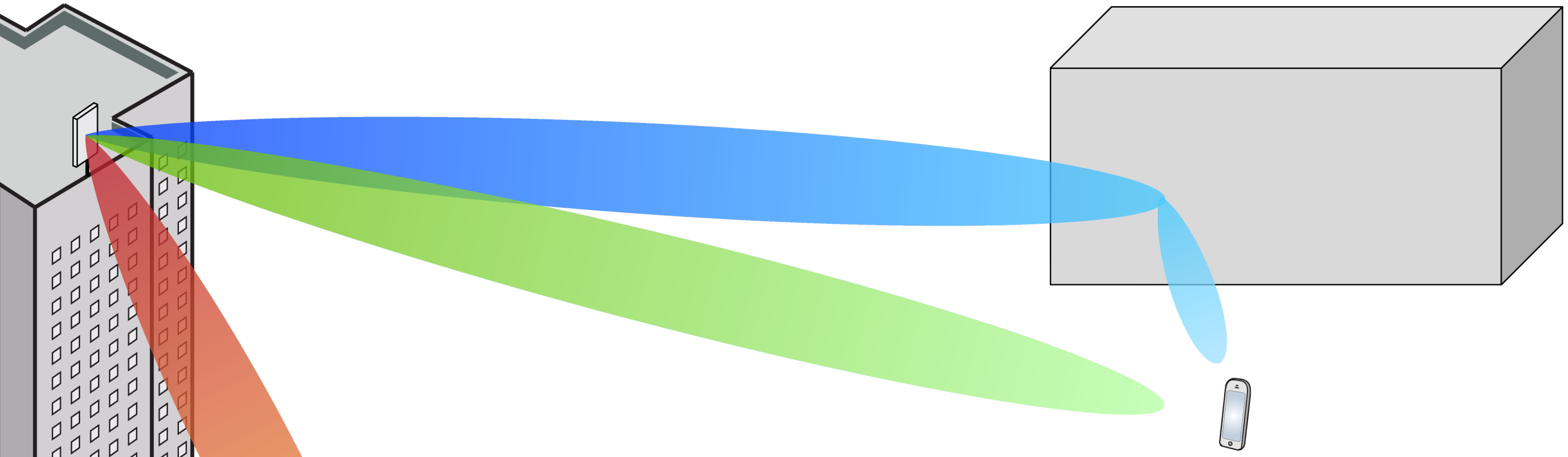
Channel hardening:
When M is large, no penalty from channel fading

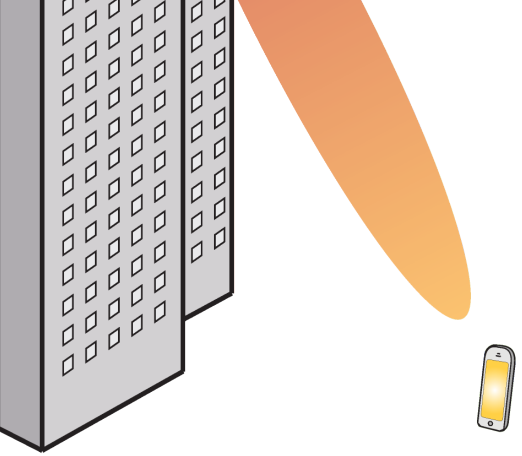


Summary

- **Slow fading:** One realization per transmission
  - Outage probability, outage capacity
  - Reliability → Large performance loss
  - Multiple antennas give more reliability

- **Fast fading:** Many realizations per transmission
  - Ergodic capacity with averaging over fading
  - No reliability issue, but performance loss
  - Multiple antennas give similar capacity as with non-fading channels





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