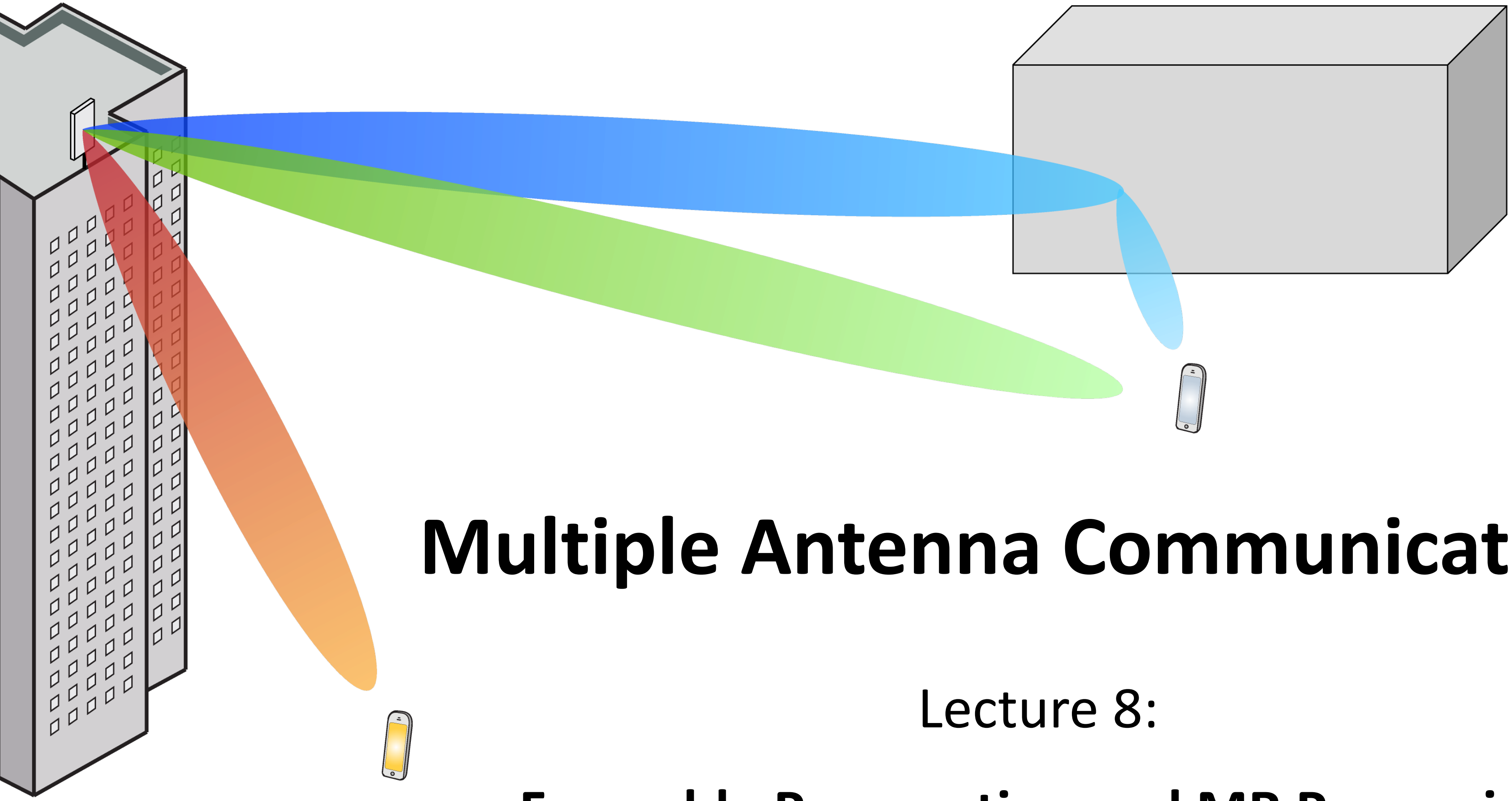


Multiple Antenna Communications

Lecture 8:

Favorable Propagation and MR Processing

Emil Björnson

Outline

- Favorable propagation
    - Motivates using simple processing
    - Maximum ratio (MR) processing

- Simple capacity lower bound
    - Use-and-then-forget technique
    - Expression with MR processing

Recall: Sum Capacity with $K = 2$

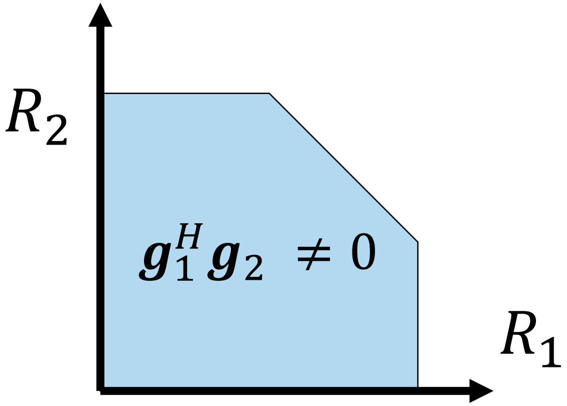
- Recall: Sum Capacity with $K = 2$ and $\boldsymbol{G} = [\boldsymbol{g}_1\ \boldsymbol{g}_2]$:

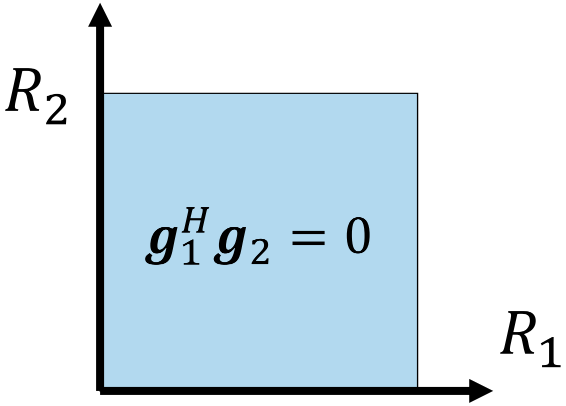
$$R_1 + R_2 = \log_2(\det(\boldsymbol{I}_2 + \rho_{ul}\boldsymbol{G}^H\boldsymbol{G})) = \log_2\left(\det\left(\boldsymbol{I}_2 + \rho_{ul}\begin{bmatrix} \|\boldsymbol{g}_1\|^2 & \boldsymbol{g}_1^H\boldsymbol{g}_2 \\ \boldsymbol{g}_2^H\boldsymbol{g}_1 & \|\boldsymbol{g}_2\|^2 \end{bmatrix}\right)\right)$$

$$= \log_2\left((1 + \rho_{ul}\|\boldsymbol{g}_1\|^2)(1 + \rho_{ul}\|\boldsymbol{g}_2\|^2) - \rho_{ul}^2\left|\boldsymbol{g}_1^H\boldsymbol{g}_2\right|^2\right)$$

$$\leq \log_2(1 + \rho_{ul}\|\boldsymbol{g}_1\|^2) + \log_2(1 + \rho_{ul}\|\boldsymbol{g}_2\|^2)$$

Equality if and only if $\boldsymbol{g}_1^H\boldsymbol{g}_2 = 0$





Favorable propagation

- A collection of channel vectors $\{\boldsymbol{g}_k\}$ are said to offer *favorable propagation* if
$$\boldsymbol{g}_k^H \boldsymbol{g}_i = 0, \qquad k, i = 1, \dots, K, \qquad k \neq i$$
    - Never satisfied exactly in practice

- *Asymptotically favorable propagation* if
$$\frac{1}{M} \boldsymbol{g}_k^H \boldsymbol{g}_i \to 0, \qquad M \to \infty, \qquad k, i = 1, \dots, K, \qquad k \neq i$$
    - One cannot physically let $M \to \infty$ in practice

Law of large numbers

- Consider a sequence $X_1, X_2, \dots$ of independent and identically distributed random variables
  - Assume $E\{X_i\} = \mu$ for $i = 1,2, \dots$
  - Assume $Var\{X_i\} = \sigma^2 < \infty$ for $i = 1,2, \dots$

Law of large numbers

The sample average

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

converges to the expected value

$$\bar{X}_n \to \mu \text{ as } n \to \infty$$

$$Var\{\bar{X}_n\} = \frac{Var\{X_1\} + \cdots + Var\{X_n\}}{n^2} = \frac{\sigma^2}{n}$$

Properties of Rayleigh fading channels

- Channels independently distributed as $\boldsymbol{g}_k \sim CN(\boldsymbol{0}, \beta_k \boldsymbol{I}_M)$

  - Offer channel hardening:

$$\frac{1}{M}\|\boldsymbol{g}_k\|^2 \to \beta_k, \qquad M \to \infty, \qquad k = 1, \dots, K$$

  - Offer favorable propagation:

$$\frac{1}{M}\boldsymbol{g}_k^H \boldsymbol{g}_i \to 0, \qquad M \to \infty, \qquad k, i = 1, \dots, K, \qquad k \neq i$$

Approximations when M is large:

$\frac{1}{M}\|\boldsymbol{g}_k\|^2 \approx \beta_k$ and $\frac{1}{M}\boldsymbol{g}_k^H \boldsymbol{g}_i \approx 0$

Recall: Estimates of channels

- MMSE estimate of g_k^m from user k to antenna m
  - Estimate: $\hat{g}_k^m = E\{g_k^m | \boldsymbol{Y}_p'\} = \frac{\sqrt{\tau_p \rho_{ul}} \beta_k}{1+\tau_p \rho_{ul} \beta_k} [\boldsymbol{Y}_p']_{mk} \sim CN(0, \gamma_k)$
  - Estimation error: $\tilde{g}_k^m = \hat{g}_k^m - g_k^m \sim CN(0, \beta_k - \gamma_k)$

  where

$$\gamma_k = \frac{\tau_p \rho_{ul} \beta_k^2}{1 + \tau_p \rho_{ul} \beta_k}$$

- Vector notation: $\hat{\boldsymbol{g}}_k = \begin{bmatrix} \hat{g}_k^1 \\ \vdots \\ \hat{g}_k^M \end{bmatrix} \sim CN(\boldsymbol{0}, \gamma_k \boldsymbol{I}_M)$, $\tilde{\boldsymbol{g}}_k = \begin{bmatrix} \tilde{g}_k^1 \\ \vdots \\ \tilde{g}_k^M \end{bmatrix} \sim CN(\boldsymbol{0}, (\beta_k - \gamma_k) \boldsymbol{I}_M)$

Properties of estimated Rayleigh fading channels

- Estimated channels independently distributed as $\hat{\boldsymbol{g}}_k \sim CN(\mathbf{0}, \gamma_k \boldsymbol{I}_M)$

  - Offer channel hardening:

$$\frac{1}{M}\|\hat{\boldsymbol{g}}_k\|^2 \to \gamma_k, \qquad M \to \infty, \qquad k = 1, \dots, K$$

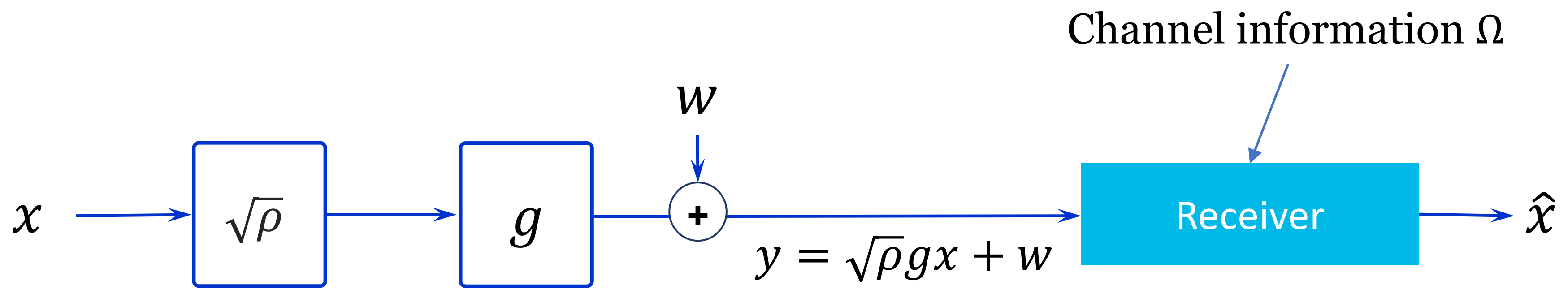
  - Offer favorable propagation:

$$\frac{1}{M}\hat{\boldsymbol{g}}_k^H \hat{\boldsymbol{g}}_i \to 0, \qquad M \to \infty, \qquad k, i = 1, \dots, K, \qquad k \neq i$$

Approximations when M is large:

$$\frac{1}{M}\|\hat{\boldsymbol{g}}_k\|^2 \approx \gamma_k \text{ and } \frac{1}{M}\hat{\boldsymbol{g}}_k^H \hat{\boldsymbol{g}}_i \approx 0$$

Recall: Capacity lower bound

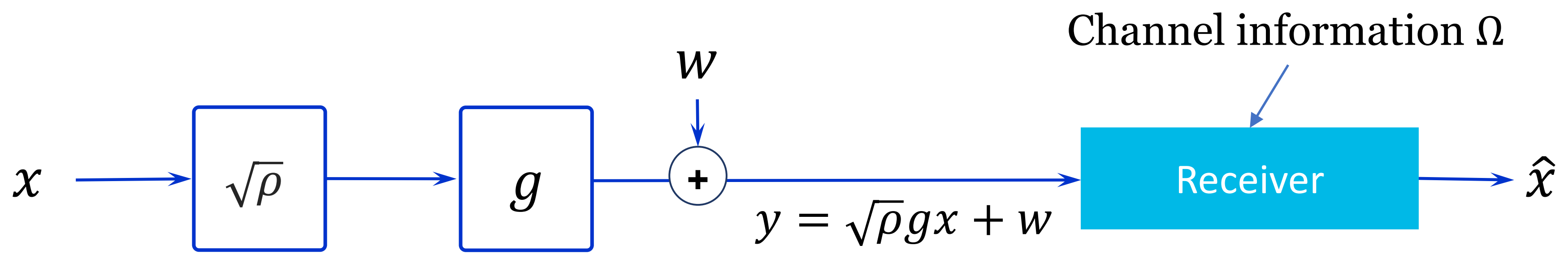


- Desired signal x, power ρ, and g and w uncorrelated
- Channel coefficient g, known channel information Ω

Capacity lower bound:

$$C \geq E\left\{\log_2\left(1 + \frac{\rho |E\{g|\Omega\}|^2}{\rho Var\{g|\Omega\} + Var\{w|\Omega\}}\right)\right\}$$

Capacity lower bound with deterministic channel



- Desired signal x with unit power, transmit power ρ
- Determinstic and known channel coefficient g $(\Omega = \{g\})$

Capacity lower bound:

$$C \geq E\left\{\log_2\left(1+\frac{\rho|E\{g|\Omega\}|^2}{\rho Var\{g|\Omega\}+Var\{w|\Omega\}}\right)\right\} = \log_2\left(1+\frac{\rho|g|^2}{Var\{w\}}\right)$$

Revisiting the received uplink signal

- Received signal:

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{g}_k \sqrt{\rho_{ul}\eta_k} q_k + \boldsymbol{w} = \underbrace{\sum_{k=1}^{K} \widehat{\boldsymbol{g}}_k \sqrt{\rho_{ul}\eta_k} q_k}_{\text{Useful part}} - \underbrace{\sum_{k=1}^{K} \widetilde{\boldsymbol{g}}_k \sqrt{\rho_{ul}\eta_k} q_k + \boldsymbol{w}}_{\boldsymbol{w}' \text{ : Unusable part}}$$

- Assign receiver filter $\boldsymbol{a}_i$ for user i:

$$\boldsymbol{a}_i^H \boldsymbol{y} = \underbrace{\boldsymbol{a}_i^H \widehat{\boldsymbol{g}}_i \sqrt{\rho_{ul}\eta_i} q_i}_{\text{Desired part}} + \underbrace{\sum_{k=1,k\neq i}^{K} \boldsymbol{a}_i^H \widehat{\boldsymbol{g}}_k \sqrt{\rho_{ul}\eta_k} q_k}_{\text{Interference}} + \boldsymbol{a}_i^H \boldsymbol{w}'$$

Focusing on the desired part

- Consider $\widehat{\boldsymbol{g}}_i \sim CN(\mathbf{0}, \gamma_i \boldsymbol{I}_M)$
- Which value of $\boldsymbol{a}_i$ maximizes the ratio $\frac{|\boldsymbol{a}_i^H \widehat{\boldsymbol{g}}_i|}{\|\boldsymbol{a}_i\|}$?

Cauchy-Schwartz inequality

$$\frac{\left|\boldsymbol{a}_i^H \widehat{\boldsymbol{g}}_i\right|}{\|\boldsymbol{a}_i\|} \leq \frac{\|\boldsymbol{a}_i\| \|\widehat{\boldsymbol{g}}_i\|}{\|\boldsymbol{a}_i\|} = \|\widehat{\boldsymbol{g}}_i\|$$

with equality if $\boldsymbol{a}_i = c\widehat{\boldsymbol{g}}_i$ for some constant $c \neq 0$

- We now call $\boldsymbol{a}_i = c\widehat{\boldsymbol{g}}_i$ maximum ratio (MR) processing
    - Same thing as MRC for $c = 1/\|\widehat{\boldsymbol{g}}_i\|$

Received signal when using MR processing

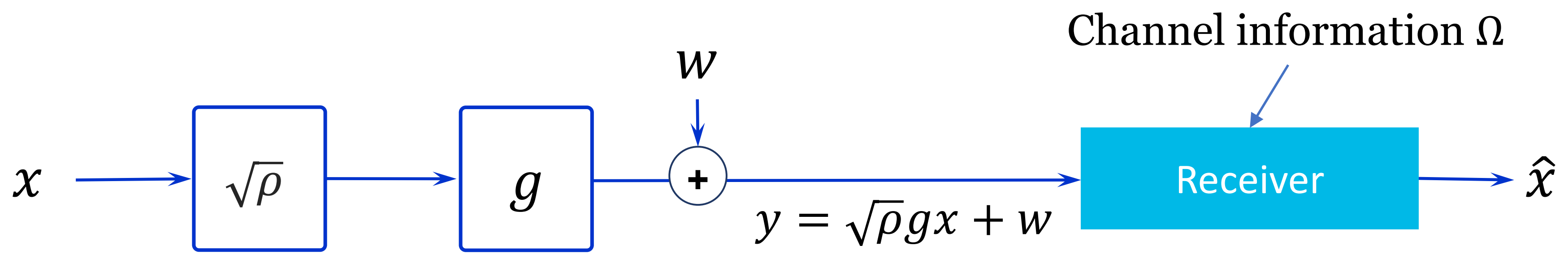
- Set $\boldsymbol{a}_i = \frac{1}{M}\widehat{\boldsymbol{g}}_i$:

$$\boldsymbol{a}_i^H \boldsymbol{y} = \underbrace{\frac{\widehat{\boldsymbol{g}}_i^H \widehat{\boldsymbol{g}}_i}{M}}_{\approx \gamma_i} \sqrt{\rho_{ul}\eta_i} q_i + \sum_{k=1, k\neq i}^{K} \underbrace{\frac{\widehat{\boldsymbol{g}}_i^H \widehat{\boldsymbol{g}}_k}{M}}_{\approx 0} \sqrt{\rho_{ul}\eta_k} q_k + \underbrace{\frac{\widehat{\boldsymbol{g}}_i^H \boldsymbol{w}'}{M}}_{\approx 0}$$

- Use-and-then-forget technique:

$$\boldsymbol{a}_i^H \boldsymbol{y} = \underbrace{\gamma_i \sqrt{\rho_{ul}\eta_i} q_i}_{\text{Desired part with deterministic channel!}} + \underbrace{\left(\frac{\widehat{\boldsymbol{g}}_i^H \widehat{\boldsymbol{g}}_i}{M} - \gamma_i\right) \sqrt{\rho_{ul}\eta_i} q_i + \sum_{k=1, k\neq i}^{K} \frac{\widehat{\boldsymbol{g}}_i^H \widehat{\boldsymbol{g}}_k}{M} \sqrt{\rho_{ul}\eta_k} q_k + \frac{\widehat{\boldsymbol{g}}_i^H \boldsymbol{w}'}{M}}_{w\text{: Uncorrelated interference and noise}}$$

Using the capacity bound with deterministic channel



- Desired signal $x = q_i$, transmit power $\rho = \rho_{ul}\eta_i$
- Determinstic and known channel coefficient $g = \gamma_i$

Capacity lower bound:

$$C \geq \log_2\left(1 + \frac{\rho|g|^2}{Var\{w\}}\right)$$

$$\rho|g|^2 = \rho_{ul}\eta_i\gamma_i^2$$

$$Var\{w\} = \cdots = \frac{\gamma_i}{M}\left(\sum_{k=1}^{K} \rho_{ul}\eta_k\beta_k + 1\right)$$

Capacity bound with MR and use-and-then-forget technique

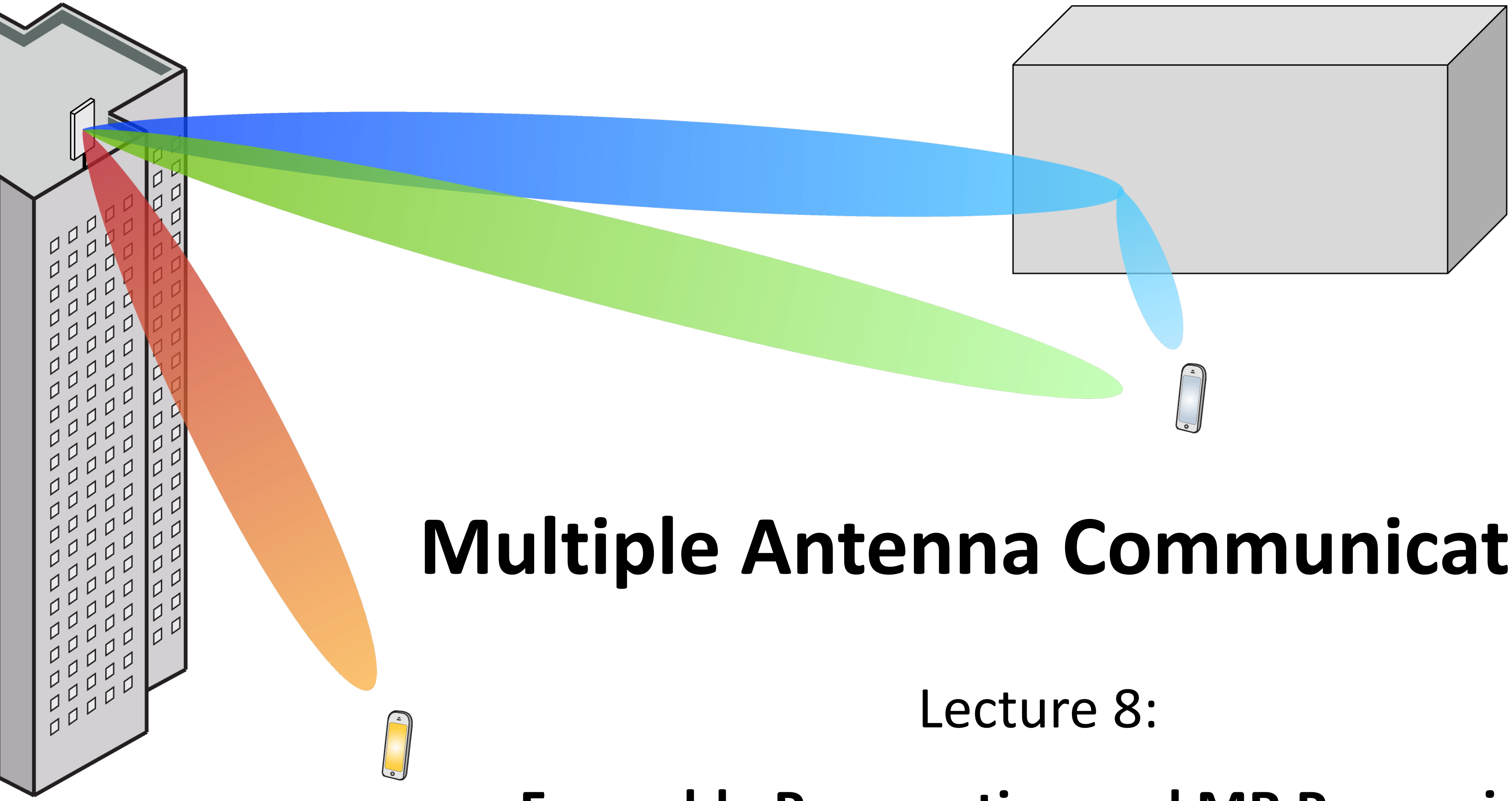
$$C \geq \log_2\left(1 + \frac{M\rho_{ul}\eta_i\gamma_i}{\sum_{k=1}^{K}\rho_{ul}\eta_k\beta_k + 1}\right)$$

- Interpretation
  - Small-scale fading is not visible in this bound
  - **Numerator:**
    Coherent beamforming gain, grows with antennas M, power $\rho_{ul}\eta_i$ and estimation quality γ_i
  - **Denominator:**
    Sum of non-coherent interference from all users plus noise variance

Summary

- When having many antennas:
  - Channel hardening: $\frac{1}{M}\|\widehat{\boldsymbol{g}}_k\|^2 \approx \gamma_k$
  - Favorable propagation: $\frac{1}{M}\widehat{\boldsymbol{g}}_k^H\widehat{\boldsymbol{g}}_i \approx 0$
  - Motivation for MR processing

- Use-and-then-forget technique
  - Pretend as if channel is deterministic
  - Compute a closed-form capacity lower bound with MR processing



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