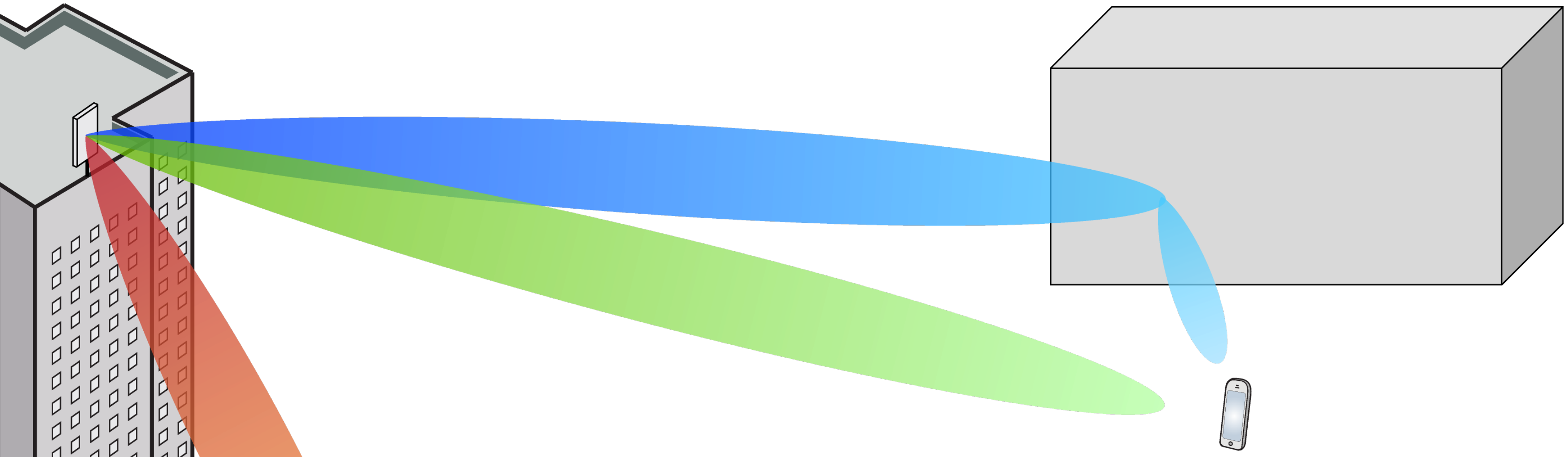
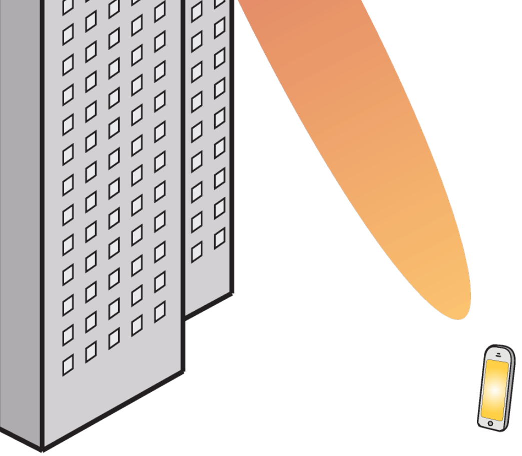




Multiple Antenna Communications

Lecture 4:

Capacity of Point-to-Point MIMO Channels

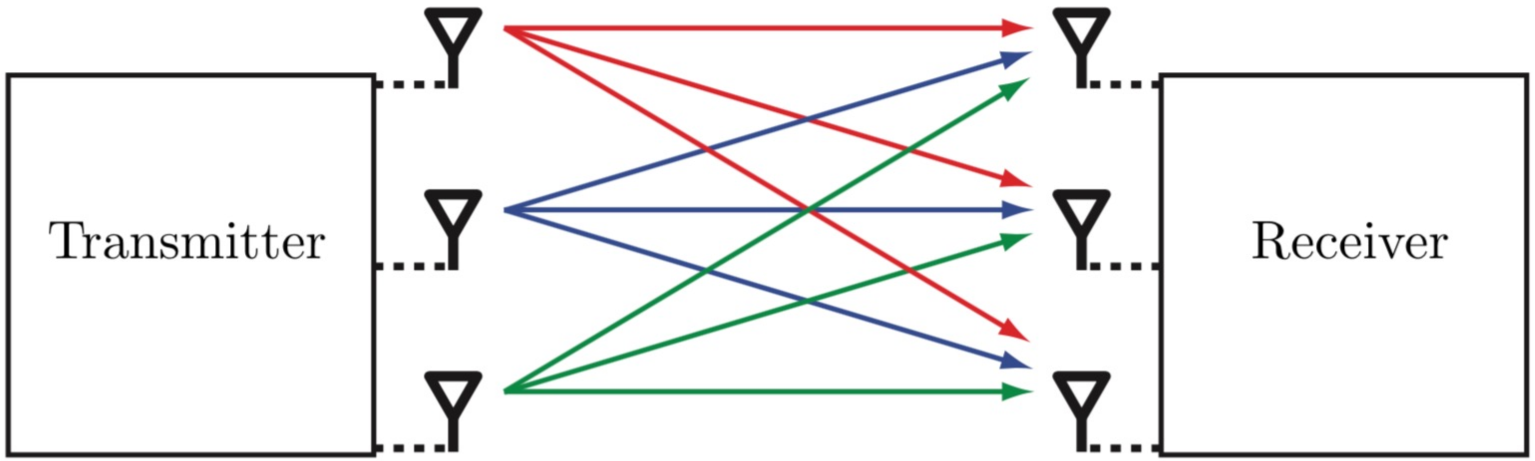
Emil Björnson

Outline

- Point-to-point MIMO channels
  - Basic formulation
  - Some linear algebra results
  - Capacity of MIMO channels
  - Examples of capacity behavior
- Transmitter diversity
  - MISO channels with slow fading

Point-to-point MIMO channel

- K transmit antennas
- M receive antennas
- Deterministic channel

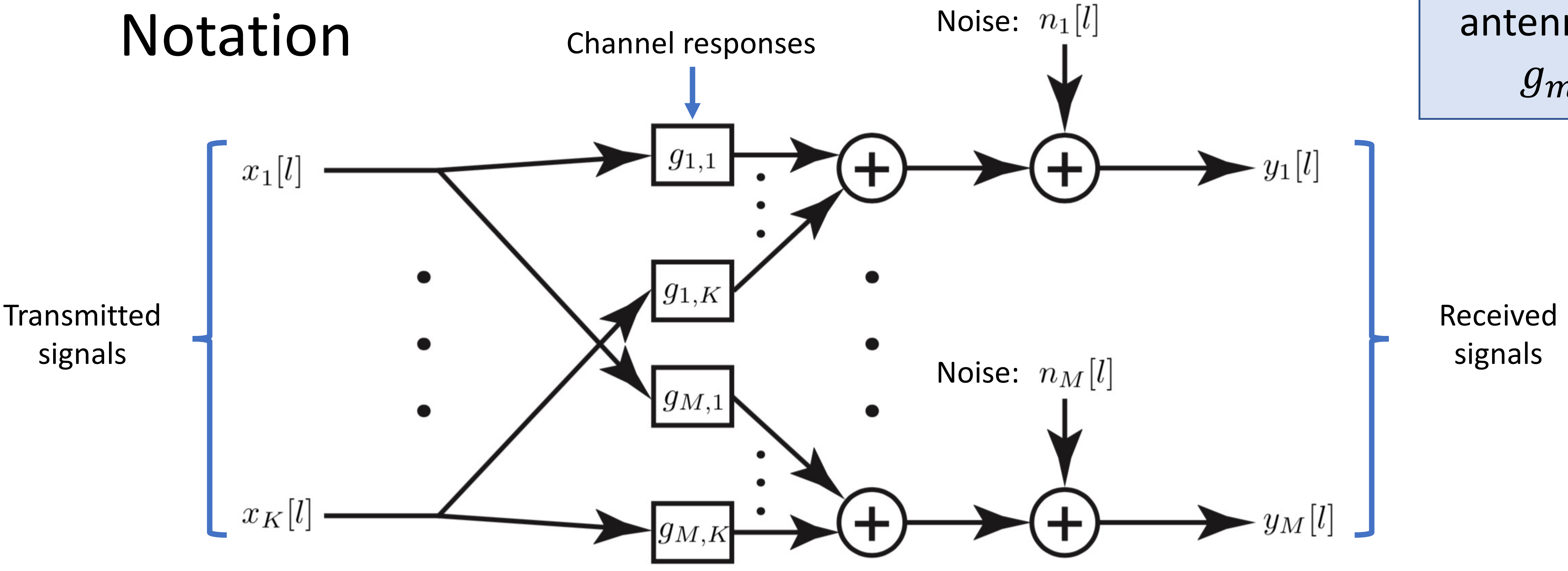


- Generalization of SIMO and MISO channels

What is the capacity of such a channel?

Notation

Channel from antenna k to antenna m: $g_{m,k}$



Received signal at antenna m: $y_m[l] = \sum_{k=1}^{K} g_{m,k} x_k[l] + n_m[l]$

Vector-Matrix Description

- Memoryless channel to antenna m:

$$y_m = \sum_{k=1}^{K} g_{m,k} x_k + n_m$$

- Matrix form:

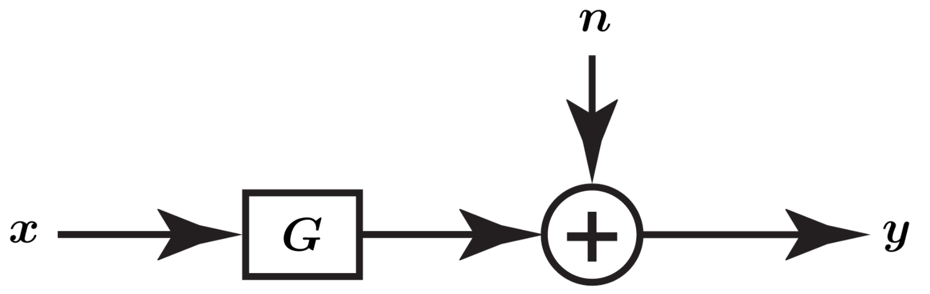
$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix}}_{\boldsymbol{y}} = \begin{bmatrix} \sum_{k=1}^{K} g_{1,k} x_k \\ \vdots \\ \sum_{k=1}^{K} g_{M,k} x_k \end{bmatrix} + \begin{bmatrix} n_1 \\ \vdots \\ n_M \end{bmatrix} = \underbrace{\begin{bmatrix} g_{1,1} & \cdots & g_{1,K} \\ \vdots & \ddots & \vdots \\ g_{M,1} & \cdots & g_{M,K} \end{bmatrix}}_{\boldsymbol{G}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_K \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} n_1 \\ \vdots \\ n_M \end{bmatrix}}_{\boldsymbol{n}}$$

Short form: $\boldsymbol{y} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{n}$

What is the channel capacity?

- Recall: Channel capacity is defined as

$$C = \max_{f(\boldsymbol{x}):\, E\{\|\boldsymbol{x}\|^2\} \leq q} I(\boldsymbol{x}; \boldsymbol{y}) \quad \text{bit/s/Hz}$$



  - Mutual information: $I(\boldsymbol{x}; \boldsymbol{y}) = h(\boldsymbol{y}) - h(\boldsymbol{y}|\boldsymbol{x})$

  - Power constraint: $E\{\|\boldsymbol{x}\|^2\} = \sum_{k=1}^{K} E\{|x_k|^2\} \leq q$
  - Independent $\mathrm{CN}(0, N_0)$ elements in $\boldsymbol{n}$

We will not compute $h(\boldsymbol{y})$ and $h(\boldsymbol{y}|\boldsymbol{x})$ directly but look for a shortcut using linear algebra!

Eigenvalues and eigenvectors

- Consider an $M \times M$ matrix $\boldsymbol{A}$
  - A non-zero vector $\boldsymbol{u}$ is an *eigenvector* of $\boldsymbol{A}$ if
  $$\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$$
  where the scalar λ is the *eigenvalue* corresponding to $\boldsymbol{u}$

Eigenvalue decomposition

- If $\boldsymbol{A}$ has M linearly independent eigenvectors, then

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$

  - $\mathbf{U}$ contains (unit-norm) eigenvectors as columns
  - $\mathbf{D}$ is the diagonal matrix with corresponding eigenvalues

- If $\mathbf{A}$ is symmetric ($\mathbf{A} = \mathbf{A}^H$) then $\mathbf{U}^{-1} = \mathbf{U}^H$ and

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^H$$

  - The matrix can be *diagonalized* as

$$\mathbf{U}^H\mathbf{A}\mathbf{U} = \mathbf{D}$$

Unitary matrix:
$\mathbf{U}\mathbf{U}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}$

Singular value decomposition

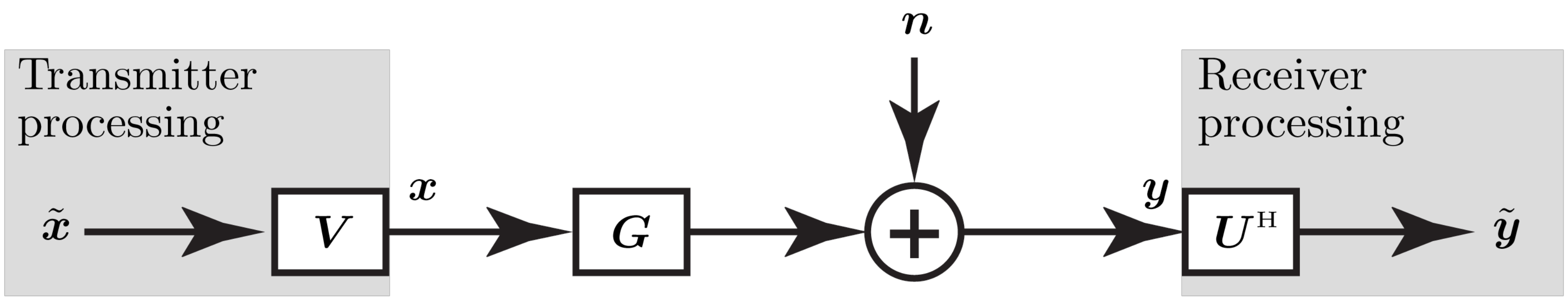
- Every complex $M \times K$ matrix $\boldsymbol{G}$ can be factorized as

$$\boldsymbol{G} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H$$

- Left singular vectors:
  - $\boldsymbol{U}$ is an $M \times M$ unitary matrix containing the eigenvectors of $\boldsymbol{G}\boldsymbol{G}^H$
- Right singular vectors:
  - $\boldsymbol{V}$ is a $K \times K$ unitary matrix containing the eigenvectors of $\boldsymbol{G}^H\boldsymbol{G}$
- Singular values:
  - $\boldsymbol{\Sigma}$ is an $M \times K$ “diagonal” matrix with $s_1 \geq s_2 \geq \cdots \geq s_{\min(M,K)} \geq 0$ on diagonal
  - $s_1^2, s_2^2, \ldots, s_{\min(M,K)}^2$ are the non-zero eigenvalues of $\boldsymbol{G}\boldsymbol{G}^H$ and $\boldsymbol{G}^H\boldsymbol{G}$

Diagonalizing the MIMO channel

- Non-destructive processing:
  - Pre-processing: $\boldsymbol{x} = \boldsymbol{V}\widetilde{\boldsymbol{x}}$
  - Post-processing: $\widetilde{\boldsymbol{y}} = \boldsymbol{U}^H\boldsymbol{y}$



$$\widetilde{\boldsymbol{y}} = \boldsymbol{U}^H(\boldsymbol{G}\boldsymbol{x} + \boldsymbol{n}) = \boldsymbol{U}^H\underbrace{\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H}_{\text{Singular value decomposition}}\boldsymbol{V}\widetilde{\boldsymbol{x}} + \underbrace{\boldsymbol{U}^H\boldsymbol{n}}_{\widetilde{\boldsymbol{n}}\text{: Rotated noise: i.i.d. } CN(0, N_0)}$$

Diagonalizing the MIMO channel (2)

- Processed received signal:

$$\tilde{\boldsymbol{y}} = \boldsymbol{\Sigma}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{n}}$$

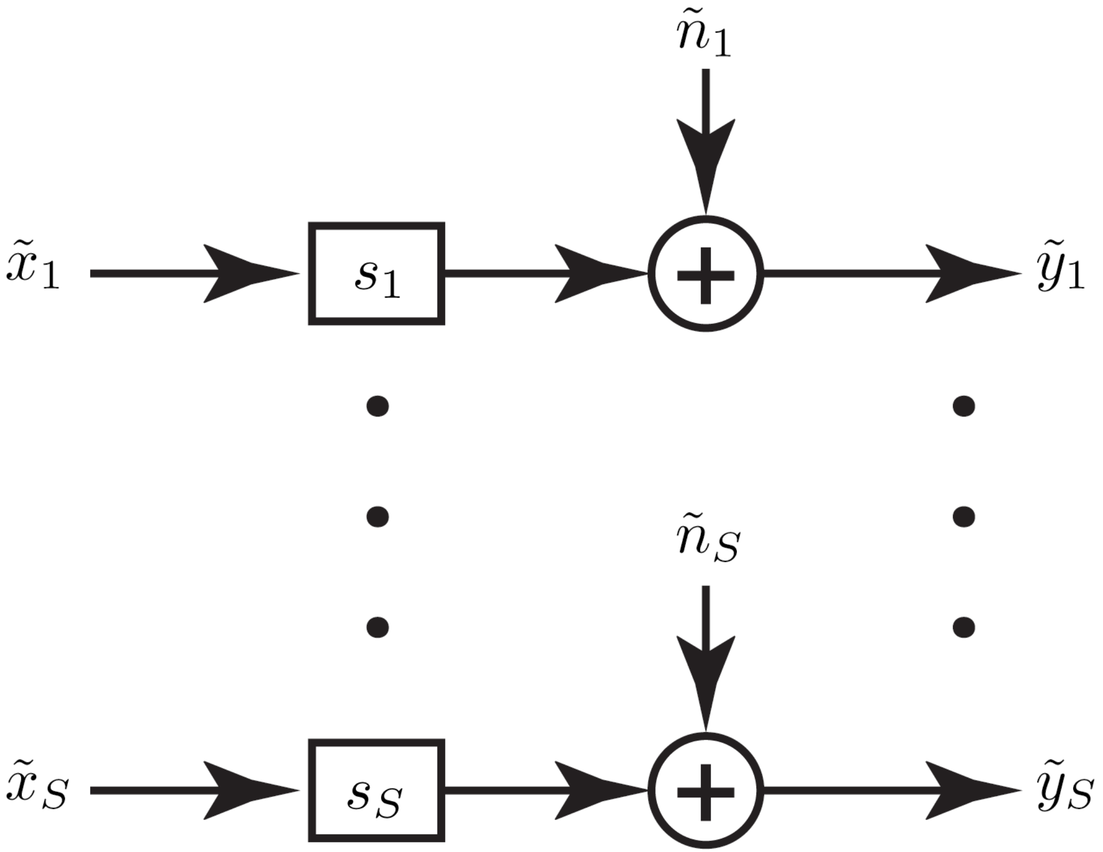
- Let S denote the number of non-zero singular values (rank of $\boldsymbol{G}$):

$$\tilde{y}_k = \begin{cases} s_k \tilde{x}_k + \tilde{n}_k, & \text{if } k = 1, \ldots, S, \\ \tilde{n}_k, & \text{if } k = S+1, \ldots, M \end{cases}$$

Useful subchannels

Useless subchannels

S parallel channels



Suppose we assign power q_k to subchannel k

- Capacity of subchannel k:

$$R_k = \log_2\left(1 + s_k^2 \frac{q_k}{N_0}\right)$$

Channel capacity:

$$C = \max_{q_1,\ldots,q_S:\, q_1+\cdots+q_S=q} \sum_{k=1}^{S} R_k$$

Optimal Power Allocation

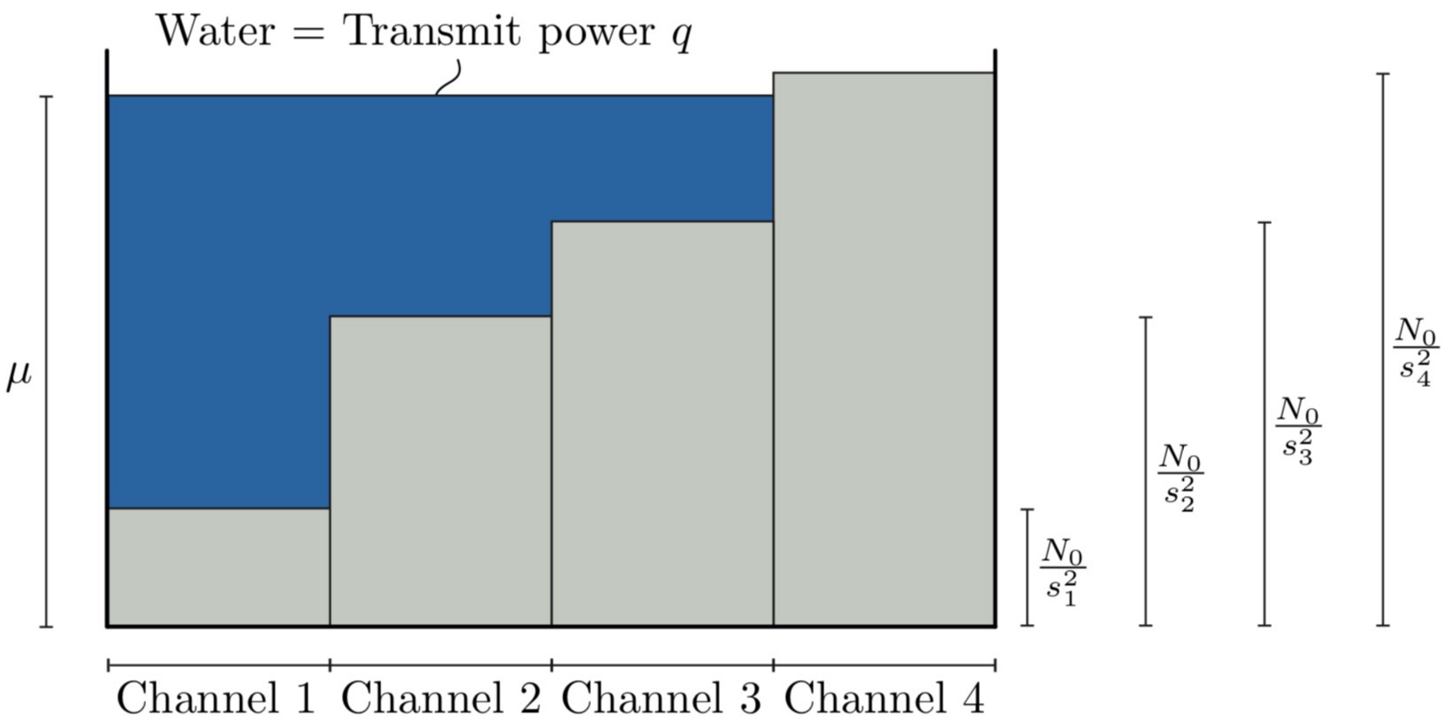
- After some optimization:

$$q_k = \max\left(\mu - \frac{N_0}{s_k^2}, 0\right)$$

where μ is selected such that $q_1 + \cdots + q_S = q$

Properties:

- Larger s_k: More power
- Some subchannels might get zero power

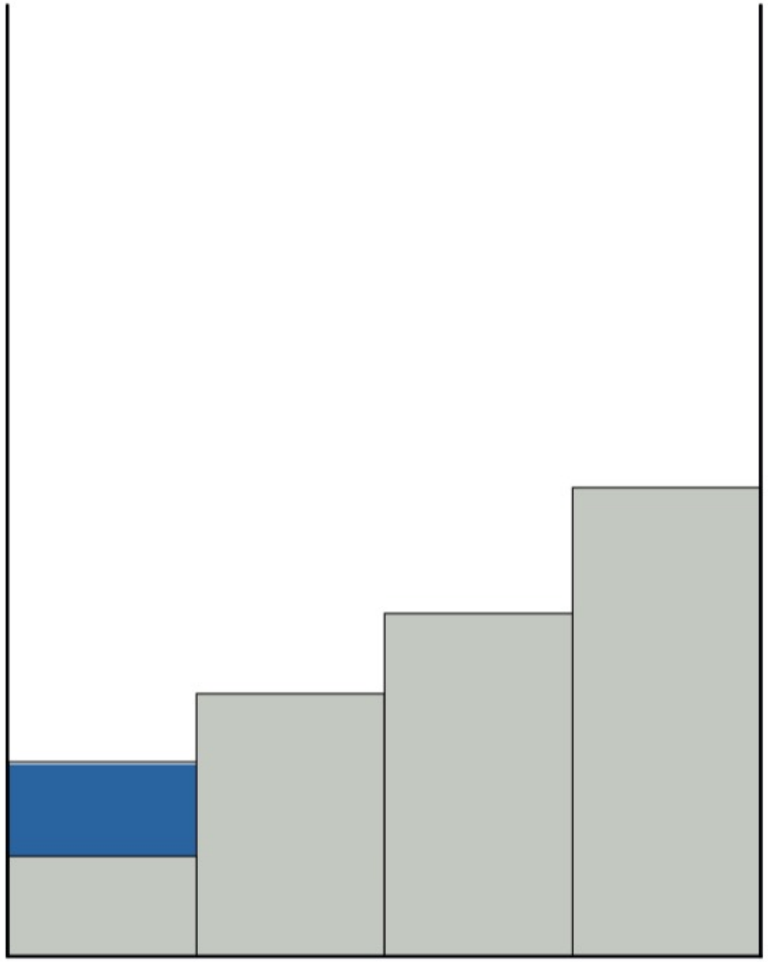


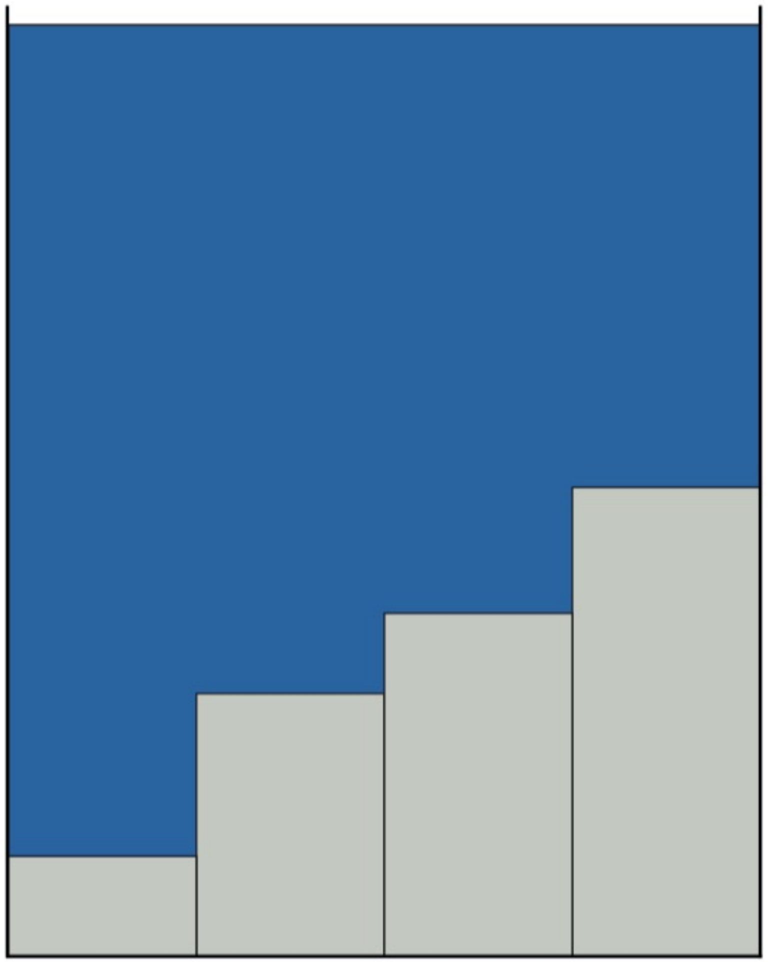
Called waterfilling

Low and high SNR

- Low SNR:
  - Only power to one subchannel

- High SNR:
  - Approximately the same power to all subchannels





Capacity behavior at high SNR

- Equal power allocation (approximately optimal):
  - $q_1 = \cdots = q_S = \frac{q}{S}$

- Capacity approximation:

$$C \approx \sum_{k=1}^{S} \log_2\left(1 + \overset{q/N_0}{\overbrace{\mathrm{SNR}}}\,\frac{s_k^2}{S}\right) \approx S\log_2(\mathrm{SNR}) + \sum_{k=1}^{S} \log_2\left(\frac{s_k^2}{S}\right)$$

Multiplexing gain:

First term proportional to $S = \mathrm{rank}(\boldsymbol{G}^H\boldsymbol{G}) \leq \min(M, K)$

Capacity behavior at low SNR

- Assume $s_1 > \cdots > s_S$
  - Low SNR: $q_1 = q$, $q_2 = \cdots = q_S = 0$

- Capacity approximation:

$$C \approx \log_2(1 + \mathrm{SNR}\, s_1^2) \approx \log_2(e)\, \mathrm{SNR}\, s_1^2$$

  - No multiplexing gain

Beamforming gain:
s_1 is the largest singular value

Capacity comparison with $|g_{m,k}| = 1$

- SISO channel:

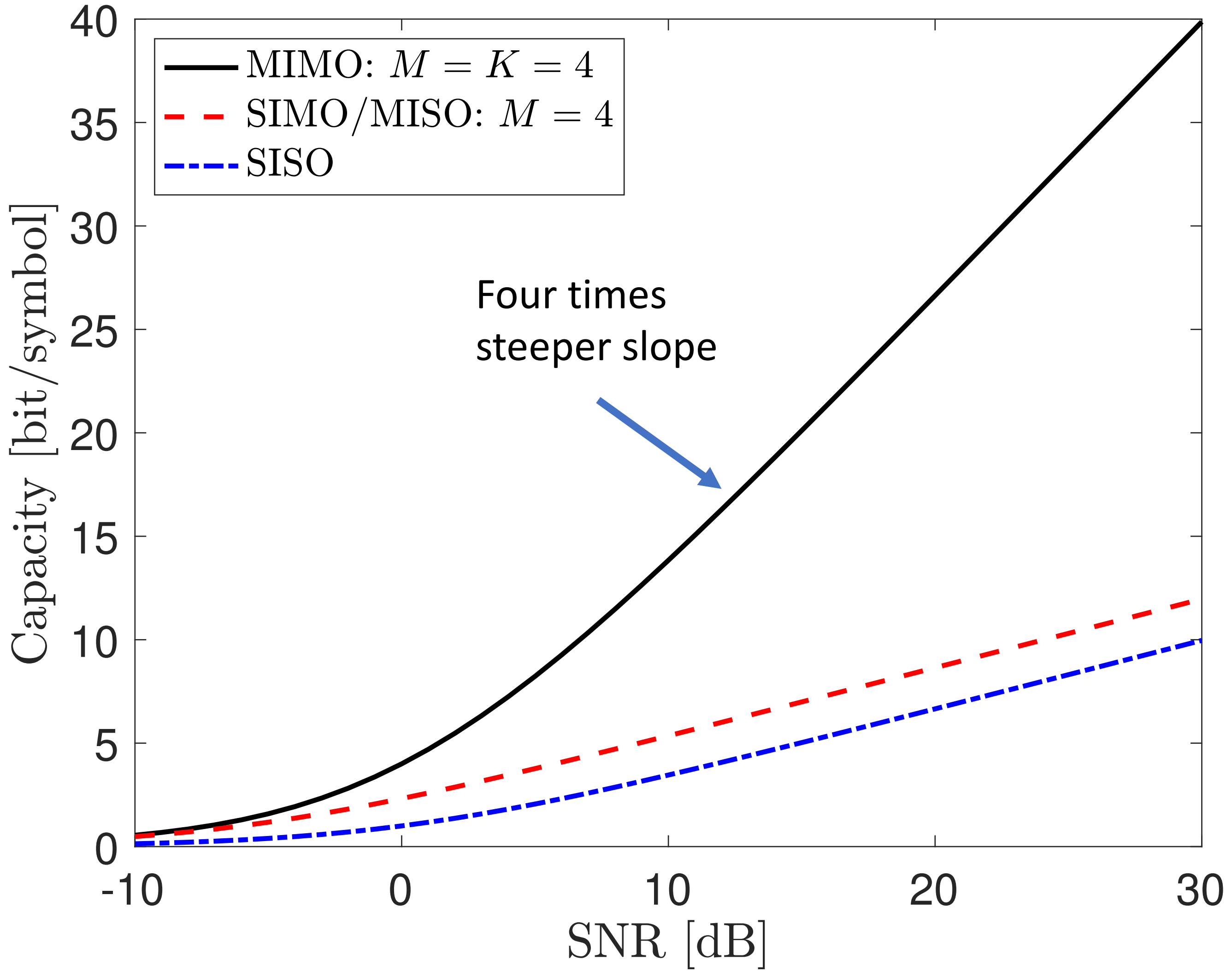
$$C = \log_2(1 + \mathrm{SNR})$$

- SIMO/MISO, M antennas:

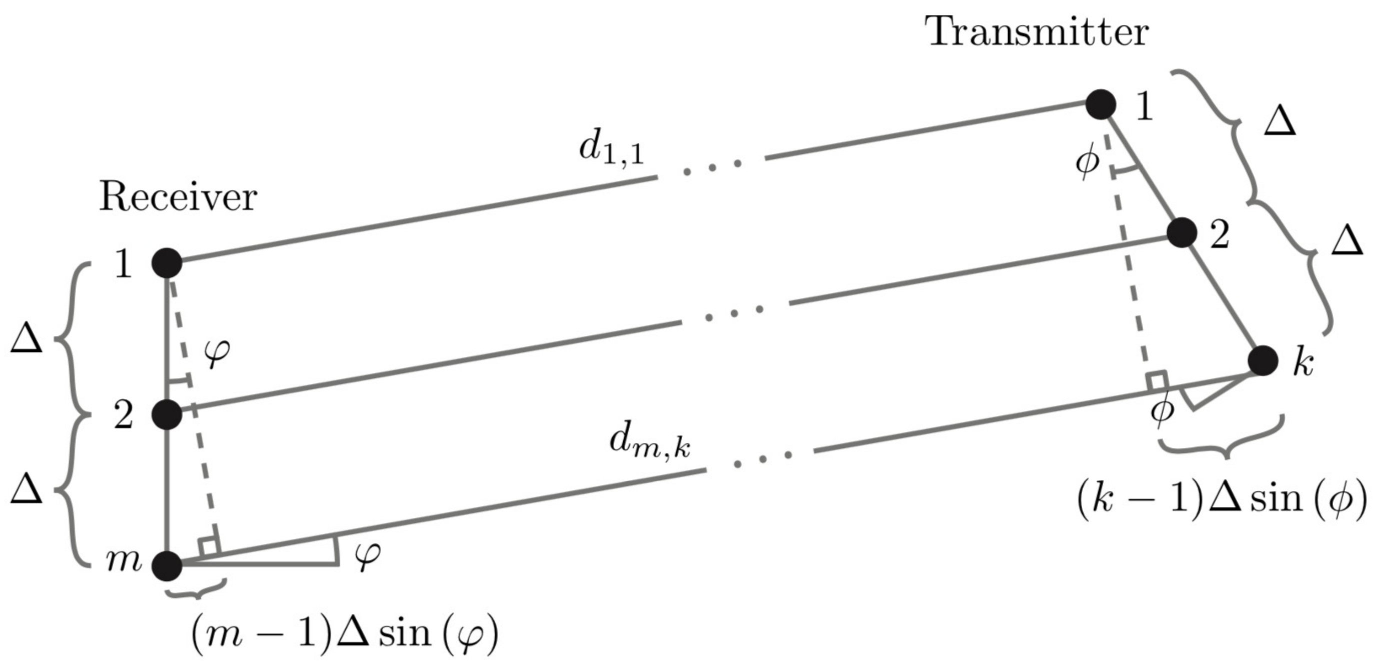
$$C = \log_2(1 + M \cdot \mathrm{SNR})$$

- MIMO, $M = K$ antennas:
  - All singular values are $\sqrt{M}$

$$C = M \cdot \log_2(1 + \mathrm{SNR})$$



Example: Line-of-sight channel



$$\boldsymbol{G} = \begin{bmatrix} g_{1,1} & \cdots & g_{1,K} \\ \vdots & \ddots & \vdots \\ g_{M,1} & \cdots & g_{M,K} \end{bmatrix} = \sqrt{\beta} \begin{bmatrix} 1 & \cdots & e^{-\mathrm{j}2\pi \frac{(K-1)\Delta \sin(\phi)}{\lambda}} \\ \vdots & \ddots & \vdots \\ e^{-\mathrm{j}2\pi \frac{(M-1)\Delta \sin(\varphi)}{\lambda}} & \cdots & e^{-\mathrm{j}2\pi \frac{(M-1)\Delta \sin(\varphi)}{\lambda}} e^{-\mathrm{j}2\pi \frac{(K-1)\Delta \sin(\phi)}{\lambda}} \end{bmatrix}$$

$$= \sqrt{\beta} \begin{bmatrix} 1 \\ \vdots \\ e^{-\mathrm{j}2\pi \frac{(M-1)\Delta \sin(\varphi)}{\lambda}} \end{bmatrix} \begin{bmatrix} 1 & \cdots & e^{-\mathrm{j}2\pi \frac{(K-1)\Delta \sin(\phi)}{\lambda}} \end{bmatrix}$$

Rank 1: $s_1 = \sqrt{\beta M K}$

Line-of-sight channels: No multiplexing gain

- We have $S = 1$:
  - $q_1 = q$

- Capacity:

$$C = \log_2(1 + \beta MK \cdot \mathrm{SNR})$$

  - Compare with SIMO and MISO:

$$C = \log_2(1 + \beta M \cdot \mathrm{SNR})$$

Beamforming gain:
Larger in the MIMO case

Slow fading and MISO channels ($M = 2$)

- Received signal

$$y[l] = [g_1 \quad g_2] \begin{bmatrix} x_1[l] \\ x_2[l] \end{bmatrix} + n[l]$$

  - Fixed channel g_1, g_2 for the entire transmission
  - Assumption: Receiver knows g_1, g_2, but not the transmitter

- Consider two time slots: $[y[1] \quad y[2]] = \underbrace{[g_1 \quad g_2]}_{=\boldsymbol{g}^{\mathrm{T}}} \underbrace{\begin{bmatrix} x_1[1] & x_1[2] \\ x_2[1] & x_2[2] \end{bmatrix}}_{=\boldsymbol{X}} + [n[1] \quad n[2]]$

Can we select $\boldsymbol{X}$ in a clever way?

Space-time block coding

Data signals $\tilde{x}[1], \tilde{x}[2]$

- Alamouti code: $$\boldsymbol{X} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{x}[1] & \tilde{x}[2] \\ -\tilde{x}^*[2] & \tilde{x}^*[1] \end{bmatrix}$$
- Consider received signal: $$\begin{bmatrix} y[1] \\ y^*[2] \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} g_1\tilde{x}[1] - g_2\tilde{x}^*[2] \\ g_1^*\tilde{x}^*[2] + g_2^*\tilde{x}[1] \end{bmatrix} + \begin{bmatrix} n[1] \\ n^*[2] \end{bmatrix}$$ $$= \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} g_1 & -g_2 \\ g_2^* & g_1^* \end{bmatrix}}_{=\tilde{\boldsymbol{G}}} \begin{bmatrix} \tilde{x}[1] \\ \tilde{x}^*[2] \end{bmatrix} + \begin{bmatrix} n[1] \\ n^*[2] \end{bmatrix}$$

$$\tilde{\boldsymbol{G}} = \underbrace{\frac{1}{\|\boldsymbol{g}\|} \begin{bmatrix} g_1 & -g_2 \\ g_2^* & g_1^* \end{bmatrix}}_{=\tilde{\boldsymbol{U}}} \underbrace{\begin{bmatrix} \frac{\|\boldsymbol{g}\|}{\sqrt{2}} & 0 \\ 0 & \frac{\|\boldsymbol{g}\|}{\sqrt{2}} \end{bmatrix}}_{=\tilde{\boldsymbol{\Sigma}}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=\tilde{\boldsymbol{V}}^{\mathrm{H}}}$$

This is like a 2×2 MIMO channel! Transmitter knows $\tilde{\boldsymbol{V}}$ without knowing channel

Transmit diversity versus receive diversity

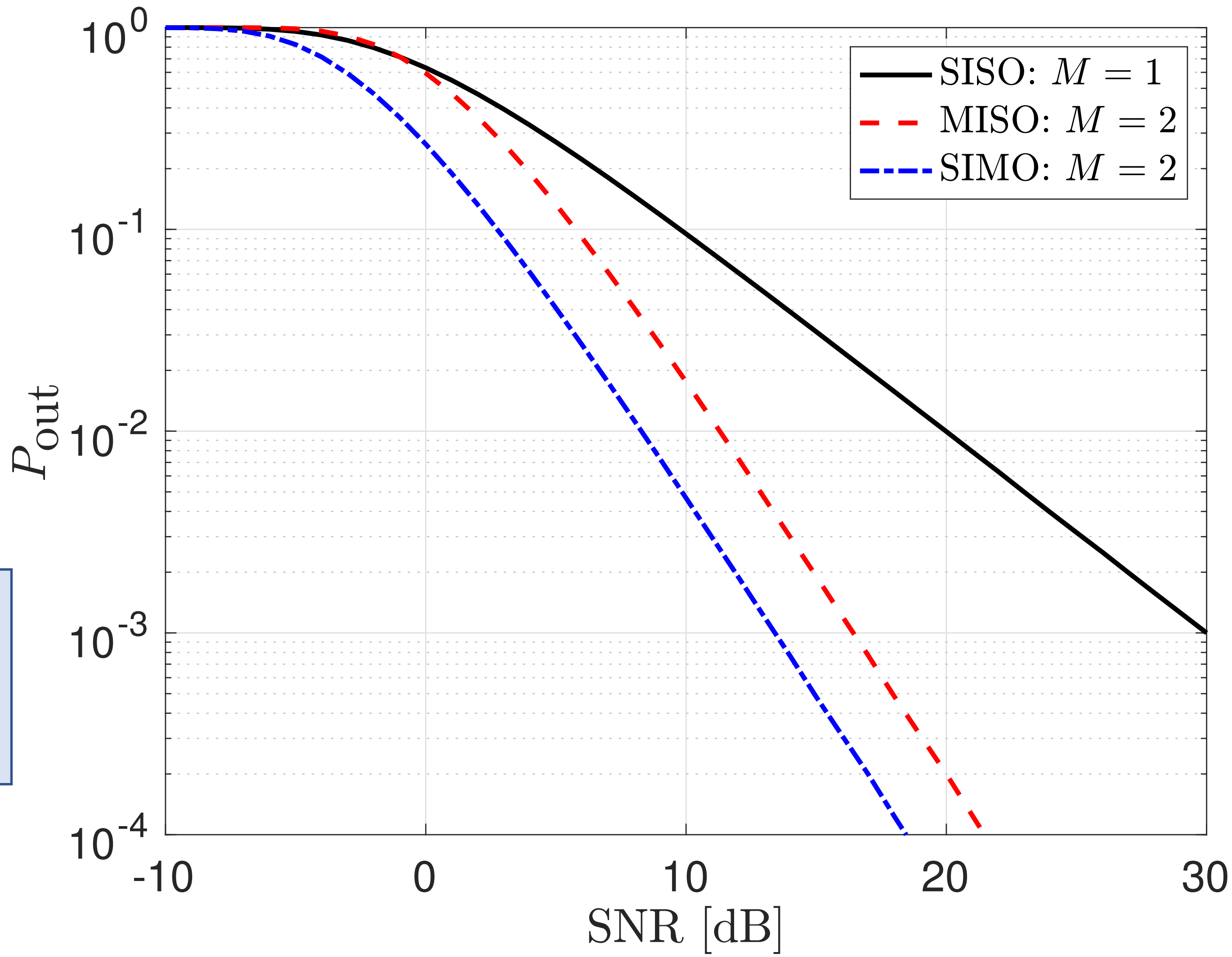
- Ideal capacity with MISO:

$$C_{\boldsymbol{g}} = \log_2\left(1 + \frac{q}{2N_0}\|\boldsymbol{g}\|^2\right)$$

- Ideal capacity with SIMO:

$$C_{\boldsymbol{g}} = \log_2\left(1 + \frac{q}{N_0}\|\boldsymbol{g}\|^2\right)$$

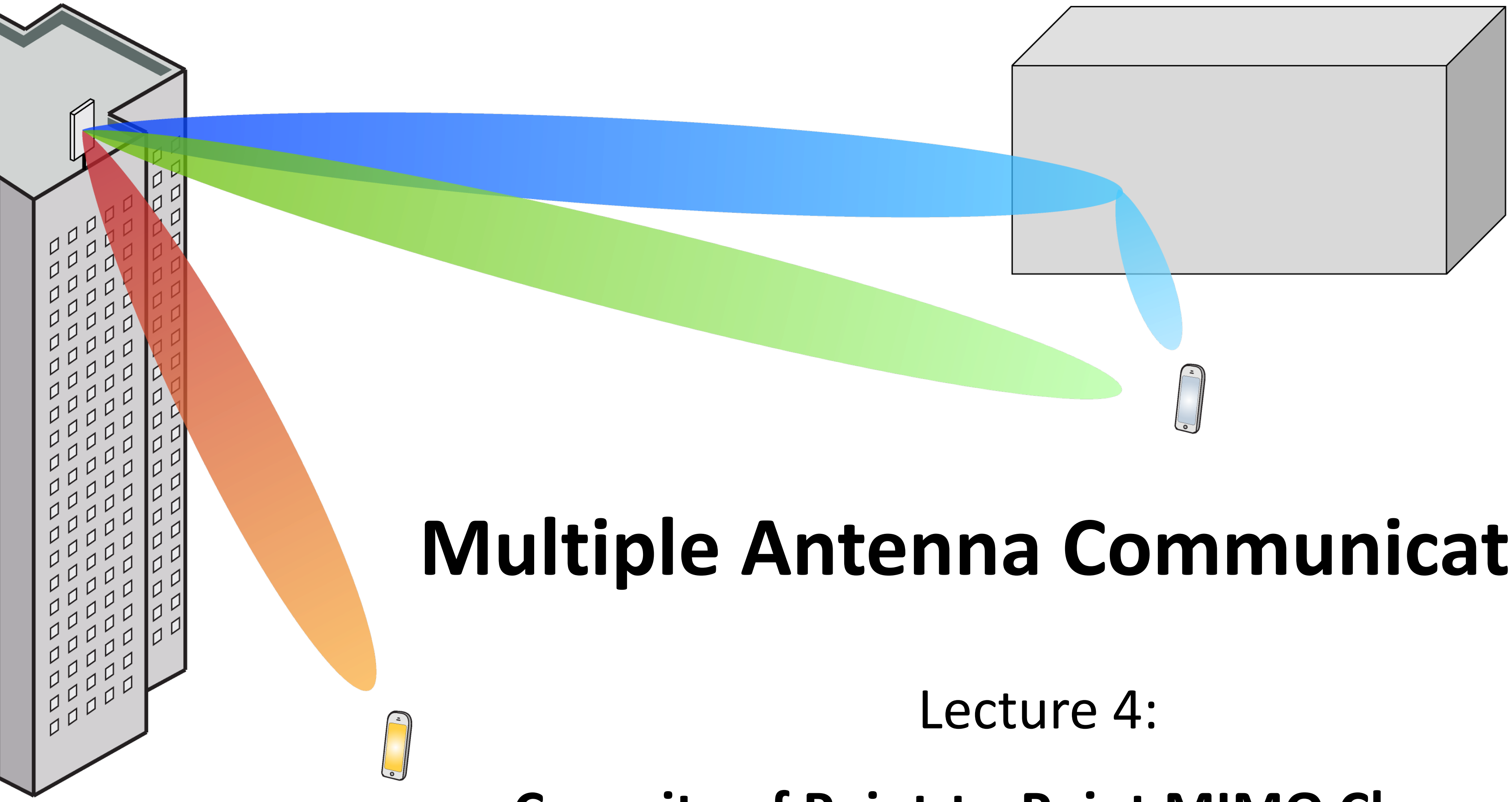
Outage probability:
Same behavior but half the SNR
Diversity gain, but no beamforming gain



Summary

- Capacity of $M \times K$ MIMO channel
  - Send K different messages along eigenvectors of $\boldsymbol{G}^H\boldsymbol{G}$
  - This creates K parallel channels
  - Divide power using waterfilling
  - High SNR: Multiplexing gain
  - Low SNR: Beamforming gain

- MISO channels with slow fading
  - No beamforming gain, but
    diversity gain can be achieved



Multiple Antenna Communications

Lecture 4:

Capacity of Point-to-Point MIMO Channels

Emil Björnson