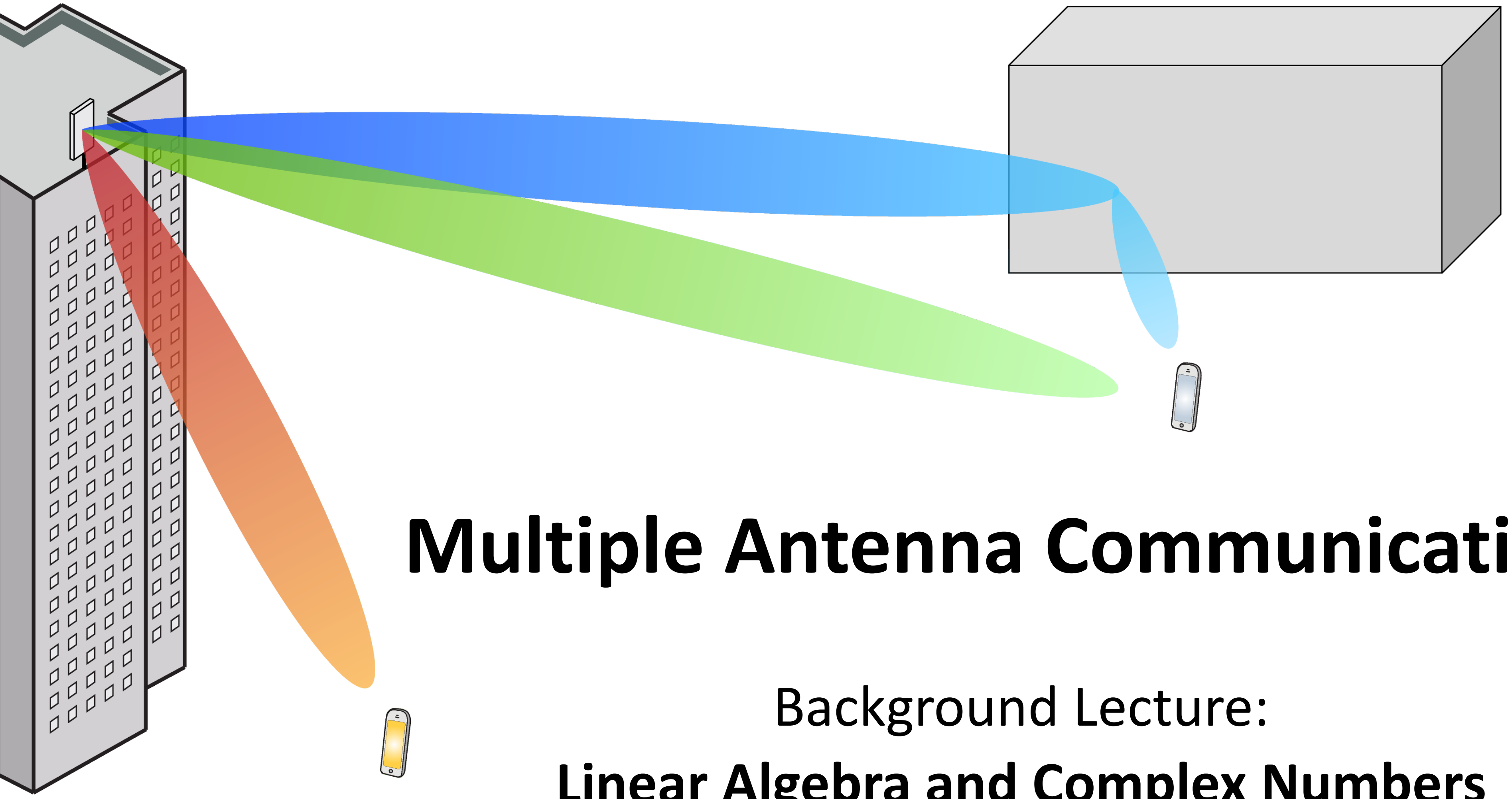


Multiple Antenna Communications

Background Lecture:

Linear Algebra and Complex Numbers

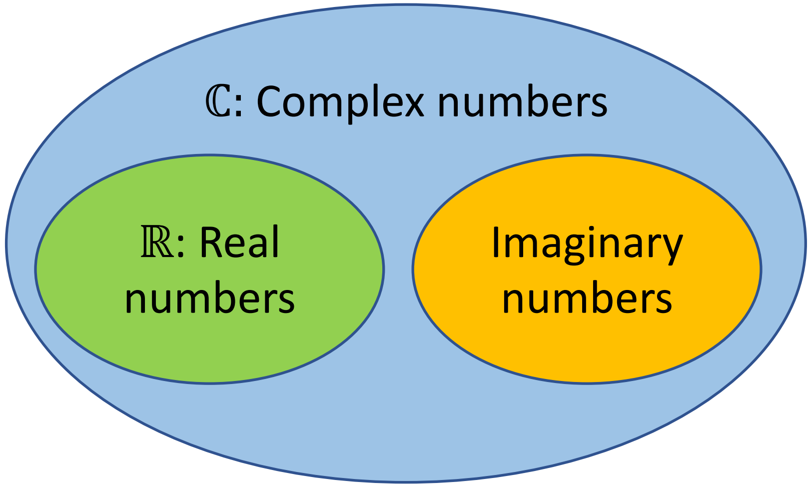
Emil Björnson

Outline

- Complex numbers

- Linear algebra
  - Vector analysis
  - Matrix analysis

Real and complex numbers

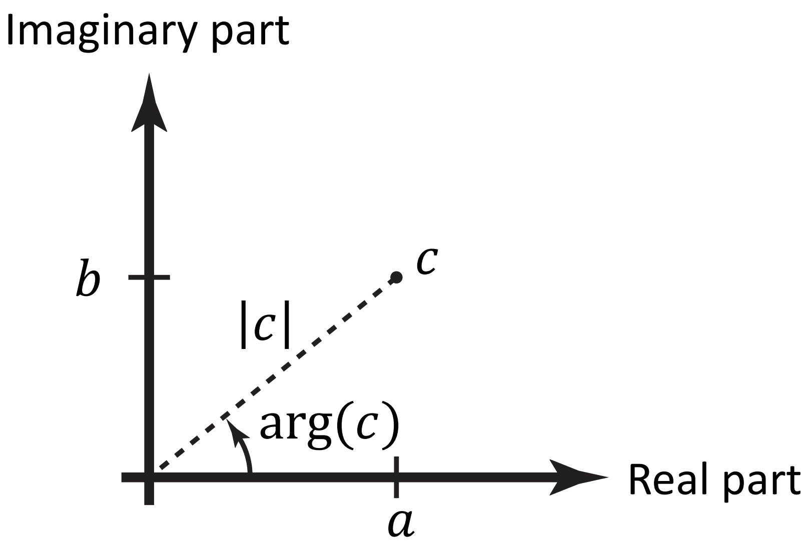


- Real number $a, b \in \mathbb{R}$
- Imaginary unit: $j = \sqrt{-1}$

- Complex number: $c = \underbrace{a}_{\text{Real part}} + j\underbrace{b}_{\text{Imaginary part}} \quad \in \mathbb{C}$

Polar form: $c = |c|e^{j\arg(c)}$

- Magnitude: $|c| = \sqrt{a^2 + b^2}$
- Argument: $\arg(c)$



Calculations with complex numbers

- Two complex numbers: $a + jb, x + jy$

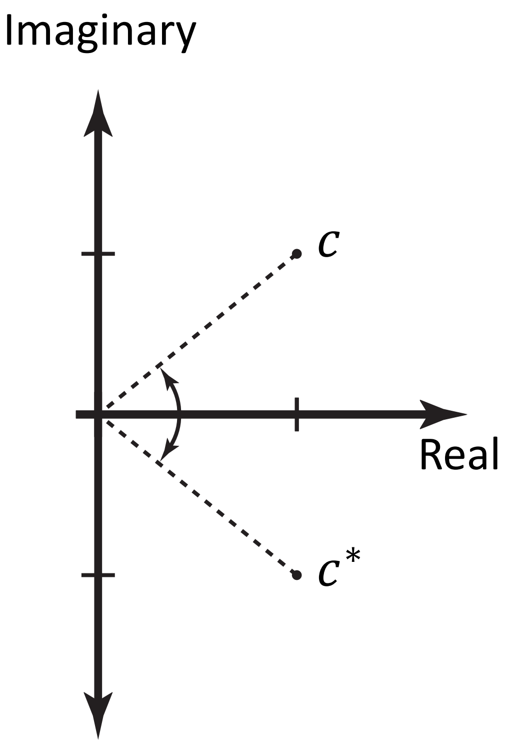
$$(a + jb)(x + jy) = ax + \underset{-1}{\underset{\uparrow}{j^2}}by + jay + jbx$$

- Complex conjugate of $c = a + jb$ is $c^* = a - jb$
  - Note that:

$$cc^* = (a + jb)(a - jb) = a^2 + b^2 + jab - jab = a^2 + b^2$$

$$= |c|^2$$

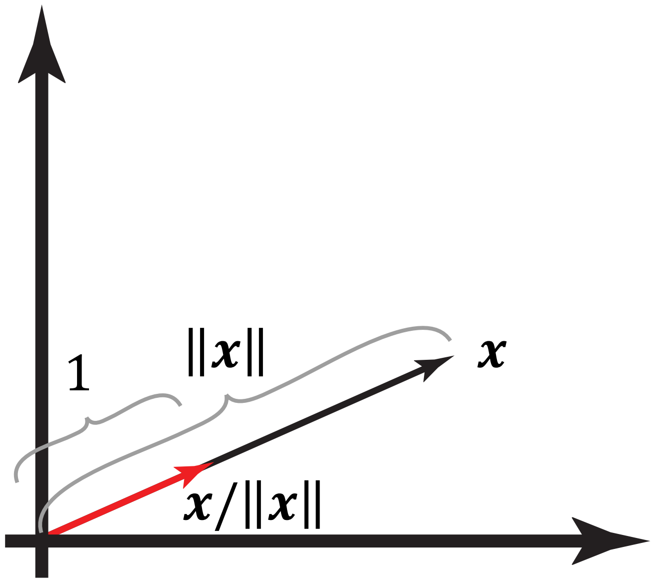
  - Change sign of argument: $c = |c|e^{j\arg(c)}, c^* = |c|e^{-j\arg(c)}$



Complex vectors

- M-dimensional vector with complex entries:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}$$



- Decomposition:

$$\boldsymbol{x} = \|\boldsymbol{x}\| \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

Direction

Length

Norm of vector

$$\|\boldsymbol{x}\| = \sqrt{|x_1|^2 + \cdots + |x_M|^2} = \sqrt{\sum_{m=1}^{M} |x_m|^2}$$

Transpose and conjugate of $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}$

- Conventional transpose:

$$\boldsymbol{x}^T = [x_1 \quad \cdots \quad x_M]$$

- Complex conjugate:

$$\boldsymbol{x}^* = \begin{bmatrix} x_1^* \\ \vdots \\ x_M^* \end{bmatrix}$$

- Conjugate transpose (Hermitian transpose):

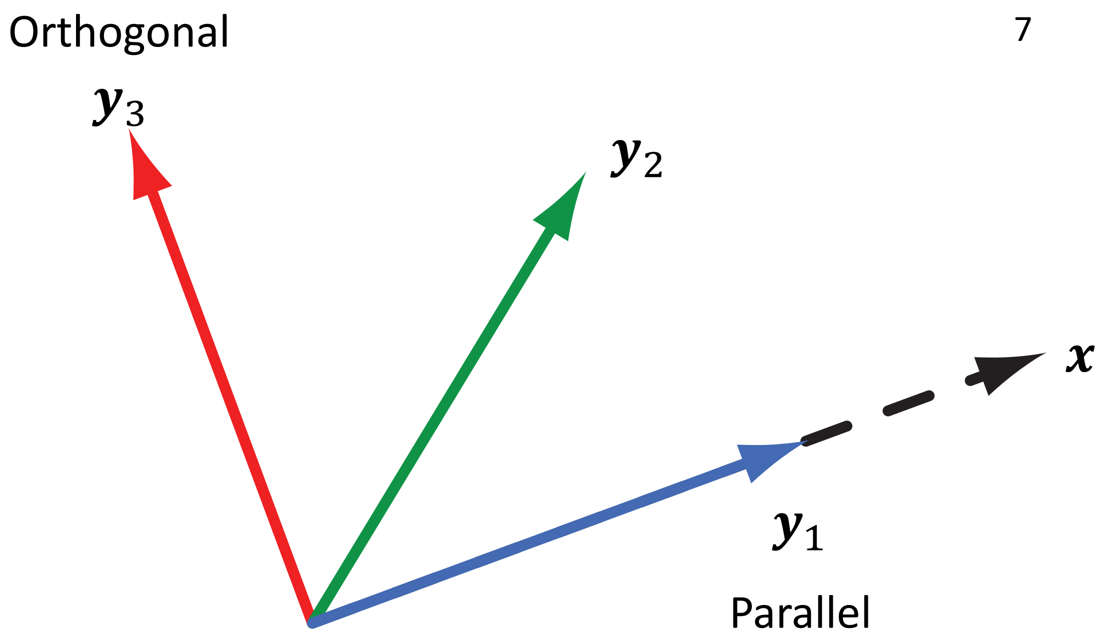
$$\boldsymbol{x}^H = [x_1^* \quad \cdots \quad x_M^*]$$

$$\boldsymbol{x}^H = (\boldsymbol{x}^*)^T$$

Inner product of vectors

- Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{x}^H\boldsymbol{y} = \sum_{m=1}^{M} x_m^* y_m$$



- Cauchy-Schwartz inequality: $|\boldsymbol{x}^H\boldsymbol{y}| \leq \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$
  - Equality if parallel: $\boldsymbol{x} = c\boldsymbol{y}$

$$\boxed{\boldsymbol{x}^H\boldsymbol{x} = \sum_{m=1}^{M} x_m^* x_m = \|\boldsymbol{x}\|^2}$$

Matrices and basic operations

- $M \times K$ matrix:

$$\boldsymbol{G} = \begin{bmatrix} g_{1,1} & \cdots & g_{1,K} \\ \vdots & \ddots & \vdots \\ g_{M,1} & \cdots & g_{M,K} \end{bmatrix}$$

Transpose:

$$\boldsymbol{G}^{\mathrm{T}} = \begin{bmatrix} g_{1,1} & \cdots & g_{M,1} \\ \vdots & \ddots & \vdots \\ g_{1,K} & \cdots & g_{M,K} \end{bmatrix}$$

Conjugate transpose:

$$\boldsymbol{G}^{\mathrm{H}} = \begin{bmatrix} g_{1,1}^* & \cdots & g_{M,1}^* \\ \vdots & \ddots & \vdots \\ g_{1,K}^* & \cdots & g_{M,K}^* \end{bmatrix}$$

Square and diagonal matrices

- Square matrix = Same number of columns and rows

- Diagonal matrix $\boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_M)$:

$$\boldsymbol{D} = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & d_M \end{bmatrix}$$

  - Example: Identity matrix, $\boldsymbol{I}_M = \mathrm{diag}(1, \ldots, 1)$

Eigenvalues and eigenvectors

- Consider an $M \times M$ matrix $\boldsymbol{A}$
  - A non-zero vector $\boldsymbol{u}$ is an *eigenvector* of $\boldsymbol{A}$ if
$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$$
    where the scalar λ is the *eigenvalue* corresponding to $\boldsymbol{u}$

Rank: Number of linearly independent columns

- Finding eigenvalues and eigenvectors
  - Solve $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ to find an eigenvalue
  - Solve $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$ to find an eigenvector (up to a scaling)

Eigenvalue decomposition

- If $\boldsymbol{A}$ has M linearly independent eigenvectors, then

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$

  - $\mathbf{U}$ contains (unit-norm) eigenvectors as columns
  - $\mathbf{D}$ is the diagonal matrix with corresponding eigenvalues

The matrix can be *diagonalized* as: $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{D}$

Eigenvalue decomposition for *Hermitian* matrices

- If $\mathbf{A}$ is *Hermitian* ($\mathbf{A} = \mathbf{A}^H$), then

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^H$$

  - $\mathbf{U} = [\boldsymbol{u}_1 \quad \ldots \quad \boldsymbol{u}_M]$ contains orthogonal eigenvectors
  - $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_M)$ with corresponding eigenvalues

Unitary matrix: $\mathbf{U}\mathbf{U}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}$

Can be *diagonalized* as $\mathbf{U}^H\mathbf{A}\mathbf{U} = \mathbf{D}$

- The matrix can be expressed as

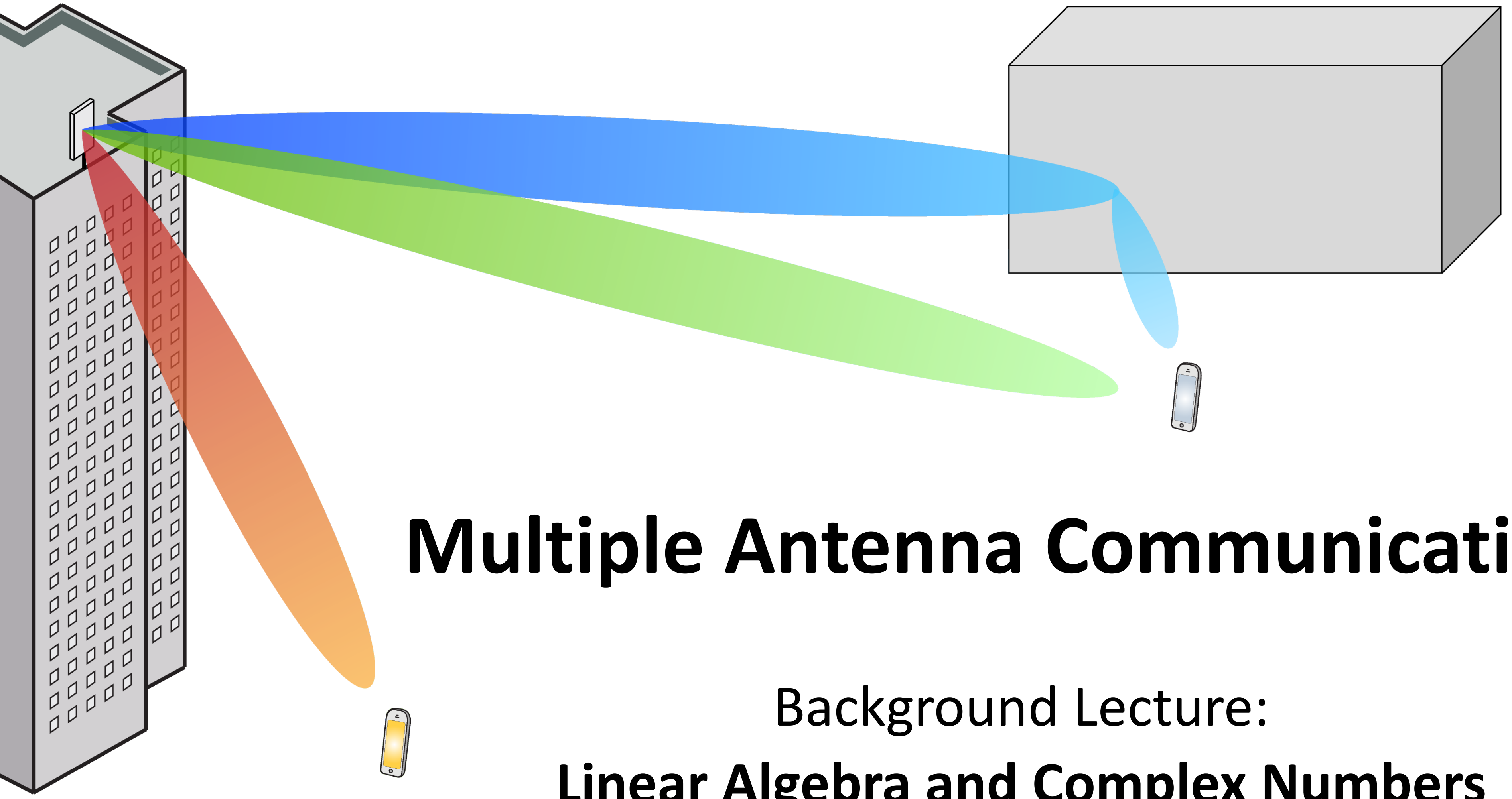
$$\boldsymbol{A} = \sum_{m=1}^{M} d_m \boldsymbol{u}_m \boldsymbol{u}_m^H$$

Summary

- Definition of complex numbers
  - Describe communication signals

- Complex vectors and matrices
  - Describe systems with multiple antennas

- Absolute values, norms, eigenvalues
  - Describe signal strength, in different directions



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