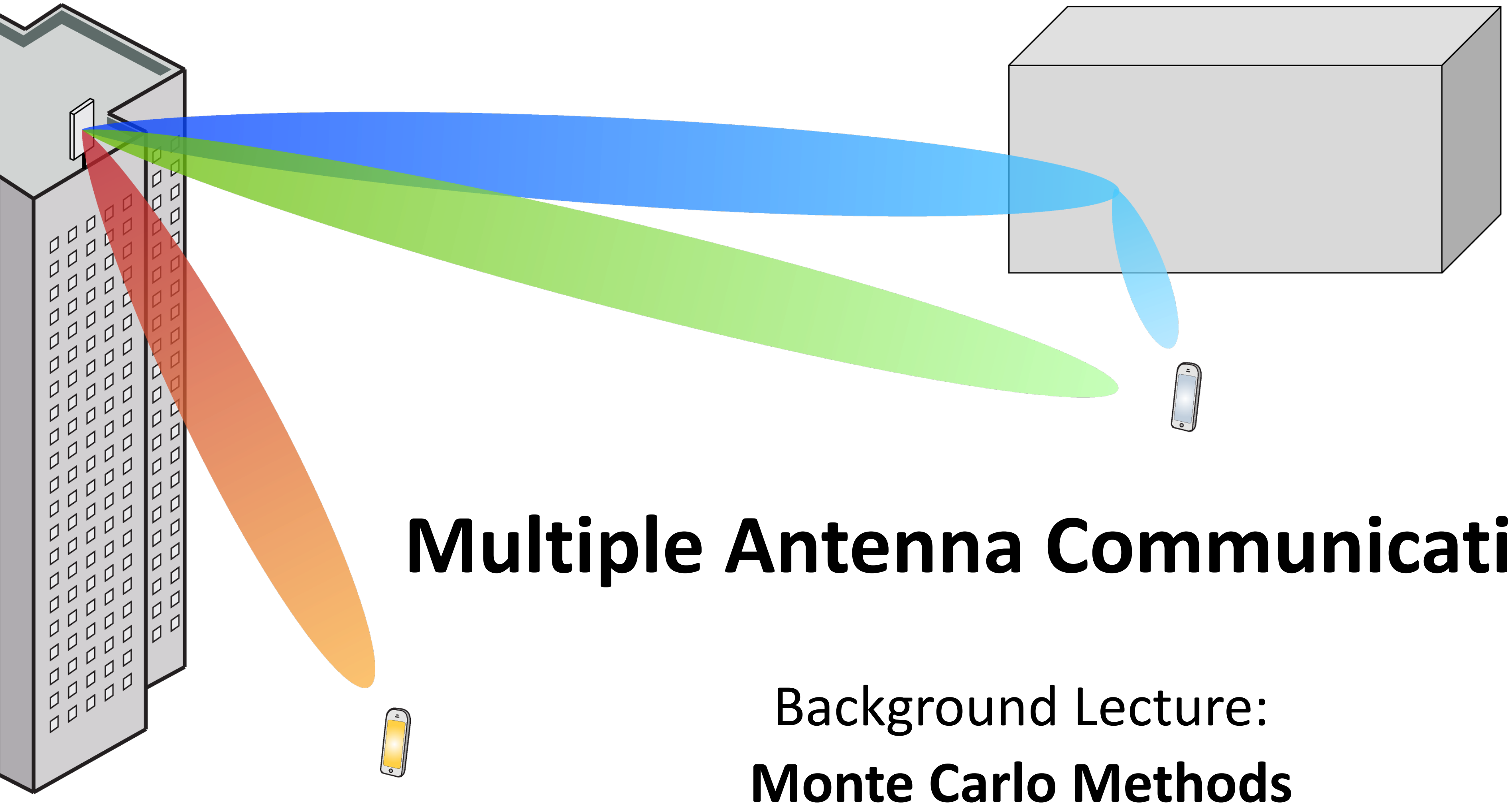


Multiple Antenna Communications

Background Lecture:

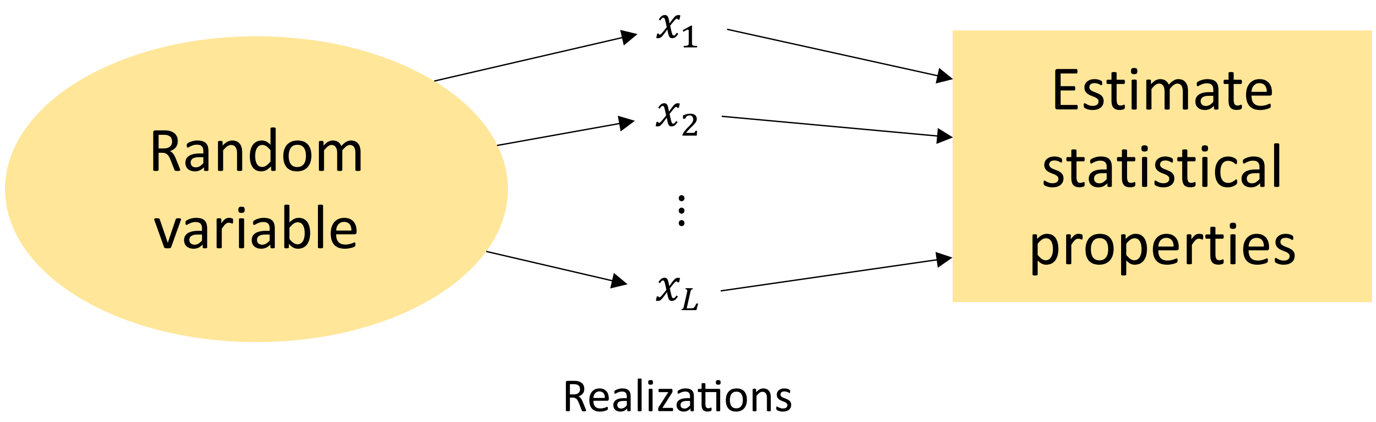
Monte Carlo Methods

Emil Björnson

Outline

- Main principle of the Monte Carlo method
- Estimate mean value of a random variable
- Estimate error probability
- Estimate entire statistical distribution

Statistical inference



Monte Carlo method

Generate many independent random realizations
Use them to estimate deterministic properties

Properties of sample average

- Consider independent realizations $x_1, \dots, x_L$
  - Mean value $\mu = E\{x_i\}$ and finite variance $\sigma^2 = Var\{x_i\}$

- Sample average $\hat{\mu}_L = \frac{1}{L}\sum_{l=1}^{L} x_l$ satisfies:

$$\boxed{\hat{\mu}_L \to \mu \ \text{ as } \ L \to \infty}$$

  - For finite L: $Var\{\hat{\mu}_L\} = \frac{\sigma^2}{L}$

95% confidence interval for μ:

$$\Pr\{\hat{\mu}_L - \delta \leq \mu \leq \hat{\mu}_L + \delta\} \geq 0.95$$

Error tolerance

How large is δ**?**

Central limit theorem:
Gaussian approximation of $\hat{\mu}_L$

Two standard deviations give 95%

$$\delta = 2\sqrt{Var\{\hat{\mu}_L\}} = 2\sigma/\sqrt{L}$$

Estimate mean value: Monte Carlo method

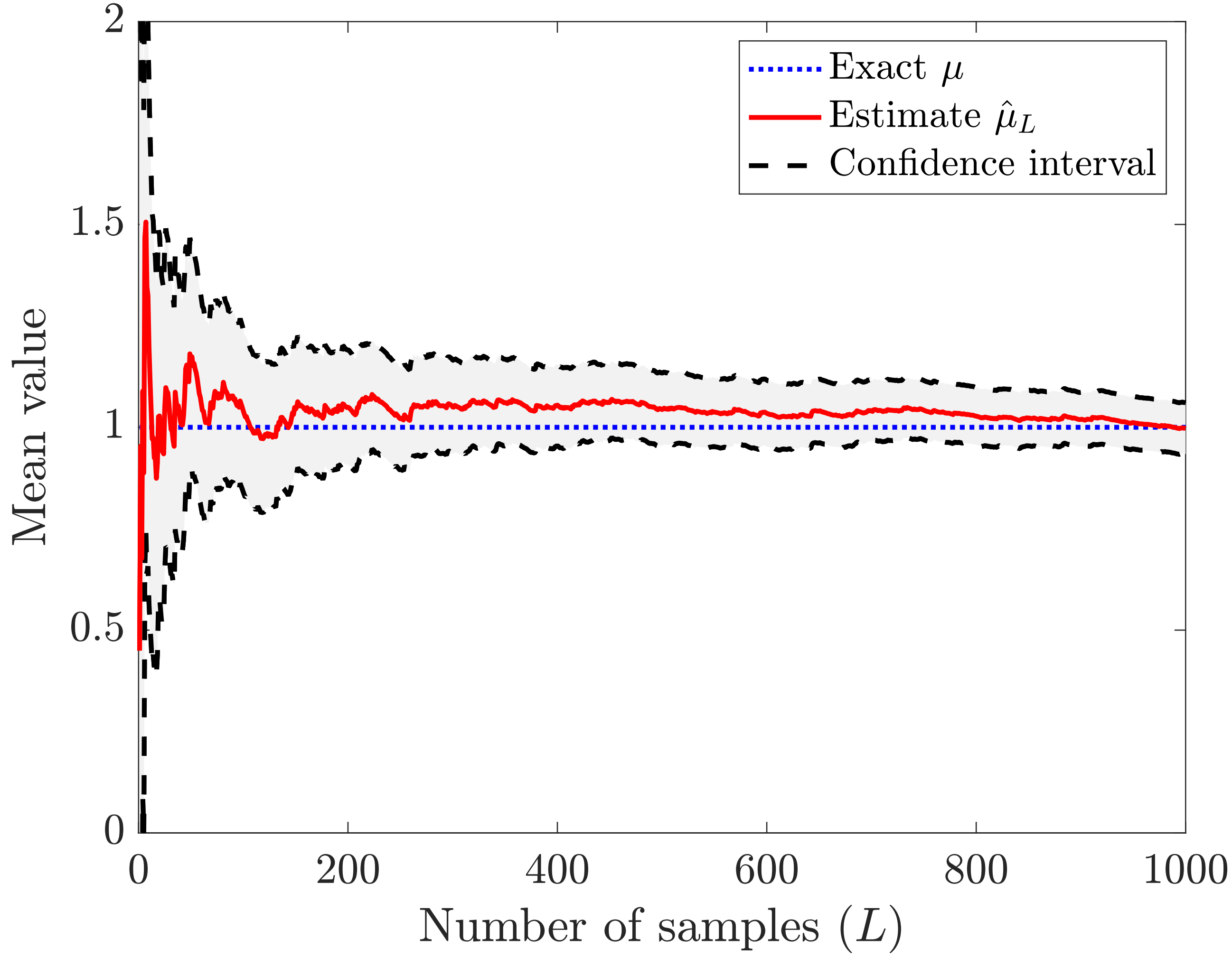
1. Select error tolerance $\delta > 0$
2. Compute required number of realizations from $\delta = 2\sigma/\sqrt{L}$: $\boxed{L = 4\sigma^2/\delta}$
3. Generate L independent realizations $x_1, \ldots, x_L$
4. Compute sample estimate

$$\hat{\mu}_L = \frac{1}{L}\sum_{l=1}^{L} x_l$$

Possible extension

Compute $E\{g(x)\} = \int_{-\infty}^{\infty} g(x) f(x) dx$ as $\frac{1}{L}\sum_{l=1}^{L} g(x_l)$

Example: Exponential distribution, Exp(1)



Confidence interval:

$$\Pr\left\{\hat{\mu}_L - \frac{2\sigma}{\sqrt{L}} \leq \mu \leq \hat{\mu}_L + \frac{2\sigma}{\sqrt{L}}\right\} \geq 0.95$$

Estimate error probability of experiment

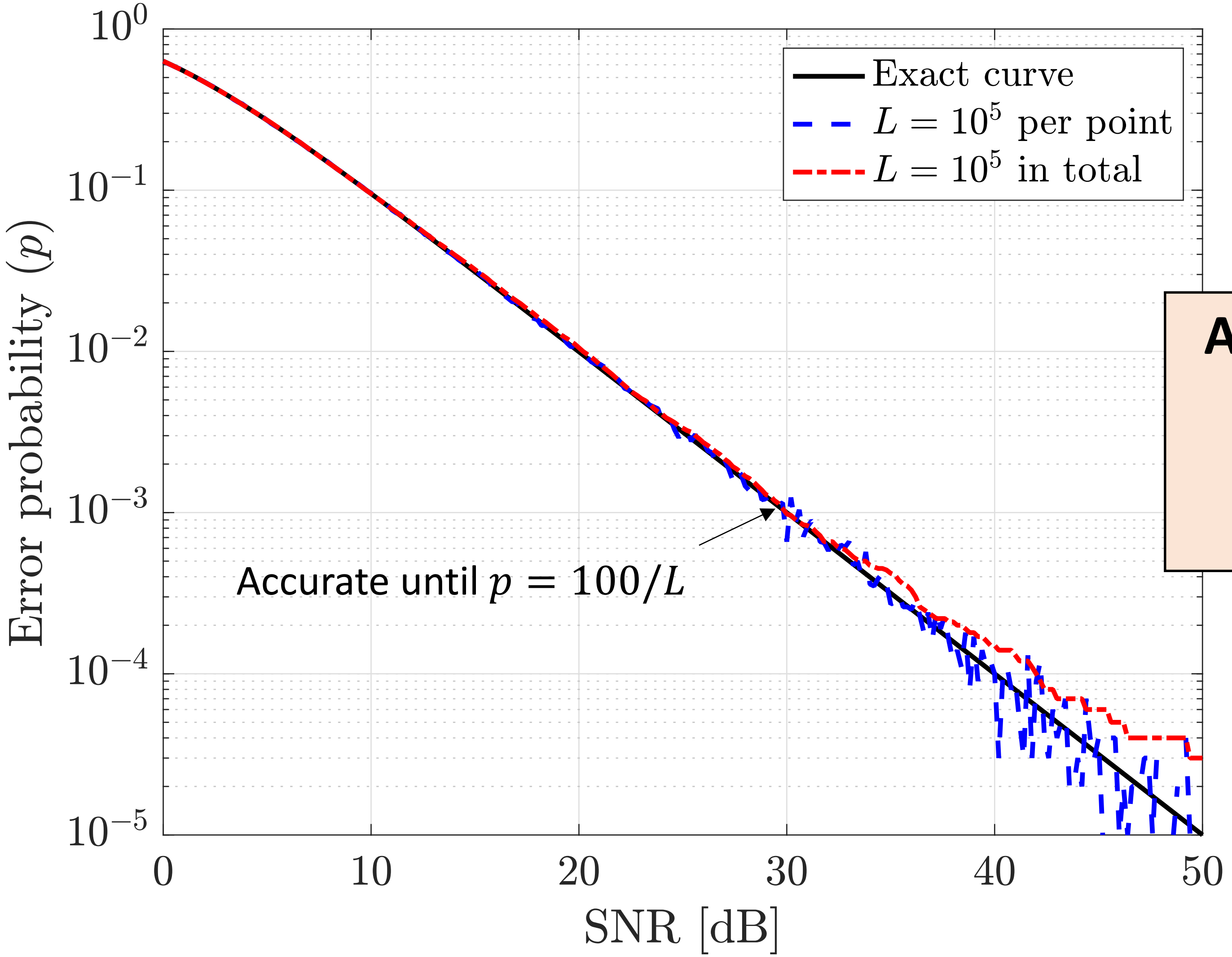
- Each experiment results in *success* or *error*
  - Bernoulli distribution: $\Pr\{x = 1\} = p$ and $\Pr\{x = 0\} = 1 - p$

- Monte Carlo method:
  - Generate L independent realizations: $x_1, \dots, x_L$
  - Compute sample estimate $\hat{p}_L = \frac{1}{L}\sum_{l=1}^{L} x_l$

Error tolerance should be proportional to p

Rule-of-thumb: $L = 100/p_{\text{smallest}}$

Example: $p = 1 - e^{-1/\mathrm{SNR}}$



Avoid systematic errors

L new realizations for each point

Empirical cumulative distribution function

- Estimate the entire statistical distribution: $F(a) = \Pr\{x \leq a\}$

- Monte Carlo method:
  - Generate L independent realizations: $x_1, \ldots, x_L$
  - Compute estimate for each a:

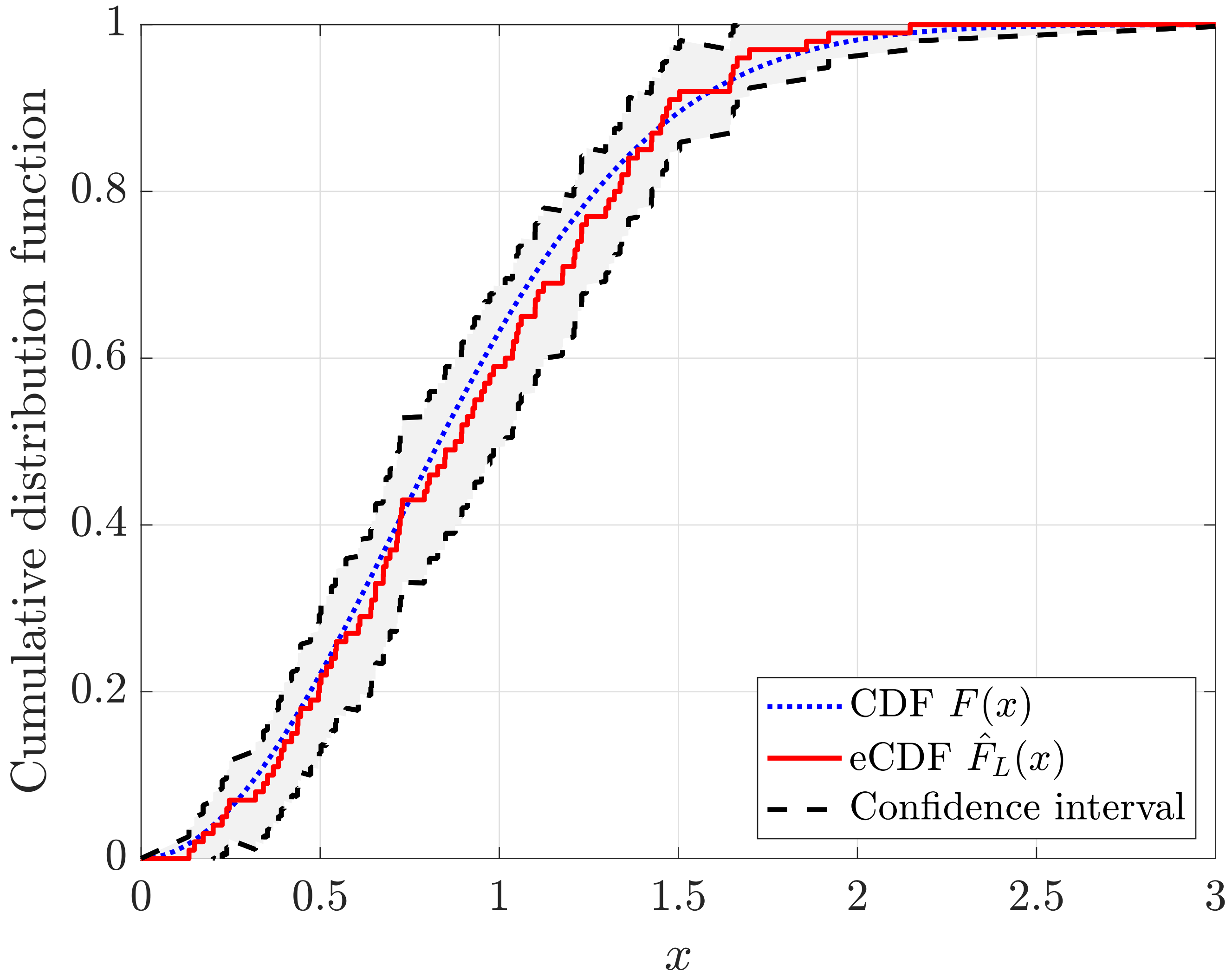
$$\hat{F}_L(a) = \frac{1}{L} \sum_{l=1}^{L} \mathbb{I}_{x_l \leq a}$$

Indicator function: $\begin{cases} 1, & \text{if } x_l \leq a \\ 0, & \text{if } x_l > a \end{cases}$

Variance depends on a:

$$Var\{\hat{F}_L(a)\} = \frac{F(a)\big(1 - F(a)\big)}{L}$$

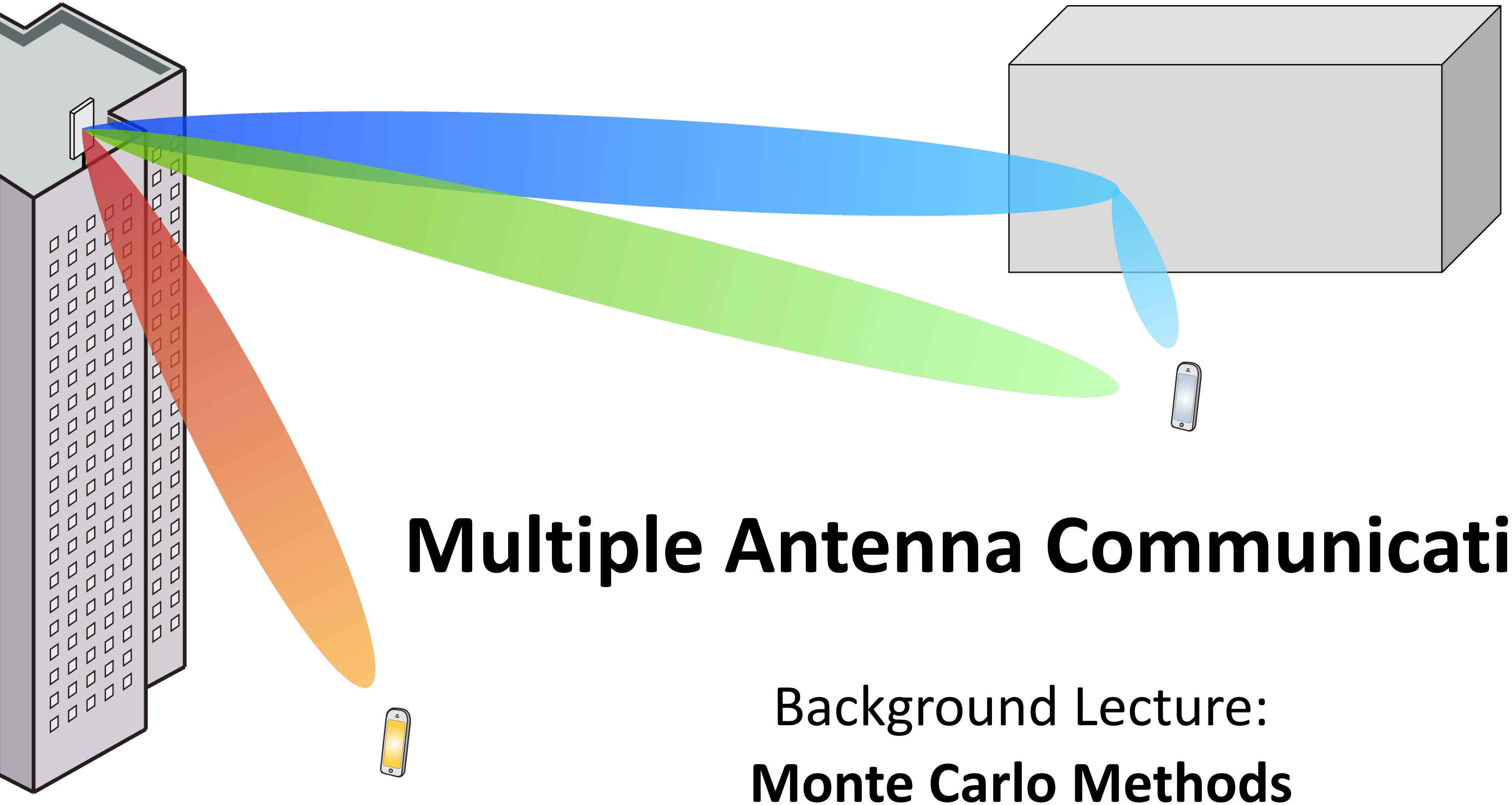
Example: $F(x) = 1 - e^{-x^2}$



Summary

- Monte Carlo method
  - Generate L independent realizations
  - Use sample average (possibly of a function) to estimate statistics
  - Important: Select L to achieve desired accuracy

- Examples:
  - Estimate mean value (e.g., error probability)
  - Empirical cumulative distribution function



Multiple Antenna Communications

Background Lecture:

Monte Carlo Methods

Emil Björnson