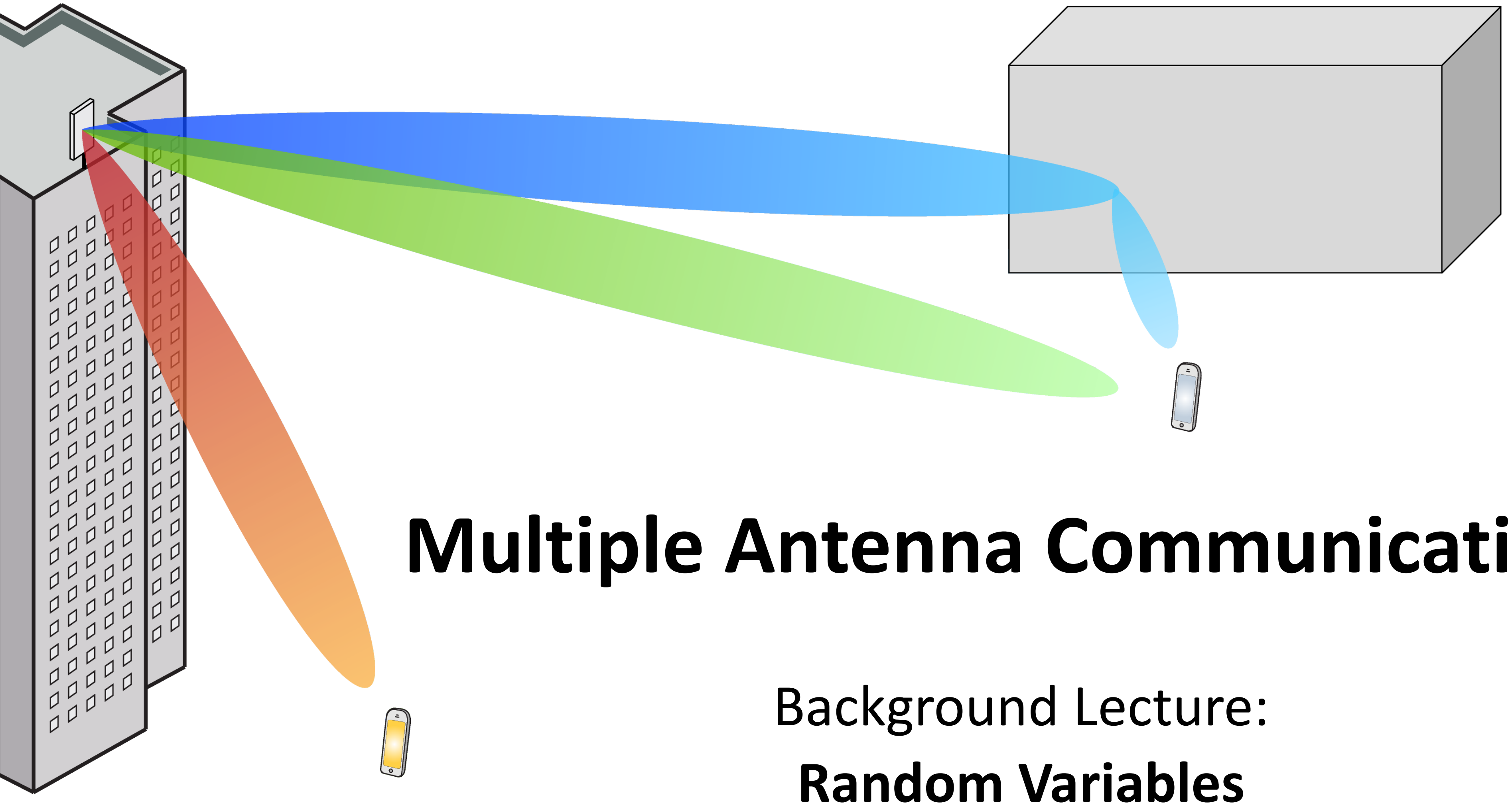


Multiple Antenna Communications

Background Lecture:

Random Variables

Emil Björnson

Outline

- Random variables
  - Real and complex
  - Mean and variance
  - Gaussian distribution
- Random vectors
- Random processes

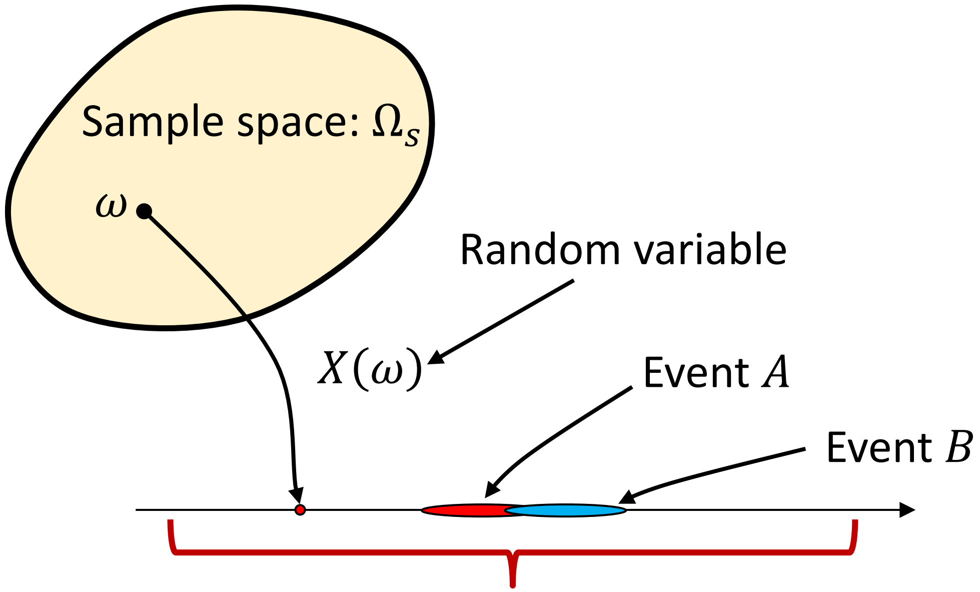
Random variable

- Total probability:

$$\Pr\{\Omega\} = 1$$

- Probability of event A:

$$\Pr\{A\} \in [0,1]$$



Measureable sample space:
$\Omega = \{X(\omega)\text{: for some } \omega \in \Omega_s\}$

Example: Thermal noise

ω = Location of all electrons

$X(\omega)$ = Measured voltage over a resistor

Example: Wireless channel

ω = Locations of all objects

$X(\omega)$ = Measured impulse response

Probability density for *real* variables

- Cumulative distribution function (CDF):

$$F_X(x) = \Pr\{\underbrace{X \le x}_{\text{An event}}\} \in [0,1]$$

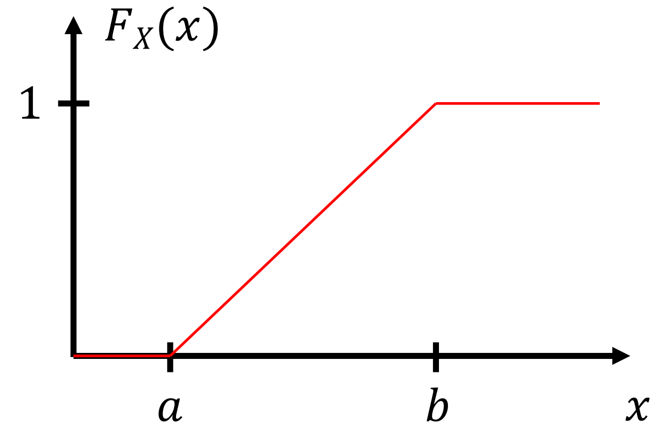
- Probability density function (PDF):

$$f_X(x) = \frac{d}{dx} F_X(x)$$

  - Properties:

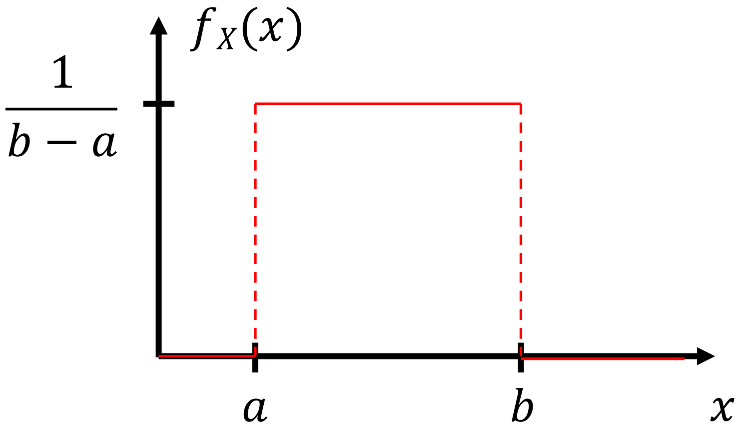
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\Pr\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)dx$$



Non-decreasing, non-zero

Uniform distribution



Non-zero, area 1

Uniform distribution:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x < b \\ 0, & \text{elsewhere} \end{cases}$$

Mean and variance

Mean value (expectation):

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$E\{X\} = \int_a^b x \frac{1}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

Quadratic mean (power):

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

$$E\{X^2\} = \int_a^b x^2 \frac{1}{b-a} dx = \frac{b^3 - a^3}{3(b-a)}$$

Variance:

$$\begin{aligned} Var\{X\} &= E\{(X - E\{X\})^2\} \\ &= E\{X^2\} - (E\{X\})^2 \end{aligned}$$

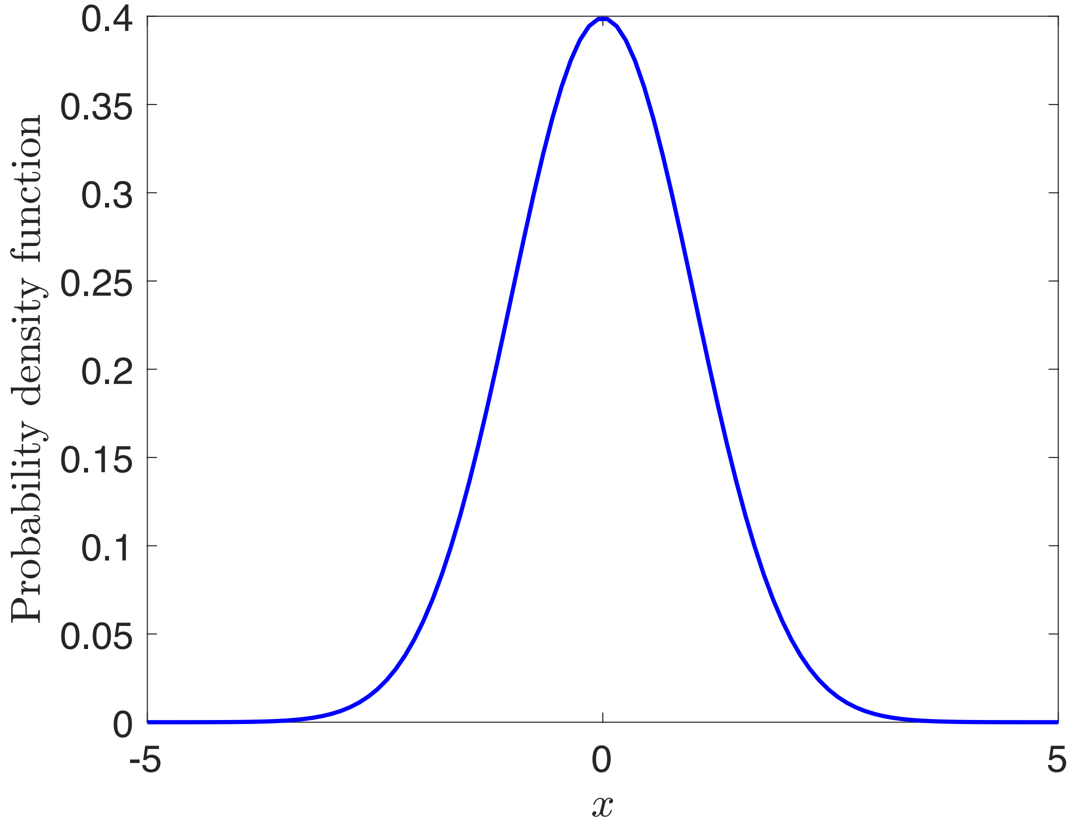
$$Var\{X\} = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Gaussian distribution

- x is zero-mean Gaussian distributed, $x \sim N(0, \sigma^2)$
  - PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

  - Properties: $E\{x\} = 0, E\{x^2\} = \sigma^2$



Probability density for *complex* variables

- Probability density function (PDF): $f_X(x)$ for $x \in \mathbb{C}$
  - Probability of event $A \subseteq \mathbb{C}$: $\Pr\{x \in A\} = \int_A f_X(x)\,dx$
  - Total probability: $\int_{\mathbb{C}} f_X(x)\,dx = 1$

- Mean: $E\{X\} = \int_{\mathbb{C}} x f_X(x)\,dx$
- Quadratic mean: $E\{|X|^2\} = \int_{\mathbb{C}} |x|^2 f_X(x)\,dx$
- Variance: $Var\{X\} = E\{|X - E\{X\}|^2\} = E\{|X|^2\} - |E\{X\}|^2$

New variable: $Y = cX$

$E\{Y\} = cE\{X\}$ $\quad$ $Var\{Y\} = |c|^2 Var\{X\}$

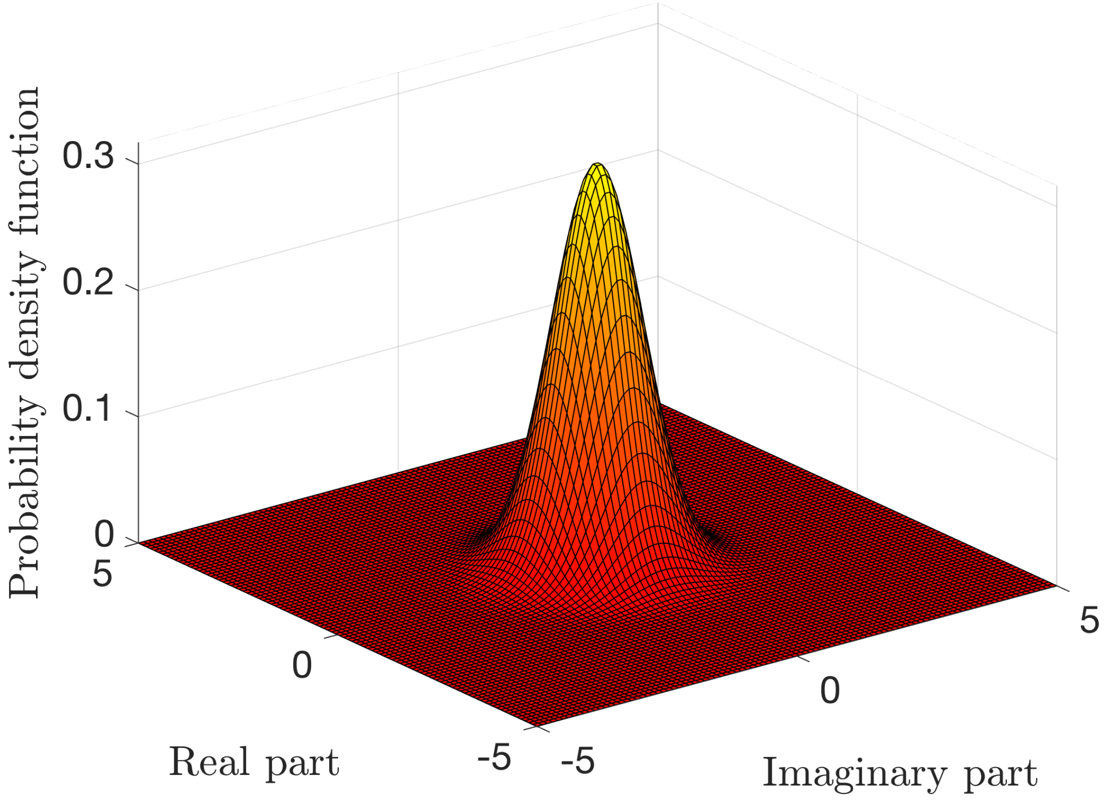
Complex Gaussian Distribution

- Consider independent $x_R, x_I \sim N(0, \sigma^2/2)$
  - $x = x_R + jx_I$ is circularly symmetric complex Gaussian distributed

Circular symmetry:
$f_X(xe^{j\psi}) = f_X(x)$

$x \sim CN(0, \sigma^2)$, with

$$f_X(x) = \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{x_R^2}{\sigma^2}} \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{x_I^2}{\sigma^2}}$$
$$= \frac{1}{\pi\sigma^2} e^{-\frac{|x|^2}{\sigma^2}}$$



Properties: $E\{x\} = 0,\ Var\{x\} = \sigma^2$

Multivariate distribution

- Vector with M random variables: $$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}$$

- Mean value: $$E\{\boldsymbol{x}\} = \begin{bmatrix} E\{x_1\} \\ \vdots \\ E\{x_M\} \end{bmatrix}$$

- Covariance matrix: $Cov\{\boldsymbol{x}\} = E\{(\boldsymbol{x} - E\{\boldsymbol{x}\})(\boldsymbol{x} - E\{\boldsymbol{x}\})^H\}$
    - Diagonal, element m: $Var\{x_m\}$
    - Element (m, n): $E\{(x_m - E\{x_m\})(x_n - E\{x_n\})^*\}$

Complex Gaussian vectors

- Consider independent $x_1, \ldots, x_M \sim CN(0, \sigma^2)$
  - Vector notation:

Covariance matrix

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix} \sim CN(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_M)$$

Mean

- New vector: $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$
  - Mean: $E\{\boldsymbol{y}\} = \boldsymbol{A}E\{\boldsymbol{x}\} = \boldsymbol{0}$
  - Covariance: $Cov\{\boldsymbol{y}\} = \boldsymbol{A}Cov\{\boldsymbol{x}\}\boldsymbol{A}^H = \sigma^2\boldsymbol{A}\boldsymbol{A}^H$

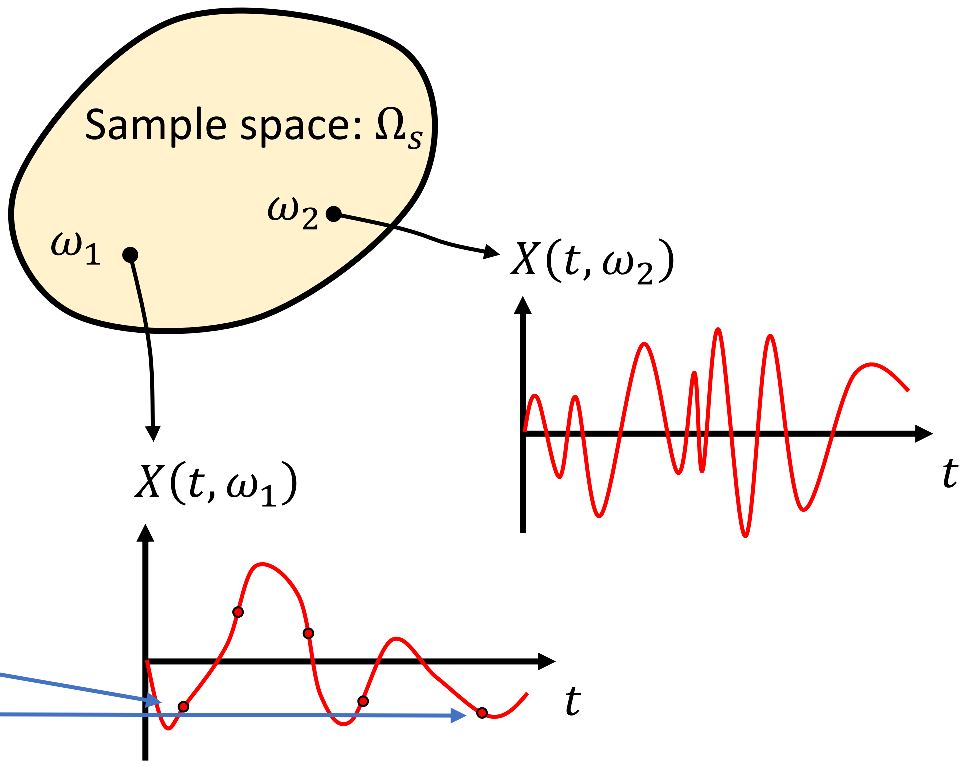
$$\boxed{\boldsymbol{y} \sim CN(\boldsymbol{0}, \sigma^2 \boldsymbol{A}\boldsymbol{A}^H)}$$

Random process

- Random continuous-time function

- Create random vector by sampling

$$\boldsymbol{x} = \begin{bmatrix} x(t_1) \\ \vdots \\ x(t_M) \end{bmatrix}$$



Mean and covariance/correlation can be defined for arbitrary times

Wide-sense stationarity:

$E\{x(t)\}$ = constant

$E\{x(t_1)x^*(t_2)\}$ = depends only on $t_2 - t_1$

White Gaussian process:

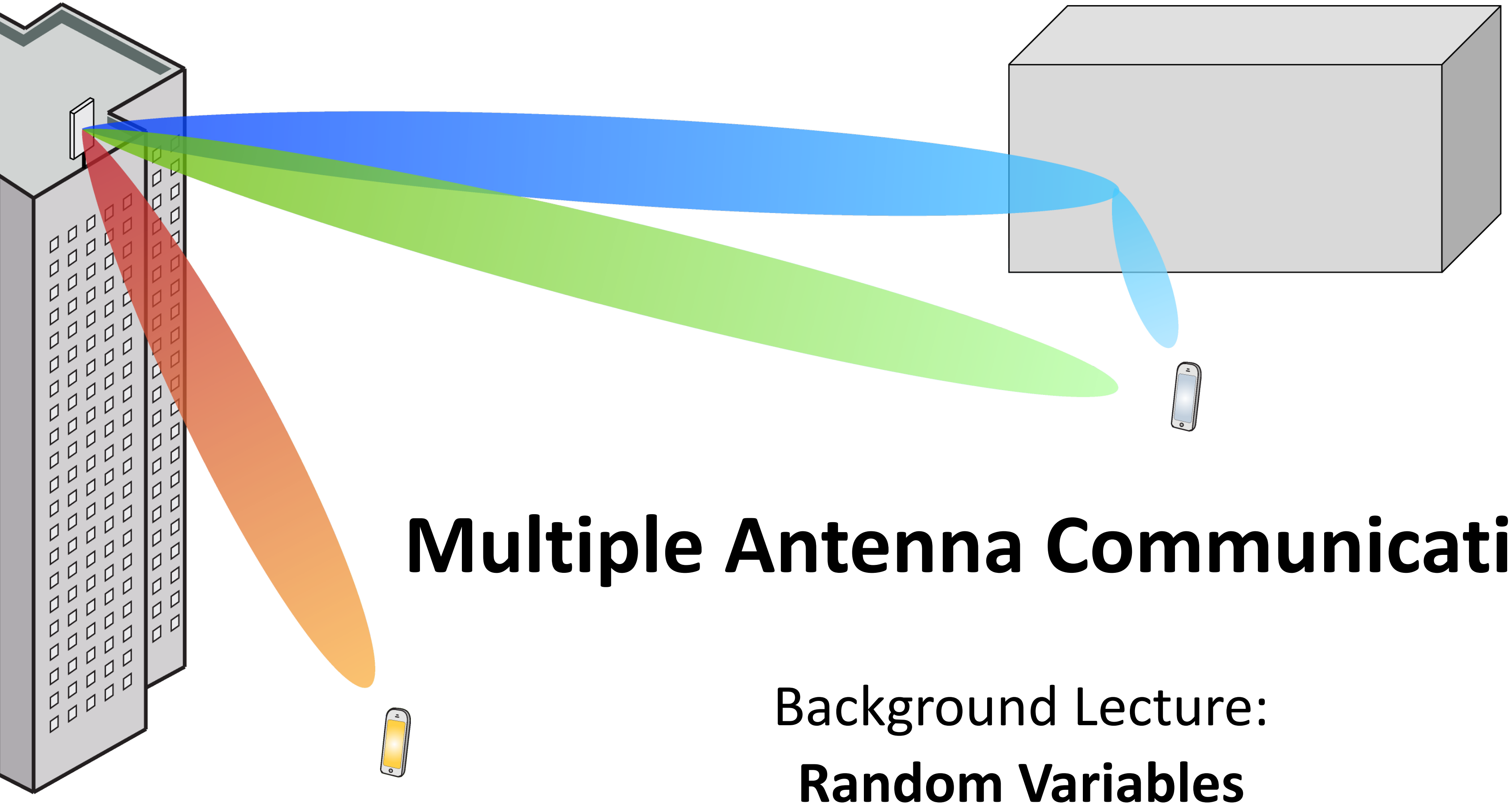
$x(t)$ has Gaussian distribution

$E\{x(t_1)x^*(t_2)\} = \delta(t_2 - t_1)$

Summary

- Scalars, vectors, and functions can be random

- Useful to model
  - Signals with random data
  - Complicated communication channels
  - Thermal noise



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Random Variables

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