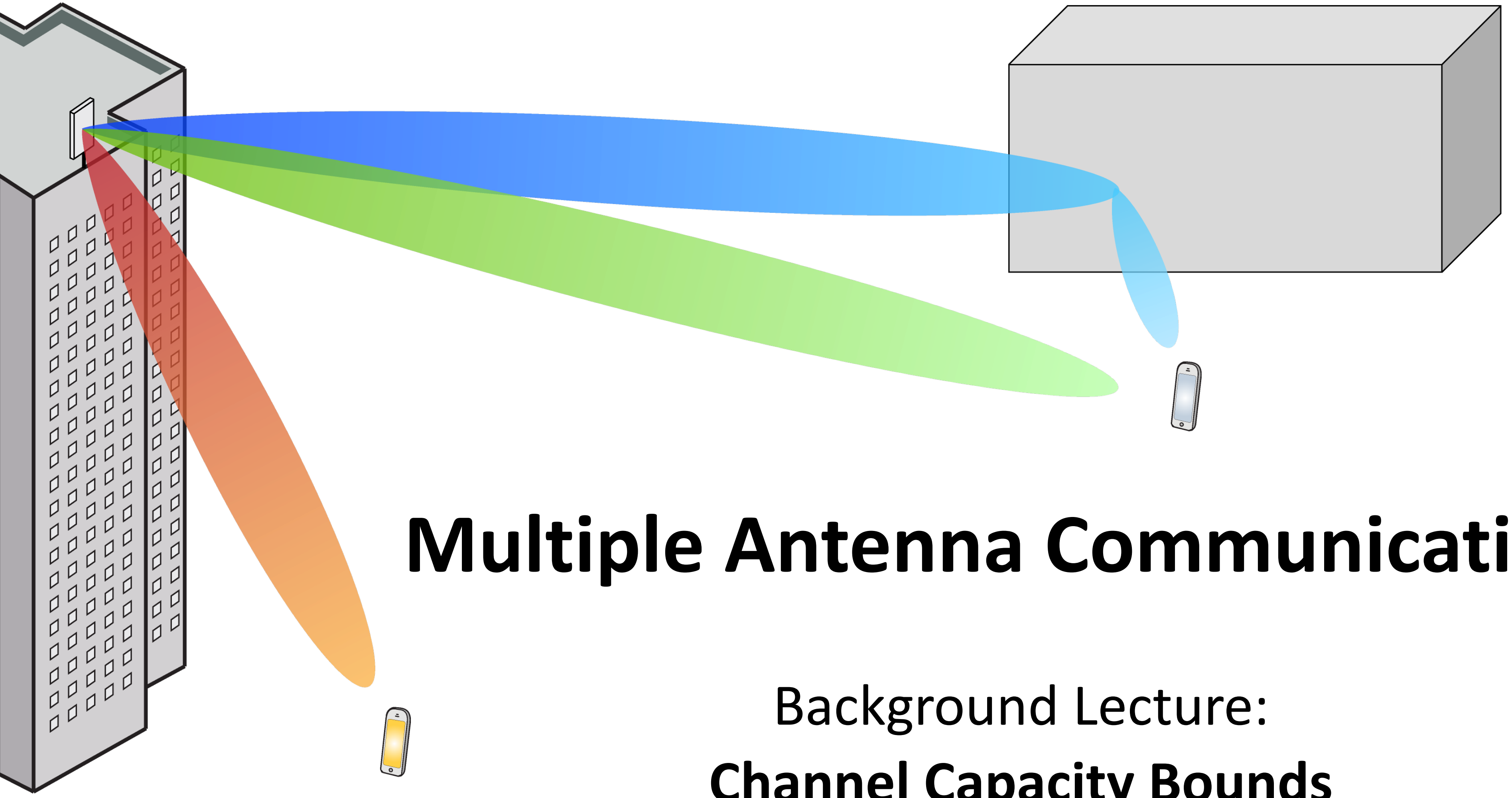


Multiple Antenna Communications

Background Lecture:

Channel Capacity Bounds

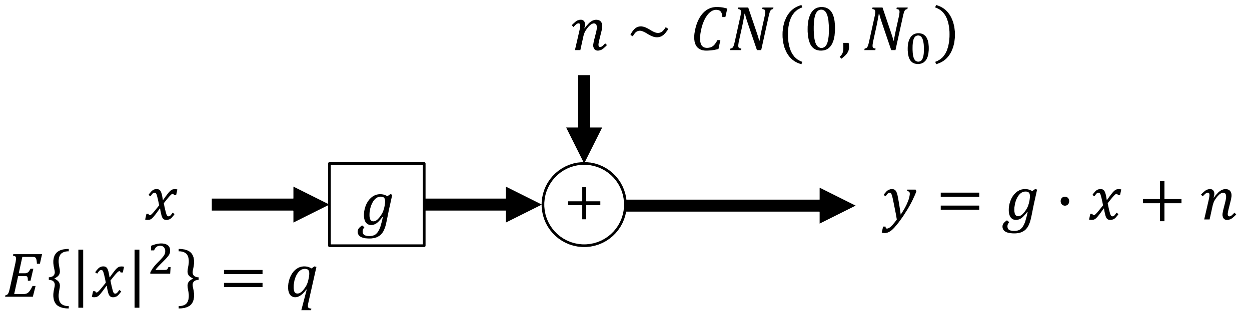
Emil Björnson

Outline

- Hard to compute channel capacity
  - With non-Gaussian noise
  - With incomplete channel knowledge

- Capacity lower bounds
  - Technical assumptions
  - Proof techniques

Recall: Capacity of complex discrete memoryless channel



Channel definition

$$C = \max_{f_X(x)} h(y) - h(y|x)$$
$$= \max_{f_X(x)} h(x) - h(x|y)$$

Maximum entropy

$$h(y) \leq \log_2(\pi e \mathrm{Var}\{y\})$$

Equality if y is complex Gaussian

- In this case:

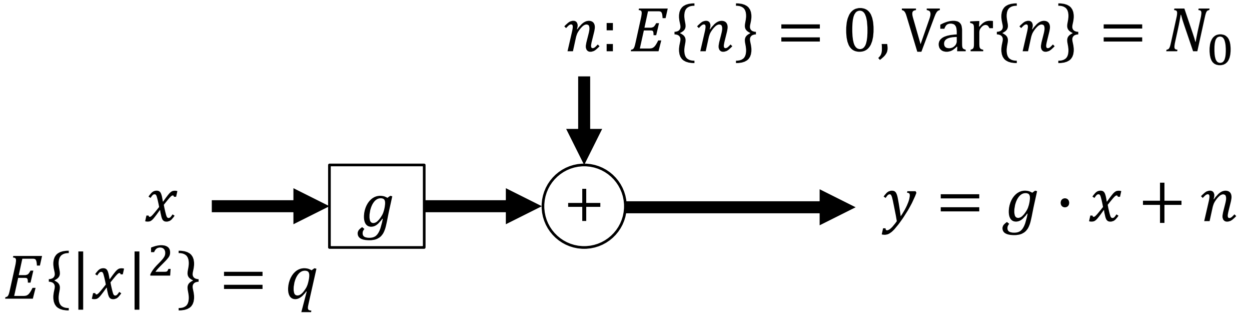
$$h(y|x) = [y - gx = n \sim CN(0, N_0)\ ] = \log_2(\pi e N_0)$$

- Entropy $h(y)$ maximized by $x \sim CN(0, q)$ since

$$h(y) \leq [y = gx + n \sim CN(0, q|g|^2 + N_0)] = \log_2(\pi e(q|g|^2 + N_0))$$

$$\boxed{C = h(y) - h(y|x) = \log_2\left(1 + \frac{q|g|^2}{N_0}\right)}$$

What if the noise is *not* Gaussian and *only* uncorrelated?



Uncorrelated noise
$E\{x^* n\} = 0$

Preparation: *Linear* estimate of x given y

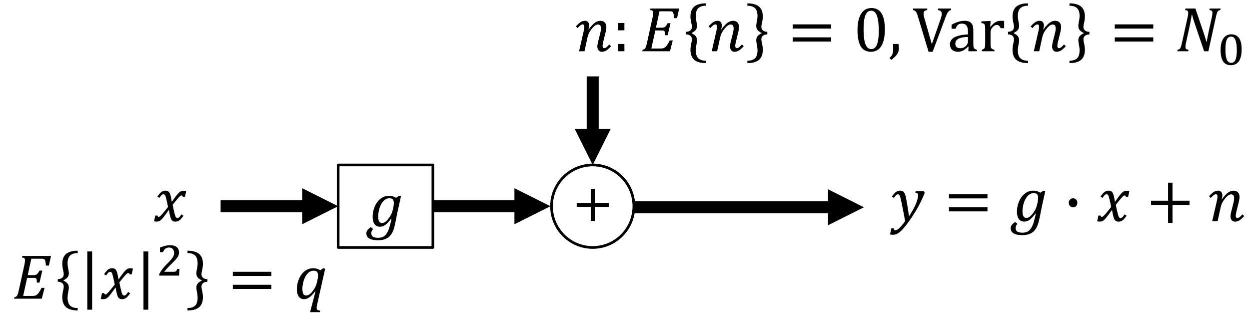
$$\hat{x} = cy$$

- MSE $E\{|x - \hat{x}|^2\} = |1 - cg|^2 q + |c|^2 N_0$ is minimized by $c = \frac{qg^*}{q|g|^2 + N_0}$
- Minimum MSE: $E\{|\hat{x} - x|^2\} = \frac{qN_0}{q|g|^2 + N_0}$

The MMSE estimator must satisfy

$$E\{|x - \hat{x}|^2\} \leq \frac{qN_0}{q|g|^2 + N_0}$$

Bound on capacity with arbitrary uncorrelated noise



Capacity definition

$$C = \max_{f_X(x)} h(x) - h(x|y)$$

Lower bound based on two steps:

1. Assume $x \sim CN(0, q)$: $\quad C \geq \log_2(\pi e q) - h(x|y)$

2. Bound the conditional entropy:

$$h(x|y) = h(x|\hat{x}) \leq h(x - \hat{x}) \leq \log_2(\pi e \mathrm{E}\{|x - \hat{x}|^2\}) = \log_2\left(\pi e \frac{qN_0}{q|g|^2 + N_0}\right)$$

$$C \geq \log_2(\pi e q) - \log_2\left(\pi e \frac{qN_0}{q|g|^2 + N_0}\right) = \log_2\left(1 + \frac{q|g|^2}{N_0}\right)$$

Gaussian versus arbitrary noise

Independent Gaussian noise

$$C = \log_2\left(1 + \frac{q|g|^2}{N_0}\right)$$

Arbitrary uncorrelated noise

$$C \geq \log_2\left(1 + \frac{q|g|^2}{N_0}\right)$$

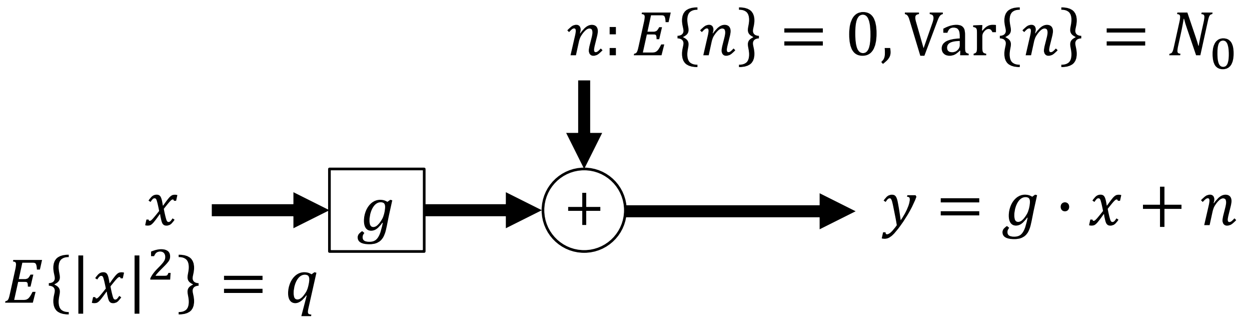
- Worst-case *uncorrelated* noise: Independent and Gaussian distributed
  - Achieve bound by Gaussian signaling: $x \sim CN(0, q)$

“Treating interference as noise”

If n contains Gaussian noise and non-Gaussian interference

No approximations: Use suboptimal transmitter and receiver

Fast-fading channel without channel knowledge



Uncorrelated noise

$$E\{x^* n\} = 0$$

$$E\{g^* x^* n\} = 0$$

New realization of g for every transmission: Independent of x

Rewrite received signal:

$$y = \underbrace{E\{g\}}_{\text{Effective channel}} x + \underbrace{(g - E\{g\})x + n}_{\text{Effective noise}}$$

Uncorrelated effective noise

$$E\{x^*((g - E\{g\})x + n)\} = (E\{g\} - E\{g\})E\{|x|^2\} + E\{x^* n\} = 0$$

Capacity lower bound

$$C \geq \log_2\left(1 + \frac{q|E\{g\}|^2}{q\mathrm{Var}\{g\} + N_0}\right)$$

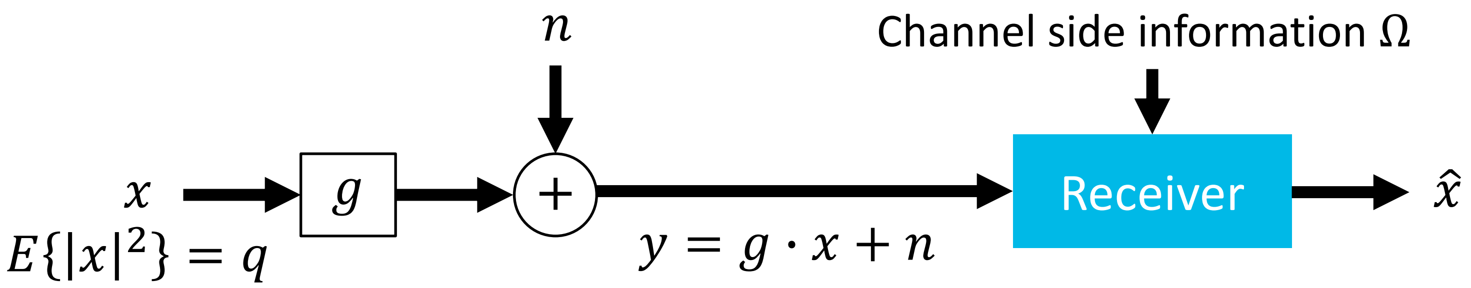
Will the channel have non-zero mean?

Capacity lower bound

$$C \geq \log_2\left(1 + \frac{q|E\{g\}|^2}{q\mathrm{Var}\{g\} + N_0}\right)$$

- Most random channel has $E\{g\} = 0$
  - But $E\{|g|\} \neq 0$
- Use-and-then-forget technique
  - Receive a signal $y' = g'x + n'$
  - Use the channel g' to preprocess: $y = \frac{(g')^*}{|g'|} y' = \underbrace{|g'|}_{g} x + \underbrace{\frac{(g')^* n'}{|g'|}}_{n}$
  - "Forget" g' and apply the capacity lower bound

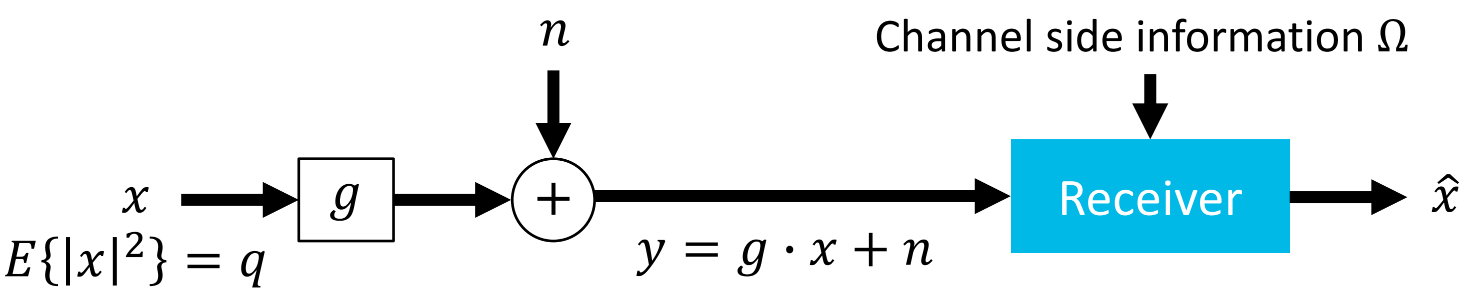
Fast-fading with partial channel knowledge



Assumptions:

- New realization of g for every transmission: Ω might be correlated with g
- Signal x is independent of g and Ω
- Zero-mean noise: $E\{n|\Omega\} = 0$
- Uncorrelated noise: $E\{x^* n|\Omega\} = 0$ and $E\{g^* x^* n|\Omega\} = 0$

Computing a capacity lower bound with side information



Capacity definition

$$C = \max_{f_X(x)} h(x) - h(x|y,\Omega)$$

Lower bound based on two steps:

1. Assume $x \sim CN(0,q)$: $\quad C \geq \log_2(\pi e q) - h(x|y,\Omega)$

2. Bound the conditional entropy:

$$h(x|y,\Omega) = h(x|\hat{x}(\Omega),\Omega) \leq h(x - \hat{x}(\Omega)|\Omega) \leq E\{\log_2(\pi e \mathrm{E}\{|x - \hat{x}(\Omega)|^2|\Omega\})\}$$

$$= E\left\{\log_2\left(\pi e \frac{q(\text{Effective noise variance})}{q|\text{Effective channel}|^2 + (\text{Effective noise variance})}\right)\right\}$$

$$= E\left\{\log_2\left(\pi e \frac{q(q\mathrm{Var}\{g|\Omega\} + \mathrm{Var}\{n|\Omega\})}{q|E\{g|\Omega\}|^2 + q\mathrm{Var}\{g|\Omega\} + \mathrm{Var}\{n|\Omega\}}\right)\right\}$$

Finalizing the *capacity bound with side information*

$$C \geq \log_2(\pi e q) - E\left\{\log_2\left(\pi e \frac{q(q\mathrm{Var}\{g|\Omega\} + \mathrm{Var}\{n|\Omega\})}{q|E\{g|\Omega\}|^2 + q\mathrm{Var}\{g|\Omega\} + \mathrm{Var}\{n|\Omega\}}\right)\right\}$$

$$= E\left\{\log_2\left(1 + \frac{q|E\{g|\Omega\}|^2}{q\mathrm{Var}\{g|\Omega\} + \mathrm{Var}\{n|\Omega\}}\right)\right\}$$

Bound achieved by $x \sim CN(0, q)$

Example:
Perfect knowledge, $\Omega = g$
Uncorrelated noise, $\mathrm{Var}\{n\} = N_0$

$$C \geq E\left\{\log_2\left(1 + \frac{q|g|^2}{N_0}\right)\right\}$$

Example:
No knowledge, $\Omega = \emptyset$

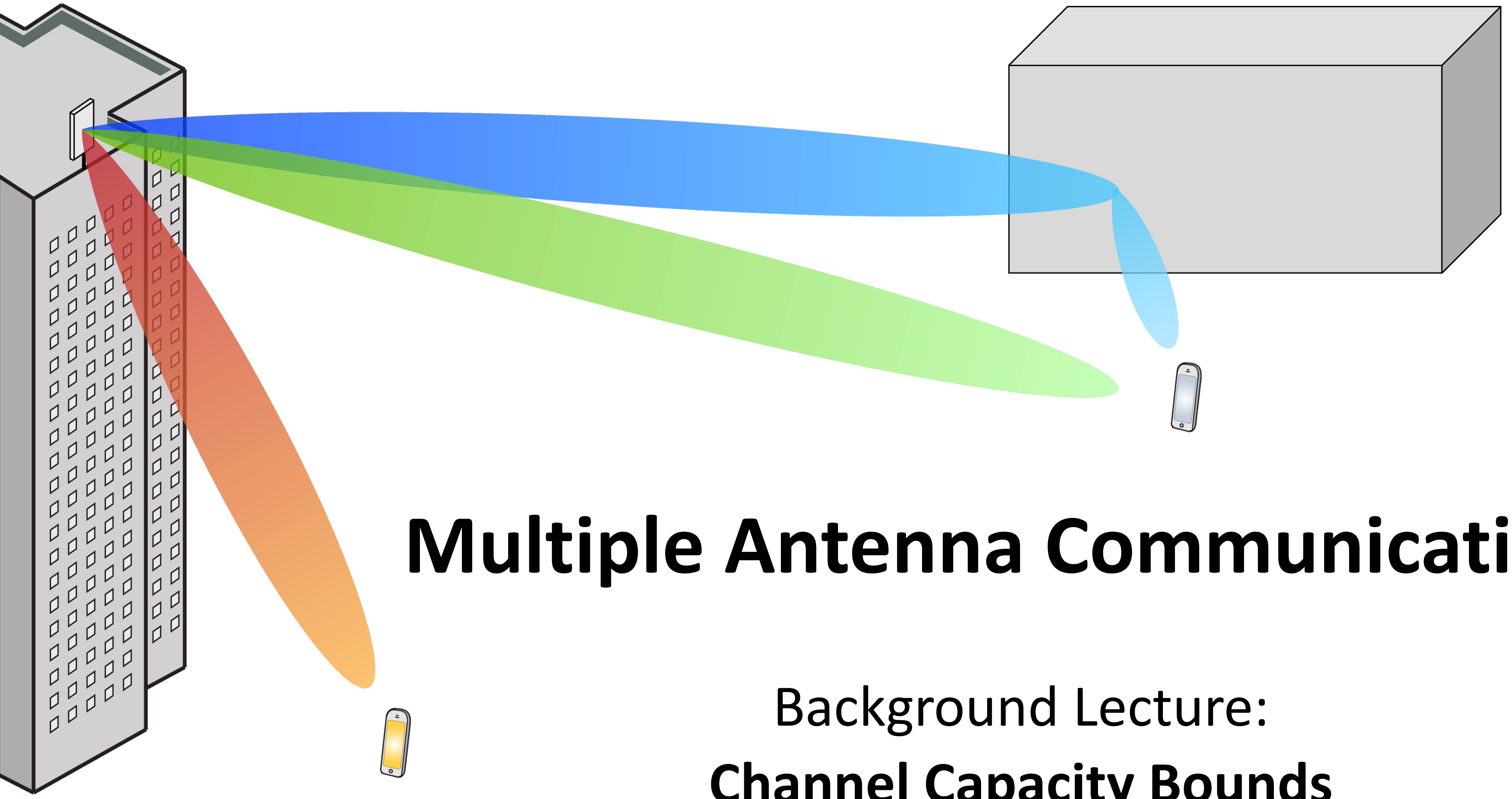
$$C \geq \log_2\left(1 + \frac{q|E\{g\}|^2}{q\mathrm{Var}\{g\} + \mathrm{Var}\{n\}}\right)$$

The general bound is particularly useful when we have partial information about g and n

Summary

- Achievable rate / spectral efficiency
  - Lower bound on channel capacity
  - Handle uncorrelated noise and interference
  - Handle partial channel knowledge

- Technical conditions must be satisfied
  - Not always the case for the capacity bound with side information



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