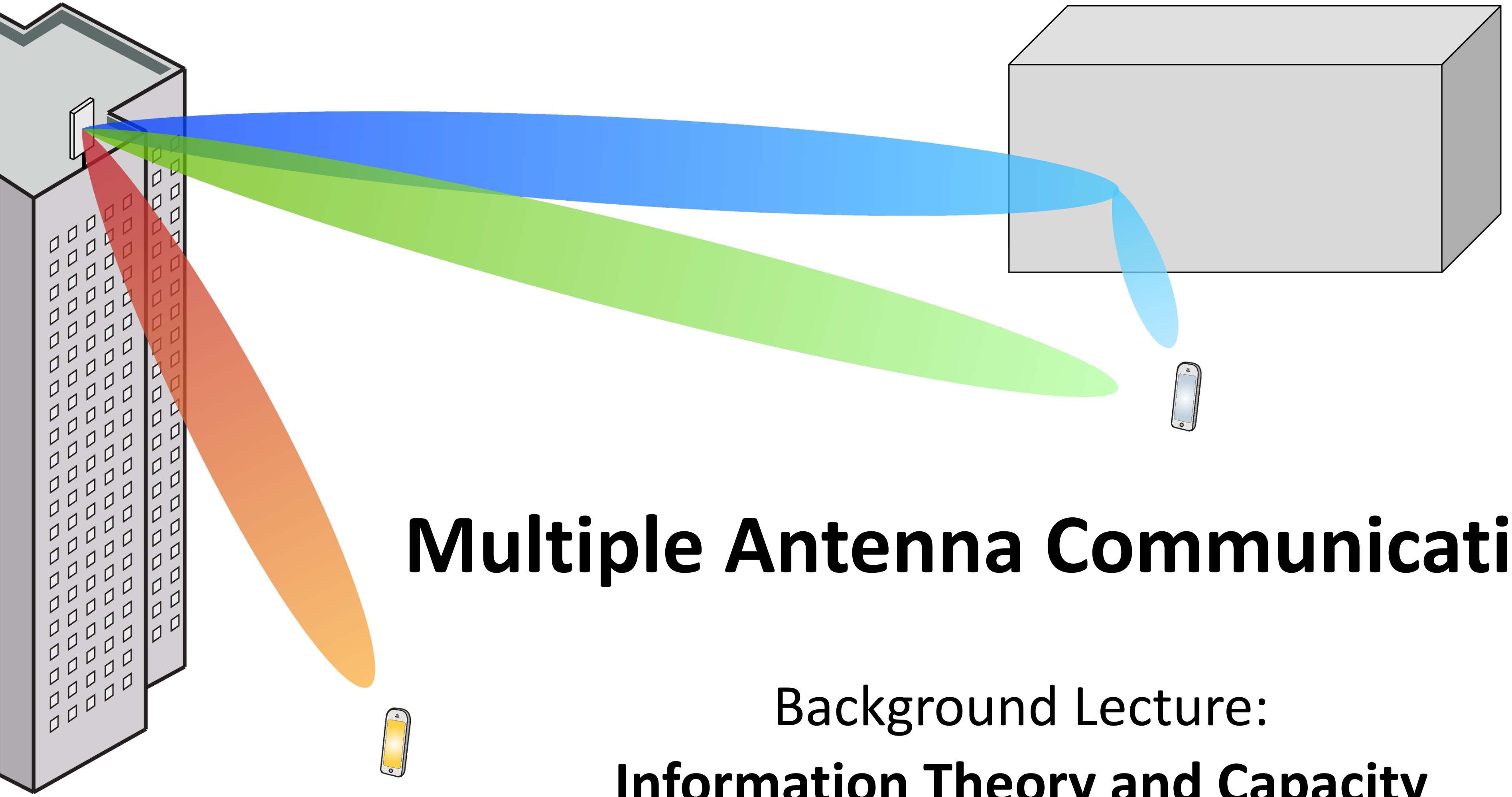


Multiple Antenna Communications

Background Lecture:

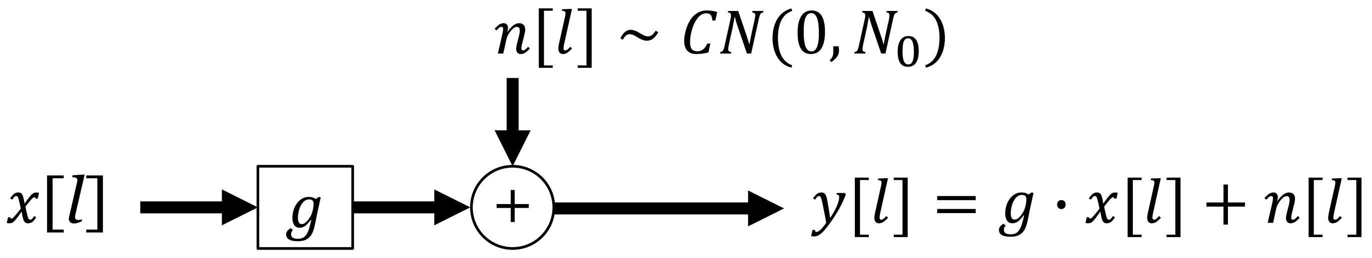
Information Theory and Capacity

Emil Björnson

Outline

- Discrete memoryless channel
- Performance metrics
- Channel capacity
    - General formulation
    - Expression for discrete memoryless channel

Discrete memoryless channel



- Transmitted *complex* signal sequence $\{x[l]\}$
  - B symbols per second (B = bandwidth)
  - Signal power P Watt, energy per symbol $q = P/B$
  - Channel response $g \in \mathbb{C}$
  - Noise with power spectral density N_0 (Watt/Hz)

How to measure communication performance?

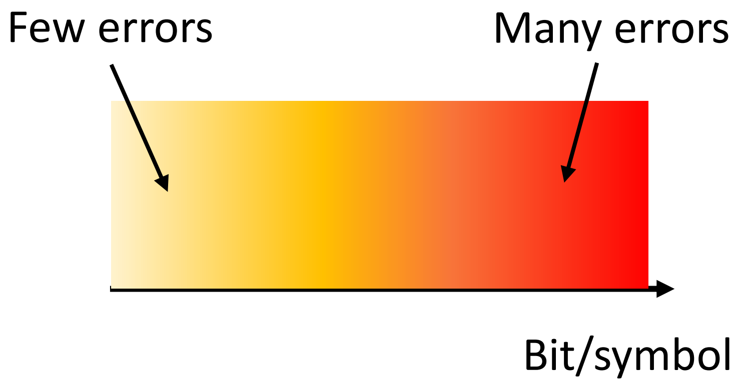
- Data packet:

$x[1]$	$x[2]$	$x[3]$	$x[4]$	…	$x[L]$

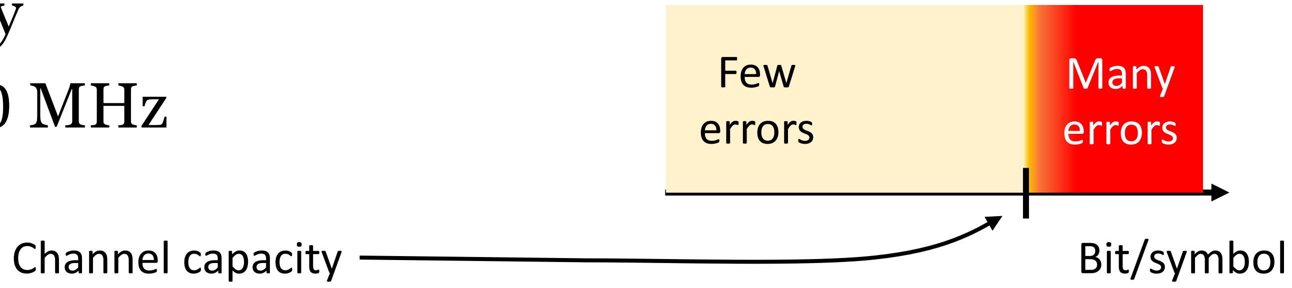
- Characterized by
  - How many L symbols the packet contains
  - How many information bits these symbols represent (determined by the modulation and coding scheme)
  - Probability of incorrect decoding at the receiver

Small or large packages

- Small package $\boxed{x[1], \dots, x[8]}$
    - Few noise realizations → Unpredictability
    - Tradeoff between bit/symbol and error probability



- Large package $\boxed{x[1], \dots, x[10000]}$
    - Many noise realizations → Statistical predictability
    - Tradeoff becomes almost binary
    - 10000 symbols = 1 ms if $B = 10$ MHz



Channel capacity

- Channel capacity
  - Random variables X and Y
  - Channel described by conditional distribution

$$f_{Y|X}(y|x)$$

Channel coding theorem

C [bit/symbol] is the capacity of the channel if:

For any given $\delta > 0$ and $\gamma > 0$,
there exist a channel coding codebook of a finite length L that has rate $R = C - \delta$ and offers an error probability $P(\text{error}) \leq \gamma$

Capacity and mutual information

- Channel capacity

$$C = \max_{f_X(x)} I(x;y)$$

$$x \longrightarrow \boxed{\text{Channel}} \longrightarrow y$$

- Mutual information: $I(x;y) = h(y) - h(y|x)$
- Differential entropy:

$$h(y) = -E\{\log_2(f_Y(y))\} \leq \log_2(\pi e \text{Var}\{y\})$$

- Conditional differential entropy:

$$h(y|x) = -E\{\log_2(f_{Y|X}(y|x))\}$$

Equality if
Complex Gaussian

Differential entropy $h(x)$ of $x \sim CN(0, p)$

$$f_X(x) = \frac{1}{\pi p} e^{-\frac{|x|^2}{p}}$$

- Direct computation

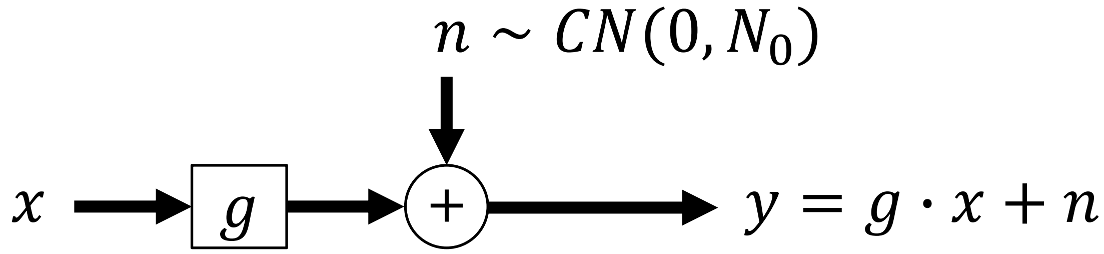
$$h(x) = -E\{\log_2(f_X(x))\} = -\int_{\mathbb{C}} \frac{1}{\pi p} e^{-\frac{|x|^2}{p}} \log_2\left(\frac{1}{\pi p} e^{-\frac{|x|^2}{p}}\right) dx$$

$$= \int_{\mathbb{C}} \frac{1}{\pi p} e^{-\frac{|x|^2}{p}} \left(\log_2(\pi p) + \frac{|x|^2}{p} \log_2(e)\right) dx$$

$$= \log_2(\pi p) \int_{\mathbb{C}} \frac{1}{\pi p} e^{-\frac{|x|^2}{p}} dx + \frac{\log_2(e)}{p} \int_{\mathbb{C}} \frac{|x|^2}{\pi p} e^{-\frac{|x|^2}{p}} dx$$

$$= \log_2(\pi p) \cdot 1 + \frac{\log_2(e)}{p} E\{|x|^2\} = \log_2(\pi e p)$$

Capacity of complex discrete memoryless channel



- Recall: $I(x; y) = h(y) - h(y|x)$

$$h(y|x) = [y - gx = n \sim CN(0, N_0)\ ] = \log_2(\pi e N_0)$$

- Mutual information maximized by $x \sim CN(0, q)$

$$h(y) \leq [y = gx + n \sim CN(0, q|g|^2 + N_0)\ ] = \log_2(\pi e(q|g|^2 + N_0))$$

Channel capacity

$$C = h(y) - h(y|x) = \log_2\left(1 + \frac{q|g|^2}{N_0}\right)$$

Different forms of same expression

- Capacity expression:

$$C = \log_2\left(1 + \frac{q|g|^2}{N_0}\right) \text{ bits per symbol}$$

  - Achieved by: $x[l] \sim CN(0, q)$
  - Abstracts away exact modulation and coding

- Alternative expressions:
  - Utilize that $q = P/B$: $\mathrm{C} = \log_2\left(1 + \frac{P|g|^2}{BN_0}\right)$ bits per symbol
  - Utilize B symbols/second: $C = B \cdot \log_2\left(1 + \frac{P|g|^2}{BN_0}\right)$ bits per second

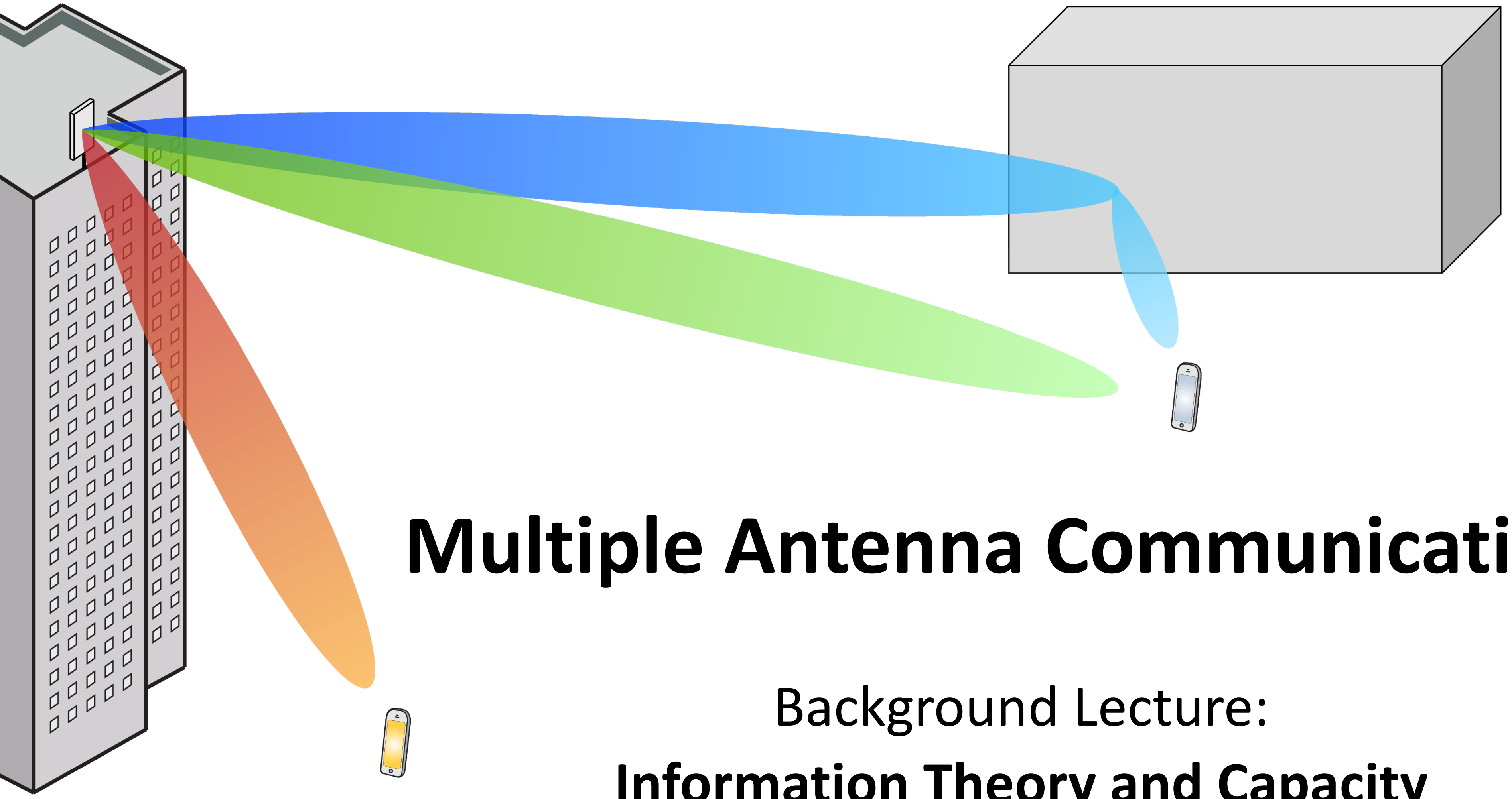
Summary

- Capacity of memoryless channel:

$$C = B \cdot \log_2\left(1 + \frac{P|g|^2}{BN_0}\right) \text{bits per symbol}$$

  - Depends on bandwidth B
  - Depends on SNR per symbol: $\frac{P}{BN_0}$

- Preferred performance metric for broadband applications



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