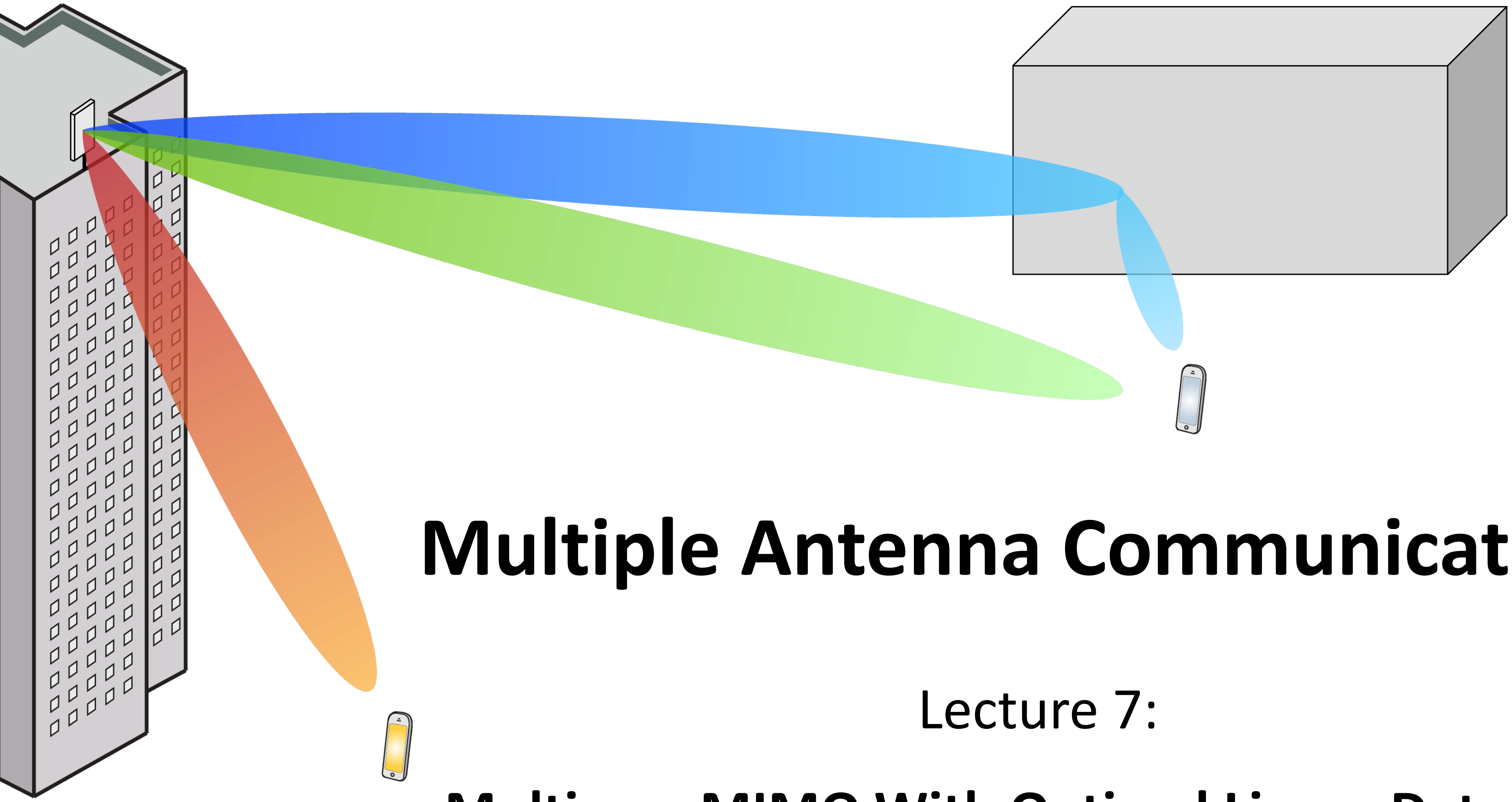


Multiple Antenna Communications

Lecture 7:

Multiuser MIMO With Optimal Linear Detection

Emil Björnson

Outline

- Uplink Massive MIMO
    - Pilot transmission
    - Channel estimation

- Capacity lower bound
    - Computation using channel estimates
    - Maximization of the bound

Recall: Uplink Massive MIMO system model

- Received signal:

$$\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{G}\boldsymbol{x} + \boldsymbol{w}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \quad \boldsymbol{G} = \begin{pmatrix} g_1^1 & \cdots & g_K^1 \\ \vdots & \ddots & \vdots \\ g_1^M & \cdots & g_K^M \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_M \end{pmatrix}$$

  - Parameters are normalized: Maximum power is ρ_{ul}
    $x_1, \ldots, x_K$ has power ≤ 1
  - Channel of user k: $g_k^1, \ldots, g_k^M \sim CN(0, \beta_k)$
  - Normalized noise: $\boldsymbol{w} \sim CN(\boldsymbol{0}, \boldsymbol{I}_M)$

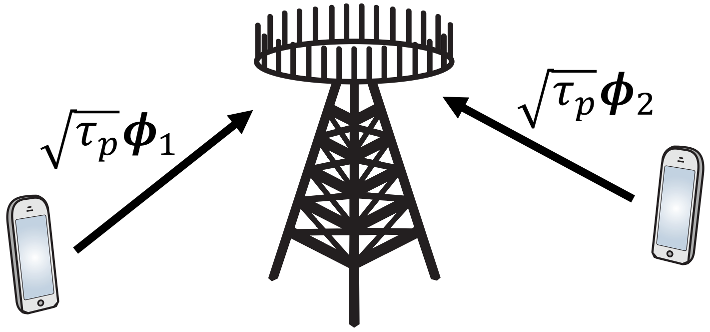
Large-scale fading coefficient

Sending pilot sequences

- Send pilot matrix $\sqrt{\tau_p}\boldsymbol{\Phi} = \sqrt{\tau_p}[\boldsymbol{\phi}_1 \ldots \boldsymbol{\phi}_K]$ over τ_p uses of the channel:

$$\boldsymbol{Y}_p = \sqrt{\tau_p \rho_{ul}}\boldsymbol{G}\boldsymbol{\Phi}^H + \boldsymbol{W}_p$$

  - Stacking received signals in $M \times \tau_p$ matrix $\boldsymbol{Y}_p$
  - Noise $\boldsymbol{W}_p$ with i.i.d. $CN(0,1)$ elements



- Despreading of pilot signal:

$$\boldsymbol{Y}'_p = \boldsymbol{Y}_p\boldsymbol{\Phi} = \sqrt{\tau_p \rho_{ul}}\boldsymbol{G}\underbrace{\boldsymbol{\Phi}^H\boldsymbol{\Phi}}_{=\boldsymbol{I}_K} + \underbrace{\boldsymbol{W}_p\boldsymbol{\Phi}}_{\text{Still i.i.d. } CN(0,1) \text{ elements}}$$

Estimating Gaussian variable in noise

- Consider $y = \sqrt{p}g + w$ where
  - p is a constant, $g \sim CN(0, \beta)$, $w \sim CN(0,1)$

Mean squared error:
$E\{|\hat{g} - g|^2\}$

Minimum mean squared error (MMSE) estimator:

$$\hat{g} = E\{g|y\} = \frac{\sqrt{p}\beta}{1 + p\beta}y$$

Estimation error: $\tilde{g} = \hat{g} - g \sim CN\left(0, \beta - \frac{p\beta^2}{1 + p\beta}\right)$

Estimate: $\hat{g} \sim CN\left(0, \frac{p\beta^2}{1 + p\beta}\right)$

Independent random variables

Estimates of the channels

$$\boxed{\left[\boldsymbol{Y}_p'\right]_{mk} = \sqrt{\tau_p \rho_{ul}} g_k^m + \left[\boldsymbol{W}_p \boldsymbol{\Phi}\right]_{mk}}$$

- MMSE estimate of g_k^m from user k to antenna m
  - Estimate:

$$\hat{g}_k^m = E\{g_k^m | \boldsymbol{Y}_p'\} = \frac{\sqrt{\tau_p \rho_{ul}} \beta_k}{1 + \tau_p \rho_{ul} \beta_k} \left[\boldsymbol{Y}_p'\right]_{mk} \sim CN(0, \gamma_k)$$

  - Estimation error:

$$\tilde{g}_k^m = \hat{g}_k^m - g_k^m \sim CN(0, \beta_k - \gamma_k)$$

where

$$\gamma_k = \frac{\tau_p \rho_{ul} \beta_k^2}{1 + \tau_p \rho_{ul} \beta_k}$$

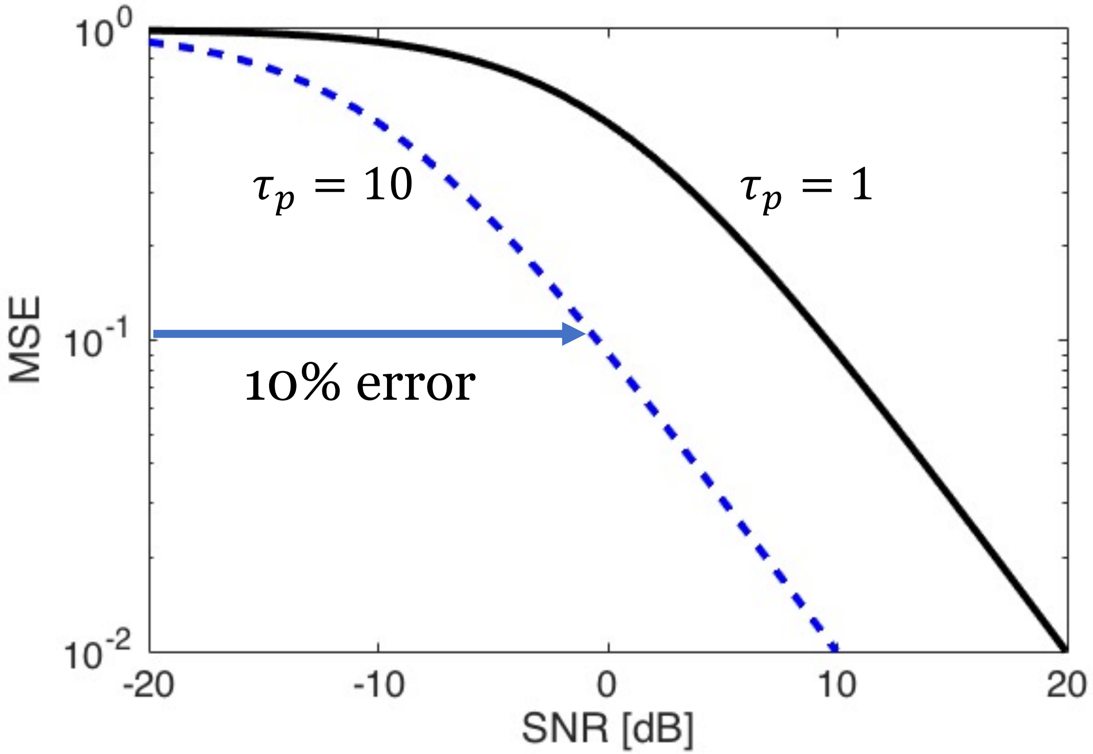
How good is the channel estimate?

- Mean squared error (MSE):

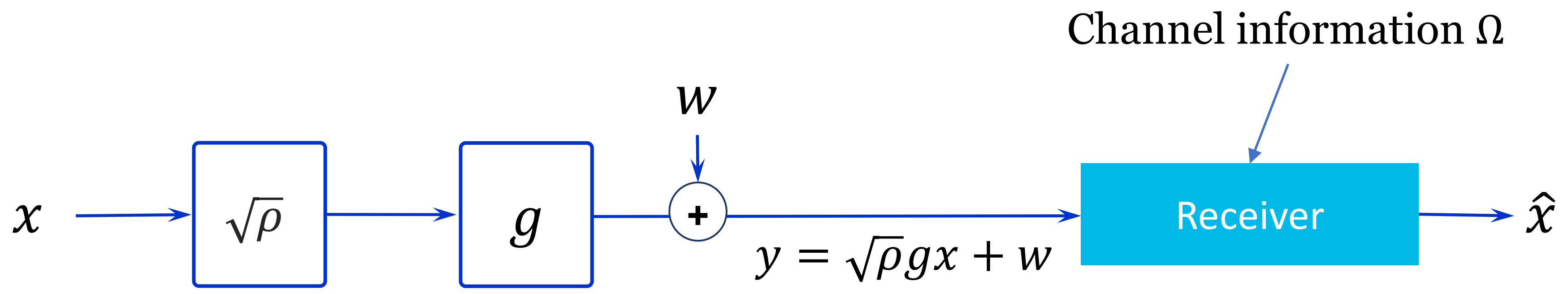
$$E\{|\hat{g}_k^m - g_k^m|^2\} = E\{|\tilde{g}_k^m|^2\} = \beta_k - \gamma_k = \beta_k - \frac{\tau_p \rho_{ul} \beta_k^2}{1 + \tau_p \rho_{ul} \beta_k}$$

  - Goes to zero as $\rho_{ul} \to \infty$ or $\tau_p \to \infty$ (perfect estimate)

Simulation:
$\beta_k = 1$
$\mathrm{SNR} = \rho_{ul}\beta_k = \rho_{ul}$



A capacity lower bound



- Desired signal x, power ρ, and g and w uncorrelated
- Channel coefficient g, known channel information Ω

Capacity lower bound:

$$C \geq E\left\{\log_2\left(1 + \frac{\rho|E\{g|\Omega\}|^2}{\rho Var\{g|\Omega\} + Var\{w|\Omega\}}\right)\right\}$$

Uplink data transmission

- Received signal:

$$\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{G}\boldsymbol{x} + \boldsymbol{w}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \quad \boldsymbol{G} = \begin{pmatrix} g_1^1 & \cdots & g_K^1 \\ \vdots & \ddots & \vdots \\ g_1^M & \cdots & g_K^M \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_M \end{pmatrix}$$

- Signals: $x_k = \sqrt{\eta_k} q_k$ where $q_k \sim CN(0,1)$ data symbol
  $0 \leq \eta_k \leq 1$ controls the power
- Channel of user k: $g_k^1, \ldots, g_k^M \sim CN(0, \beta_k)$
- Normalized noise: $\boldsymbol{w} \sim CN(\boldsymbol{0}, \boldsymbol{I}_M)$

Large-scale fading coefficient

Linear receiver processing

- Received signal:

$$\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{w}$$

where $\boldsymbol{D}_{\boldsymbol{\eta}} = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \eta_K \end{pmatrix}$, $\boldsymbol{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_K \end{pmatrix}$

No successive interference cancelation

- Assign receiver filter $\boldsymbol{a}_i$ for user i
  - Select it to make

$$\begin{aligned}\boldsymbol{a}_i^H\boldsymbol{y} &= \sqrt{\rho_{ul}}\boldsymbol{a}_i^H\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{a}_i^H\boldsymbol{w} \\ &= \sum_{k=1}^{K} \boldsymbol{a}_i^H\boldsymbol{g}_k\sqrt{\rho_{ul}\eta_k}q_k + \boldsymbol{a}_i^H\boldsymbol{w} \approx q_i\end{aligned}$$

In some "good" sense

Capacity bound for User i using Channel Estimates

- General capacity lower bound:

$$C \geq E\left\{\log_2\left(1 + \frac{\rho|E\{g|\Omega\}|^2}{\rho Var\{g|\Omega\} + Var\{w'|\Omega\}}\right)\right\}$$

In our case:

- $\rho = \rho_{ul}\eta_i$
- $g = \boldsymbol{a}_i^H \boldsymbol{g}_i$
- $x = q_i$
- $\Omega = \{\widehat{\boldsymbol{g}}_1, \dots, \widehat{\boldsymbol{g}}_K\}$
- $w' = \sum_{k=1,k\neq i}^{K} \boldsymbol{a}_i^H \boldsymbol{g}_k \sqrt{\rho_{ul}\eta_k} q_k + \boldsymbol{a}_i^H \boldsymbol{w}$

We need to compute each term!

Computing the expectation in the numerator

$$
\begin{aligned}
E\{g|\Omega\} &= E\left\{\boldsymbol{a}_i^H \boldsymbol{g}_i | \widehat{\boldsymbol{g}}_1, \ldots, \widehat{\boldsymbol{g}}_K\right\} \\
&= \boldsymbol{a}_i^H E\{\widehat{\boldsymbol{g}}_i | \widehat{\boldsymbol{g}}_1, \ldots, \widehat{\boldsymbol{g}}_K\} - \boldsymbol{a}_i^H E\{\widetilde{\boldsymbol{g}}_i | \widehat{\boldsymbol{g}}_1, \ldots, \widehat{\boldsymbol{g}}_K\} \\
&= \boldsymbol{a}_i^H \widehat{\boldsymbol{g}}_i
\end{aligned}
$$

We have used

- $g = \boldsymbol{a}_i^H \boldsymbol{g}_i$
- $\boldsymbol{a}_i$ selected based on $\Omega = \{\widehat{\boldsymbol{g}}_1, \ldots, \widehat{\boldsymbol{g}}_K\}$
- $\boldsymbol{g}_i = \widehat{\boldsymbol{g}}_i - \widetilde{\boldsymbol{g}}_i$ where $\widehat{\boldsymbol{g}}_i$ and $\widetilde{\boldsymbol{g}}_i$ are independent
- $E\{\widetilde{\boldsymbol{g}}_i | \widehat{\boldsymbol{g}}_1, \ldots, \widehat{\boldsymbol{g}}_K\} = E\{\widetilde{\boldsymbol{g}}_i\} = \boldsymbol{0}$

$$
\boxed{C \geq E\left\{\log_2\left(1 + \frac{\rho |E\{g|\Omega\}|^2}{\rho Var\{g|\Omega\} + Var\{w|\Omega\}}\right)\right\}}
$$

Computing the first term in the denominator

$$Var\{g|\Omega\} = E\{|g|^2|\Omega\} - |E\{g|\Omega\}|^2$$

- We have already computed $E\{g|\Omega\}$
- Note that

$$E\{|g|^2|\Omega\} = E\left\{\left|\boldsymbol{a}_i^H \boldsymbol{g}_i\right|^2|\Omega\right\}$$
$$= \boldsymbol{a}_i^H E\{\widehat{\boldsymbol{g}}_i\widehat{\boldsymbol{g}}_i^H + \widetilde{\boldsymbol{g}}_i\widetilde{\boldsymbol{g}}_i^H - \widehat{\boldsymbol{g}}_i\widetilde{\boldsymbol{g}}_i^H - \widetilde{\boldsymbol{g}}_i\widehat{\boldsymbol{g}}_i^H|\Omega\}\boldsymbol{a}_i$$
$$= \boldsymbol{a}_i^H\left(\widehat{\boldsymbol{g}}_i\widehat{\boldsymbol{g}}_i^H + (\beta_i - \gamma_i)\boldsymbol{I}_M - \boldsymbol{0} - \boldsymbol{0}\right)\boldsymbol{a}_i = \boldsymbol{a}_i^H\left(\widehat{\boldsymbol{g}}_i\widehat{\boldsymbol{g}}_i^H + (\beta_i - \gamma_i)\boldsymbol{I}_M\right)\boldsymbol{a}_i$$

We used the same properties as before.

$$C \geq E\left\{\log_2\left(1 + \frac{\rho|E\{g|\Omega\}|^2}{\rho Var\{g|\Omega\} + Var\{w|\Omega\}}\right)\right\}$$

Computing the second term in the denominator

$$Var\{w'|\Omega\} = Var\left\{\sum_{k=1,k\neq i}^{K} \boldsymbol{a}_i^H \boldsymbol{g}_k \sqrt{\rho_{ul}\eta_k} q_k + \boldsymbol{a}_i^H \boldsymbol{w}|\Omega\right\}$$

- $E\{w'|\Omega\} = 0$ since $E\{q_k\} = 0, E\{\boldsymbol{w}\} = \boldsymbol{0}$

$$\begin{aligned} Var\{w'|\Omega\} &= E\{|w'|^2|\Omega\} \\ &= \sum_{k=1,k\neq i}^{K} E\left\{\left|\boldsymbol{a}_i^H \boldsymbol{g}_k\right|^2|\Omega\right\} \rho_{ul}\eta_k E\{|q_k|^2|\Omega\} + E\left\{\left|\boldsymbol{a}_i^H \boldsymbol{w}\right|^2|\Omega\right\} \\ &= \sum_{k=1,k\neq i}^{K} \boldsymbol{a}_i^H \left(\widehat{\boldsymbol{g}}_k \widehat{\boldsymbol{g}}_k^H + (\beta_k - \gamma_k)\boldsymbol{I}_M\right)\boldsymbol{a}_i\, \rho_{ul}\eta_k + \boldsymbol{a}_i^H \boldsymbol{I}_M \boldsymbol{a}_i \end{aligned}$$

Result: Capacity lower bound of User i

$$C \geq E\left\{\log_2\left(1 + \frac{\rho_{ul}\eta_i\left|\boldsymbol{a}_i^H\widehat{\boldsymbol{g}}_i\right|^2}{\boldsymbol{a}_i^H\boldsymbol{B}_i\boldsymbol{a}_i}\right)\right\}$$

where

$$\boldsymbol{B}_i = \sum_{k=1,k\neq i}^{K} \rho_{ul}\eta_k\widehat{\boldsymbol{g}}_k\widehat{\boldsymbol{g}}_k^H + \sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_M + \boldsymbol{I}_M$$

How to pick $\boldsymbol{a}_i$?

Generalized Rayleigh Quotient

The ratio

$$\frac{|\boldsymbol{a}^H\boldsymbol{b}|^2}{\boldsymbol{a}^H\boldsymbol{B}\boldsymbol{a}}$$

for given $\boldsymbol{b}$ and invertible $\boldsymbol{B}$ is maximized by $\boldsymbol{a} = \boldsymbol{B}^{-1}\boldsymbol{b}$

- Intuition:
  - The ratio $\frac{|\boldsymbol{a}^H\boldsymbol{b}|^2}{\boldsymbol{a}^H\boldsymbol{a}}$ is maximized by $\boldsymbol{a} = \boldsymbol{b}$
  - The extra term $\boldsymbol{B}^{-1}$ is “whitening”

Extension of maximum ratio combining

Maximizing the capacity lower bound

$$C \geq E\left\{\log_2\left(1 + \frac{\left|\boldsymbol{a}_i^H \boldsymbol{b}_i\right|^2}{\boldsymbol{a}_i^H \boldsymbol{B}_i \boldsymbol{a}_i}\right)\right\}$$

where $\boldsymbol{b}_i = \sqrt{\rho_{ul}\eta_i}\widehat{\boldsymbol{g}}_i$ and

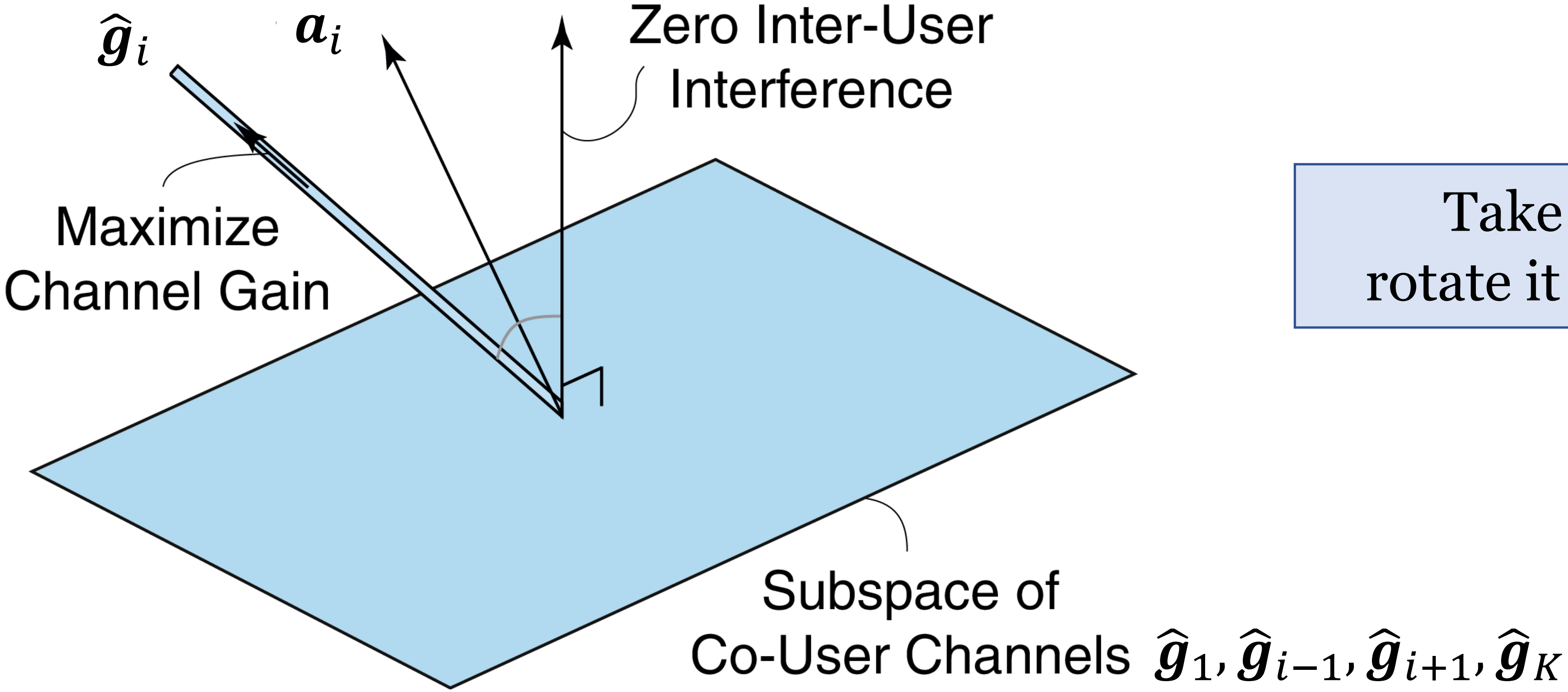
$$\boldsymbol{B}_i = \sum_{k=1,k\neq i}^{K} \rho_{ul}\eta_k \widehat{\boldsymbol{g}}_k \widehat{\boldsymbol{g}}_k^H + \sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_M + \boldsymbol{I}_M$$

Maximized by selecting

$$\boldsymbol{a}_i = \boldsymbol{B}_i^{-1}\boldsymbol{b}_i = \sqrt{\rho_{ul}\eta_i}\boldsymbol{B}_i^{-1}\widehat{\boldsymbol{g}}_i$$

Called MMSE combining

Interpretation of $\boldsymbol{a}_i = \sqrt{\rho_{ul}\eta_i}\boldsymbol{B}_i^{-1}\widehat{\boldsymbol{g}}_i$ (whitening)

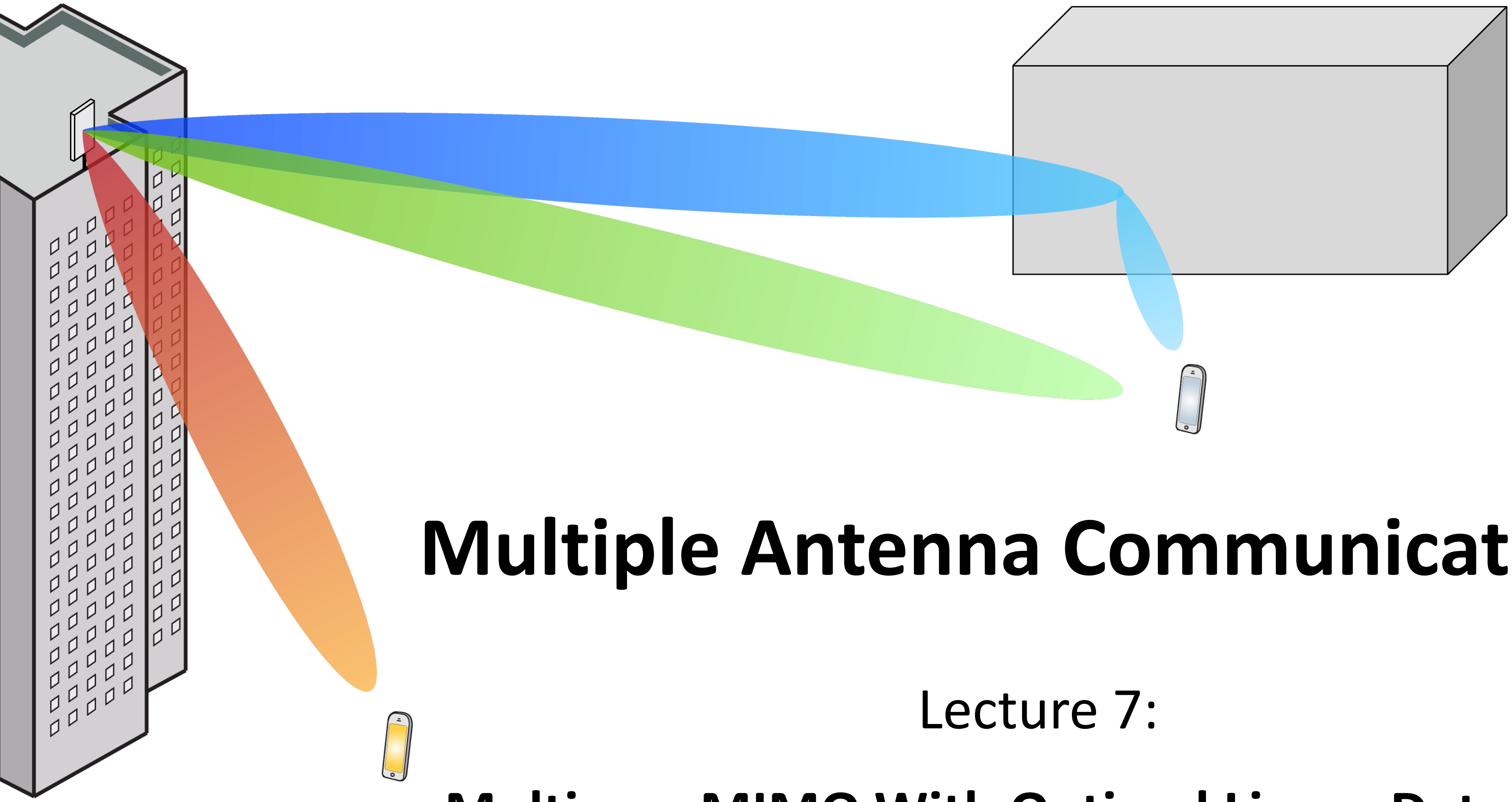


Take $\widehat{\boldsymbol{g}}_i$ and
rotate it using $\boldsymbol{B}_i^{-1}$

Summary

- Channel estimation
  - Send pilot sequences of length $\tau_p \geq K$ in the uplink
  - Estimate channels using MMSE estimation

- Data transmission
  - General capacity lower bound
  - Expresssion when the
    MMSE estimates are known
  - Optimized linear receiver processing



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