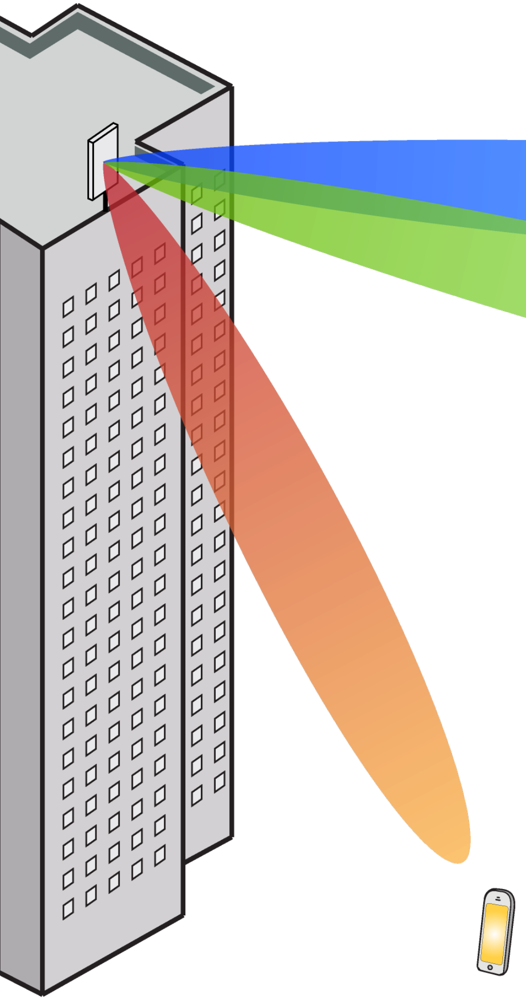


Multiple Antenna Communications

Lecture 9b:

Zero-Forcing Processing in Uplink and Downlink

Emil Björnson

Outline

- Revisit MMSE processing
    - Asymptotic behaviors
- Zero-forcing processing
    - Key properties
    - Closed-form expression
- Performance comparison: Different processing schemes

Recall: Uplink Massive MIMO with linear processing

- Received signal:

$$\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{w}$$

where $\boldsymbol{D}_{\boldsymbol{\eta}} = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \eta_K \end{pmatrix}$, $\boldsymbol{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_K \end{pmatrix}$

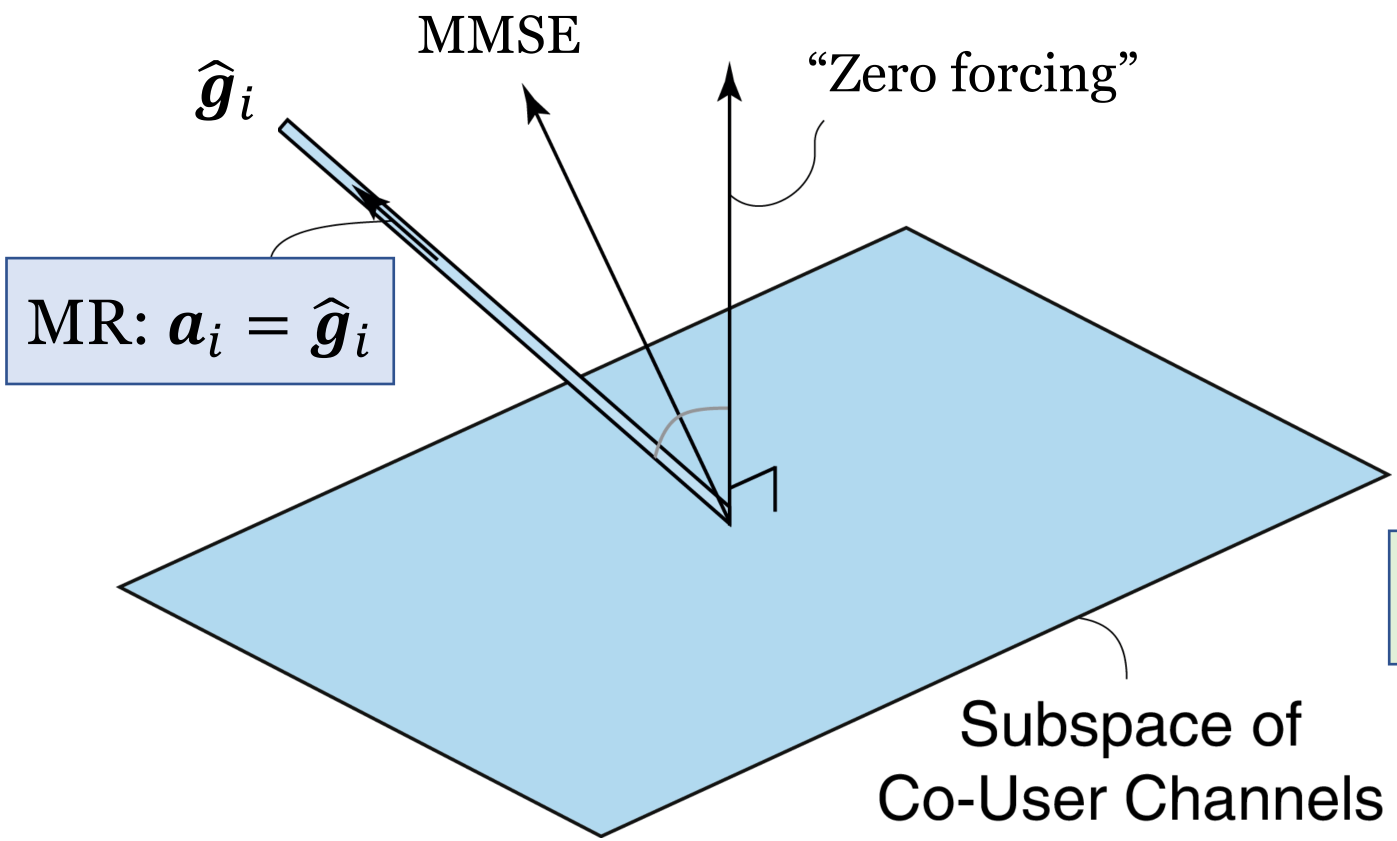
- Assign receiver filter $\boldsymbol{a}_i$ for user i: $\boldsymbol{a}_i^H\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{a}_i^H\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{a}_i^H\boldsymbol{w}$

  - For all users $\boldsymbol{A} = [\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_K]$:

$$\boldsymbol{A}^H\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{A}^H\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{A}^H\boldsymbol{w} \approx \boldsymbol{q}$$

In some “good” sense

MMSE receiver processing



MMSE

$$\boldsymbol{a}_i = \sqrt{\rho_{ul}\eta_i}\boldsymbol{B}_i^{-1}\widehat{\boldsymbol{g}}_i$$

$$= \text{constant} \cdot \left(\boldsymbol{B}_i + \rho_{ul}\eta_i\widehat{\boldsymbol{g}}_i\widehat{\boldsymbol{g}}_i^H\right)^{-1}\widehat{\boldsymbol{g}}_i$$

$$\boldsymbol{B}^{-1}\boldsymbol{a} = (1 + \boldsymbol{a}^H\boldsymbol{B}^{-1}\boldsymbol{a})(\boldsymbol{B} + \boldsymbol{a}\boldsymbol{a}^H)^{-1}\boldsymbol{a}$$

$$\boldsymbol{B}_i = \sum_{k=1,k\neq i}^{K} \rho_{ul}\eta_k\widehat{\boldsymbol{g}}_k\widehat{\boldsymbol{g}}_k^H + \sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_M + \boldsymbol{I}_M$$

What happens to MMSE asymptotically?

Reformulating MMSE processing

- Notation: $\widehat{\boldsymbol{G}} = [\widehat{\boldsymbol{g}}_1 \; \dots \; \widehat{\boldsymbol{g}}_K], \boldsymbol{D}_\eta = \operatorname{diag}(\eta_1, \dots, \eta_K)$
  - For one user:

$$\boldsymbol{a}_i = \sqrt{\rho_{ul}\eta_i}\left(\sum_{k=1}^{K} \rho_{ul}\eta_k \widehat{\boldsymbol{g}}_k \widehat{\boldsymbol{g}}_k^H + \sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_M + \boldsymbol{I}_M\right)^{-1} \widehat{\boldsymbol{g}}_i$$

  - For all users, $\boldsymbol{A} = [\boldsymbol{a}_1 \; \dots \; \boldsymbol{a}_K]$:

$$\boldsymbol{A} = \sqrt{\rho_{ul}}\left(\rho_{ul}\widehat{\boldsymbol{G}}\boldsymbol{D}_\eta\widehat{\boldsymbol{G}}^H + \sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_M + \boldsymbol{I}_M\right)^{-1} \widehat{\boldsymbol{G}}\boldsymbol{D}_\eta^{1/2}$$

$$= \sqrt{\rho_{ul}}\widehat{\boldsymbol{G}}\boldsymbol{D}_\eta^{1/2}\left(\rho_{ul}\boldsymbol{D}_\eta^{1/2}\widehat{\boldsymbol{G}}^H\widehat{\boldsymbol{G}}\boldsymbol{D}_\eta^{1/2} + \sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_K + \boldsymbol{I}_K\right)^{-1}$$

$$\boxed{(\boldsymbol{B}\boldsymbol{D} + \boldsymbol{I})^{-1}\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{D}\boldsymbol{B} + \boldsymbol{I})^{-1}}$$

Asymptotic: High SNR ($M \geq K$)

- MMSE processing:

$$\boldsymbol{A} = \sqrt{\rho_{ul}}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2}\left(\rho_{ul}\boldsymbol{D}_{\eta}^{1/2}\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2} + \sum_{k=1}^{K}\rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_K + \boldsymbol{I}_K\right)^{-1}$$

- High SNR ($\rho_{ul} \rightarrow \infty$): $\gamma_k = \frac{\tau_p \rho_{ul} \beta_k^2}{1+\tau_p \rho_{ul} \beta_k} \rightarrow \beta_k$, $\widehat{\boldsymbol{G}} \rightarrow \boldsymbol{G} = [\boldsymbol{g}_1 \ \dots \ \boldsymbol{g}_K]$

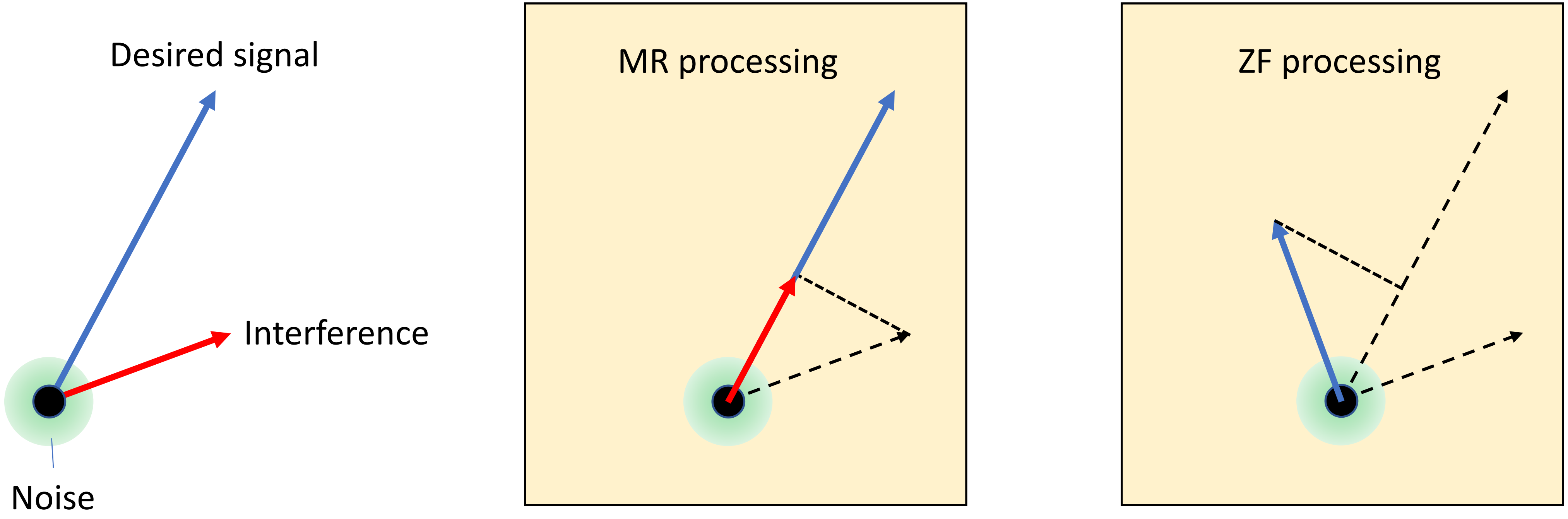
$$\sqrt{\rho_{ul}}\boldsymbol{A} = \widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2}\left(\boldsymbol{D}_{\eta}^{1/2}\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2} + \sum_{k=1}^{K}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_K + \frac{1}{\rho_{ul}}\boldsymbol{I}_K\right)^{-1}$$

$$\rightarrow \boldsymbol{G}(\boldsymbol{G}^{H}\boldsymbol{G})^{-1}\boldsymbol{D}_{\eta}^{-1/2}$$

$$\boxed{\boldsymbol{A}^{H}\boldsymbol{y} = \boldsymbol{D}_{\eta}^{-1/2}(\boldsymbol{G}^{H}\boldsymbol{G})^{-1}\boldsymbol{G}^{H}\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{A}^{H}\boldsymbol{w}/\sqrt{\rho_{ul}} = \boldsymbol{q} + \boldsymbol{A}^{H}\boldsymbol{w}/\sqrt{\rho_{ul}}}$$

Zero-forcing (ZF):
Cancel all interference

Geometry of MR and ZF



Use all signal power
Get part of the interference

Remove all interference
Get part of the signal

Asymptotic: Many Antennas

- MMSE processing:

$$\boldsymbol{A} = \sqrt{\rho_{ul}}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2}\Bigg(\underbrace{\rho_{ul}\boldsymbol{D}_{\eta}^{1/2}\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2}}_{\text{Diagonal contains } \|\widehat{\boldsymbol{g}}_k\|^2 \text{, grows with } M} + \underbrace{\sum_{k=1}^{K}\rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_K + \boldsymbol{I}_K}_{\text{Independent of } M}\Bigg)^{-1}$$

Large M

$$\approx \sqrt{\rho_{ul}}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2}\left(\rho_{ul}\boldsymbol{D}_{\eta}^{1/2}\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}}\boldsymbol{D}_{\eta}^{1/2}\right)^{-1}$$

$$= \frac{1}{\sqrt{\rho_{ul}}}\widehat{\boldsymbol{G}}\left(\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}}\right)^{-1}\boldsymbol{D}_{\eta}^{-1/2}$$

Zero-forcing with imperfect channel estimates

Complex Wishart Matrices

- $\boldsymbol{Z}$ is an $M \times K$ matrix with i.i.d. $CN(0,1)$ random variables ($M \geq K$)

$\boldsymbol{Z}^H\boldsymbol{Z}$ has a complex Wishart distribution

- Properties:
  - $E\{\boldsymbol{Z}^H\boldsymbol{Z}\} = M\boldsymbol{I}_K$
  - $E\{(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\} = \frac{1}{M-K}\boldsymbol{I}_K$

ZF: $$\boldsymbol{A} = \frac{1}{\sqrt{\rho_{ul}}}\widehat{\boldsymbol{G}}\left(\widehat{\boldsymbol{G}}^H\widehat{\boldsymbol{G}}\right)^{-1}\boldsymbol{D}_\eta^{-1/2} = \frac{1}{\sqrt{\rho_{ul}}}\boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\boldsymbol{D}_\gamma^{-1/2}\boldsymbol{D}_\eta^{-1/2}$$

$$\widehat{\boldsymbol{G}} = \boldsymbol{Z}\boldsymbol{D}_\gamma^{1/2}$$

Equivalent ZF: $\boldsymbol{A} = \boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}$

Received signal when using ZF processing

$$\boxed{\boldsymbol{G} = \boldsymbol{Z}\boldsymbol{D}_{\gamma}^{1/2} - \widetilde{\boldsymbol{G}}}$$

- Received signal: $\boldsymbol{y} = \sqrt{\rho_{ul}}\boldsymbol{G}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{w} = \sqrt{\rho_{ul}}\boldsymbol{Z}\boldsymbol{D}_{\gamma}^{1/2}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + \boldsymbol{w} - \sqrt{\rho_{ul}}\widetilde{\boldsymbol{G}}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q}$
  - Apply ZF ($\boldsymbol{A} = \boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}$):

$$\boldsymbol{A}^H\boldsymbol{y} = \sqrt{\rho_{ul}}\underbrace{(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\boldsymbol{Z}^H\boldsymbol{Z}}_{=\boldsymbol{I}_K}\boldsymbol{D}_{\gamma}^{1/2}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q} + (\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\boldsymbol{Z}^H\left(\boldsymbol{w} - \sqrt{\rho_{ul}}\widetilde{\boldsymbol{G}}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q}\right)$$

  - For user i:

$$[\boldsymbol{A}^H\boldsymbol{y}]_i = \underbrace{\sqrt{\rho_{ul}\gamma_i\eta_i}}_{\text{Deterministic channel}}q_i + \overbrace{\left[(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\boldsymbol{Z}^H\left(\boldsymbol{w} - \sqrt{\rho_{ul}}\widetilde{\boldsymbol{G}}\boldsymbol{D}_{\boldsymbol{\eta}}^{1/2}\boldsymbol{q}\right)\right]_i}^{w\text{: Uncorrelated interference and noise}}$$

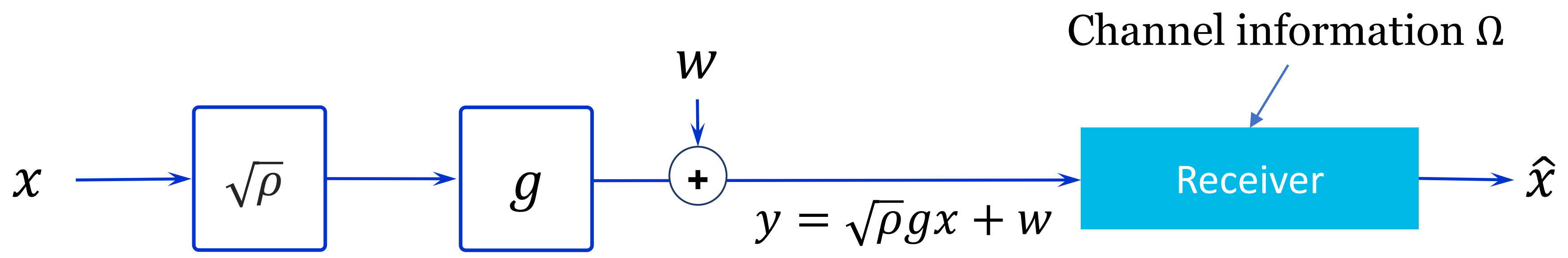
Covariance:

$$E\{(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\boldsymbol{Z}^H\boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\} = \frac{1}{M-K}\boldsymbol{I}_K$$

Covariance:

$$\sum_{k=1}^{K}\rho_{ul}\eta_k(\beta_k - \gamma_k)\boldsymbol{I}_K + \boldsymbol{I}_K$$

Using the capacity bound with deterministic channel



- Desired signal $x = q_i$, transmit power $\rho = \rho_{ul}\eta_i$
- Determinstic and known channel coefficient $g = \sqrt{\gamma_i}$

Capacity lower bound:

$$C \geq \log_2\left(1 + \frac{\rho|g|^2}{Var\{w\}}\right)$$

$$\rho|g|^2 = \rho_{ul}\eta_i\gamma_i$$

$$Var\{w\} = \frac{1}{M-K}\left(\sum_{k=1}^{K} \rho_{ul}\eta_k(\beta_k - \gamma_k) + 1\right)$$

Capacity bound with ZF and use-and-then-forget technique

$$C \geq \log_2\left(1 + \frac{(M-K)\rho_{ul}\eta_i\gamma_i}{\sum_{k=1}^{K}\rho_{ul}\eta_k(\beta_k - \gamma_k) + 1}\right)$$

- Interpretation
  - Small-scale fading is not visible in this bound
  - **Numerator:**
    Coherent beamforming gain, grows with antennas as $M-K$, power $\rho_{ul}\eta_i$ and estimation quality γ_i. Interference suppression removes dimensions
  - **Denominator:**
    Sum of non-coherent interference from all users, proportional to estimation error variance $\beta_k - \gamma_k$, plus noise variance

MR versus ZF processing

- MR: $\log_2\left(1+\frac{M\rho_{ul}\eta_i\gamma_i}{\sum_{k=1}^{K}\rho_{ul}\eta_k\beta_k+1}\right)$
- ZF: $\log_2\left(1+\frac{(M-K)\rho_{ul}\eta_i\gamma_i}{\sum_{k=1}^{K}\rho_{ul}\eta_k(\beta_k-\gamma_k)+1}\right)$

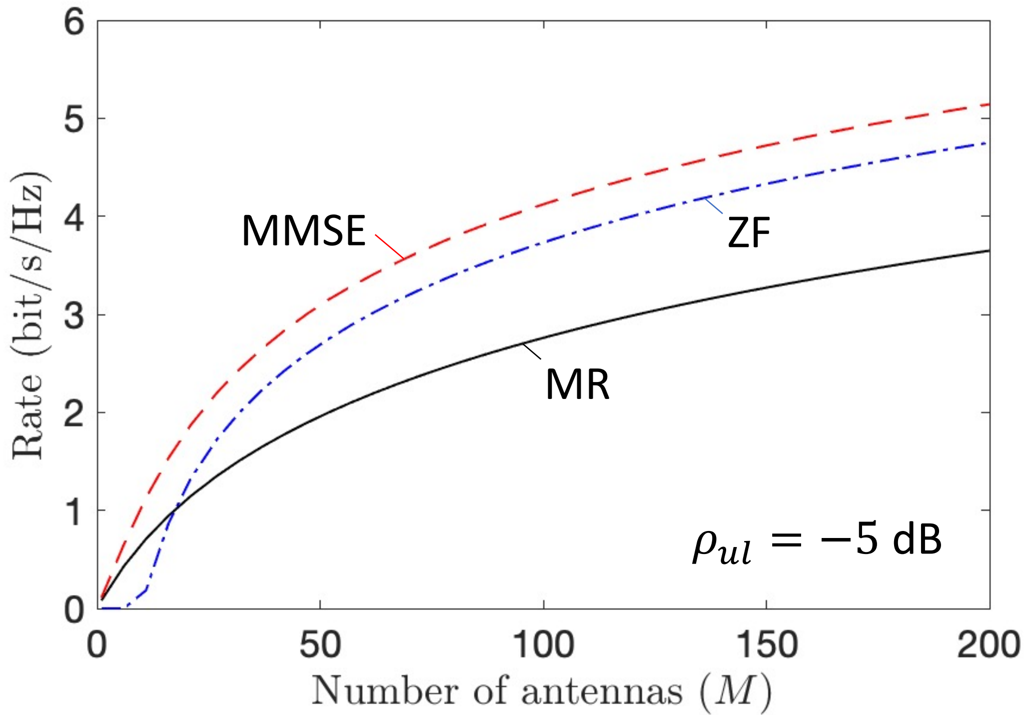
Assumptions

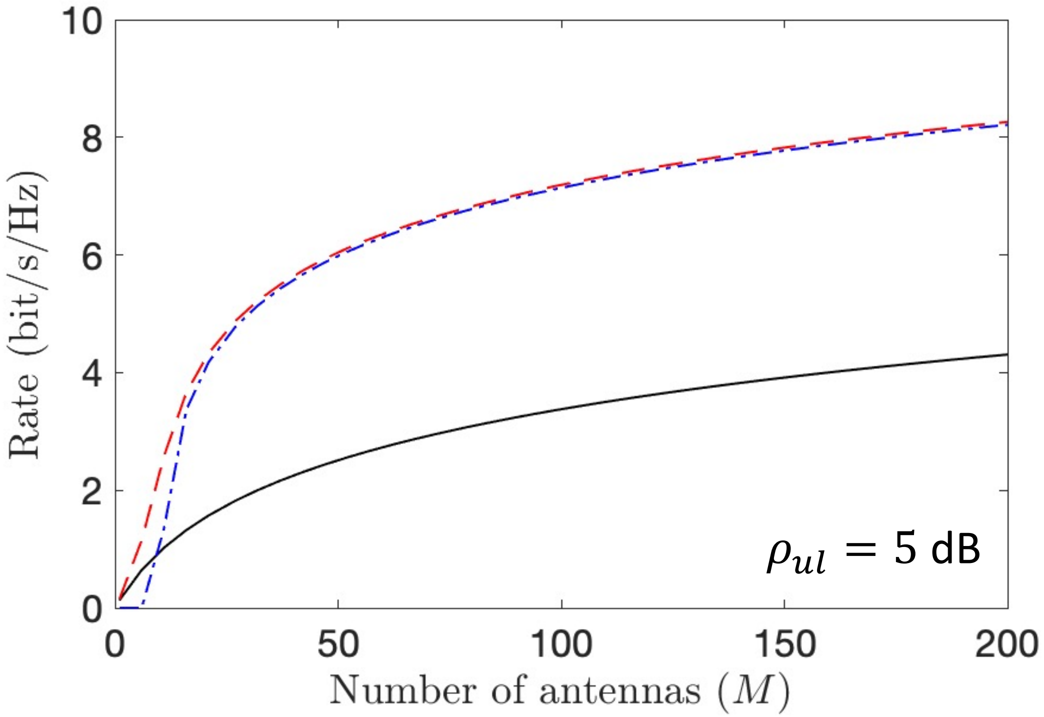
$K=10$

$\beta=1$

$\tau_p=K$

$\eta_k=1\ \forall k$

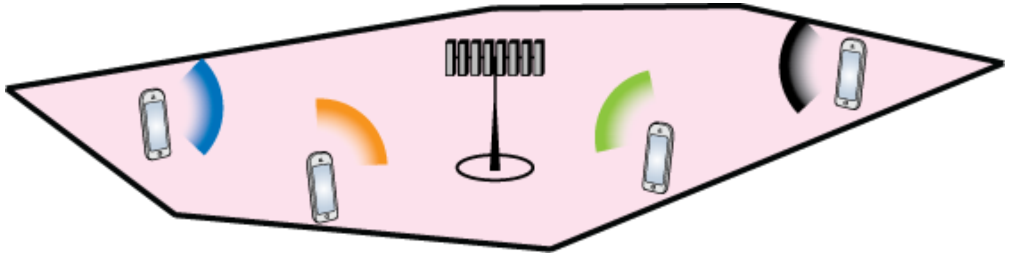




Zero-forcing precoding

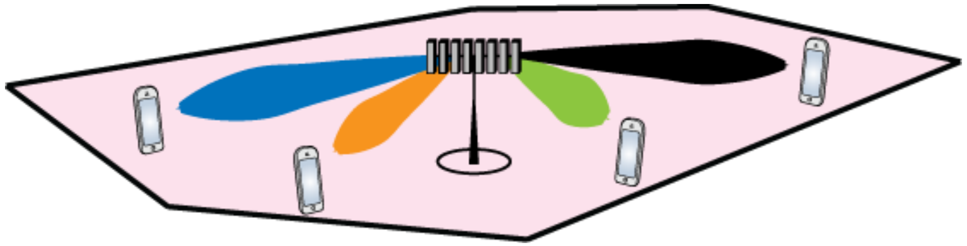
- Uplink ZF expression

$$\boldsymbol{A} = \boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}$$



Precoding principle

Transmit in the direction where you heard the users “most clearly”



- Downlink ZF expression

$$\boldsymbol{A} = \sqrt{M-K}\left(\boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\right)^*$$

since $E\{(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\boldsymbol{Z}^H\boldsymbol{Z}(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\} = E\{(\boldsymbol{Z}^H\boldsymbol{Z})^{-1}\} = \frac{1}{M-K}\boldsymbol{I}_K$

Comparing uplink and downlink (with ZF)

Uplink:

$$\log_2\left(1 + \frac{(M-K)\rho_{ul}\eta_i\gamma_i}{\sum_{k=1}^{K}\rho_{ul}\eta_k(\beta_k - \gamma_k) + 1}\right)$$

Downlink:

$$\log_2\left(1 + \frac{(M-K)\rho_{dl}\eta_i\gamma_i}{(\beta_i - \gamma_i)\sum_{k=1}^{K}\rho_{dl}\eta_k + 1}\right)$$

Similarities

- Same structure (beamforming gain $M - K$, powers $\rho_{ul/dl}\eta_i$)

Differences

- Uplink interference: From users ($\beta_1 - \gamma_1, \dots, \beta_K - \gamma_K$)
- Downlink interference: From base station ($\beta_i - \gamma_i$)

What processing method to utilize?

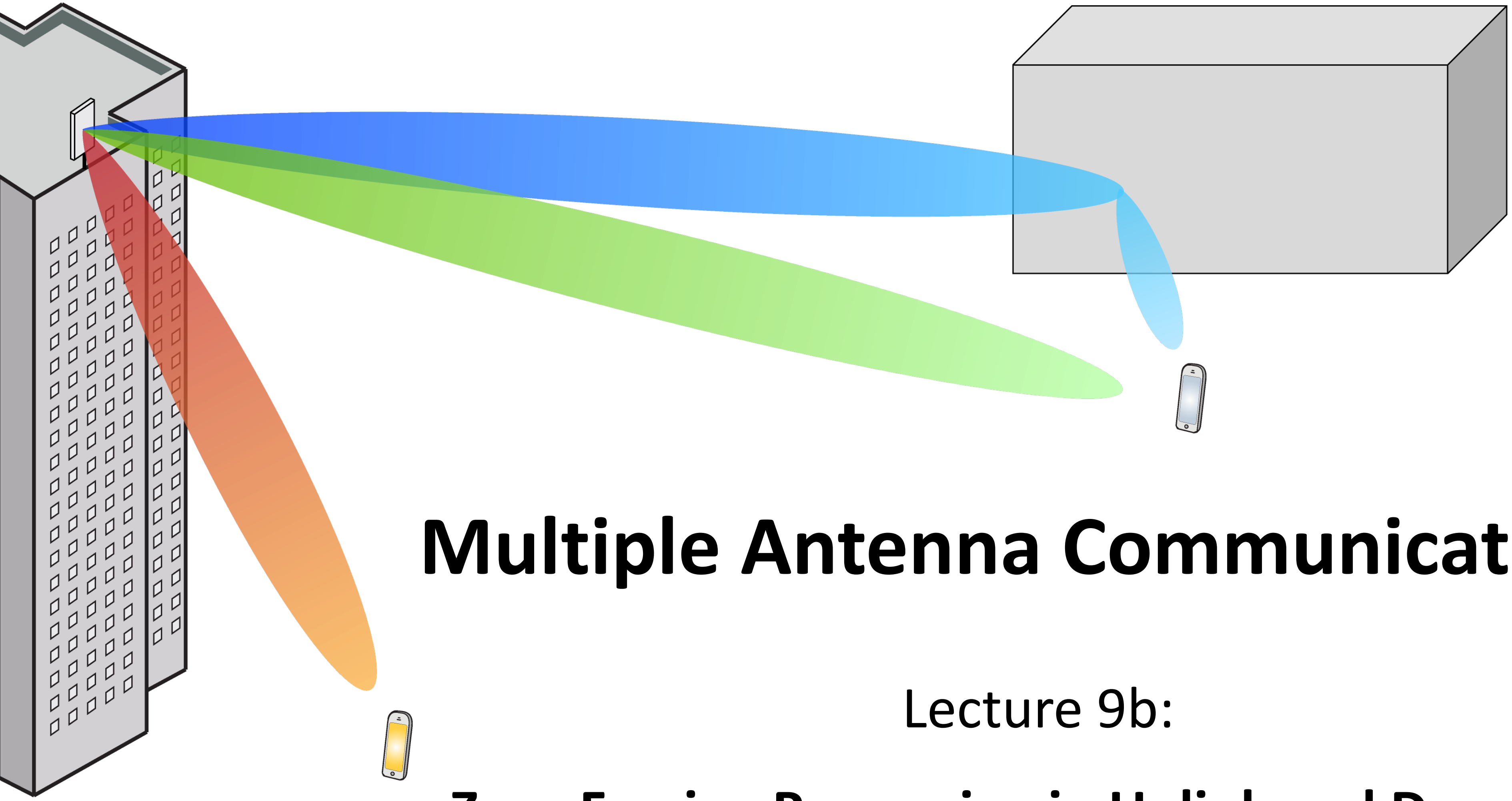
- MMSE processing
  - Highest performance, complicated capacity expression

- ZF processing
  - Closed-form expression, requires $M > K$
  - Same complexity as MMSE processing, inferior performance

- MR processing
  - Closed-form expression, *greatly* inferior performance
  - Lowest computational complexity

Summary

- Zero-forcing processing
  - Focus on interference cancellation
  - Equivalent to MMSE at high SNR or many antennas
- Closed-form expression
  - Remaining interference depends on estimation errors
  - Sacrifice antenna dimensions



Multiple Antenna Communications

Lecture 9b:

Zero-Forcing Processing in Uplink and Downlink

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