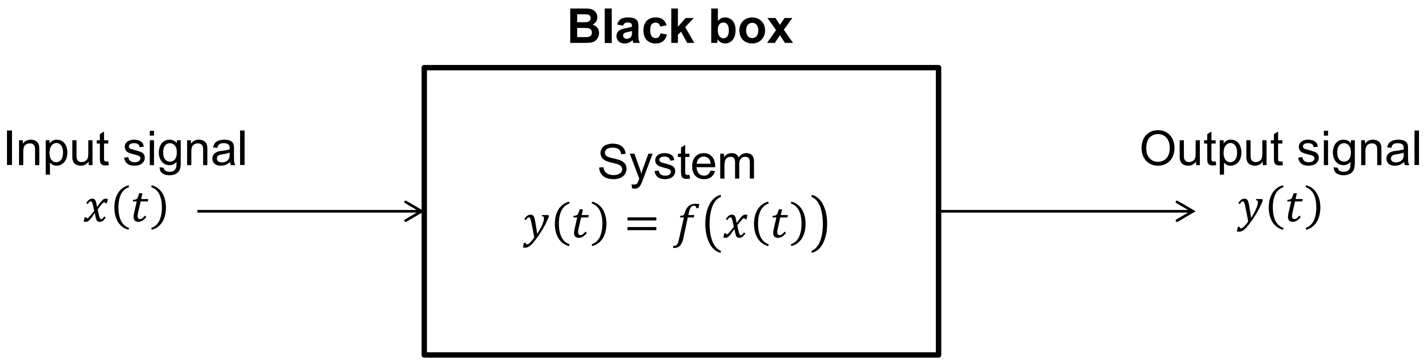
KTH ROYAL INSTITUTE OF TECHNOLOGY

Introduction to Mobile Networks and Services

# Linear Time-Invariant Systems

# Input and output signal

- System models can be used in many applications
  - For electrical communication, the signal is represented by a voltage or a current

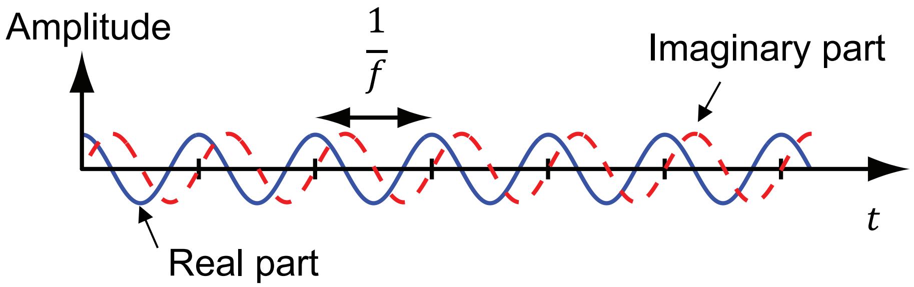



System: Manipulate/filter signals

# Examples of signals

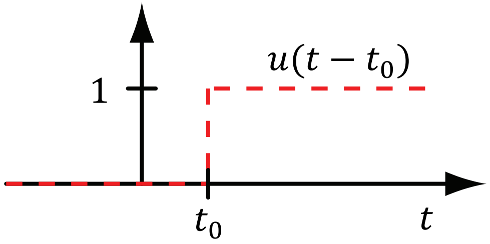
**Complex exponential:**

$$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$$




**Unit step:**

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



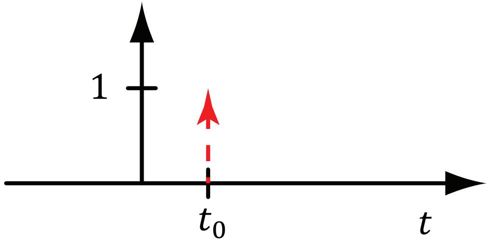

**Unit impulse:**

$$\delta(t): \int_{-\infty}^{\infty} x(t)\,\delta(t-a)\,dt = x(a)$$

– **Properties:**

$$\delta(t) = \frac{d}{dt}u(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$




Dirac delta function

# Properties of systems

General case:

$x(t) \longrightarrow$ **Energy-free system** $\longrightarrow y(t)$

*Energy-free system: No transients, constant input → constant output*

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Impulse response:

$\delta(t) \longrightarrow$ **Energy-free system** $\longrightarrow h(t)$

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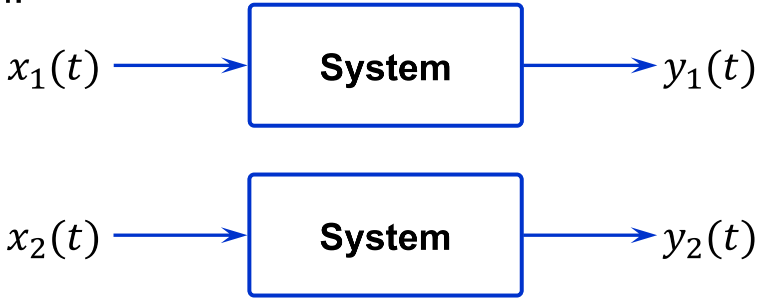
**Linear time-invariant (LTI) system**

- Linear: Output is scaled, time-delayed versions of input
- Time-invariant: Always reacts in the same way

# Linear system

- Consider a system:




- The system is **linear** if


for all $x_1(t)$, $x_2(t)$, $a_1$, and $a_2$

Otherwise the system is **non-linear**

# Linear system: Example 1

Is the following system linear?

$$x(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y(t) = x(t-1)$$

$$x_1(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y_1(t) = x_1(t-1)$$

$$x_2(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y_2(t) = x_2(t-1)$$

- Consider $x(t) = a_1 x_1(t) + a_2 x_2(t)$
- Then: $y(t) = a_1 x_1(t-1) + a_2 x_2(t-1) = a_1 y_1(t) + a_2 y_2(t)$

Linear combination of outputs of $x_1(t)$ and $x_2(t)$: ***Linear system***

# Linear system: Example 2

Is the following system linear?

$$x(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y(t) = \big(x(t)\big)^2$$

$$x_1(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y_1(t) = \big(x_1(t)\big)^2$$

$$x_2(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y_2(t) = \big(x_2(t)\big)^2$$

- Consider $x(t) = a_1 x_1(t) + a_2 x_2(t)$
- Then: $y(t) = (a_1 x_1(t) + a_2 x_2(t))^2 = a_1^2 x_1^2(t) + a_2^2 x_2^2(t) + 2 a_1 a_2 x_1(t) x_2(t)$
- If linear: $a_1 y_1(t) + a_2 y_2(t) = a_1 x_1^2(t) + a_2 x_2^2(t) \leftarrow$ This is different!

Not a linear combination of outputs of $x_1(t)$ and $x_2(t)$: ***Non-linear***

# Time-invariant system

- Consider a system:

$$x(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y(t)$$

- The system is **time-invariant** if

$$x_1(t) = x(t+\tau) \longrightarrow \boxed{\textbf{System}} \longrightarrow y_1(t) = y(t+\tau)$$

for any $x(t)$, $t$, and $\tau$

Otherwise the system is **time-variant**

# Time-invariant system: Example 1

Is the following system time-invariant?

$x(t) \longrightarrow$ **System** $\longrightarrow y(t) = x(t-1)$

$x_1(t) = x(t+\tau) \longrightarrow$ **Time-Invariant system** $\longrightarrow y_1(t) = y(t+\tau)$

- True output: $y_1(\mathrm{t}) = x_1(t-1) = x(t+\tau-1)$
- Output if time-invariant: $y(t+\tau) = x(t+\tau-1)$

Since $y_1(t) = y(t+\tau)$ : **Time-invariant**

# Time-invariant system: Example 2

Is the following system time-invariant?

$$x(t) \longrightarrow \boxed{\textbf{System}} \longrightarrow y(t) = \cos(2\pi f t)x(t)$$

$$x_1(t) = x(t+\tau) \longrightarrow \boxed{\textbf{Time-Invariant system}} \longrightarrow y_1(t) = y(t+\tau)$$

- True output: $y_1(t) = \cos(2\pi f t)\, x_1(t) = \cos(2\pi f t)\, x(t+\tau)$
- Output if time-invariant: $y(t+\tau) = \cos(2\pi f(t+\tau))x(t+\tau)$

Since $y_1(t) \neq y(t+\tau)$ : **Time-varying**

# Linear Time-Invariant (LTI) Systems

**Definition:** A system that is both linear and time-invariant is referred to as a *linear time-invariant (LTI) system*.

**Example:** $y(t) = x(t-1)$ is an *LTI system*

**Property:** The input and output of LTI systems are related as

$$\begin{matrix} x(t) \\ X(f) \end{matrix} \longrightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \longrightarrow \begin{matrix} y(t) = (x * h)(t) \\ Y(f) = X(f)H(f) \end{matrix}$$

# Thank you for watching!