True Online Temporal-Difference Learning

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joint work with

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Outline

- Part 1: true online temporal-difference learning
- Part 2: effective multi-step learning for non-linear FA

Outline

Markov Decision Processes

A *Markov decision process* (MDP) can be described by 5-tuple: $\langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, with

- $\mathcal{S}$: the set of all states
- $\mathcal{A}$: the set of all actions
- $p(s'|s,a)$: the transition probability function
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state-value function: $v^{\pi}(s) = \mathbb{E}\{G_t \mid S_t = s, \pi\}$

action-value function: $q^{\pi}(s,a) = \mathbb{E}\{G_t \mid S_t = s, A_t, = a, \pi\}$

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Stochastic gradient descent (sampling from the stationary distribution):

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unbiased estimate of $v_\pi(S_t)$: $U_t = G_t$

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TD(λ)

- update equations for linear function approximation:

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\delta_t &= R_{t+1} + \gamma \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_{t+1} - \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t \,, \\
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- This update is different from the general TD update rule.

the traditional forward view of TD(λ)

- the λ-return algorithm:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \alpha\big(G_t^\lambda - \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t\big)\boldsymbol{\phi}_t$$

where G_t^λ is the λ-return, defined as:

$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$

with $G_t^{(n)} = \sum_{k=1}^{n} \gamma^{k-1} R_{t+k} + \gamma^n \boldsymbol{\theta}^\top \boldsymbol{\phi}_{t+n}$

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note:

$\lambda = 0 \quad : \quad G_t^\lambda = G_t^{(1)}$

$\lambda = 1 \quad : \quad G_t^\lambda = G_t$

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- **theorem*:** for small step-sizes, $\boldsymbol{\theta}_T$ computed by TD(λ) is approximately the same as $\boldsymbol{\theta}_T$ computed by the λ-return algorithm.

*see: Bertsekas, D. P. and Tsitsiklis, J. N. (1996). Neuro-Dynamic Programming.

How to set λ?

- λ controls a trade-off between variance and bias of the update target, in general the best value of λ will differ from domain to domain.

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- theoretical results for TD(λ) (Peter Dayan, 1992):
  - for $\lambda = 1$: convergence to LMS solution
  - for $\lambda < 1$: convergence to a different fixed point

online vs offline methods

- **online method:** the value of each visited state is updated at the time step immediately after the visit.
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*Is it possible to construct an online version of the λ-return algorithm that approximates TD(λ) at **all** time steps?*

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- To compute $\boldsymbol{\theta}_t$, no data beyond time t should be used.
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the trick:

Use update targets that grow with the data-horizon.

interim update target

- normal update targets:

$$\phi_t \quad \rightarrow \quad U_t$$

- interim update targets:

$$\phi_t \quad \rightarrow \quad U_t^1,\ U_t^2,\ U_t^3, \ldots,\ U_t^h$$

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data-horizon: time step up to which data is observed

interim λ-return

- λ-return: $$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

- interim λ-return: replace all *n*-step returns with $n > h\text{-}t$ with the $(h\text{-}t)$-step return

$$\begin{aligned}
G_t^{\lambda|h} &= (1-\lambda)\sum_{n=1}^{h-t-1}\lambda^{n-1}G_t^{(n)} + (1-\lambda)\sum_{n=h-t}^{\infty}\lambda^{n-1}G_t^{(h-t)} \\
&= (1-\lambda)\sum_{n=1}^{h-t-1}\lambda^{n-1}G_t^{(n)} + G_t^{(h-t)}\cdot\left[(1-\lambda)\sum_{n=h-t}^{\infty}\lambda^{n-1}\right] \\
&= (1-\lambda)\sum_{n=1}^{h-t-1}\lambda^{n-1}G_t^{(n)} + G_t^{(h-t)}\cdot\left[\lambda^{h-t-1}(1-\lambda)\sum_{k=0}^{\infty}\lambda^{k}\right] \\
&= (1-\lambda)\sum_{n=1}^{h-t-1}\lambda^{n-1}G_t^{(n)} + \lambda^{h-t-1}G_t^{(h-t)}
\end{aligned}$$

update sequences

$$t = 1: \quad \boldsymbol{\theta}_1^1 = \boldsymbol{\theta}_0^1 + \alpha\Big(G_0^{\lambda|1} - (\boldsymbol{\theta}_0^1)^\top \boldsymbol{\phi}_0\Big)\boldsymbol{\phi}_0$$

$$\begin{aligned} t = 2: \quad \boldsymbol{\theta}_1^2 &= \boldsymbol{\theta}_0^2 + \alpha\Big(G_0^{\lambda|2} - (\boldsymbol{\theta}_0^2)^\top \boldsymbol{\phi}_0\Big)\boldsymbol{\phi}_0 \\ \boldsymbol{\theta}_2^2 &= \boldsymbol{\theta}_1^2 + \alpha\Big(G_1^{\lambda|2} - (\boldsymbol{\theta}_1^2)^\top \boldsymbol{\phi}_1\Big)\boldsymbol{\phi}_1 \end{aligned}$$

$$\begin{aligned} t = 3: \quad \boldsymbol{\theta}_1^3 &= \boldsymbol{\theta}_0^3 + \alpha\Big(G_0^{\lambda|3} - (\boldsymbol{\theta}_0^3)^\top \boldsymbol{\phi}_0\Big)\boldsymbol{\phi}_0 \\ \boldsymbol{\theta}_2^3 &= \boldsymbol{\theta}_1^3 + \alpha\Big(G_1^{\lambda|3} - (\boldsymbol{\theta}_1^3)^\top \boldsymbol{\phi}_1\Big)\boldsymbol{\phi}_1 \\ \boldsymbol{\theta}_3^3 &= \boldsymbol{\theta}_2^3 + \alpha\Big(G_2^{\lambda|3} - (\boldsymbol{\theta}_2^3)^\top \boldsymbol{\phi}_2\Big)\boldsymbol{\phi}_2 \end{aligned}$$

with $\boldsymbol{\theta}_0^t := \boldsymbol{\theta}_{init}$

online lambda-return algorithm.

$$\boldsymbol{\theta}_t := \boldsymbol{\theta}_t^t$$

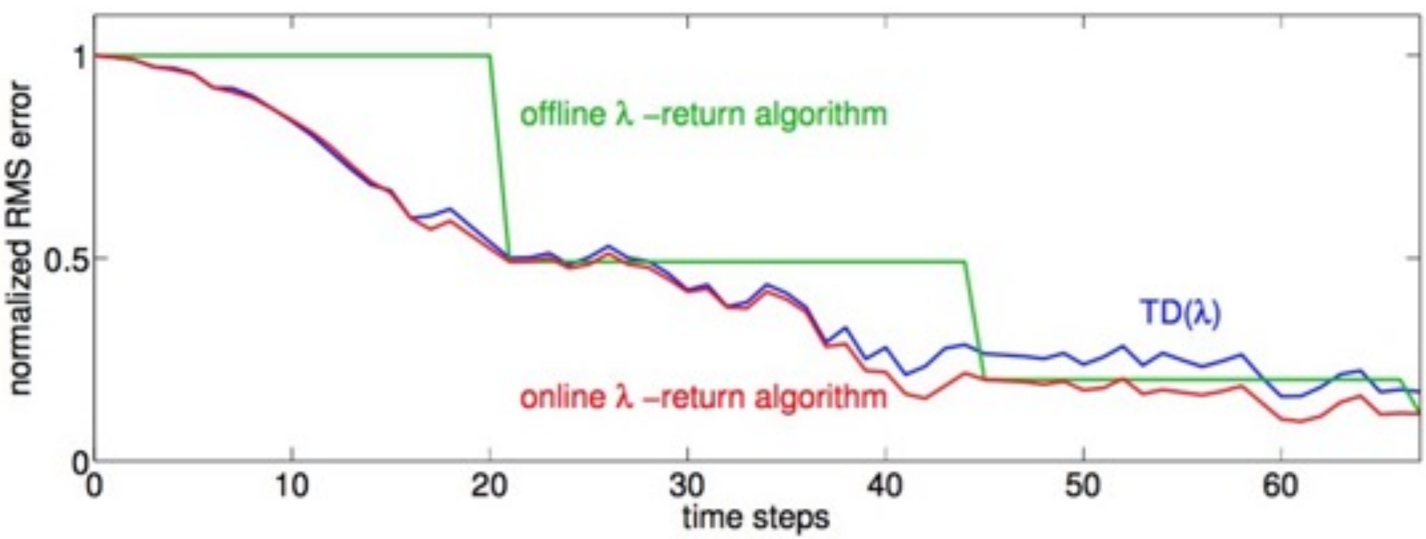
$$\boldsymbol{\theta}_{k+1}^t := \boldsymbol{\theta}_k^t + \alpha\Big(G_k^{\lambda|t} - (\boldsymbol{\theta}_k^t)^\top \boldsymbol{\phi}_k\Big)\boldsymbol{\phi}_k\,, \qquad \text{for } 0 \le k < t$$

with

$$G_k^{\lambda|t} := (1-\lambda)\sum_{n=1}^{t-k-1} \lambda^{n-1} G_k^{(n)} + \lambda^{t-k-1} G_k^{(t-k)}$$

online vs. offline λ-return algorithm

- performance on a 10-state random walk task for the first 3 episodes ($\lambda = 1$, $\alpha = 0.2$)

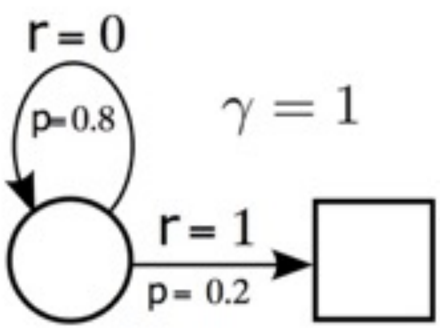


Theorem*

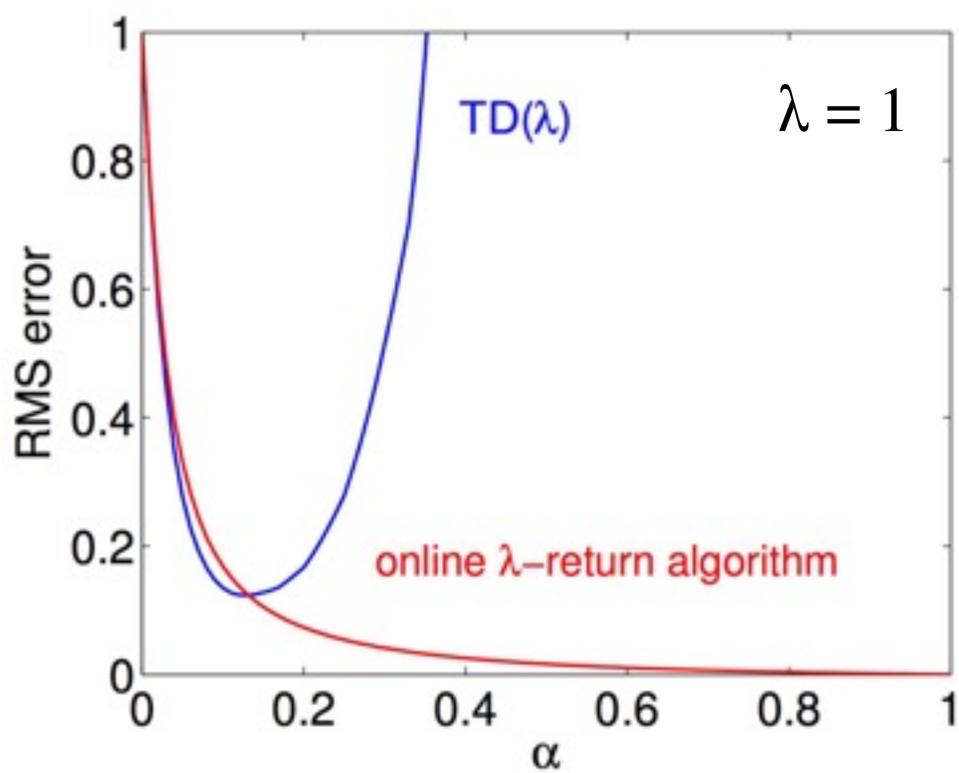
> "For small step-size, the online λ-return algorithm behaves like TD(λ) at all time steps"

*see Theorem 1: *van Seijen, H., Mahmood, A. R., Pilarski, P. M., Machado, M. C., and Sutton, R. S. True online temporal-difference learning. Journal of Machine Learning Research, 17(145):1–40, 2016.*

Sensitivity of TD(λ) to Divergence



RMS error during early learning



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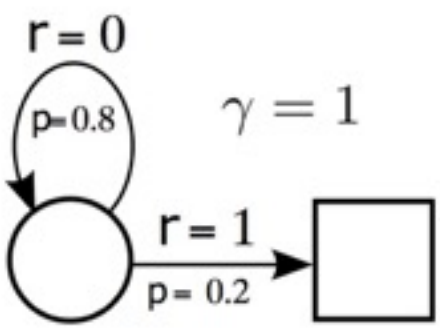

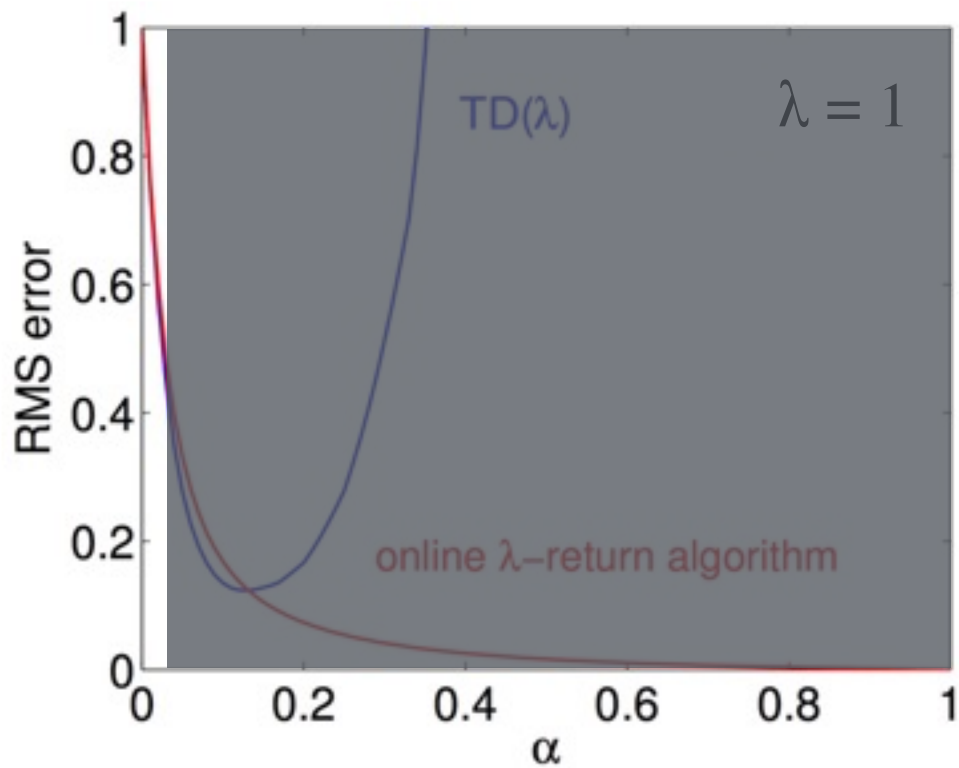

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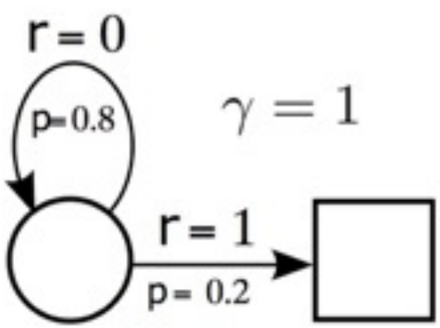

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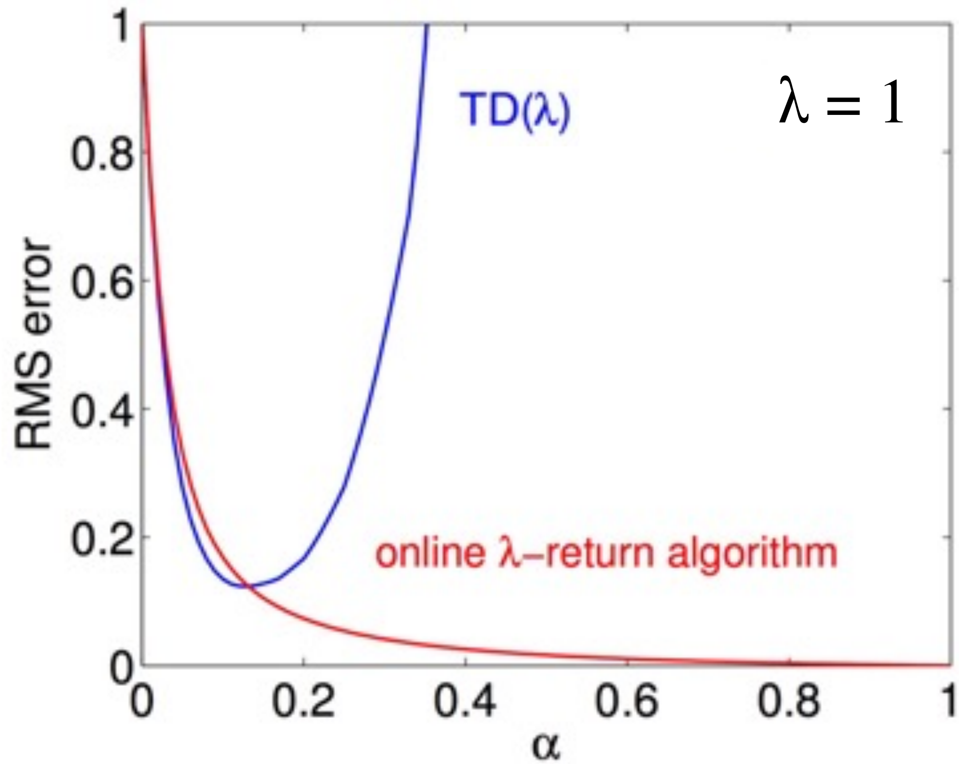

Computational Complexity

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True online TD(λ)

- true online TD(λ) is an efficient implementation of the online λ-return algorithm

$$
\begin{aligned}
\delta_t &= R_{t+1} + \gamma \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_{t+1} - \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t \\
\mathbf{e}_t &= \gamma\lambda \mathbf{e}_{t-1} + \boldsymbol{\phi}_t - \alpha\gamma\lambda[\mathbf{e}_{t-1}^\top \boldsymbol{\phi}_t]\,\boldsymbol{\phi}_t \\
\boldsymbol{\theta}_{t+1} &= \boldsymbol{\theta}_t + \alpha\delta_t\,\mathbf{e}_t + \alpha[\boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t - \boldsymbol{\theta}_{t-1}^\top \boldsymbol{\phi}_t][\mathbf{e}_t - \boldsymbol{\phi}_t]
\end{aligned}
$$

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dutch trace

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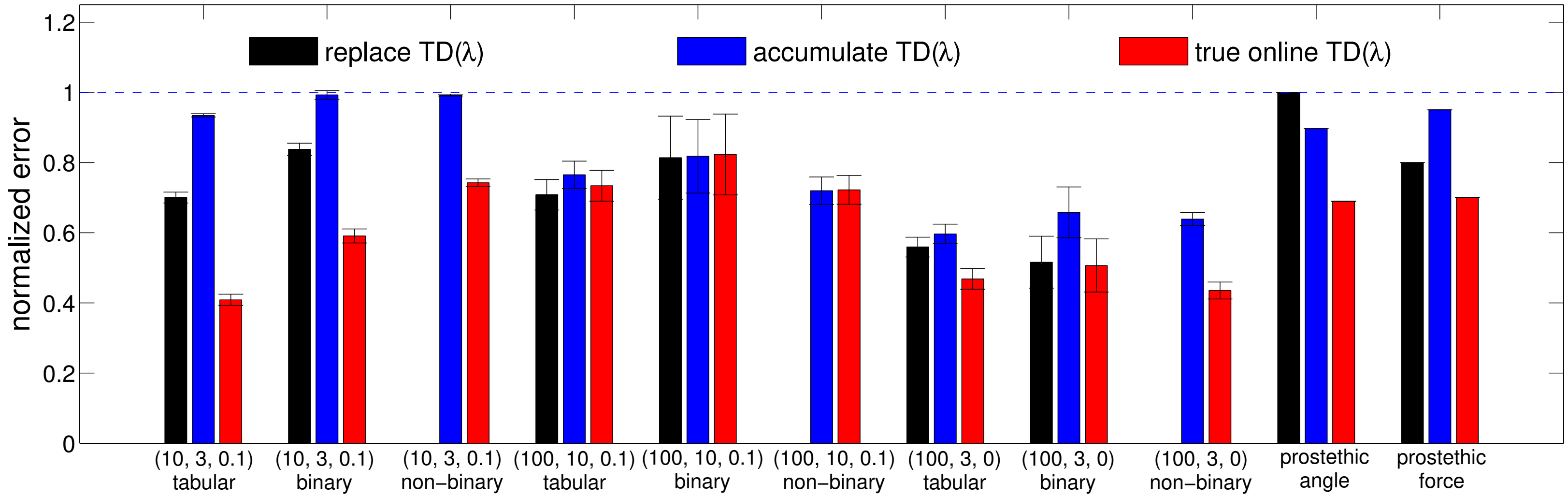
\end{aligned}$$

Empirical Comparison



- in all domains, true online TD(λ) performs at least as good as replace/accumulate TD(λ)

Outline

- Part 1:   true online temporal-difference learning
- **Part 2:   effective multi-step learning for non-linear FA**

Computational Cost

- Implementing the online forward view is computationally very expensive.
  - Memory as well as computation time per time step grows over time.

- In the case of linear FA there is an efficient backward view with exact equivalence: true online TD(λ).
  - Computational cost is span-independent and linear in the number of features.

- In the case of non-linear FA such an efficient backward view does not appear to exist.

New Research Question

Is it possible to construct a different online forward view, with a performance close to that of the online λ-return algorithm, that can be implemented efficiently?

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Answer: Yes

forward TD(λ)

- Uses online λ-return with fixed horizon, K steps ahead: $G_t^{\lambda|t+K}$
- As a consequence, updates occur with a delay of K time steps.
- Computational cost is span-independent and efficient (computational complexity equal to TD(0)).

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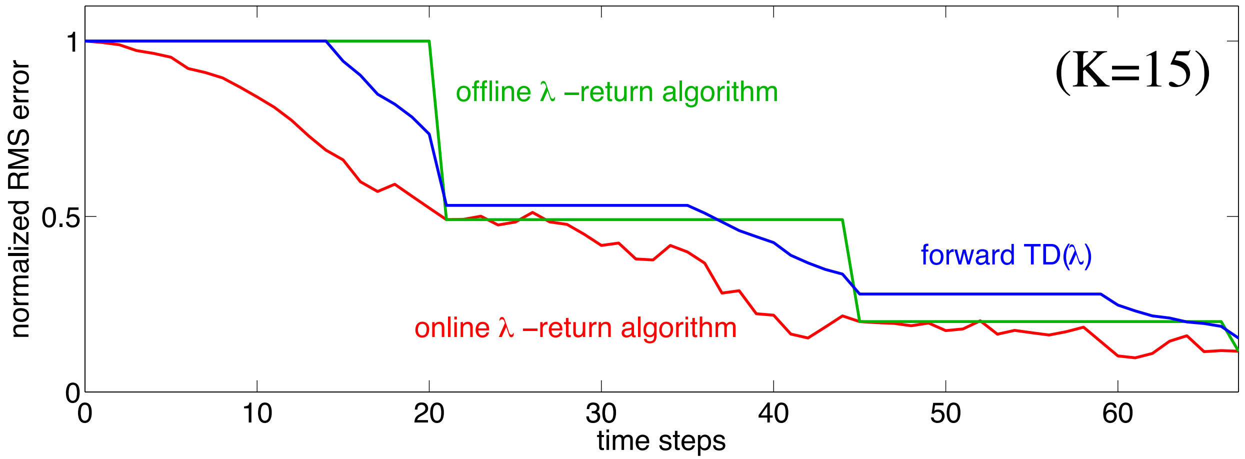

How to set K?

- Setting K involves a trade-off:
  - small K : less delay in updates
  - large K : better approximation of the λ-return
- How well $G_t^{\lambda|t+K}$ approximates G_t^{λ} depends on K, but also on $\gamma\lambda$.
- Whereas the weight of R_{t+1} in G_t^{λ} is 1, the weight of R_{t+n} is only $\gamma\lambda^{n-1}$.
  - Example: $\gamma\lambda = 0.5$ and $n = 20$, then $\gamma\lambda^{n-1}$ is about 10^{-6}.
- Strategy: set K such that $\gamma\lambda^{K-1}$ is just below η, with η some tiny number like 0.01

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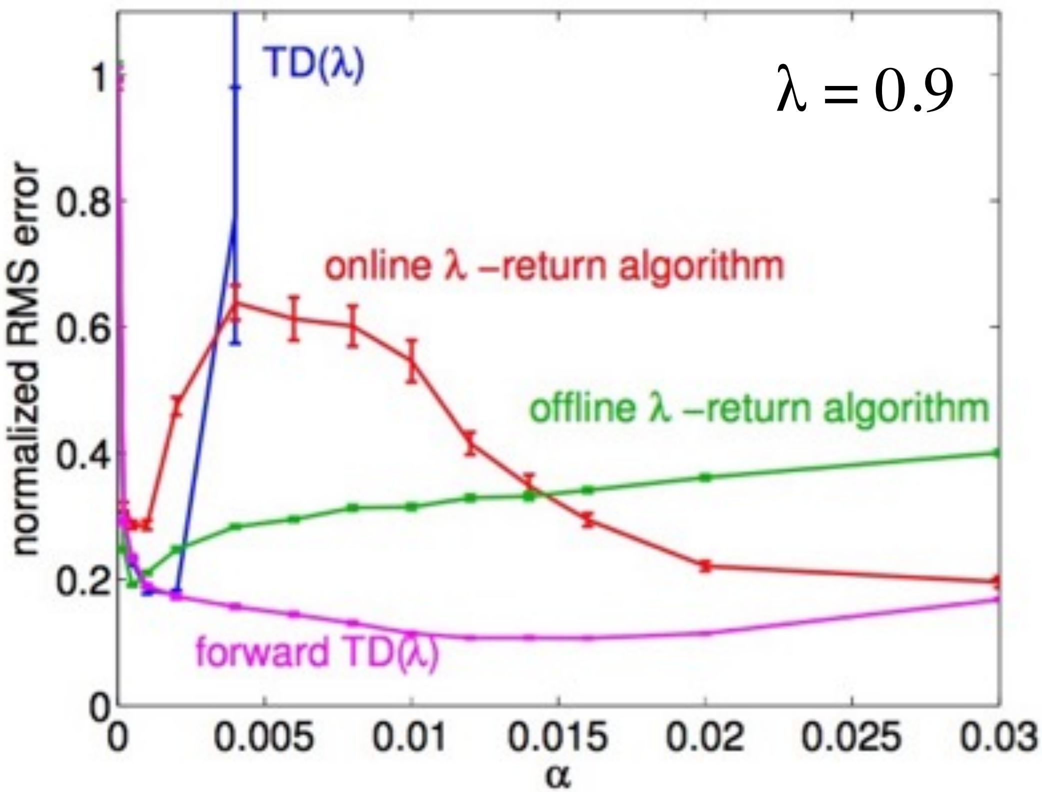
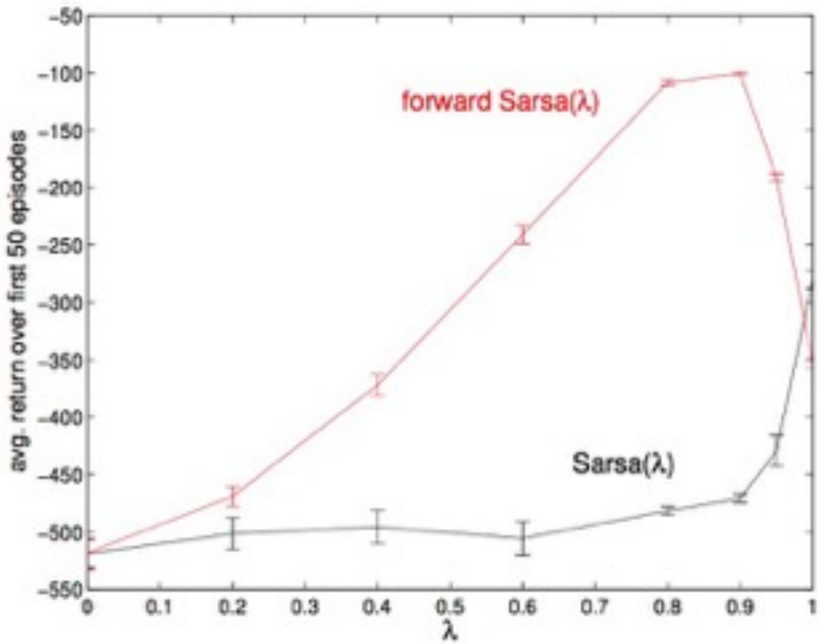
Results on Prediction Task

mountain-car task with non-linear FA

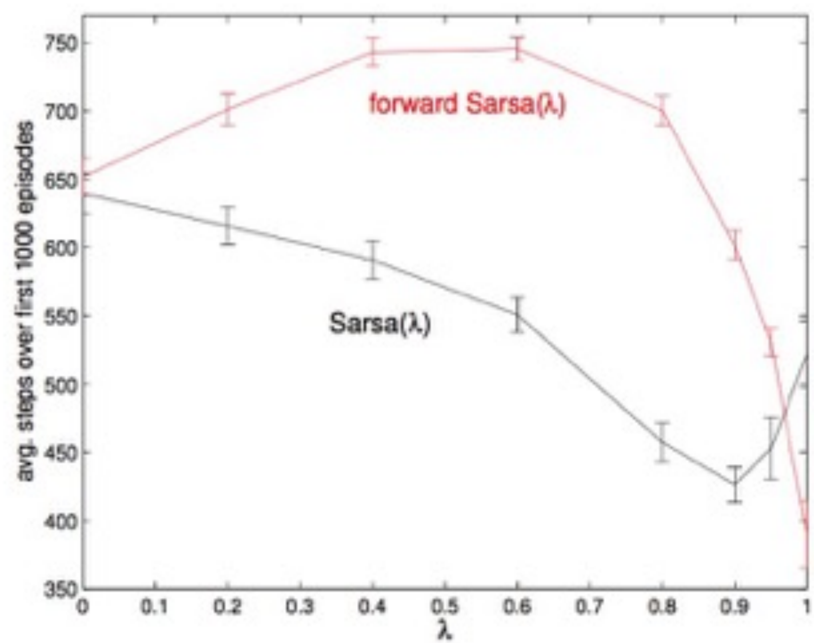


Results on 2 Control Tasks

mountain-car task



cart-pole task



Summary

- The online λ-return algorithm outperforms TD(λ), but is computationally very expensive.
- For linear FA, an efficient backward view exists with exact equivalence: true online TD(λ).
- For non-linear FA, such an efficient backward view does not appear to exist.
- Forward TD(λ) approximates the online λ-return algorithm and can be implemented efficiently for non-linear FA.
- The price that forward TD(λ) pays is a delay in the updates.
- Empirically, forward TD(λ) outperforms TD(λ)

Thank you!

References:
1) *van Seijen, H., Mahmood, A. R., Pilarski, P. M., Machado, M. C., and Sutton, R. S.*
True online temporal-difference learning. Journal of Machine Learning Research, 17(145):1–40, 2016.
2) *van Seijen, H. Effective multi-step temporal-difference learning for non-linear function approximation.*
arXiv:1608.05151, 2016.