Normal and Poisson Distributions

Textbook Sections 3.3–3.4

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Learning Objectives

By the end of today's lecture, you will be able to:

1. Describe the characteristics of the Normal distribution
2. Calculate and interpret Z-scores to standardize observations
3. Use the Empirical Rule (68-95-99.7) to estimate probabilities
4. Calculate probabilities using `pnorm()` and find percentiles using `qnorm()`
5. Apply the Normal approximation to the Binomial distribution
6. Recognize when to use a Poisson distribution and calculate probabilities

Roadmap for Today

Part 1: Normal Distribution Basics

- Continuous distributions
- Normal parameters (μ, σ)
- Standard Normal distribution

Part 2: Z-scores & The Empirical Rule

- Standardization
- 68-95-99.7 rule
- Identifying unusual observations

Part 3: Calculating with R

- `pnorm()` for probabilities
- `qnorm()` for percentiles
- Real-world examples

Part 4: Normal Approximation

- When Binomial → Normal
- Conditions: $np \geq 10, n(1-p) \geq 10$
- Continuity correction

Part 5: Poisson Distribution

- Modeling rare events
- Rate × time = λ
- Poisson approximation to Binomial

Part 6: Wrap-up

- Summary
- Next steps

Last time: Discrete vs. Continuous Random Variables

Discrete Random Variable

A **discrete r.v.** takes on:

- A finite number of values, OR
- A countably infinite number of values

Examples:

- Number of heads in 10 coin flips
- Number of students in a class
- Number of COVID cases per day

Continuous Random Variable

A **continuous r.v.** can take:

- Any real value in an interval
- Any value in a union of intervals

Examples:

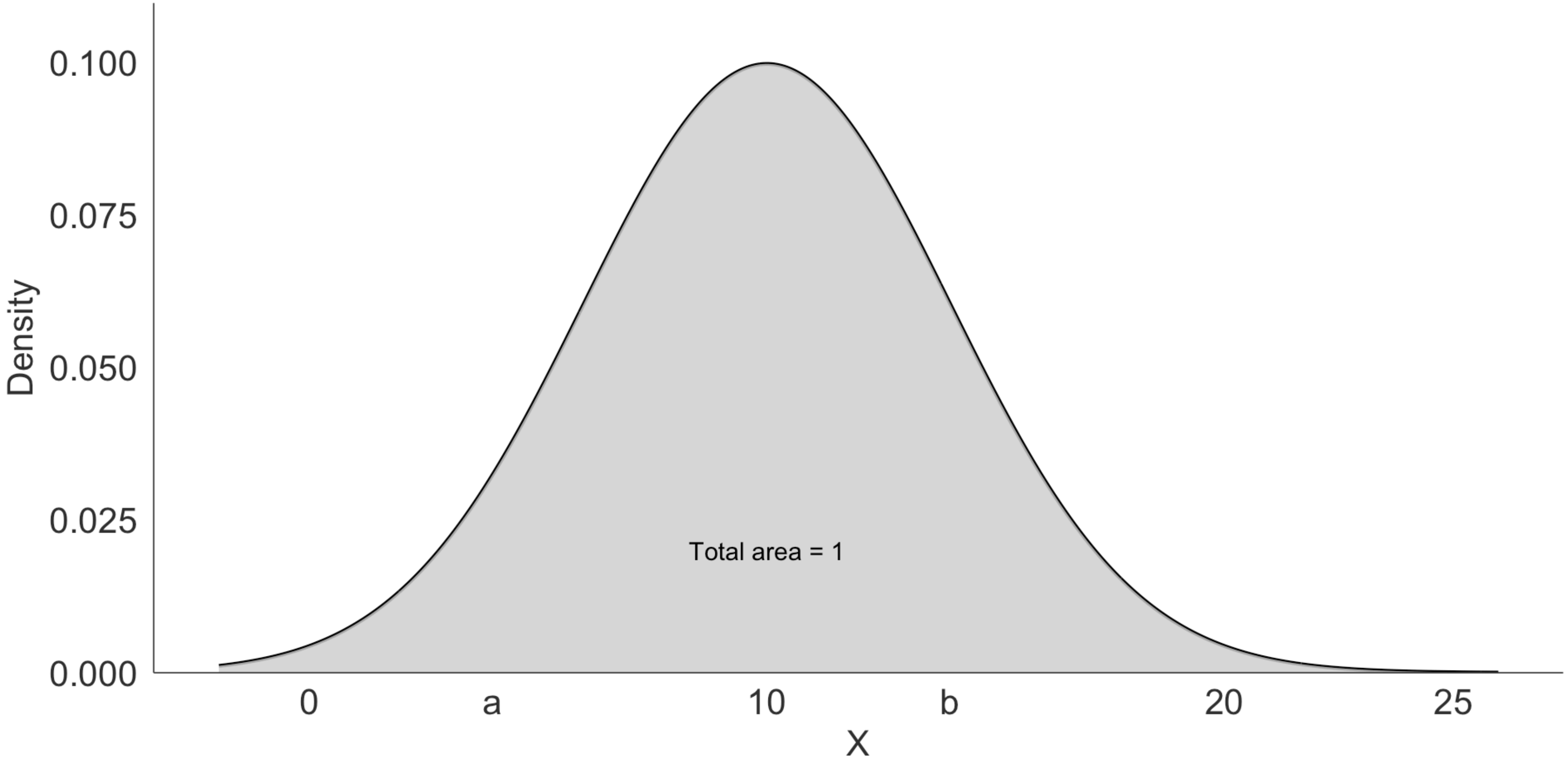
- Height
- Blood pressure
- Time until an event occurs

Today's focus: Continuous random variables

Continuous random variables

Continuous rv in general

- The distribution of a continuous rv is governed by a *density function/curve*.
- Probabilities are calculated as *area* under the curve over an *interval*.
- Total area beneath the density function/curve is 1.

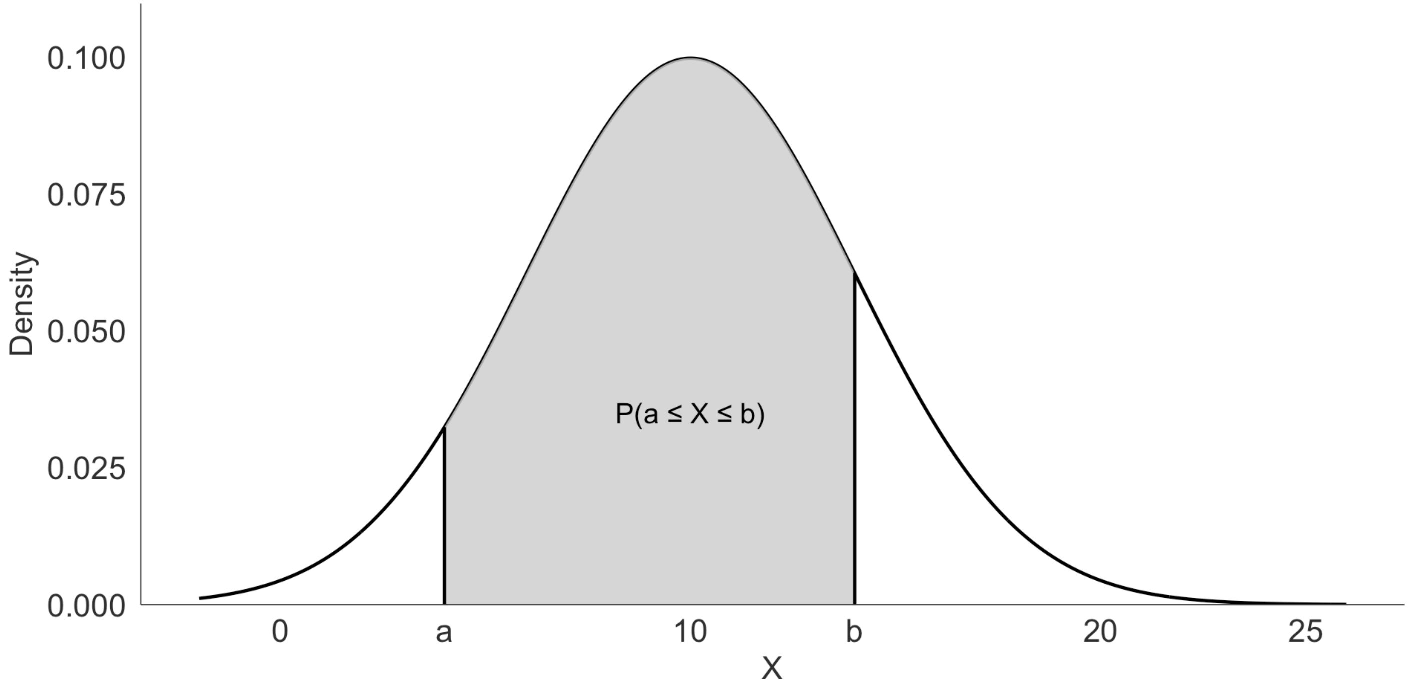


Probability between *a* and *b*

The area beneath the density curve/function between two points *a* and *b* represents the probability that the random variable (X, say) will assume some value between *a* and *b*

When working with continuous random variables, probability is found for **intervals of values** rather than **individual values**.

- The probability that a continuous r.v. X takes on any single individual value is 0
- That is, $P(X = x) = 0$.
- Thus, $P(a < X < b)$ is equivalent to $P(a \leq X \leq b)$



Probability at exactly *a*

For a **continuous** random variable X:

- Probability comes from **areas under the curve**
- Areas require **width**
- A single point has **no width**

So:

- $P(X = a) = 0$
- Only intervals have non-zero probability (for example, $P(a \leq X \leq b)$)

Intuition: you can shade an interval, but you can't shade a point.

All the following are equivalent

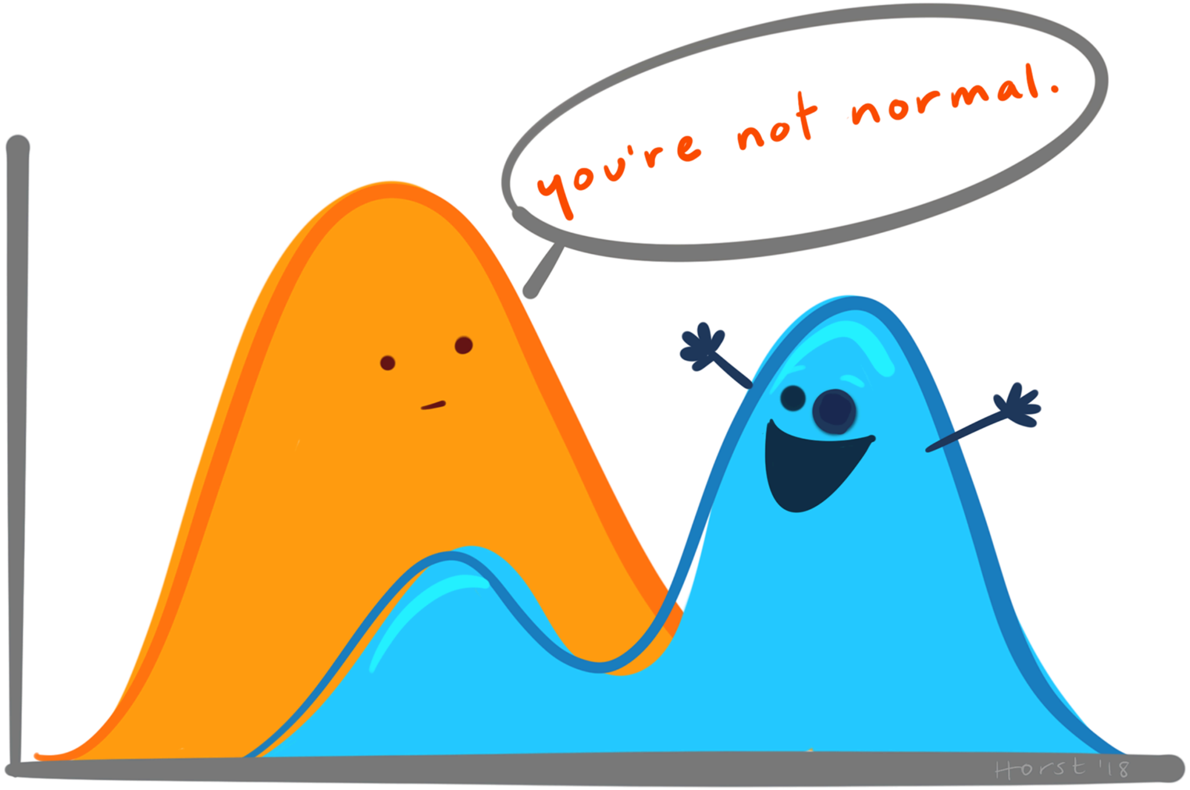
For a **continuous** r.v. X, all of the following are equivalent:

$$P(a \leq X \leq b) = P(a < X \leq b)$$

$$= P(a \leq X < b) = P(a < X < b)$$

That is, we can safely ignore equality when defining the endpoints of a given interval. **[This is certainly not true for discrete rv]**

Normal distribution



Artwork by Allison Horst

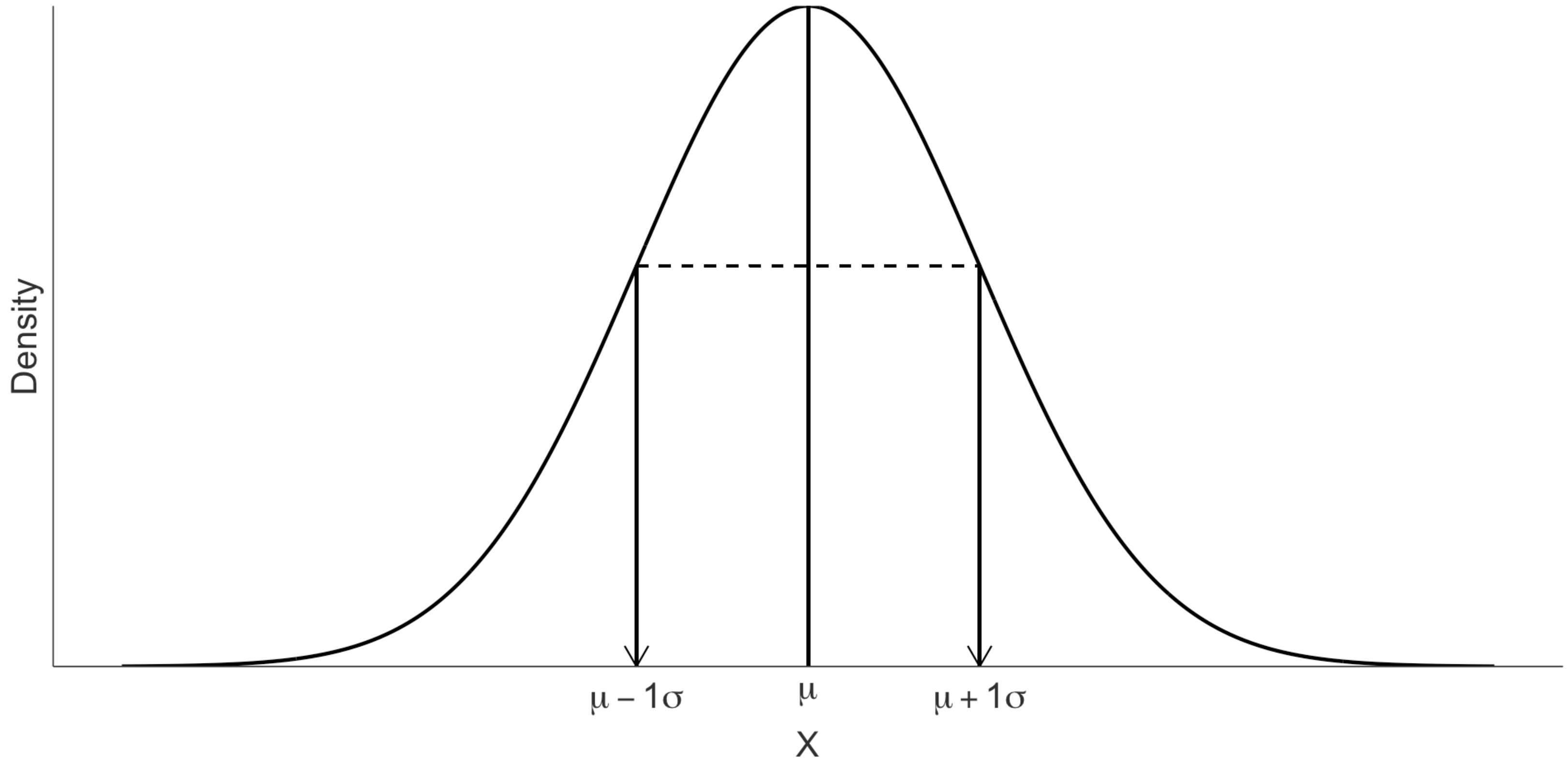
Normal distribution

- A random variable X is modeled with a normal distribution:
- **Shape:** symmetric, unimodal bell curve
- **Center:** mean μ
- **Spread (variability):** standard deviation σ
- Shorthand for a random variable, X, that has a Normal distribution:

$$X \sim \text{Normal}(\mu, \sigma)$$

- **Example:** We recorded the high temperature in the past 100 years for today. The mean high is 19°C (66.2°F)

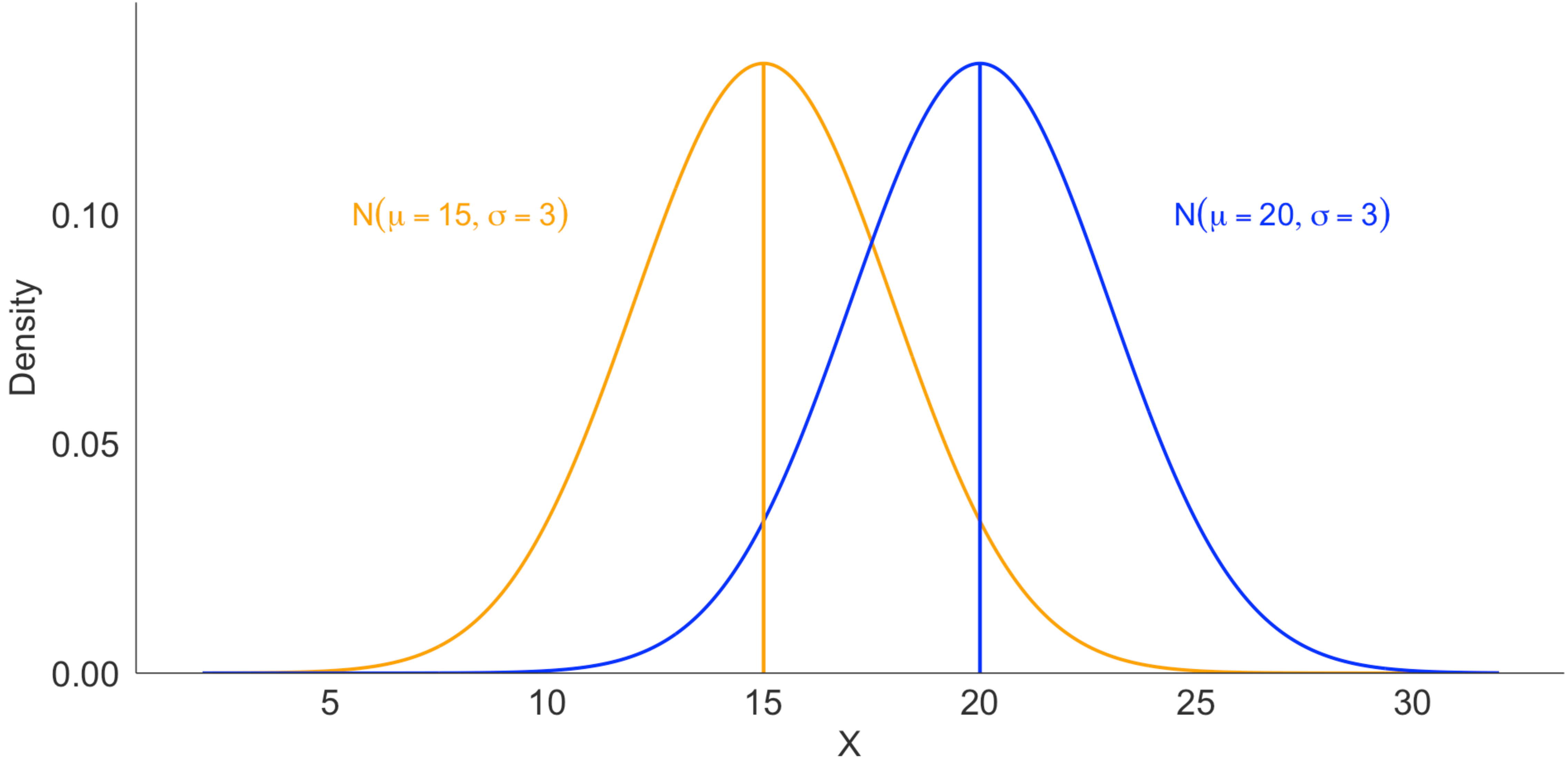
Anatomy of the Normal curve



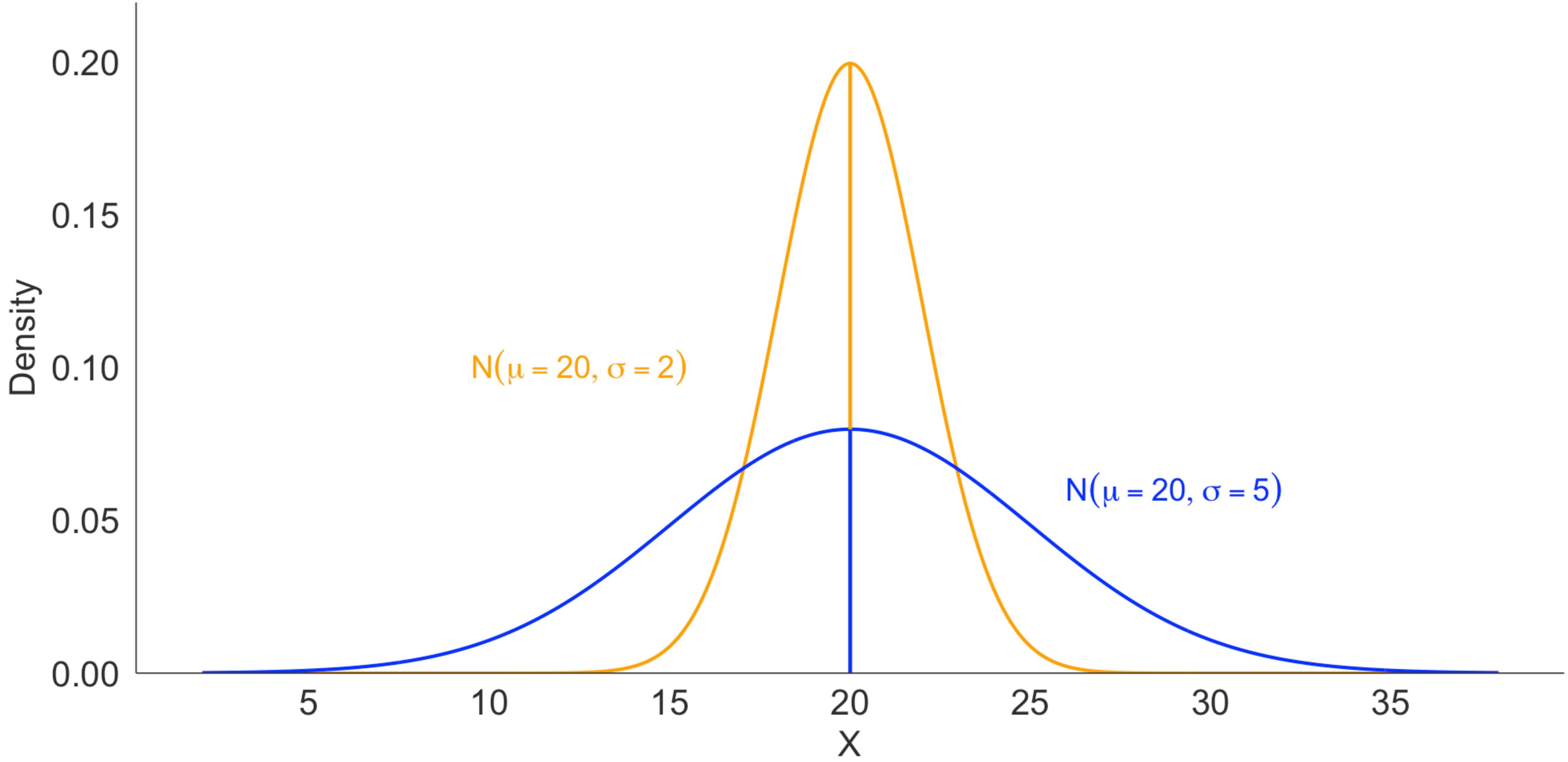
The **Normal distribution has a closed-form equation**, but for this course:

- You do **not** need to memorize it
- You do **not** need to compute it by hand
- What matters is how μ and σ shape the curve

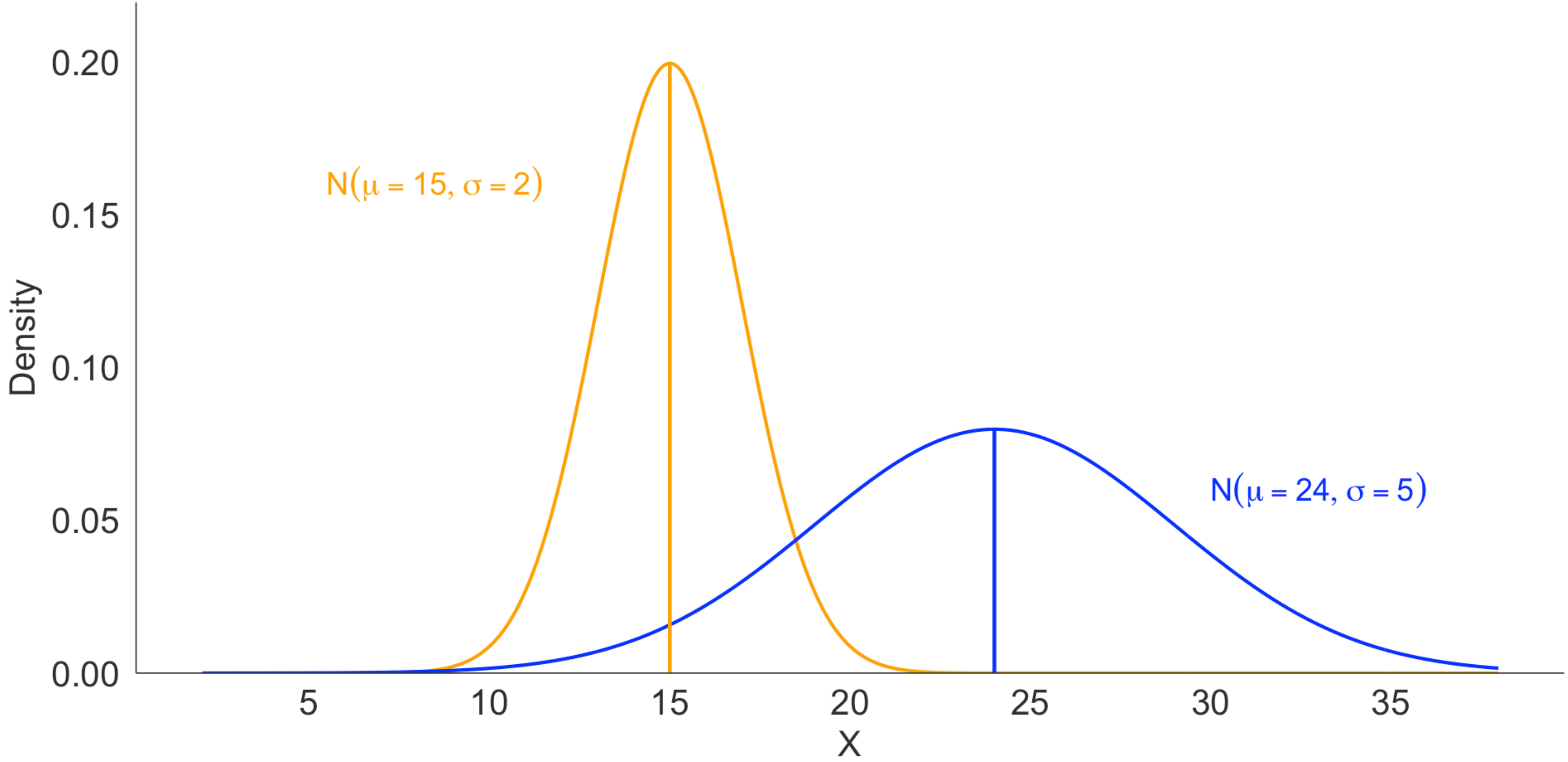
Different means, Same SD



Same mean, Different SD



Different means, different SD



Z-scores & Standardization

Why standardize?

Different normal distributions have different scales:

- Heights measured in centimeters: $X \sim N(170, 10)$
- Test scores: $Y \sim N(75, 8)$
- Blood pressure: $Z \sim N(120, 15)$

Problem: How do we compare observations across different distributions?

Solution: Convert to a common scale using **Z-scores**

What is a Z-score?

The **Z-score** tells you how many standard deviations an observation is from the mean:

Suppose X is an arbitrary random variable that follows a normal distribution with mean μ and standard deviation σ, i.e. $X \sim N(\mu, \sigma)$

$$Z = \frac{X - \mu}{\sigma} \iff X = \mu + Z\sigma$$

Notation:

- X = original observation
- μ = mean of the distribution
- σ = standard deviation of the distribution
- Z = standardized score

Interpreting Z-scores

Z-score	Interpretation
$Z = 0$	Exactly at the mean
$Z = 1$	One SD above the mean
$Z = -1$	One SD below the mean
$Z = 2.5$	2.5 SDs above the mean
$Z = -1.8$	1.8 SDs below the mean

Rule of thumb:

- Z-scores between -2 and 2 are common
- Z-scores beyond ±3 are rare (unusual observations)

Example: Height standardization

Suppose adult male heights follow $N(\mu = 175, \sigma = 7)$ cm.

Question: What is the Z-score for a man who is 189 cm tall?

$$Z = \frac{X - \mu}{\sigma} = \frac{189 - 175}{7} = \frac{14}{7} = 2$$

Interpretation: This man is 2 standard deviations above the mean height.

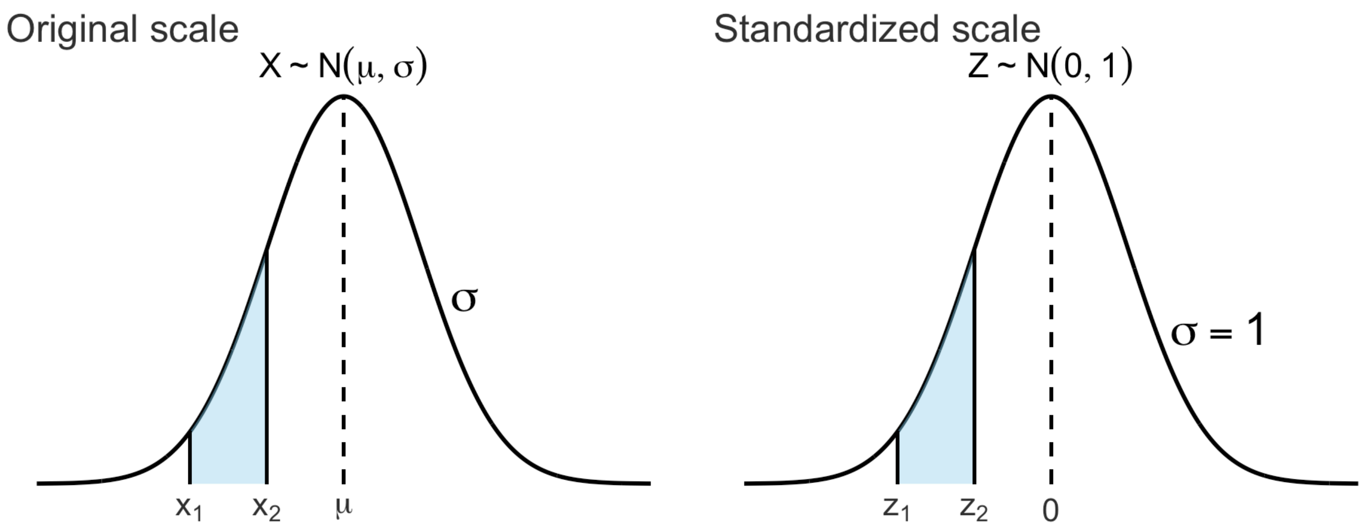
Standard Normal Distribution

When we standardize a normal random variable, we get the **Standard Normal distribution**:

$$Z \sim N(\mu = 0, \sigma = 1)$$

Properties:

- Mean = 0
- Standard deviation = 1
- Denoted by Z
- All normal distributions can be converted to this standard form

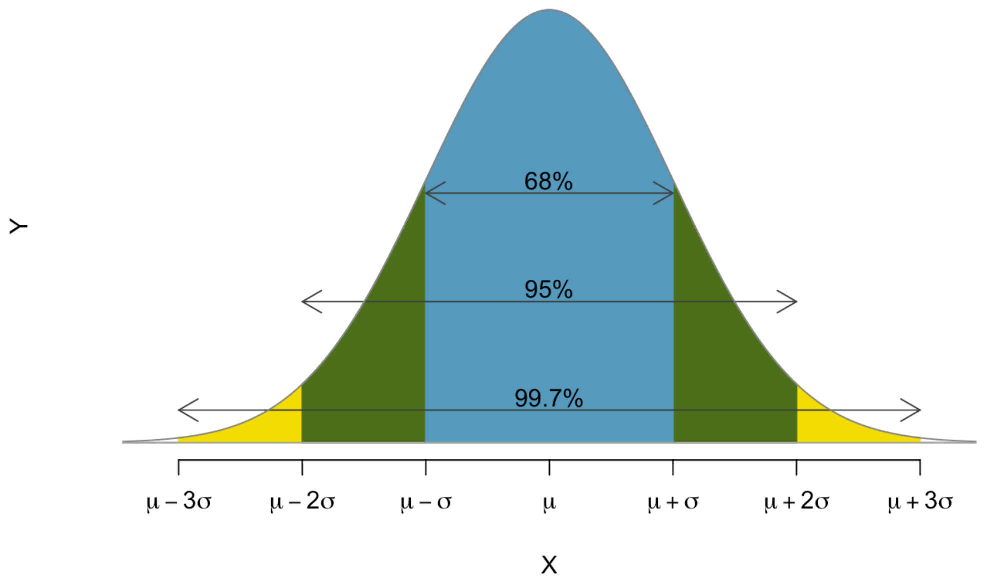


The Empirical Rule

The 68-95-99.7 Rule

For **any** normal distribution:

- **68%** of observations fall within **1 SD** of the mean
- **95%** of observations fall within **2 SDs** of the mean
- **99.7%** of observations fall within **3 SDs** of the mean



Empirical rule (68–95–99.7).

Source: *OpenIntro Biostatistics*.

Using the Empirical Rule

Example: IQ scores follow $N(\mu = 100, \sigma = 15)$

Questions:

1. What percentage of people have IQ between 85 and 115?

- 85 = 100 - 15 = $\mu - \sigma$
- 115 = 100 + 15 = $\mu + \sigma$
- **Answer: About 68%.** So about 2/3 of people fall within 15 points of 100.

2. What percentage have IQ between 70 and 130?

- 70 = 100 - 30 = $\mu - 2\sigma$
- 130 = 100 + 30 = $\mu + 2\sigma$
- **Answer: About 95%**

Why is the Empirical Rule useful?

1. **Quick estimates** without calculations
2. **Identify unusual observations**
   - Values beyond 2 SDs are uncommon (<5%)
   - Values beyond 3 SDs are very rare (<0.3%)
3. **Check data quality**
   - If your data doesn't follow this pattern, it may not be normally distributed

Calculating Probabilities with R

Calculating probabilities from a Normal distribution

There are several ways to calculate probabilities from a normal distribution:

1. **Calculus**
   (not for us!)
2. **Normal probability tables**
   - Included in the **textbook (Appendix B.1)**
   - Helpful historically, but not required for this course
3. **R commands (what we will use)**
   - $P(Z \leq q) =$ `pnorm(q, mean = 0, sd = 1)`
   - $P(Z > q) =$ `pnorm(q, mean = 0, sd = 1, lower.tail = FALSE)`

> In this course, we will calculate normal probabilities using **R**.

R functions for the Normal distribution

Four key functions

Function	Purpose	Example
`dnorm()`	**Density** at a point	Height of curve at x
`pnorm()`	**Cumulative probability**	$P(X \leq x)$
`qnorm()`	**Quantile/percentile**	What x gives $P(X \leq x) = p$?
`rnorm()`	**Random** samples	Generate random normal data

Most common: `pnorm()` and `qnorm()`

`pnorm()`: Cumulative probabilities

`pnorm(q, mean, sd, lower.tail = TRUE)`

Cumulative probability is the total area under the curve to the left of a value, i.e. the probability that a random variable is **less than or equal to a value**: $P(X \leq q)$.

- `q` = the value you're interested in
- `mean` $= \mu$
- `sd` $= \sigma$
- `lower.tail = TRUE` $\longrightarrow P(X \leq q)$
- `lower.tail = FALSE` $\longrightarrow P(X > q)$

Example: For standard normal $Z \sim N(0, 1)$:

```
# P(Z ≤ 1.96)
pnorm(1.96, mean = 0, sd = 1)
```

```
[1] 0.9750021
```

Interpretation: About 97.5% of observations fall below Z = 1.96

Example: Standard normal probabilities

Let $Z \sim N(0, 1)$. Calculate:

1. $P(Z < 2.67)$

```r
pnorm(2.67)  # mean=0, sd=1 is the default
```

```
[1] 0.9962074
```

2. $P(Z > -0.37)$

```r
pnorm(-0.37, lower.tail = FALSE)
```

```
[1] 0.6443088
```

Or

```r
1 - pnorm(-0.37)
```

```
[1] 0.6443088
```

Example: Standard normal probabilities (continued)

3. $P(-2.18 < Z < 2.46)$

```r
pnorm(2.46) - pnorm(-2.18)
```

```
[1] 0.9784244
```

4. P(Z = 1.53)

```r
# For continuous distributions, P(X = x) = 0
0
```

```
[1] 0
```

Remember: For continuous random variables, the probability of any single exact value is 0!

Example: General normal distribution

Suppose the distribution of diastolic blood pressure (DBP) in 35- to 44-year old men is normally distributed with mean 80 mm Hg and variance 144 mm Hg.

$$X \sim N(\mu = 80, \sigma = 12)$$

(Note: $variance = 144$, so $SD = \sqrt{144} = 12$)

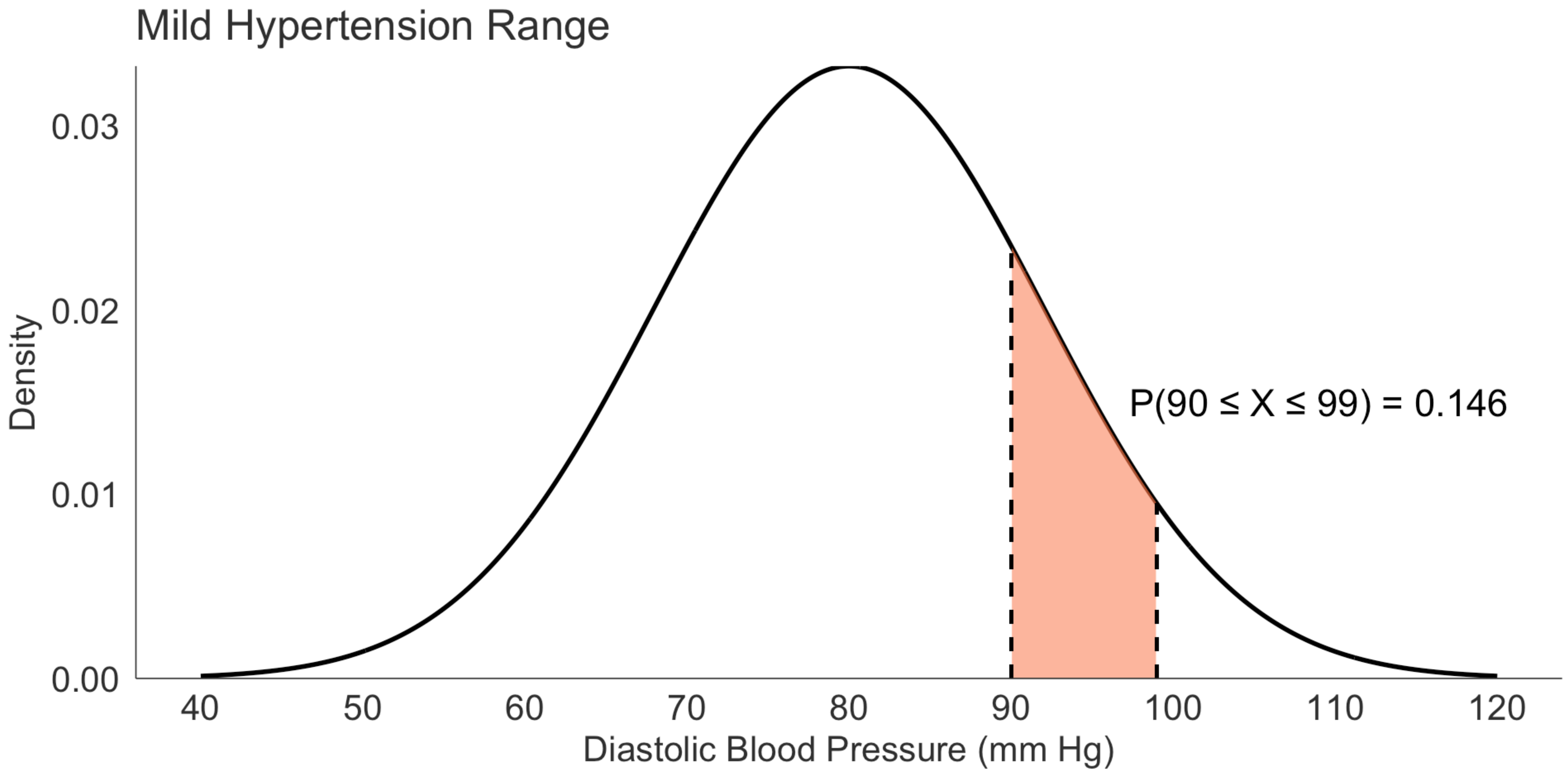
Question 1: What proportion has mild hypertension (DBP between 90 and 99)?

```r
# P(90 ≤ X ≤ 99)
pnorm(99, mean = 80, sd = 12) - pnorm(90, mean = 80, sd = 12)
```

```
[1] 0.1456556
```

About 14.6% have mild hypertension.

Visualizing the probability



Same probability, different scale

Recall the Z-score transformation:

$$Z = \frac{X - \mu}{\sigma}$$

This means any probability from a normal distribution can be computed in two equivalent ways:

Original scale

```r
pnorm(99, mean = 80, sd = 12)
```

```
[1] 0.9433272
```

Standardized scale

```r
pnorm((99 - 80) / 12, mean = 0, sd = 1)
```

```
[1] 0.9433272
```

- These two probabilities are **identical**.
- Standardization changes the **scale**, not the probability.

Finding Percentiles

`qnorm()`: Finding percentiles

```
qnorm(p, mean, sd, lower.tail = TRUE)
```

A **percentile** is the value below which a given percentage of observations fall;
`qnorm()` returns the **value on the x-axis** corresponding to a cumulative probability.

- `p` = the probability/percentile (as a decimal)
- `mean` = μ
- `sd` = σ
- Returns the **value** where $P(X \leq value) = p$

Example: What Z-score has 97.5% of data below it?

```r
qnorm(0.975, mean = 0, sd = 1)
```

```
[1] 1.959964
```

This is the famous **1.96** critical value!

Example: Blood pressure percentiles

DBP: $X \sim N(80, 12)$

Question 2: What is the 10th percentile?

```r
qnorm(0.10, mean = 80, sd = 12)
```

```
[1] 64.62138
```

10% of men have DBP below about 64.6 mm Hg.

Question 3: What is the 95th percentile?

```r
qnorm(0.95, mean = 80, sd = 12)
```

```
[1] 99.73824
```

95% of men have DBP below about 99.7 mm Hg.

Connecting percentiles to Z-scores

Any percentile question can be solved with Z-scores:

What is the 10th percentile of N(80, 12)?

Step 1: Find Z-score for 10th percentile

```r
z <- qnorm(0.10)
z
```

```
[1] -1.281552
```

Step 2: Convert back to original scale

```r
x <- 80 + z * 12
x
```

```
[1] 64.62138
```

Formula: $X = \mu + Z \cdot \sigma$

Normal Approximation to Binomial

When is a binomial approximately normal?

Recall that a binomial random variable X counts the total number of successes in n independent trials, each with probability p of a success.

Probability function for $k = 0, 1, \ldots, n$:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

As *n* gets larger, the binomial distribution becomes more **symmetric** and can be approximated by a normal distribution.

Note

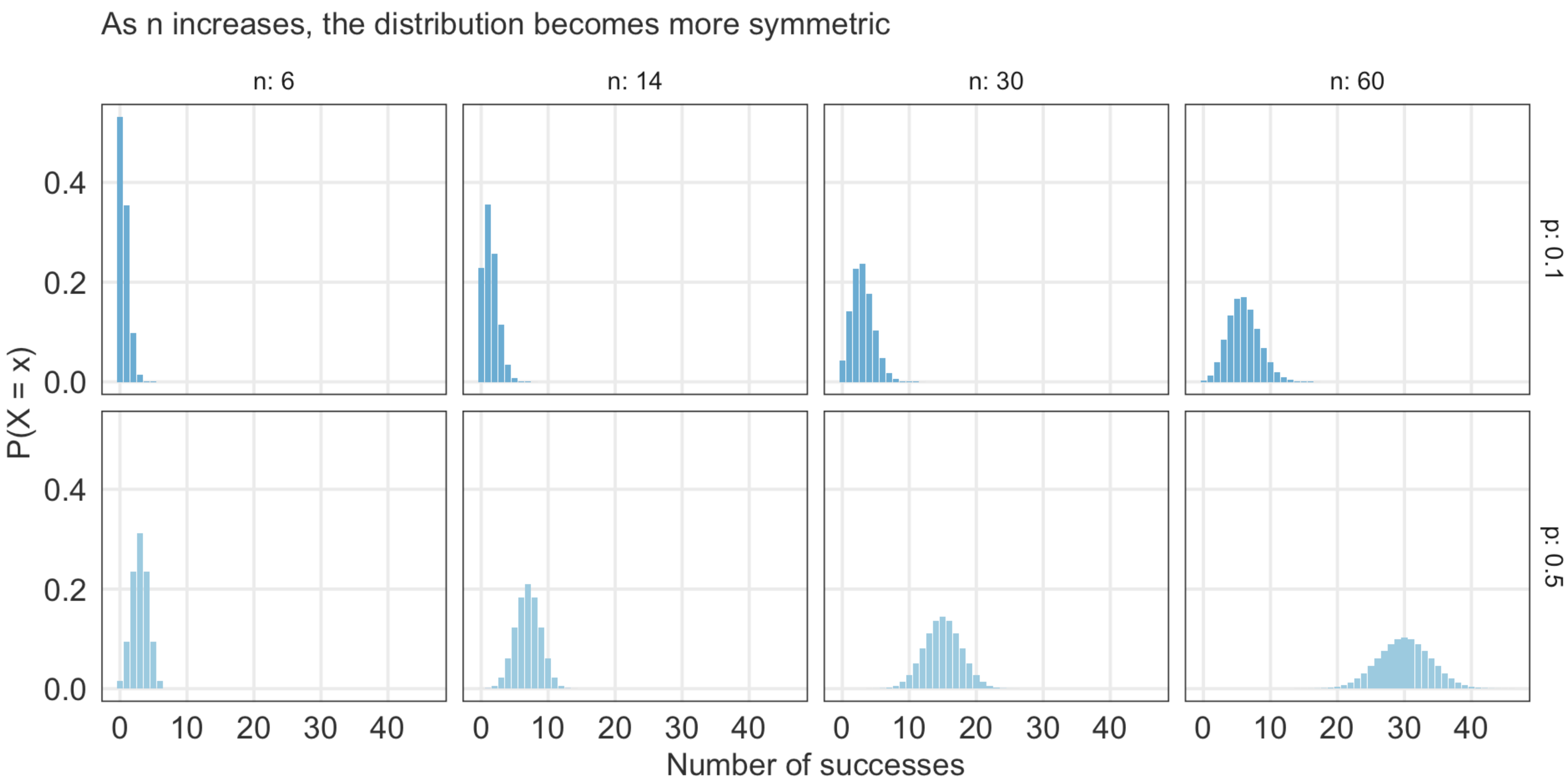
Rule of thumb: Normal approximation works well when:

$$np \geq 10 \quad \text{and} \quad n(1 - p) \geq 10$$

- Ensures sample size (n) is moderately large and the p is not too close to 0 or 1
- Other resources use other criteria (like $npq > 5$ or $np > 5$)

Visual: Binomial approaching Normal

Binomial distributions for different n (columns) and p (rows)



Normal approximation parameters

If $X \sim \text{Binomial}(n, p)$ and the conditions are met:

$$X \approx N(\mu, \sigma)$$

where:

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{np(1-p)}$$

These are the same formulas for the mean and SD of a binomial distribution!

Continuity correction

The binomial is **discrete** (0, 1, 2, …), but the Normal is **continuous**.

So we “nudge” the cutoff by 0.5 when using the Normal approximation.

- For **left-tail** probabilities ($<$ or $\leq$): **add 0.5**
  - $P(X \leq k)$ becomes $P(Y \leq k + 0.5)$
  - $P(X < k)$ becomes $P(Y \leq k - 0.5)$ (since $X < k$ means $X \leq k - 1$)
- For **right-tail** probabilities ($>$ or $\geq$): **subtract 0.5**
  - $P(X \geq k)$ becomes $P(Y \geq k - 0.5)$
  - $P(X > k)$ becomes $P(Y \geq k + 0.5)$ (since $X > k$ means $X \geq k + 1$)

Where X is binomial, and Y is the Normal approximation.

Example: COVID vaccination status

About 25% of people that test positive for Covid-19 are vaccinated for it. Suppose 100 people have tested positive for Covid-19 (independently of each other). Let X denote the number of people that are vaccinated among the 100 that tested positive. What is the probability that fewer than 20 of the people that tested positive are vaccinated?

Let X = number vaccinated among the 100.

Question: What is $P(X < 20)$?

Check conditions:

```r
n <- 100
p <- 0.25

n * p  # Should be ≥ 10
```

```
[1] 25
```

```r
n * (1 - p)  # Should be ≥ 10
```

```
[1] 75
```

✓ Conditions met!

Exact vs. Approximate probability (1/2)

Method 1: Exact (Binomial)

$$P(X < 20) = P(X \leq 19) = P(X = 0) + P(X = 1) + \cdots + P(X = 19)$$

```r
pbinom(19, size = 100, prob = 0.25)
```

```
[1] 0.09953041
```

Method 2: Normal Approximation

```r
mu <- n * p
sigma <- sqrt(n * p * (1 - p))

pnorm(19, mean = mu, sd = sigma)
```

```
[1] 0.08292833
```

Very close! The normal approximation is 0.083 vs. exact 0.100.

Exact vs. Approximate probability (2/2)

Method 3: Normal Approximation with continuity correction

Because we want $P(X < 20) = P(X \leq 19)$, we use 19.5 in the normal approximation.

```r
mu <- n * p
sigma <- sqrt(n * p * (1 - p))

pnorm(19 + 0.5, mean = mu, sd = sigma)
```

```
[1] 0.1020119
```

With continuity correction: 0.102

What's really happening?

- A binomial counts the number of successes
- When n is large, this count behaves like a normal variable
- The normal approximation lets us:
  - Use Z-scores
  - Use `pnorm()`
  - Reuse everything we just learned

When to use each method?

Use Binomial (exact):

- When n is small or moderate
- When you need exact probabilities
- R handles this easily with `pbinom()`

Use Normal approximation:

- When n is very large (computing binomial probabilities becomes slow)
- For theoretical understanding
- Historically important (before computers!)

In practice: With modern computers, we usually just use the exact binomial.

Checking whether Normal is reasonable (1/2)

In practice, we check normality visually:

- Histogram or density plot (shape: roughly symmetric, unimodal)
- Q-Q plot (points close to a straight line)
- If using a normal model: check residuals, not raw outcomes

We usually avoid hypothesis tests (e.g., Shapiro-Wilk) because:

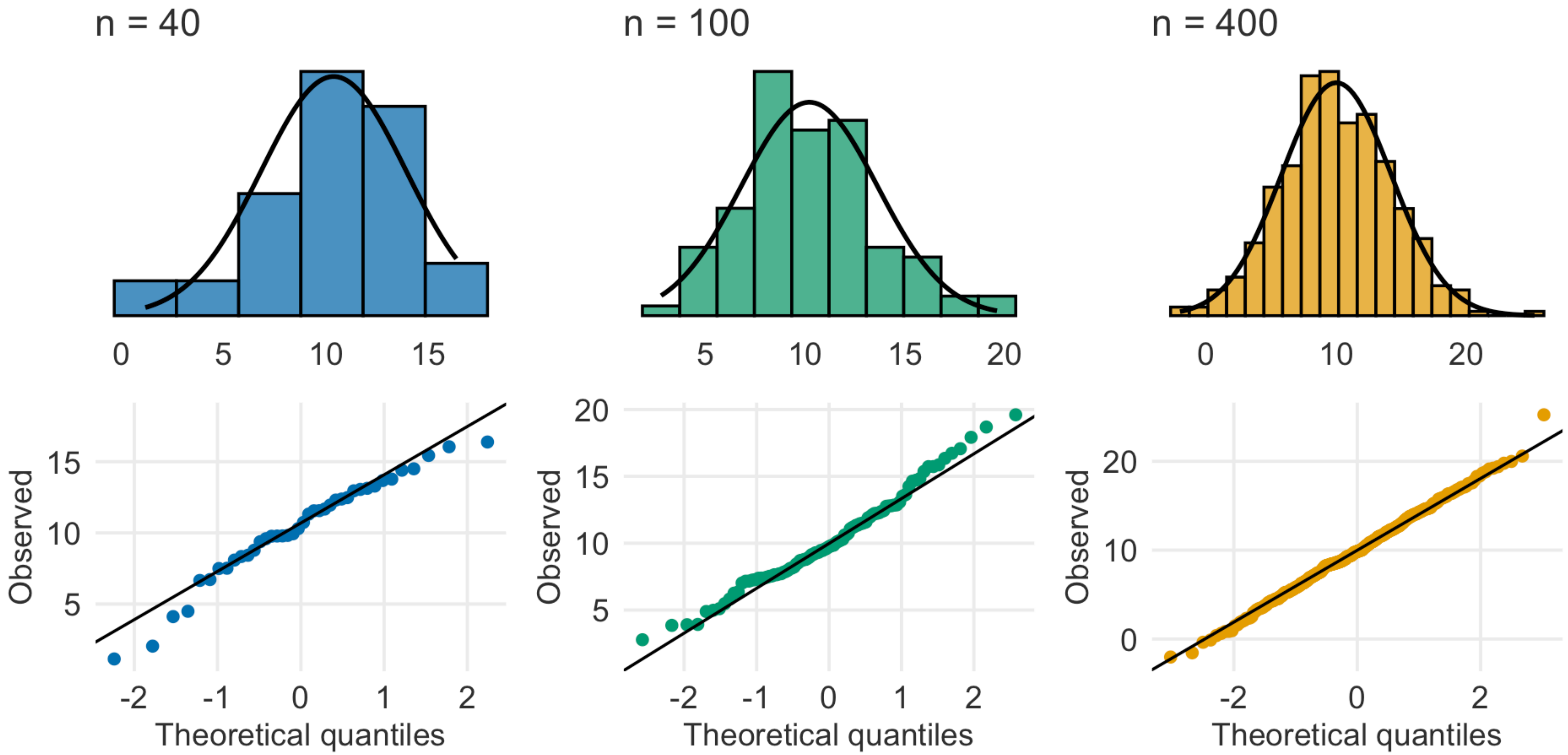
- With large n, tiny deviations look “significant”
- With small n, tests have low power
- Visuals + context are more informative

Quantile-quantile plots in ggplot2

```r
# Example code
ggplot(data,
       aes(sample = x)) +
  stat_qq() +
  stat_qq_line()
```

Checking whether Normal is reasonable (2/2)

A Q-Q plot compares your sample quantiles to theoretical Normal quantiles; straight-line agreement means the distributional shape matches Normal (especially in the tails).



Poisson Distribution

Poisson distribution (counts)

Use a Poisson model for the number of events in a fixed interval when:

- Events occur independently
- The event rate is roughly constant over the interval
- We are counting events (0, 1, 2, …)

Notation:

$$X \sim \mathrm{Pois}(\lambda)$$

Interpretation:

- λ = expected number of events per interval (rate × time)
- Mean = λ
- SD = $\sqrt{\lambda}$

Poisson distribution: when you see it, think Poisson

> **Note**
>
> Poisson models counts of events in a fixed interval.
>
> Ask yourself:
>
> “How many times does something happen in a given amount of time or space?”

Common examples:

- Number of ER arrivals per hour
- Number of emails received per day
- Number of mutations per gene
- Number of accidents per mile of highway per year

Key clues:

- Counting events: 0, 1, 2, 3, …
- Fixed interval (time, distance, area)
- Events are relatively rare and independent

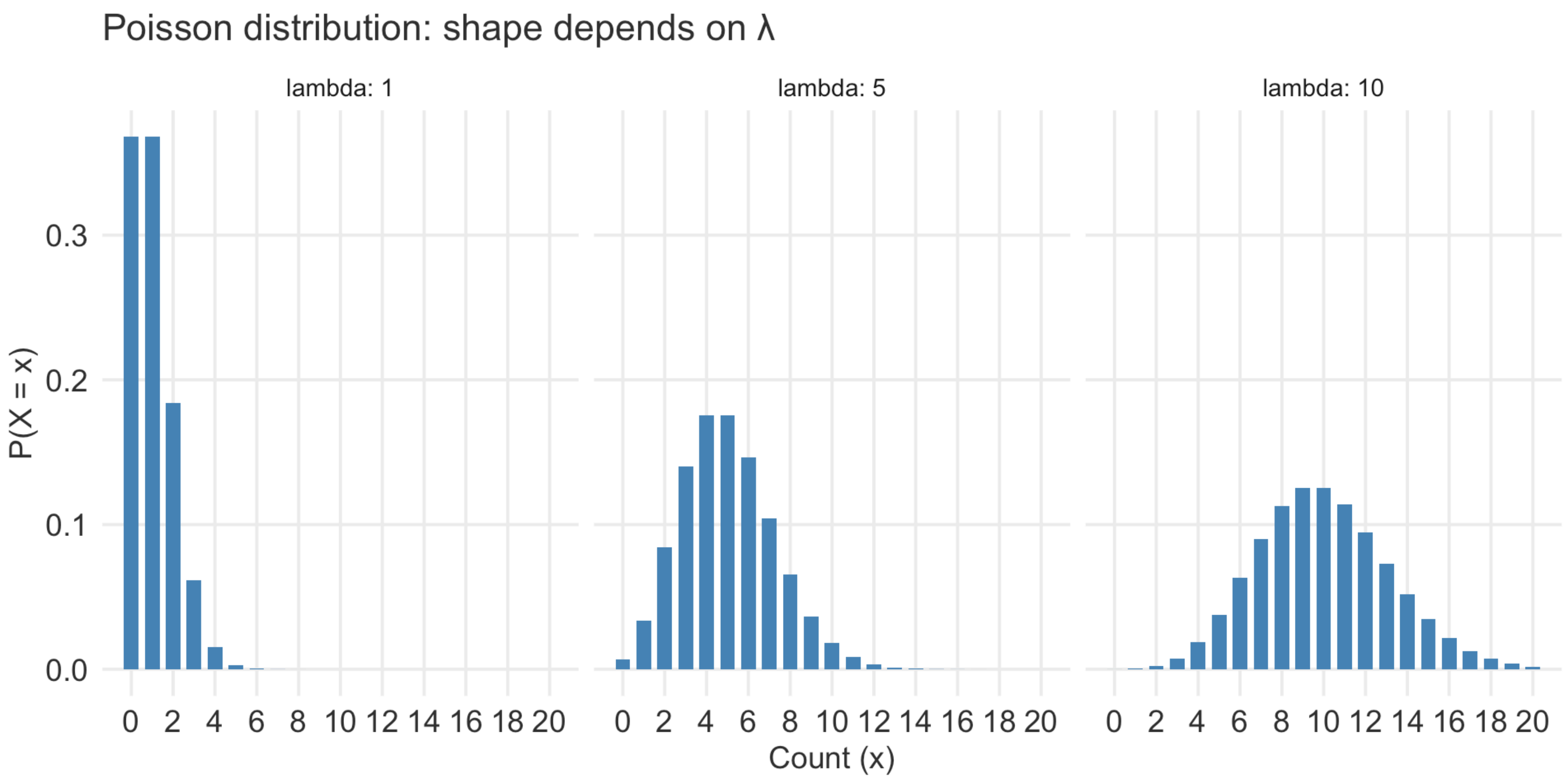
R functions for Poisson

Four key functions

Function	Purpose	Example
`dpois()`	**Probability** at a value	P(X = x)
`ppois()`	**Cumulative probability**	P(X ≤ x)
`qpois()`	**Quantile**	What x gives P(X ≤ x) = p?
`rpois()`	**Random** samples	Simulate counts

Most common: `dpois()` and `ppois()`

Shape depends on λ



As λ increases:

- The distribution shifts to the right (larger expected counts)
- The distribution becomes more symmetric
- For large λ, it starts to look approximately Normal

Example: Typhoid deaths (λ scaling)

Suppose there are on average 5 deaths per year.

1. Probability of exactly 3 deaths in 1 year:

```r
dpois(x = 3, lambda = 5)
```

```
[1] 0.1403739
```

2. Probability of exactly 2 deaths in 0.5 years:

Rate scales with time, so $\lambda_{0.5} = 5 \times 0.5 = 2.5$

```r
dpois(x = 2, lambda = 2.5)
```

```
[1] 0.2565156
```

3. Probability of more than 12 deaths in 2 years:

$\lambda_2 = 5 \times 2 = 10$

```r
ppois(q = 12, lambda = 10, lower.tail = FALSE)
```

```
[1] 0.2084435
```

```r
# or: 1 - ppois(12, lambda = 10)
```

Poisson approximation to Binomial (rare events)

When n is large and p is small, a Binomial can be approximated by a Poisson:

If $X \sim \text{Binomial}(n, p)$ and p is small, then

$$X \approx \text{Pois}(\lambda = np)$$

This is most useful when the **Normal approximation is not appropriate** (because p is too close to 0).

Quick example:

```r
n <- 1000
p <- 0.002
lambda <- n * p

# Approx P(X <= 3)
pbinom(3, size = n, prob = p)
```

```
[1] 0.8573042
```

```r
ppois(3, lambda = lambda)
```

```
[1] 0.8571235
```

Very close! (0.857 vs 0.857)

Summary: Poisson distribution

Key characteristics:

- Models **count data** for rare events
- Parameter: λ = rate × time
- Mean = λ, SD = $\sqrt{\lambda}$

R functions:

- `dpois(x, lambda)` → P(X = x)
- `ppois(q, lambda)` → P(X ≤ q)

Important: Remember to adjust λ when changing the time interval!

Approximation: Can approximate Binomial when n is large and p is small.

Summary & Next Steps

What you need to know: Normal distribution

Conceptual understanding:

- Normal distributions are symmetric, bell-shaped, continuous
- Fully characterized by mean (μ) and standard deviation (σ)
- Z-scores standardize to N(0, 1): $Z = \frac{X-\mu}{\sigma}$

Empirical Rule (68-95-99.7):

- 68% of data within 1 SD, 95% within 2 SDs, 99.7% within 3 SDs
- Values beyond ±3 SDs are very rare

R skills:

- `pnorm(q, mean, sd)` → P(X ≤ q)
- `qnorm(p, mean, sd)` → value at pth percentile
- Normal approximation when $np \geq 10$ AND $n(1-p) \geq 10$

What you need to know: Poisson distribution

When to use Poisson:

- Counting rare events in a fixed interval
- Events occur independently at rate λ

Key concepts:

- Parameter: λ = rate × time
- Mean = λ, SD = $\sqrt{\lambda}$
- Adjust λ when changing time intervals

R skills:

- `dpois(x, lambda)` → $P(X = x)$
- `ppois(q, lambda)` → $P(X \leq q)$
- Poisson approximates Binomial when n is large and p is small

Key formulas (for reference)

You don't need to memorize these, but understand what they represent:

Z-score transformation:

$$Z = \frac{X - \mu}{\sigma} \quad \text{or} \quad X = \mu + Z\sigma$$

Normal approximation to Binomial:

When $np \geq 10$ and $n(1-p) \geq 10$:

$$X \sim \text{Binomial}(n, p) \approx N\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

Poisson approximation to Binomial:

When n is large and p is small:

$$X \sim \text{Binomial}(n, p) \approx \text{Pois}(\lambda = np)$$