Mathematical Problem Solving Exercises

Emile NIYITANGA

July 28, 2025

Sum of p-th powers

It is said that, at the age of seven or eight, the famous mathematician Carl Gauss was asked to find the sum of the numbers from 1 to 100. His teacher expected this task to keep him occupied for some time but Gauss surprised him by writing down the correct answer almost immediately. His method involved adding the numbers in pairs.

1. **Show that:**

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

RESPONSE:

Proof of:

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$$

Here we have:

$$S_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n \tag{1.1}$$

It can then be written as:

$$S_n = n + (n-1) + (n-2) + \cdots + 2 + 1 \tag{1.2}$$

Add the above two expressions, then we get:

$$2S_n = (1+n) + (2+(n-1)) + (3+(n-2)) + \cdots + ((n-1)+2) + (n+1)$$
$$2S_n = (1+n) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

We see that $2S_n$ is the sum of $(n+1)$ n-times.

Then, by simplifying we have:

$2S_n = n(n+1)$

$S_n = \frac{1}{2}n(n+1)$

Therefore,

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

2. **Prove that:**

$$\sum_{k=1}^{n}\left[(k+1)^3-k^3\right]=n(n^2+3n+3)$$

and

$$\sum_{k=1}^{n}\left[(k+1)^3-k^3\right]=3\sum_{k=1}^{n}\left(k^2+k+\frac{1}{3}\right)$$

PROOF 1:

$$\sum_{k=1}^{n}\left[(k+1)^3-k^3\right]=n(n^2+3n+3)$$

Let's expand $(k+1)^3$: Then we get

$$(k+1)^3=k^3+3k^2+3k+1$$

Then we remain with,

$$\sum_{k=1}^{n}\left[(k^3+3k^2+3k+1)-k^3\right]$$

By evaluating the above expression, k^3 terms cancel out and we get,

$$\sum_{k=1}^{n}\left[3k^2+3k+1\right]$$

Let's now split this sum into small pieces,

$$\sum_{k=1}^{n}\left[3k^2+3k+1\right]=3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1 \tag{2.1}$$

Since we actually have

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n}1=n$$

By substituting in equation(2.1) we obtain,

$$\sum_{k=1}^{n}\left[3k^2+3k+1\right]=3\left[\frac{n(n+1)(2n+1)}{6}\right]+3\left[\frac{n(n+1)}{2}\right]+n$$

And then, by simplifying, we have,

$$\frac{n(n+1)(2n+1)}{2}+\frac{3n(n+1)}{2}+n,$$

$$= \frac{n(n+1)(2n+1) + 3n(n+1) + 2n}{2},$$

$$= \frac{2n^3 + n^2 + 2n^2 + n + 3n^2 + 3n + 2n}{2},$$

$$= \frac{2n^3 + 6n^2 + 6n}{2},$$

$$= n^3 + 3n^2 + 3n,$$

$$= n(n^2 + 3n + 3)$$

Therefore,

$$\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) = n(n^2 + 3n + 3)$$

PROOF 2:

$$\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) = 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \frac{n}{3}$$

From step 2 in PROOF 1, we have,

$$\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) = \sum_{k=1}^{n} \left(3k^2 + 3k + 1\right)$$

Rewrite the expression,

$$3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1,$$

$$= 3\sum_{k=1}^{n} \left(k^2 + k + \frac{1}{3}\right)$$

Therefore,

$$\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) = 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \frac{n}{3}$$

3. **Deduce that**

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

Let's deduce that,

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

Since from our previous results, we have,

$$\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) = n(n^2 + 3n + 3)$$

and also we have,

$$\sum_{k=1}^{n}\left((k+1)^3-k^3\right)=3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k+\frac{n}{3}$$

by equating the above two expressions we get,

$$n(n^2+3n+3)=3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k+\frac{n}{3}$$

Let's now use below equation, i.e,

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2},$$

in the above expression here we get,

$$n(n^2+3n+3)=3\sum_{k=1}^{n}k^2+\frac{n(n+1)}{2}+\frac{n}{3}$$

Now, let's solve for $\sum_{k=1}^{n}k^2$,

$$3\sum_{k=1}^{n}k^2=n(n^2+3n+3)-\frac{n(n+1)}{2}-\frac{n}{3}$$

by evaluating the right side we have,

$$3\sum_{k=1}^{n}k^2=\frac{2n^3+6n^2+6n-n^2-n-2n}{2},$$

$$=\frac{2n^3+5n^2+3n}{2},$$

By dividing both sides by 3:

$$\sum_{k=1}^{n}k^2=\frac{2n^3+5n^2+3n}{6}$$

$$=\frac{n(2n^2+5n+3)}{6}$$

$$=\frac{n(n+1)(2n+1)}{6}$$

Therefore,

$$\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1)$$

4. **Find polynomial P_4 (of degree 5) such that:**

$$\sum_{k=1}^{n}k^4=P_4(n)$$

Let's evaluate:

$$(k+1)^4 - k^4 = (k+1)^4 - 4k(k+1)^3 + 6k^2(k+1)^2 - 4k^3(k+1) + k^4,$$

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

Test the expression for different values of k

for $k = 0$:

$$1^4 - 0^4 = 1$$

for $k = 1$:

$$2^4 - 1^4 = 4(1^3) + 6(1^2) + 4(1) + 1$$

for $k = 2$:

$$3^4 - 2^4 = 4(2^3) + 6(2^2) + 4(2) + 1$$

$$\vdots$$

for k=n-2:

$$(n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1$$

for k=n-1:

$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$

For k=n:

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

From k=0, the addition of left hand side is $(n+1)^4$ and the right side is

$$4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + n + 1$$

Therefore,

$$(n+1)^4 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + n + 1$$

$$(n+1)^4 = 4\sum_{k=1}^{n} k^3 + 6(\frac{1}{6}n(2n+1)(n+1)) + 4(\frac{1}{2}n(n+1)) + n + 1$$

let

$$P_3(n) = \sum_{k=1}^{n} k^3,$$

$$(n+1)^4 = 4P_3(n) + n(2n+1)(n+1) + 2n(n+1) + (n+1),$$

$$P_3(n) = \frac{1}{4}[(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)],$$

$$P_3(n) = \frac{1}{4}(n+1)[(n+1)^3 - (n(2n+1) + (2n+1))],$$

$$P_3(n) = \frac{1}{4}(n+1)$$

To Find polynomial P_4 (of degree 5) such that:

$$\sum_{k=1}^{n} k^4 = P_4(n).$$

We have to evaluate the following polynomial,

$$(k+1)^4 - k^4 = (k+1)^4 - 4k(k+1)^3 + 6k^2(k+1)^2 - 4k^3(k+1) + k^4$$

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1,$$

for $k = 0$:

$$1^4 - 0^4 = 1$$

for $k = 1$:

$$2^4 - 1^4 = 4(1^3) + 6(1^2) + 4(1) + 1$$

For $k = 2$:

$$3^4 - 2^4 = 4(2^3) + 6(2^2) + 4(2) + 1$$

$\vdots$

for k=n-2:

$$(n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1$$

for k=n-1:

$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$

for k=n:

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

From k=0, The addition of left hand side is $(n+1)^4$ and the right side is

$$4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + n + 1$$

Therefore,

$$(n+1)^4 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + n + 1$$

$$(n+1)^4 = 4\sum_{k=1}^{n} k^3 + 6(\frac{1}{6}n(2n+1)(n+1)) + 4(\frac{1}{2}n(n+1)) + n + 1$$

let

$$P_3(n) = \sum_{k=1}^{n} k^3$$

$$(n+1)^4 = 4P_3(n) + n(2n+1)(n+1) + 2n(n+1) + (n+1),$$

$$P_3(n) = \frac{1}{4}[(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)],$$

$$P_3(n) = \frac{1}{4}(n+1)[(n+1)^3 - (n(2n+1) + (2n+1))],$$

$$P_3(n) = \frac{1}{4}(n+1)[(n+1)^3 - (n+1)(2n+1)],$$

$$P_3(n) = \frac{1}{4}(n+1)^2[(n+1)^2 - (2n+1)],$$

$$P_3(n) = \frac{1}{4}(n+1)^2(n^2 + 2n + 1 - 2n - 1),$$

$$P_3(n) = \frac{n^2}{4}(n+1)^2$$

$$P_3(n) = \sum_{k=1} nk^3 = \frac{n}{4}(n+1)^2$$

Then,

$$(k+1)^5 - k^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k^5$$

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

for k=0,

$$1^5 - 0^5 = 1$$

for k=1,

$$2^5 - 1^5 = 5(1^4) + 10(1^3) + 10(1^2) + 5(1) + 1$$

for k=2,

$$3^5 - 2^5 = 5(2^4) + 10(2^3) + 10(2^2) + 5(2) + 1$$

$$\vdots$$

For k=(n-1),

$$n^5 - (n-1)^5 = 5(n-1)^4 + 10(n-1)^3 + 10(n-1)^2 + 5(n-1) + 1$$

For k=n,

$$(n+1)^5 - (n^5) = 5n^4 + 10n^3 + 10n^2 + 5n + 1$$

By summing all above differences we should get,

$$(n+1)^5 = 5\sum_{k=1}^{n} k^4 + 10\sum_{k=1}^{n} k^3 + 10\sum_{k=1}^{n} k^2 + 5\sum_{k=1}^{n} k + n + 1$$

$$5\sum_{k=1}^{n} k^4 = (n+1)^5 - 10\frac{n^2}{4}(n+1)^2 - 10\frac{1}{6}n(2n+1)(n+1) - 5\frac{n(n+1)}{2} - (n+1)$$

$$5\sum_{k=1}^{n} k^4 = (n+1)^5 - \frac{5}{2}n^2(n+1)^2 - \frac{5}{3}n(n+1)(2n+1) - \frac{5}{2}n(n+1) - (n+1)$$

$$5P_4(n) = (n+1)[(n+1)^4 - \frac{5}{2}n^2(n+1) - \frac{5}{3}n(2n+1) - \frac{5}{2}n - 1]$$

Therefore the polynomial P4 (degree 5) required will be,

$$\frac{1}{5}(n+1)[(n+1)^4 - \frac{5}{2}n^2(n+1) - \frac{5}{3}n(2n+1) - \frac{5}{2}n - 1]$$

5. **Show that in general, we can find a polynomial P of degree $p+1$ such that:**

$$\sum_{k=1}^{n} k^p = P(n).$$

Now, we can ask,

$$\sum_{k=1}^{n} k^p$$

for positive integer p. Give the answer for p=0. or more, what if p is negative?

Let's show that it is possible to derive a polynomial P of degree $P+1$ such that:

$$\sum_{k=1}^{n} k^P = P(n)$$

Now, let's compute $(k+1)^{p+1} - k^{p+1}$

$$(k+1)^{p+1} = (k+1) \times (k+1) \times \cdots \times (k+1)$$

Using the binomial expansion formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Applying this principle,

$$(k+1)^{p+1} = \sum_{i=0}^{p+1} \binom{p+1}{i} k^i, \text{(with } b=1\text{)}$$

Then,

$$(k+1)^{p+1} - k^{p+1} = \sum_{i=0}^{p} \binom{p+1}{i} k^i + \binom{p+1}{p+1} k^{p+1} - k^{p+1}$$

This simplifies to,

$$\sum_{i=0}^{p} \binom{p+1}{i} k^i$$

Summing over all values from $k=1$ to n,

$$\sum_{k=1}^{n} \left[(k+1)^{p+1} - k^{p+1}\right] = \sum_{k=1}^{n} \sum_{i=0}^{p} \binom{p+1}{i} k^i$$

We can define now,

$$\sum_{k=1}^{n} k^i = P_i(n)$$

So,

$$\sum_{k=1}^{n} (k+1)^{p+1} - \sum_{k=1}^{n} k^{p+1} = \sum_{k=1}^{n} \sum_{i=0}^{p} \binom{p+1}{i} k^i$$

This implies,

$$\sum_{k=1}^{n} k^{p+1} = P(n) \cdot \sum_{i=0}^{P} (P+1)$$

Finally, the expression for $P_p(n)$ becomes,

$$P_p(n) = \frac{1}{(p+1)}\left[(n+1)^{p+1} - 1 - \sum_{i=0}^{p}\binom{p+1}{i}P_i(n)\right]$$

For $p = 5$, the formula is given by,

$$P_5(n) = \frac{1}{6}\left[(n+1)^6 - 1 - \sum_{i=0}^{4}\binom{6}{i}P_i(n)\right]$$

Given that:

$$P_3 = \frac{n^2(n+1)^2}{4}, \quad P_2 = \frac{1}{6}n(n+1)(2n+1), \quad P_1 = \frac{n}{2}(n+1)$$

The resulting expression for P_5 is,

$$P_5 = \frac{1}{6}\left[n^6 + 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n + 1 - 1 - n - 3n^2 - 15n^3 - \dots\right]$$

Let's explore what happens when P takes negative values. Specifically, we will evaluate P_{-1} and P_{-2}.

$$\begin{aligned} P_{-1} &= \sum_{k=1}^{n}\frac{1}{k} \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}, \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \end{aligned}$$

As $n \to \infty$, the series P_{-1} behaves asymptotically like $\log(n)$.

$$\begin{aligned} P_{-2} &= \sum_{k=1}^{n}\frac{1}{k^2} \\ &= 1^{-2} + 2^{-2} + 3^{-2} + 4^{-2} + \cdots + n^{-2}, \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \end{aligned}$$

As n tends to infinity, the value of P_{-2} converges to $\frac{\pi^2}{6}$.

For a general $m \in \mathbb{C}$, we can express P_i as,

$$P_i = \sum_{k=1}^{n}\frac{1}{k^i}$$

This expression is related to the Riemann zeta function.

To summarize, for positive values of P, the sums are typically polynomial-like. On the other hand, for negative values of P, the sums no longer resemble polynomials and instead involve functions like the zeta function.