Mathematical Problem Solving Exercise

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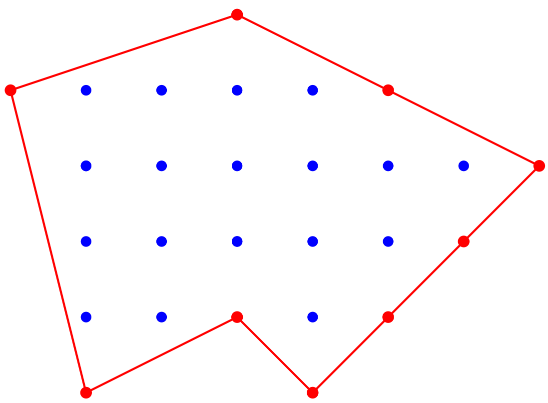
Polygons on the grid

Consider a square grid as in the figure below; the distance between consecutive parallel lines is 1. The intersection points, which are precisely the points whose coordinates are integers, are referred to as rectangular points. Now we draw a polygon whose vertices are lattice points. Let p be the number of lattice points inside the polygon and q the number of lattice points on the boundary of the polygon. In the example, p = 18 and q = 9.



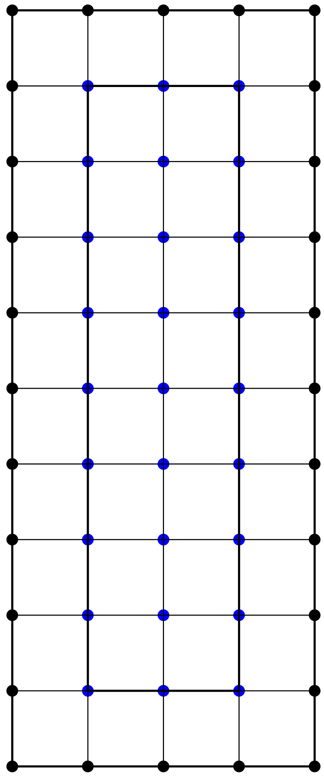
Given p and q, what can be said about the area of the polygon?

1. **Prove the formula for rectangles**

   Let's now take a right-angled triangle with base B and height H, then its area is given by

$$A_T = \frac{1}{2} * B * H \tag{1}$$

   28 lattice points at the boundary and 27 points in the inside:



We have p = the number of lattice points inside the polygon

and q, the number of lattice points on the boundary of the polygon For this this example our rectangle has length L=10 and width,W=4 so, its area A_R is given by:

$$A_R = L * W = 40u^2 \tag{2}$$

Since q equals the number of points at the boundary of the triangle, then we have: q_R is the perimeter of the rectangle and is calculated as follows:

$$q_R = (L+1) + (W+1) + ((L+1) + (W+1) - 4 \tag{3}$$

Here, we subtract 4 because the points at each vertex are counted twice. By evaluating the above equation, we remain with:

$$q_R = 2(W+L) \tag{4}$$

And also the number of points inside the rectangle p_R is given by:

$$q_R = (L-1)(W-1) \tag{5}$$

This is because, for counting the inside lattice points, each side of the rectangle is reduced by 1 unit. Let's now take a right-angled triangle with base B and height H then its area is given by

$$A_T = \frac{1}{2} * B * H \tag{6}$$

Proof of the rectangle's formula

$$\begin{align}
A_R &= L * W \tag{7}\\
q_R &= 2(L+1) + 2(W+1) - 4 \tag{8}\\
q_R &= 2L + 2 + 2W + 2 - 4 \tag{9}\\
q_R &= 2L + 2W \tag{10}\\
P_R &= (L-1)(W-1) \tag{11}\\
\text{or } W &= \frac{q_B}{2} - L \tag{12}
\end{align}$$

Since we have: $2W = q_R - 2L$

$$W = \frac{q_R}{2} - L \tag{13}$$

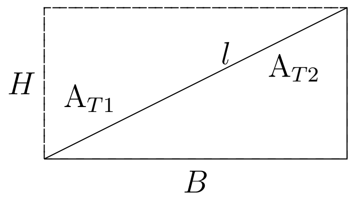
$$\begin{align}
p_R &= (L-1)\left(\frac{q_R}{2} - L - 1\right) \tag{14}\\
p_R &= \frac{q_R}{2}L - L^2 - L - \frac{q_R}{2} + L + 1 \tag{15}\\
p_R &= L\left(\frac{q_R}{2} - L\right) - \frac{q_R}{2} + 1 \tag{16}\\
p_R &= LW - \frac{q_R}{2} + 1 \tag{17}
\end{align}$$

Since the area of a rectangle is given by: $A_R = L \cdot W \quad (18)$

$$\begin{align}
p_R &= A_R - \frac{q_R}{2} + 1 \tag{19}\\
A_R &= p_R + \frac{q_R}{2} - 1 \tag{20}
\end{align}$$

2. **Prove the formula for a right-angled triangle**

Let's now draw two triangles with areas A_{T1} and A_{T2} respectively so that they form a rectangle with area A_R and l is the number of lattice points located at the boundary of each triangle as shown in the following figure:



Since it is a right-angled triangle with base B and height H, then its area is given by:

$$A_T = \frac{1}{2} * B * H \tag{21}$$

Since the two triangles are similar and also have the same dimensions, then we have:

A_{T1}=A_{T2}=A_T

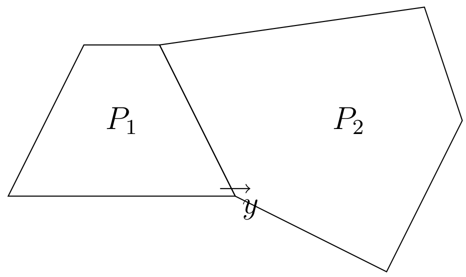
$$A_R = A_T + A_T = 2A_T \tag{22}$$

Here,

$$q_R = q_T - l + q_T - l - 2 \tag{23}$$
$$q_R = 2q_T - 2l - 2 \tag{24}$$
$$p_R = p_T + p_T + l \tag{25}$$

$$A_T = \frac{1}{2}\left(\frac{q_R}{2} + p_R - 1\right) \tag{26}$$
$$A_T = \frac{1}{2}\left(q_N - y - 1 + p_T + y - 1\right) \tag{27}$$
$$= \frac{1}{2}\left(q_T - 2 - 2p_T\right) \tag{28}$$
$$= \frac{q_T}{2} - 1 + p_T \tag{29}$$
$$A_T = p_T + \frac{q_2}{2} - 1 \tag{30}$$

3. **Suppose the formula is true for P$_1$, P$_2$(polygon) (adjacent), prove that the formula is also true for** $P = P_1 + P_2$



$$P_p = A_{p_1} + A_{p_2} \tag{31}$$

$$\text{Since:} A_{p_1} = \frac{q_1}{2} + p_{p_1} - 1 \tag{32}$$

$$\text{and } A_{p_2} = p_{p_2} + q_2 - 1 \tag{33}$$

$$q_{p_2} = q_{p_1} + q_{p_2} - 2y - 2 \tag{34}$$

$$q_p + 2y + 2 = q_{p_1} + q_{p_2} \tag{35}$$

$$P_{p_p} = P_{p_1} + P_{p_2} + y \tag{36}$$

$$P_{p_1} + p_{p_2} = p_{p_2} - y \tag{37}$$

$$A_p = \frac{q_{p_1}}{2} + P_{p_1} - 1 + \frac{q_p}{2} + P_{p_2} - 1 \tag{38}$$

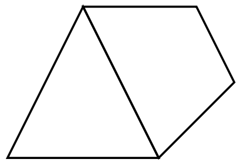
$$A_p = \frac{1}{2}\left(\underline{q_1} + q_{p_2}\right) + (p_{p_1} + p_{p_2}) - 2 \tag{39}$$

$$A_p = \frac{1}{2}\left(q_p + 2y + 2\right) + (p_p - y) - 2 \tag{40}$$

$$A_p = \frac{1}{2}q_p + l + 1 + p_p - y - 2 \tag{41}$$

$$A_p = p_p + \frac{1}{2}q_p - 1 \tag{42}$$

4. **Suppose that the formula is true for $P = P_1 + P_2$ and P_1. Prove that it is also true for P_2.**



Let the number of lattice points on the adjacent side be j, then we have:

$$P_2 \geq P_{P_2} + \frac{q_P}{2} - 1 \tag{43}$$

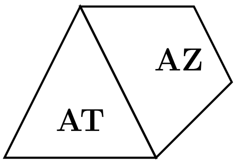
$$A_p = p_p + \frac{1}{2}q_p - 1 \tag{44}$$
$$A_{p_1} = \frac{q_{p_1}}{2} + p_{p_1} - 1 \tag{45}$$
$$A_p = A_{p_1} + A_{p_2} \tag{46}$$
$$q_p = q_{p_1} + q_{p_2} - 2j - 2 \tag{47}$$
$$p_p = p_{p_1} + p_{p_2} + j \tag{48}$$

$$\frac{q_p}{2} + p_p - 1 = \frac{q_{p_1}}{2} + p_{p_1} - 1 + Ap_2 \tag{49}$$
$$\frac{1}{2}\left(q_{p_1} + q_{p_2} - 2j - 2\right) + p_{p_1} + p_{p_2} + j - 1 = \frac{q_{p_1}}{2} + p_{p_1} - 1 + A_{p_2} \tag{50}$$
$$\frac{q_{p_1}}{2} = \frac{q_{p_2}}{2} - j - 1 + p_{p_1} + p_{p_2} + j - 1 = \frac{q_{p_1}}{2} + p_{p_1} - 1 + Ap_{p_2} \tag{51}$$
$$\frac{q}{\frac{P_2}{2}} + p_{p_2} - 1 = A_{p_2} \tag{52}$$
$$Ap_2 = p_{p_2} + \frac{q_{p_2}}{2} - 1 \tag{53}$$

Therefore, it holds for A_{p_2} as proved above.

5. **Prove the formula for any triangle T**

Consider the following figure composed of a triangle T and a trapezium-like polygon Z, and the number of lattice points located on the adjacent sides is f



For any polygon p that can be represented as the sum of one or more polygons with known areas, the area of p can be expressed as,

$$A_p = \frac{1}{2} q_p + p_p - 1 \tag{54}$$
$$A_{p_1} = \frac{1}{2} q_{p_1} + p_{p_1} - 1 \tag{55}$$
$$p_P = p_{p_1} + p_Z + f \tag{56}$$
$$q_p = (q_p + q_Z - 2f - 2) \tag{57}$$
$$A_p = A_{p_1} + A_Z \tag{58}$$
$$\frac{1}{2} q_p + p_p - 1 = \frac{1}{2} q_{p_1} + p_{p_1} - 1 + A_Z \tag{59}$$
$$\frac{1}{2} (q_{p_1} + q_Z - 2f - 2) + p_p - 1 = \frac{1}{2} q_{p_1} + p_{p_1} - 1 + A_Z \tag{60}$$

$$\frac{1}{2} q_{p_1} + \frac{1}{2} q_Z - f - 1 + p_p - 1 = \frac{1}{2} q_{p_1} + p_{p_1} - 1 + A_Z \tag{61}$$
$$\frac{1}{2} q_Z - f - 1 + p_{p_1} + p_Z + f - 1 = p_{p_1} - 1 A_Z \tag{62}$$
$$A_Z = \frac{1}{2} q_Z + p_Z - 1. \tag{63}$$

Hence proved. Therefore, the theorem holds for any triangle.

By concluding, we can say that "For any polygon p that can be represented as the sum of one or more polygons with known areas, the area of the polygon can be expressed as:

$$A_p = p_p + \frac{1}{2} q_p - 1 \tag{64}$$