Algebraic groups

Emile T. Okada,
Søren Gammelgaard,
Esteban Gomezllata Marmolejo

February 19, 2019

Contents

0.1 Conventions

Categories are usually denoted by *sans-serifs*, i.e., $\mathsf{Grp}, \mathsf{Top}, \mathsf{Sch}$ are the categories of groups, topological spaces, and schemes, respectively.

After a while, given a Hopf algebra A, we may use $\operatorname{Spec} A$ for the affine group scheme represented by A, and we may similarly write "the coordinate ring" to mean the Hopf algebra representing some affine group scheme.

The notation $(-)^\vee$ means *dual*, where $(-)$ stands for a group scheme, vector space, line bundle &c.

CHAPTER 1

Affine group schemes

1.1 Affine schemes

Definition 1.1.1. Let k be a commutative ring. An *affine scheme over k* is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$.

Let us try to motivate this definition a bit. Let k be a commutative ring and $\{f_\alpha\} \subseteq k[x_1, \ldots, x_n]$. For any k-algebra R we can consider the set of points in R^n satisfying $f_\alpha = 0$ for all α. Let us call this set $I(R)$. It is clear that if we have a k-algebra homomorphism $f : R \to S$ we obtain a map $I(f) : I(R) \to I(S)$, and that this turns I into a functor $I : \mathsf{Alg}_k \to \mathsf{Set}$.

Theorem 1.1.2. *Let J be the ideal of $k[x_1, \ldots, x_n]$ generated by the f_α and $A = k[x_1, \ldots, x_n]/J$. Then I is a representable functor with representative A.*

Proof. Let $a = (a_1, \ldots, a_n) \in I(R)$ and let $f_a : k[x_1, \ldots, x_n] \to R$ be the k-algebra homomorphism given by $x_i \mapsto a_i$. By definition of the point a, the ideal J must map to 0 under f_a. Thus f_a factors through A and we obtain a map $\bar{f}_a : A \to R$.

Conversely, given a map $f : A \to R$, let $a = (f(\bar{x}_1), \ldots, f(\bar{x}_n)) \in R^n$ where $\bar{x}_i$ is the image of x_i in A. Since f is a homomorphism it is clear that a lies in $I(R)$.

These two maps give a bijection between the sets $I(R)$ and $\mathrm{Hom}_k(A, R)$. It is straightforward to check this is a natural bijection and so I is naturally isomorphic to the functor $\mathrm{Hom}(A, -)$. ■

This theorem says that the affine schemes are exactly what we expect them to be.

1.1.1 The Yoneda lemma

Recall from category theory the following result on representable functors.

Theorem 1.1.3. *(Yoneda's lemma). Let $F : \mathcal{C} \to \mathsf{Set}$ be a functor, $\mathcal{C}$ be locally small and $c \in \mathcal{C}$. Then there is a natrual bijection*

$$\begin{array}{rcl} \mathrm{Hom}(\mathcal{C}(c,-),F) \leftrightarrow Fc & & \\ \eta & \to & \eta_c(\mathrm{id}_c) \\ \eta_x : f \mapsto F(f)(y) & \leftarrow & y. \end{array} \tag{1.1}$$

Corollary 1.1.4. *(Yoneda embedding). The functor*

$$\mathcal{C}(\bullet,-) : \mathcal{C}^{op} \to \mathsf{Fun}(\mathcal{C},\mathsf{Set}) \tag{1.2}$$

is full and faithful.

Proof. Let $c,d \in \mathcal{C}$. From the Yoneda lemma we have a bijection

$$\mathrm{Hom}(\mathcal{C}(d,-),\mathcal{C}(c,-)) \leftrightarrow \mathcal{C}(c,d). \tag{1.3}$$

Let $f \in \mathcal{C}(c,d)$. Under the bijection this gets sent to the natural transformation $\eta_x : g \mapsto \mathcal{C}(c,-)(g)(f) = g \circ f$. Thus the backwards map in the Yoneda bijection is just $\mathcal{C}(\bullet,-)$ and so the result follows. ■

Remark 1.1.5. It follows that $\mathcal{C}^{op}$ is equivalent to the category of representable functors from $\mathcal{C}$. In fact there exists a functor $P : \mathsf{Fun}^{rep}(\mathcal{C},\mathsf{Set}) \to \mathcal{C}^{op}$ such that $P \circ \mathcal{C}(\bullet,-) = \mathrm{id}_{\mathcal{C}^{op}}$, $\mathcal{C}(\bullet,-) \circ P \cong \mathrm{id}_{\mathsf{Fun}^{rep}(\mathcal{C},\mathsf{Set})}$ and $\mathcal{C}(\bullet,-) \circ P|_{\mathrm{im}(\mathcal{C}(\bullet,-)} = \mathrm{id}_{\mathrm{im}(\mathcal{C}(\bullet,-)}$.

Corollary 1.1.6. *Let $c,d \in \mathcal{C}$. Then $c \cong d$ if and only if $\mathcal{C}(c,-) \cong \mathcal{C}(d,-)$.*

Proof. ($\Rightarrow$) This follows by functoriality.

($\Leftarrow$) Let $\alpha : \mathcal{C}(c,-) \Rightarrow \mathcal{C}(d,-)$ be an isomorphism and β its inverse. Let $a : d \to c$ and $b : c \to d$ be the corresponding maps. By naturality of the Yoneda bijection

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(d,-)) \arrow[r] \arrow[d, "\beta\circ"] & \mathcal{C}(d,c) \arrow[d, "\beta_c"] \\ \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(c,-)) \arrow[r] & \mathcal{C}(c,c) \end{tikzcd} \tag{1.4}$$

commutes. So $\beta_c(\alpha_c(\mathrm{id}_c)) = \mathrm{id}_c$. But $\beta_c(\alpha_c(\mathrm{id}_c)) = a \circ b$ and so $a \circ b = \mathrm{id}_c$. Similarly $b \circ a = \mathrm{id}_b$ and so the result follows.

Alternatively note use the remark. ■

This last corollary implies that two k-algebras are isomorphic if and only if the corresponding affine schemes are.

1.2 Affine group schemes

Definition 1.2.1. An *affine group scheme over k* is a functor $F : \mathsf{Alg}_k \to \mathsf{Grp}$ such that F composed with the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is representable.

Example 1.2.2. Let $k = \mathbb{Z}$ (so $\mathsf{Alg}_k = \mathsf{Ring}$). Then $\mathbf{SL}_n : \mathsf{Ring} \to \mathsf{Grp}$, $R \mapsto \mathbf{SL}_n(R)$ is an affine group scheme over $\mathbb{Z}$.

Proposition 1.2.3. *Let E, F, G be affine group schemes over k represented by A, B, C respectively. If we have morphisms $E \to G$ and $F \to G$ then the pullback exists and is represented by $A \otimes_C B$.*

Proof. Existence follows from the fact that we can compute the pullback pointwise and Grp has pullbacks. Explicitly

$$(E \times_G F)(R) = \{(e, f) : \text{ } e \text{ and } f \text{ have same image in } G(R)\}. \tag{1.5}$$

For the second part of the proposition note that the pullback in Alg_k^{op} is of the required form and that we have an equivalence of categories between Alg_k^{op} and affine schemes over k. ■

Definition 1.2.4. Let F be an affine group scheme over k and let $\phi : k \to k'$ be a ring homomorphism. Any k'-algebra can be turned into a k by composing by ϕ and so we can turn F into a functor on k'-algebras. Call this new functor $F_{k'}$.

Proposition 1.2.5. *If F is represented by A then $F_{k'}$ then is represented by $A \otimes_k k'$.*

Proof. There is a natural bijection

$$\mathrm{Hom}_{k'}(A \otimes_k k', S) \leftrightarrow \mathrm{Hom}_k(A, S). \tag{1.6}$$

■

1.2.1 Hopf algebras

Another way to define an affine group scheme is that it is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$ together with natural transformations $\mu : F \times F \to F$, $i : F \to F$ and $u : e \to G$ where e is the functor $\mathrm{Hom}_k(k, -)$ such that

$$\begin{tikzcd} F \times F \times F \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"] & F \times F \arrow[d, "\mu"] \\ F \times F \arrow[r, "\mu"] & F \end{tikzcd} \tag{1.7}$$

$$\begin{tikzcd} e \times F \arrow[r, "u \times \mathrm{id}"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} F \times e \arrow[r, "\mathrm{id} \times u"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \tag{1.8}$$

$$\begin{tikzcd} F \arrow[r, shift left, "\mathrm{id}\times i"] \arrow[r, shift right, "i\times \mathrm{id}"'] \arrow[d] & F \times F \arrow[d, "\mu"] \\ e \arrow[r, "u"] & F \end{tikzcd} \tag{1.9}$$

all commute. If F is represented by A then translating back to Alg_k we obtain maps $\Delta : A \to A \otimes_k A$, $S : A \to A$ and $\varepsilon : A \to k$. Satisfying the same commutative diagrams, but with arrows reversed and μ replaced with Δ etc. The algebra A together with these maps is what we call a Hopf algebra.

Definition 1.2.6. Let $\psi : H \to G$ be a morphism of affine groups schemes. We say ψ is a *closed embedding* if the corresponding map on algebras is surjective. H is then isomorphic to a closed subgroup of G represented by the corresponding quotient of the algebra of A.

Theorem 1.2.7. *Affine group schemes over k correspond to Hopf algebras over k.*

Definition 1.2.8. Let A be a k-algebra. We call an ideal I of A a *Hopf ideal* if A/I inherits the structure of a Hopf algebra.

Proposition 1.2.9. *Let A be a k-algebra and $I \lhd A$. Then I is a hopf ideal if and only if $\Delta(I) \subseteq I \otimes A + A \otimes I$, $S(I) \subseteq I$ and $\varepsilon(I) = 0$.*

Proof. Todo. ∎

Definition 1.2.10. Let $\Phi : G \to H$ be a morphism of affine groups schemes. Then $\ker \Phi(R) = \ker(G(R) \to H(R))$ or alternatively $\ker \Phi = G \times_H \{e\}$. It follows that if G, H are represented by A, B respectively then $\ker \Phi$ is represented by $A \otimes_B k \cong A/AI_B$.

1.2.2 Characters

Definition 1.2.11. A homomorphism $G \to G_m$ is called a character.

thm:characters_are_gp-like_elems **Theorem 1.2.12.** *The characters of an affine group scheme G represented by A correspond to the group-like elements of A.*

1.3 Diagonalisable group schemes

sec:diag_gp_sch

Let M be an abelian group, k a ring, and $k[M]$ the group algebra. We construct a Hopf algebra structure on $k[M]$ by making the group elements group-like, i.e., we set

$$\Delta(m) = m \otimes m, \quad \varepsilon(m) = 1, \quad S(m) = m^{-1},$$

and we extend by linearity to all $k[M]$. This is a Hopf algebra (easily checked on elements)

def:diag_gp_sch **Definition 1.3.1.** A group scheme constructed in this way is called a *diagonalisable group scheme.*

The name is not immediately a good one, but we will later see that most affine group schemes can be embedded into $\mathbf{GL}_n$ for some n (Theorem 2.2.5), and that the diagonalisable group schemes will be precisely those corresponding to groups of diagonalisable matrices.

We can describe these schemes completely:

hm:struct_of_diag_gps

Theorem 1.3.2. *Let G be a diagonalisable group scheme, represented by A, and suppose that A is a finitely generated k-algebra. Then*

$$G = \left(\prod_{i=1}^{n} \mathbb{G}_m\right) \times \prod_{j=1}^{n'} \boldsymbol{\mu}_{m_j}$$

for some integers n, n', m_j.

Proof. Take a finite set of generators of A. These are finite linear combinations of elements of M, so we have a finite number of elements of M generating A, Then these elements must also generate M. So M is a finitely generated abelian group. But such groups are direct sums of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, so we need only check two cases.

Case 1: $M = \mathbb{Z}$. Then $k[\mathbb{Z}]$ has basis $\{e_n | n \in \mathbb{Z}\}$, with multiplication $e_i e_j = e_{i+j}$. So $k[\mathbb{Z}] \xrightarrow{\sim} k[X, X^{-1}]$, the affine coordinate ring of $\mathbb{G}_m$.

Case 2: $M = \mathbb{Z}/n\mathbb{Z}$. The basis is now $\{1 = e_0, e_1, \ldots, e_{n-1} = e^{n-1}\}$. So $e_1^n = 1$, and the Hopf algebra is $k[X]/(X^n - 1)$, which represents $\boldsymbol{\mu}_n$. ■

We will now really exploit the Hopf algebra structure. First, we have

lem:gplikes_lin_indep

Lemma 1.3.3. *The group-like elements in a Hopf algebra over a field k are linearly independent.*

Proof. Take group-like elements $b, \{b_i\}$ such that

$$b = \sum \lambda_i b_i \qquad \text{(a finite sum)},$$

and assume the b_i to be independent. Then

$$1 = \varepsilon(b) = \sum \lambda_i \varepsilon(b_i) = \sum \lambda_i.$$

But

$$\begin{aligned} \Delta(b) = b \otimes b &= \sum \lambda_i \lambda_j b_i \otimes b_j \\ &= \sum \lambda_i \Delta(b_i) = \sum \lambda_i b_i \otimes b_i \end{aligned}$$

so all $\lambda_i \lambda_j = 0$ for $j \neq i$, $\lambda_i^2 = \lambda_i$, so all λ_i are either 0 or 1. Since their sum is 1, exactly one $\lambda_i = 1$, and so $b = b_i$. ■

Furthermore:

_implies_cring_is_span_of_gplikes

Theorem 1.3.4. *An affine group scheme over a field is diagonalisable iff its representing algebra (coordinate ring!) is spanned by group-like elements. There is a natural coequivalence of categories*

$$\{\textit{Diagonalisable group schemes}/k\} \xrightarrow{\sim} \mathsf{AbGrp}.$$

Proof. By the lemma, if A is spanned by group-like elements, they are a basis of A. The group formed by them is the character group X_G. So $k[X_G] \xrightarrow{\sim} A$, at least as sets. Checking on basis elements shows that this is an isomorphism of Hopf algebras. If M is any abelian group, its elements are the only group-like elements of $k[M]$. Thus M is the character group of $\operatorname{Spec} k[M]$ (the group scheme rep. by $k[M]$). Also, if $G \to H$ is a homomorphism of affine algebraic groups, we have an induced map $X_H \to X_G$. This determines the Hopf algebra structure, since X_H spans $k[X_H]$. In the opposite direction, a map $X_H \to X_G$ clearly induces a map $k[X_H] \to k[X_G]$. ■

We will develop this correspondence further in Section 1.5.

1.4 Finite constant groups

sec:fin_const_gps

Let Γ be a finite group, and define a functor

$$\overline{\Gamma}\colon \mathsf{k\text{-}alg} \to \mathsf{Grp}$$

sending every ring to Γ, every map to the identity map. This functor is unrepresentable (not sure why), but we'll do something almost as good.

Let $A \coloneqq Mor(\Gamma \to k)$. Let e_σ be the indicator function for $\sigma \in \Gamma$, so $\{e_\sigma\}$ is a basis for A. We define a ring structure on A by $e_\sigma^2 = e_\sigma, e_\sigma e_\tau = 0$ for $\sigma \neq \tau$, $\sum e_\sigma = 1$.

Let R be a k-algebra where the only idempotents are 0 and 1. Then a homomorphism $\phi\colon A \to R$ is completely determined by which σ is sent to 1, so $\operatorname{Hom}(A, R) \xrightarrow{\sim} \Gamma$ as sets. If we define a comultiplication on A by $\Delta(e_\rho) = \sum_{\rho=\sigma\tau} e_\sigma \otimes e_\tau$, this induces the same group structure on $\operatorname{Hom}(A, R)$ as that of Γ. If we define $S(e_\sigma) = e_{\sigma^{-1}}$, and $\varepsilon(e_\sigma) = 1$ if σ is the unit of Γ, otherwise 0, A becomes a Hopf algebra. We say that it represents the *constant group scheme* Γ.

1.4.1 Scheme-theoretic aside

As a scheme, the construction is simpler: Just take $\operatorname{Spec}(\prod_{g\in\Gamma} k) = \coprod_{g\in\Gamma} \operatorname{Spec} k$, and define the group structure by $G \times G \ni (\operatorname{Spec} k)_{g,h} \mapsto (\operatorname{Spec} k)_{gh} \in G$. Chasing through the definitions and various categorical equivalences shows that this really is the same construction.

1.5 Cartier Duality

sec:Cartier_duality

Let k be a ring, N a finite rank free k-module. Recall that then: $N^\vee$ is also free; if M is also finite rank free, $(M \otimes N)^\vee \xrightarrow{\sim} M^\vee \otimes N^\vee$; $\mathrm{Hom}_k(M^\vee, N^\vee) \xrightarrow{\sim} \mathrm{Hom}_k(M, N)^\vee$; and Hom, $\otimes$ commute with finite direct sums. Hence the same identities hold for finitely generated projective modules, as they are summands of free modules.

f:finite_group_scheme

Definition 1.5.1. Call a group scheme *finite* if it is represented a k-algebra A which is a finitely generated projective[1] module.

Note that being finitely generated as a module is a much stronger condition than being finitely generated as a k-algebra.

Remark 1.5.2. The scheme-theoretic statement would be: G is a group object in the category of schemes over k with a finite structure morphism.

So take such a group scheme G. Then we have five Hopf structure maps:

$$\begin{aligned}
\Delta\colon A &\to A \otimes A && \text{comultiplication}\\
\varepsilon\colon A &\to k && \text{counit}\\
S\colon A &\to A && \text{coinverse}\\
m\colon A \times A &\to A && \text{multiplication}\\
u\colon k &\to A && \text{unit.}
\end{aligned}$$

If we dualise these, we end up with something very similar:

$$\begin{aligned}
m^\vee : A^\vee &\to A^\vee \otimes A^\vee\\
u^\vee : A^\vee &\to k\\
S^\vee : A^\vee &\to A^\vee\\
\Delta^\vee : A^\vee \times A^\vee &\to A^\vee\\
\varepsilon^\vee : k &\to A^\vee
\end{aligned}$$

and we are immediately led to

thm:Cartier_duality

Theorem 1.5.3 (Cartier duality 1)**.** *Let G be a finite abelian group scheme represented by A. Then $A^\vee$ represents another finite abelian group scheme $G^\vee$, and $(G^\vee)^\vee \xrightarrow{\sim} G$.*

Sketch of proof. This is reasonably straightforward - just check that all the axioms are satisfied, dualise diagrams if needed. The only tricky bit is to show that $S^\vee$ is a homomorphism: We must show that

$$\Delta^\vee(S^\vee \otimes S^\vee) = S^\vee \Delta^\vee,$$

[1]The projectivity hypothesis is probably not part of the definition found elsewhere. But we will need it in these notes.

which is equivalent to $\Delta S = (S \otimes S)\Delta$. As group functors, this means that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \big\downarrow{\scriptstyle S} & & \big\downarrow{\scriptstyle S\otimes S} \\ A & \xrightarrow{\Delta} & A \otimes A \end{array} \quad \text{or equivalently} \quad \begin{array}{ccc} G & \xleftarrow[mult.]{} & G \times G \\ \big\uparrow{\scriptstyle inv} & & \big\uparrow{\scriptstyle inv\times inv} \\ G & \xleftarrow[mult.]{} & G \times G \end{array} \quad \text{which commutes}$$

iff $a^{-1}b^{-1} = (ab)^{-1}$ – that is, iff G is abelian. ■

How do we compute $G^\vee(k)$? This is the set of maps $\phi\colon A^\vee \to k$, which by duality are all of the form $\phi_b\colon f \mapsto f(b)$. We have

$$\phi_b(fg) = \phi_b\Delta^\vee(f \otimes g) = \Delta(f \otimes g)(b) = (f \otimes g)(\Delta b),$$

and

$$\phi_b(f)\phi_b(g) = f(b)g(b) = (f \otimes g)(b \otimes b)$$

so ϕ_b preserves products iff $\Delta b = b \otimes b$. A similar argument for ε leads us to conclude that

$$G^\vee(k) = \{\text{group-like elements of } A\}$$

which we know from before (Theorem 1.2.12) is the character group of G.

We can evaluate $G^\vee(R)$ by base change. G_R is the functor represented by $A \otimes R$, and so $G_R{}^\vee$ is represented by $(A \otimes R)^\vee = A^\vee \otimes R$, which also represents $(G^\vee)_R$. Hence

$$G^\vee(R) = (G^\vee)_R(R) = (G_R)^\vee(R) = \{\text{group-like elements of } A \otimes R\}.$$

We have

cor:cartier_duality_2

Corollary 1.5.4. *Forming the dual group scheme commutes with base change, and $G^\vee(R) = \mathrm{Hom}(G_R, \mathbb{G}_{m\,R})$.*

Note also that in this case, the group functor $\mathrm{Hom}(G, H)$ is representable. This is not in general true.

CHAPTER 2

Representations of algebraic groups

2.1 Actions & linear representations

Let G be a group functor, X a set functor. An *action* of G on X (written $G \curvearrowright X$) is the only sensible thing: a natural transformation $G \times X \to X$ such that all the maps $G(R) \times X(R) \to X(R)$ are group actions.

For now, we will take X of the form $R \mapsto V \times R$, for some k-module R. If the induced action of $G(R)$ is linear, we say that we have a *linear representation of G on V*.

Examples

no examples yet.

write some!

Our first theorem characterizes these:

racterize_linear_reps

Theorem 2.1.1. *Let G be an affine group scheme, represented by A. Then linear representations of G on V correspond to k-linear maps $V \to V \otimes A$ such that*

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes A \arrow[d, "\mathrm{id} \otimes \Delta"] \\ V \otimes A \arrow[r, "\rho \otimes \mathrm{id}"] & V \otimes A \otimes A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & V \arrow[dl, "\rho"'] \arrow[dr, two heads] & \\ V \otimes A \arrow[rr, "\mathrm{id} \otimes \varepsilon"] & & V \otimes k \end{tikzcd}$$

commute.

Proof. Omitted (it's super boring). You write out all the required diagrams and find out that you need exactly these to commute. ∎

Patterned on this, we make the following

def:comodule

Definition 2.1.2. A k-module V, with a k-linear map $\rho \colon V \to V \otimes A$ satisfying

$$(\mathrm{id} \otimes \varepsilon)\rho = \mathrm{id}, \quad (\mathrm{id} \otimes \Delta)\rho = (\rho \otimes \mathrm{id})\rho$$

is an A-*comodule*.

Example 2.1.3. If $V = A, \rho = \Delta$, this gives the *regular representation* of G.

Standard constructions on modules also work for comodules - tensor products and direct sums make sense, and we can speak of subcomodules and quotient comodules.[1]

sec:finiteness

2.2 Finiteness

prop:findim_cosubmodules

Proposition 2.2.1. *Let k be a field, A/k a Hopf algebra. Every A-comodule V is a direct limit of finite-dimensional subcomodules.*

Proof. We show that every $v \in V$ is contained in some finite-dimensional subcomodule. Let $\{a_i\}$ be a basis for A, so that $\rho(v) = \sum v_i \otimes a_i$, where only finitely many $v_i \neq 0$. Write $\Delta(a_i) = \sum r_{ijk} a_j \otimes a_k$. Then

$$\sum \rho(v_i) \otimes a_i = (\rho \otimes \mathrm{id})\rho(v) = (\mathrm{id} \otimes \Delta)(\rho(v)) = \sum v_i \otimes r_{ijk} a_j \otimes a_k.$$

We compare the coefficients of a_k to see that $\rho(v_k) = \sum v_i \otimes r_{ijk} a_j$. Hence $\mathrm{Span}\langle v, \{v_i\}\rangle$ is a finite-dimensional subcomodule. ■

We can extend this further to show:

thm:hopfalgs_are_direct_limits

Theorem 2.2.2. *Any Hopf algebra A over a field k equals $\lim_{\to} A_\alpha$, where the A_α are finitely generated k-Hopf-subalgebras.*

Proof sketch. We show that every finite subset of A is in some A_α. From Proposition 2.2.1, we know that the finite subset is in a finite-dimensional V, with $\Delta(V) \subseteq V \otimes A$. Let then $\{v_j\}$ be a basis for V. Then $\Delta(v_j) = \sum v_i \otimes a_{ij}$. Then $U \coloneqq \mathrm{Span}\langle \{v_j\}, \{a_{ij}\}\rangle$ satisfies $\Delta(U) \subset U \otimes U$. Also, $L \coloneqq \mathrm{Span}\langle U, S(U)\rangle$ satisfies $\Delta(L) \subset L \otimes L$, $S(L) \subseteq L$. Hence $A_\alpha = k[L]$ will work. ■

To make full use of this, we first need a

def:finite_type

Definition 2.2.3. An affine group scheme is *of finite type* if its Hopf algebra is finitely generated. [2]

Note that being *of finite type* is a weaker condition than being *finite*. Then we have:

cor:finite_type_is_inverse_limit

Corollary 2.2.4. *Every affine group scheme G over a field is an inverse limit of affine group schemes G_α of finite type.*

Proof. This is immediate, simply let G_α be the group schemes corresponding to the Hopf algebras A_α. (Or just $\mathrm{Spec}\, A_\alpha$.) ■

We conclude with today's final theorem:

[1] The category of comodules is not necessarily abelian, since kernels do not always exist. They will apparently exist if the Hopf algebra is flat. (see this MathOverflow discussion.)

[2] Waterhouse calls this *algebraic*, but *finite type* is more in line with standard scheme-theoretic terminology.

thm:everything_in_GL **Theorem 2.2.5.** *Every affine group scheme G of finite type over a field is isomorphic to a closed subgroup of some $\mathbf{GL}_n$.*

Proof. Let A be the Hopf algebra (the coordinate ring) of G. By Proposition 2.2.1, there is a finite-dimensional subcomodule $V \subseteq A$ containing the algebra generators. Let $\{v_j\}$ be a basis for V, such that $\Delta v_j = \sum v_i \otimes a_{ij}$.

Note that $\Delta|V\colon V \to V \otimes A$, so A acts on V. The corresponding map of Hopf algebras is

$$k[\{x_{ij}\}_{i,j\leq n}, 1/\mathrm{det}] \to A,$$

defined by $x_{ij} \mapsto a_{ij}$. But $v_j = (\varepsilon \otimes \mathrm{id})\Delta(v_j) = \sum \varepsilon(v_i) a_{ij}$, so the image contains V, thus all the algebra generators, so the image is all of A. Then we have a surjective map from the Hopf algebra of $\mathbf{GL}_{\dim V}$ onto A, which means that $G \hookrightarrow \mathbf{GL}_{\dim V}$ and is closed. ∎

CHAPTER 3

Milne's book

3.1 The Identity Component of an Algebraic Group

Definition 3.1.1. The *identity/neutral component* G° of G is the connected component of G containing e.

We aim to show that G° is an algebraic subgroup of G. For this, we first look at general algebraic schemes.

Definition 3.1.2. And *tale k-algebra* is a finite product of finite separable extensions of k.

A few properties of tale k-algebras.

- Finite products of tale k-algebras are tale.
- Quotients of an tale k-algebra are tale.
- Composition $A_1 \cdots A_n$ of tale k-subalgebras $A_1, \ldots, A_n$ of a k-algebra A is an tale subalgebra of A. (Due to being a quotient of $A_1 \times \cdots \times A_n$)

Definition 3.1.3. An *tale k-scheme* X is the spectrum of an tale k-algebra. They are characterized by $|X|$ being a finite set of points, and the local rings $\mathcal{O}_{X,x}$ are finite separable field extensions of k.

Lemma 3.1.4. *Let A be a ring and $f \in A$ be idempotent $f^2 = f$ which is nontrivial (neither 0 nor 1). Then $(1 - f)$ is idempotent, and* $\operatorname{Spec}(A)$ *is the disjoint union of two open-closed subsets $D(f)$ and $D(1 - f)$.*

Proof. First, $(1 - f)^2 = 1^2 - 2f + f^2 = 1 - 2f + f = (1 - f)$, so $(1 - f)$ is idempotent. If f is nontrivial, then so is $(1 - f)$.

Now, note that $f(1 - f) = f - f^2 = 0$. Therefore all idempotents in an integral domain are trivial. By considering prime ideals P such that A/P is an integral domain, we deduce that for all prime P, exactly one of $\{f, (1 - f)\}$ is contained in P. Therefore, each point in $\operatorname{Spec}(A)$ is contained in exactly one of $D(f)$ or $D(1 - f)$.

Both are open, and mutual complements, so both form an open-closed disjoint cover of $\operatorname{Spec}(A)$. ∎

Proposition 3.1.5 (1.29)**.** *Let X be an algebraic scheme over k. Then there exists a largest tale k-subalgebra $\pi(X)$ in $\mathcal{O}(X)$.*

Proof. Given an tale subalgebra A of $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$, we know $A \otimes_k k^s \cong (k^s)^n$ for some $n \in \mathbb{N}$.

In particular, we have idempotents $f_1, \ldots, f_n \in \mathcal{O}(X_{k^s})$, (given by $(0, \ldots, 1, \ldots, 0)$) which are orthogonal ($f_i f_j = 0$ for $i \neq j$), and which add up to 1.

This implies that for $i \neq j$ and for any prime P, $f_i f_j = 0 \in P$. Therefore at least one of f_i, f_j is in P. By considering this for all i, $f_i \notin P$ for at most one $1 \leq i \leq n$. But $\sum f_i = 1 \notin P$, so exactly one of the f_i is not in P.

Therefore each point in $|X_{k^s}|$ belongs in exactly one of the $D(f_i)$, which form an open-closed partition of $|X_{k^s}|$.

In particular, $n = [A : k] = [A \otimes k^s : k]$ is bounded above by the number of connected components of $|X_{k^s}|$. It follows that the composite of all tale subalgebras of $\mathcal{O}(X)$ is an tale k-subalgebra which contains all others. ■

Now, let $\pi_0(X) = \mathrm{Spec}(\pi(X))$. By the adjunction

$$\mathrm{Hom}_{k\text{-sch}}(X, \mathrm{Spec}(A)) \cong \mathrm{Hom}_{k\text{-alg}}(A, \mathcal{O}(X))$$

the inclusion of $\pi(X)$ into $\mathcal{O}(X)$ induces a morphism $X \to \pi_0(X)$. This is universal among morphisms of X into an tale k-scheme (since $\pi(X)$ is the maximal tale k-algebra of $\mathcal{O}(X)$).

Proposition 3.1.6 (1.30)**.** *Let X be an algebraic scheme over k.*

For all fields $k' \supset k$,

$$\pi_0(X_{k'}) \simeq \pi_0(X)_{k'}$$

If Y is a second algebraic scheme over k, then

$$\pi_0(X \times Y) \simeq \pi_0(X) \times \pi_0(Y)$$

Proof. Let $\pi = \pi(X)$ and $\pi' = \pi(X_{k'})$. Then $\pi \otimes k'$ is an tale subalgebra of $\mathcal{O}(X_{k'})$, and so by definition $\pi \otimes k' \subset \pi'$. The first part of the proposition requires us to prove we have equality.

First suppose $k' = k^s$. Let Γ be the Galois group of k' over k. Then π' is stable under Γ. From Galois Theory (A.62), we have π'^{Γ} is tale over k and $\pi'^{\Gamma} \otimes k' \simeq \pi'$. We also have that $\pi \subset \pi'^{\Gamma}$ and so by maximality $\pi = \pi'^{\Gamma}$. Thus $\pi \otimes k' = \pi'$.

Secondly, consider $k = k^s$ and $k' = k^a$, of characteristic $p \neq 0$ (Otherwise $k^s = k^a$). Let $e_1, \ldots, e_m$ be a basis of idempotents of $\pi' \cong (k')^m$ as a k'-vector space. Write $e_j = \sum a_i \otimes c_i$ for $a_i \in \mathcal{O}(X)$ and $c_i \in k'$ (since $\mathcal{O}(X_{k'}) \simeq \mathcal{O}(X) \otimes_k k'$). For some r, all $c_i^{p^r}$ lie in k, so $e_j = e_j^{p^r} = \sum a^{p^r} \otimes c_i^{p^r}$ (since e_j is idempotent, and the characteristic is $p \neq 0$). This lies in $\mathcal{O}(X)$ so $\mathcal{O}(X)$ contains idempotent generators of π', so $\pi \otimes k' = \pi'$.

Thirdly, consider k and k' algebraically closed. Then $\pi \simeq k^n$ and $\pi' \simeq k'^m$; we aim to show $n = m$. If $m = 1$, then $\pi \otimes k' \subset \pi' = k'$, so $n = 1$. If $n = 1$, then

X is connected and so $X_{k'}$ is connected since $|X|$ is dense in $|X_{k'}|$. Thus X is connected if and only if $X_{k'}$. Thus $n = 1$ iff $m = 1$. Somehow this extends?

In the general case, if $\pi \otimes k' \neq \pi'$, then $\pi \otimes_k k' \otimes_{k'} k'^a = (\pi \otimes_k k^a) \otimes_{k^a} k'^a \neq \pi' \otimes k'^a$, which is a contradiction with the previous statements.

For the other part of the proposition, we can suppose that $k = k^a$, and then $X \times Y$ is the union of connected subvarieties $x \times Y \cup X \times y$ with $x \in |X|$ and $y \in |Y|$.

■

Proposition 3.1.7 (1.31)**.** *Let X be an algebraic scheme over k.*

The fibres of the map $\phi : X \to \pi_0(X)$ are the connected components of X.

For all $x \in |\pi_0(X)|$, $\phi^{-1}(x)$ is a geometrically connected scheme over $\kappa(x)$.

Proof. First, if $x \in |\pi_0(X)|$, and $X_x = \phi^{-1}(x)$, then $\pi(X_x) = \kappa(x)$, which is a finite field extension of k.

Now, if $\pi(X)$ is a field, then $\mathcal{O}(X)$ has no non-trivial idempotents (as a non-trivial idempotent would give us a larger tale k-algebra). This means that X is connected, as otherwise $\mathcal{O}(X)$ would have a nontrivial idempotent.

Finally, if $\pi(X) = k'$ for a finite extension $k' \supset k$, then $\pi(X_{k^a}) = k' \otimes_k k^a = k^a$, so X is geometrically connected. (Not sure of this argument, we might actually have $\pi(X) = k$, not k') ■

Corollary 3.1.8 (1.32)**.** *Let X be a connected algebraic scheme over k, such that $X(k) \neq \varnothing$. Then X is geometrically connected (X_{k^a} is connected), and $X \times Y$ is connected for any connected algebraic scheme Y over k.*

Proof. Let $A = \pi(X)$. By definition, A is a finite product of finite separable extensions of k. If the product contains more than one term, then X is not connected, so A is a finite separable field extension of k. If $X(k) \cong \mathrm{Hom}(\mathcal{O}(X), k)$ is non-empty, then there is a k-morphism $A \to k$ (inclusion followed by map). Therefore $A = k$, so $\pi(X_{k^a}) = k^a$ and so X_{k^a} is connected, so X is geometrically connected.

Moreover $\pi_0(X \times Y) = \pi_0(X) \times \pi_0(Y)$, by considering the maximal tale k-algebra in $\mathcal{O}(X \times Y) \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$. Since $\pi(X) = k$, then $\pi_0(X) \times \pi_0(Y) = \pi_0(Y)$, so $X \times Y$ is connected. ■

Proposition 3.1.9 (1.34)**.** *Let G be an algebraic group. The identity component of G is an algebraic subgroup of G. Its formation commutes with extension of the base field $(G^\circ)_{k'} \simeq (G_{k'})^\circ$. In particular, the algebraic group G° is geometrically connected.*

Proof. The identity component G° of G contains a k-point; e. Therefore it is both geometrically connected and $G^\circ \times G^\circ$ is a connected component of $G \times G$ (1.32). As the multiplication maps $(e, e) \mapsto e$, then it maps $G^\circ \times G^\circ$ to G°. Similarly, the inversion maps $G^\circ \to G^\circ$, so G° is an algebraic subgroup of G.

Since the formation of $G \to G^\circ$ commutes with the extension of the base field (1.30), then so does its fibre over e (By composition of fibre squares).

In particular, $(G^\circ)_{k'} = (G_{k'})^\circ$, and taking $k' = k^a$ we deduce that G° is geometrically connected. ■

Corollary 3.1.10 (1.35)**.** *Every connected component of an algebraic group is irreducible.*

Proof. Suppose for contradiction that a connected component is reducible. Then there must exist a point lying in at least two irreducible components (otherwise they'd be in different connected components). Thus there exists such a point in G_{k^a}.

Now, in G_{k^a}, no irreducible component is contained in the union of the remainder (by definition), so there exists a point in G_{k^a} which is contained in exactly one irreducible component.

By homogeneity, all points in G_{k^a} are contained in a single irreducible component, which contradicts the statement before. ■

Remark 3.1.11. This means that for algebraic groups, being irreducible, connected or geometrically connected are equivalent. This is not the case for schemes in general.

If G is affine, then the above are equivalent to the quotient of $\mathcal{O}(G)$ by its nilradical being an integral domain.

3.2 The Dimension of an Algebraic Group

Definition 3.2.1. The *dimension* of an algebraic group G is the common dimension of its connected components, equal to the common Krull dimension of each local rings $\mathcal{O}_{G,x}$ for $x \in |G|$.

This is well-defined. The dimension $\dim(X)$ of an irreducible algebraic scheme X is the common Krull dimension of the local rings $\mathcal{O}_{X,x}$, for x in $|X|$. Over k^a, such irreducible scheme X becomes a finite union of irreducible algebraic schemes, all of the same dimension $\dim(X)$.

For G, its irreducible components are its connected components (1.35). By homogeneity over G_{k^a}, we deduce each irreducible component of G_{k^a} has the same dimension, and thus that each irreducible component of G has the same dimension (by our previous statement). ■

Proposition 3.2.2 (1.37)**.** *For an algebraic group G,*

$$\dim \operatorname{Tgt}_e(G) \geq \dim G$$

with equality if and only if G is smooth.

Proof. In general, for an algebraic k-scheme G with a point e such that $\kappa(e) = k$, $\dim \operatorname{Tgt}_e(G) \geq \dim G$, with equality if and only if e is smooth on G (A.52).

If this is the case, then G is smooth (1.28). ■

3.3 Algebraic Subgroups

We aim to know when is G_{red} an algebraic subgroup.

Definition 3.3.1. A k-algebra A is *affine* if $k^a \otimes A$ is reduced (no non-zero nilpotent elements). If B is a reduce k-algebra then $A \otimes B$ is reduced. If k is perfect (every algebraic extension is separable), every reduced k-algebra is affine.

Definition 3.3.2. An algebraic scheme X is *geometrically reduced* if X_{k^a} is reduced. If X is geometrically reduced and Y is reduced, then $X \times Y$ is reduced. If k is perfect, all reduced schemes are geometrically reduced.

Proposition 3.3.3 (1.38-1.39)**.** *Let (G,m) be an algebraic group. If G_{red} is geometrically reduced, then it is an algebraic subgroup. Alternatively, if k is perfect, then G_{red} is always an algebraic subgroup.*

Proof. Recall that if $\phi : Y \to X$ is a morphism of schemes, where Y is reduced and Z is a closed subscheme of X, then ϕ factors through Z_{red} if and only if $|\phi|$ factors through $|Z|$. (A.30)

If G_{red} is geometrically reduced, then $G_{\text{red}} \times G_{\text{red}}$ is reduced (by the above), so the multiplication restricted to $G_{\text{red}} \times G_{\text{red}}$ factors through G_{red}. Similarly the unit and inversion restricted to G_{red} factor through G_{red}.

It follows that $(G_{\text{red}}, m_{\text{red}})$ is an algebraic subgroup of (G, m).

If k is perfect, G_{red} is geometrically reduced. ■

Note that, in general, G_{red} is not an algebraic subgroup. If it is, even if k is perfect, it need not be a normal algebraic subgroup.

An example would be $k = \mathbb{F}_p(t)$, and an algebraic subgroup of $\mathbb{G}_a$ given by $X^{p^2} - tX^p = 0$. Then its G_{red} would be given by $X(X^{p(p-1)} - t)$, which is not geometrically reduced since it is $X(X^{p-1} - t^{1/p})^p$ in k^a.

Lemma 3.3.4 (1.40)**.** *Let G be an algebraic group, and S an abstract subgroup of $G(k)$. Then its Zariski closure $\bar{S}$ is an abstract subgroup of $G(k)$.*

Proof. Take $a, b \in \bar{S}$, we wish to show $ab \in S$. Take a neighborhood U of ab. Since multiplication is a homeomorphism, we can take neighborhoods A and B of a and b such that $AB \subset U$. Then both A and B meet S (since a, b are in the closure of S), so U must meet S. Since this happens for all neighborhoods of ab, ab must lie in the Zariski closure of S.

Since inversion is a homeomorphism, then for $a \in \bar{S}$, the inverse must lie in the closure of $S^{-1} = S$. ■

Proposition 3.3.5 (1.41)**.** *Algebraic subgroups of algebraic groups are closed (in the Zariski topology).*

Proof. By noting that the map $|G_{k^a}| \to |G|$ is a quotient map (surjective, continuous and closed), we note that $|H|$ is closed in $|G|$ if and only if $|H_{k^a}|$ is closed in $|G_{k^a}|$. Thus we may consider k to be algebraically closed.

Now, since the underlying topological spaces are preserved, we might consider both G and H to be reduced.

Now, by definition of closure $|H|$ is open in $\overline{|H|}$. By lemma (1.40), $\overline{|H|}$ is also an abstract subgroup, so $\overline{|H|}$ is a disjoint union of cosets of $|H|$. Therefore $|H|$ is closed in $\overline{|H|}$, so by definition of closure we deduce $|H| = \overline{|H|}$, i.e. $|H|$ is closed in $|G|$. ■

Corollary 3.3.6 (1.42)**.** *The algebraic subgroups of an algebraic group satisfy the descending chain condition.*

Proof. This is true for closed subschemes of an algebraic scheme. ■

Corollary 3.3.7 (1.43)**.** *Algebraic subgroups of an affine algebraic group are affine.*

Proof. This is true for closed subschemes of an algebraic scheme. ■

Corollary 3.3.8. *Say H_1, H_2 are algebraic subvarieties (separated, geometrically reduced) of an algebraic group. If $H_1(k^s) = H_2(k^s)$, then $H_1 = H_2$.*

Proof. Since H_1 and H_2 are closed, we can apply (1.18): We say $X(k')$ is *dense* in X if the only closed subscheme such that $Z(k') = X(k')$ is X itself. If X is geometrically reduced and $|X(k')|$ is dense in $|X|$, then $X(k')$ is dense in X. This is because any $Z(k') = X(k')$ would have $|Z_{k'}| = |X_{k'}|$, which, since $X_{k'}$ is reduced by X geometrically reduced, implies $Z_{k'} = X_{k'}$, ie $Z = X$.

By considering $H_1 \cap H_2 \subset H_1$ and $H_1 \cap H_2 \subset H_2$, we deduce that since $H_1(k^s) = H_2(k^s)$ implies $(H_1 \cap H_2)(k^s) = H_1(k^s)$, then

$$H_1 = H_1 \cap H_2 = H_2$$

■

3.4 Normal and Characteristic Subgroups

3.5 Descent of Subgroups