Algebraic groups

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CHAPTER 1

Affine group schemes

1.1 Affine schemes

Definition 1.1.1. Let k be a commutative ring. An *affine scheme over k* is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$.

Let us try to motivate this definition a bit. Let k be a commutative ring and $\{f_\alpha\} \subseteq k[x_1, \ldots, x_n]$. For any k-algebra R we can consider the set of points in R^n satisfying $f_\alpha = 0$ for all α. Let us call this set $I(R)$. It is clear that if we have a k-algebra homomorphism $f : R \to S$ we obtain a map $I(f) : I(R) \to I(S)$, and that this turns I into a functor $I : \mathsf{Alg}_k \to \mathsf{Set}$.

Theorem 1.1.2. *Let J be the ideal of $k[x_1, \ldots, x_n]$ generated by the f_α and $A = k[x_1, \ldots, x_n]/J$. Then I is a representable functor with representative A.*

Proof. Let $a = (a_1, \ldots, a_n) \in I(R)$ and let $f_a : k[x_1, \ldots, x_n] \to R$ be the k-algebra homomorphism given by $x_i \mapsto a_i$. By definition of the point a, the ideal J must map to 0 under f_a. Thus f_a factors through A and we obtain a map $\bar{f}_a : A \to R$.

Conversely, given a map $f : A \to R$, let $a = (f(\bar{x}_1), \ldots, f(\bar{x}_n)) \in R^n$ where $\bar{x}_i$ is the image of x_i in A. Since f is a homomorphism it is clear that a lies in $I(R)$.

These two maps give a bijection between the sets $I(R)$ and $\mathrm{Hom}_k(A, R)$. It is straightforward to check this is a natural bijection and so I is naturally isomorphic to the functor $\mathrm{Hom}(A, -)$. ■

This theorem says that the affine schemes are exactly what we expect them to be.

1.1.1 The Yoneda lemma

Recall from category theory the following result on representable functors.

Theorem 1.1.3. *(Yoneda's lemma). Let $F : \mathcal{C} \to \mathsf{Set}$ be a functor, $\mathcal{C}$ be locally small and $c \in \mathcal{C}$. Then there is a natrual bijection*

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}(c,-),F) &\leftrightarrow Fc \\ \eta \quad &\rightarrow \quad \eta_c(\mathrm{id}_c) \\ \eta_x : f \mapsto F(f)(y) \quad &\leftarrow \quad y. \end{aligned} \tag{1.1}$$

Corollary 1.1.4. *(Yoneda embedding). The functor*

$$\mathcal{C}(\bullet,-) : \mathcal{C}^{op} \to \mathsf{Fun}(\mathcal{C},\mathsf{Set}) \tag{1.2}$$

is full and faithful.

Proof. Let $c,d \in \mathcal{C}$. From the Yoneda lemma we have a bijection

$$\mathrm{Hom}(\mathcal{C}(d,-),\mathcal{C}(c,-)) \leftrightarrow \mathcal{C}(c,d). \tag{1.3}$$

Let $f \in \mathcal{C}(c,d)$. Under the bijection this gets sent to the natural transformation $\eta_x : g \mapsto \mathcal{C}(c,-)(g)(f) = g \circ f$. Thus the backwards map in the Yoneda bijection is just $\mathcal{C}(\bullet,-)$ and so the result follows. ■

Remark 1.1.5. It follows that $\mathcal{C}^{op}$ is equivalent to the category of representable functors from $\mathcal{C}$. In fact there exists a functor $P : \mathsf{Fun}^{rep}(\mathcal{C},\mathsf{Set}) \to \mathcal{C}^{op}$ such that $P \circ \mathcal{C}(\bullet,-) = \mathrm{id}_{\mathcal{C}^{op}}$, $\mathcal{C}(\bullet,-) \circ P \cong \mathrm{id}_{\mathsf{Fun}^{rep}(\mathcal{C},\mathsf{Set})}$ and $\mathcal{C}(\bullet,-) \circ P|_{\mathrm{im}(\mathcal{C}(\bullet,-)} = \mathrm{id}_{\mathrm{im}(\mathcal{C}(\bullet,-)}$.

Corollary 1.1.6. *Let $c,d \in \mathcal{C}$. Then $c \cong d$ if and only if $\mathcal{C}(c,-) \cong \mathcal{C}(d,-)$.*

Proof. ($\Rightarrow$) This follows by functoriality.

($\Leftarrow$) Let $\alpha : \mathcal{C}(c,-) \Rightarrow \mathcal{C}(d,-)$ be an isomorphism and β its inverse. Let $a : d \to c$ and $b : c \to d$ be the corresponding maps. By naturality of the Yoneda bijection

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(d,-)) \arrow[r] \arrow[d, "\beta\circ"] & \mathcal{C}(d,c) \arrow[d, "\beta_c"] \\ \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(c,-)) \arrow[r] & \mathcal{C}(c,c) \end{tikzcd} \tag{1.4}$$

commutes. So $\beta_c(\alpha_c(\mathrm{id}_c)) = \mathrm{id}_c$. But $\beta_c(\alpha_c(\mathrm{id}_c)) = a \circ b$ and so $a \circ b = \mathrm{id}_c$. Similarly $b \circ a = \mathrm{id}_b$ and so the result follows.

Alternatively note use the remark. ■

This last corollary implies that two k-algebras are isomorphic if and only if the corresponding affine schemes are.

1.2 Affine group schemes

Definition 1.2.1. An *affine group scheme over* k is a functor $F : \mathsf{Alg}_k \to \mathsf{Grp}$ such that F composed with the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is representable.

Example 1.2.2. Let $k = \mathbb{Z}$ (so $\mathsf{Alg}_k = \mathsf{Ring}$). Then $\mathbf{SL}_n : \mathsf{Ring} \to \mathsf{Grp}$, $R \mapsto \mathbf{SL}_n(R)$ is an affine group scheme over $\mathbb{Z}$.

Proposition 1.2.3. *Let E, F, G be affine group schemes over k represented by A, B, C respectively. If we have morphisms $E \to G$ and $F \to G$ then the pullback exists and is represented by $A \otimes_C B$.*

Proof. Existence follows from the fact that we can compute the pullback point-wise and Grp has pullbacks. Explicitly

$$(E \times_G F)(R) = \{(e, f) : \ e \text{ and } f \text{ have same image in } G(R)\}. \tag{1.5}$$

For the second part of the proposition note that the pullback in Alg_k^{op} is of the required form and that we have an equivalence of categories between Alg_k^{op} and affine schemes over k. ■

Definition 1.2.4. Let F be an affine group scheme over k and let $\phi : k \to k'$ be a ring homomorphism. Any k'-algebra can be turned into a k by composing by ϕ and so we can turn F into a functor on k'-algebras. Call this new functor $F_{k'}$.

Proposition 1.2.5. *If F is represented by A then $F_{k'}$ then is represented by $A \otimes_k k'$.*

Proof. There is a natural bijection

$$\mathrm{Hom}_{k'}(A \otimes_k k', S) \leftrightarrow \mathrm{Hom}_k(A, S). \tag{1.6}$$

■

1.2.1 Hopf algebras

Another way to define an affine group scheme is that it is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$ together with natural transformations $\mu : F \times F \to F$, $i : F \to F$ and $u : e \to G$ where e is the functor $\mathrm{Hom}_k(k, -)$ such that

$$\begin{tikzcd} F \times F \times F \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"] & F \times F \arrow[d, "\mu"] \\ F \times F \arrow[r, "\mu"] & F \end{tikzcd} \tag{1.7}$$

$$\begin{tikzcd} e \times F \arrow[r, "u \times \mathrm{id}"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} F \times e \arrow[r, "\mathrm{id} \times u"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \tag{1.8}$$

$$\begin{tikzcd} F \arrow[r, shift left, "\mathrm{id}\times i"] \arrow[r, shift right, "i\times\mathrm{id}"'] \arrow[d] & F\times F \arrow[d, "\mu"] \\ e \arrow[r, "u"] & F \end{tikzcd} \tag{1.9}$$

all commute. If F is represented by A then translating back to Alg_k we obtain maps $\Delta : A \to A \otimes_k A$, $S : A \to A$ and $\epsilon : A \to k$. Satisfying the same commutative diagrams, but with arrows reversed and μ replaced with Δ etc. The algebra A together with these maps is what we call a Hopf algebra.

Definition 1.2.6. Let $\psi : H \to G$ be a morphism of affine groups schemes. We say ψ is a *closed embedding* if the corresponding map on algebras is surjective. H is then isomorphic to a closed subgroup of G represented by the corresponding quotient of the algebra of A.

Theorem 1.2.7. *Affine group schemes over k correspond to Hopf algebras over k.*

Definition 1.2.8. Let A be a k-algebra. We call an ideal I of A a *Hopf ideal* if A/I inherits the structure of a Hopf algebra.

Proposition 1.2.9. *Let A be a k-algebra and $I \lhd A$. Then I is a hopf ideal if and only if $\Delta(I) \subseteq I \otimes A + A \otimes I$, $S(I) \subseteq I$ and $\epsilon(I) = 0$.*

Proof. Todo. ∎

Definition 1.2.10. Let $\Phi : G \to H$ be a morphism of affine groups schemes. Then $\ker \Phi(R) = \ker(G(R) \to H(R))$ or alternatively $\ker \Phi = G \times_H \{e\}$. It follows that if G, H are represented by A, B respectively then $\ker \Phi$ is represented by $A \otimes_B k \cong A/AI_B$.

1.2.2 Characters

Definition 1.2.11. A homomorphism $G \to G_m$ is called a character.

thm:characters_are_gp-like_elems **Theorem 1.2.12.** *The characters of an affine group scheme G represented by A correspond to the group-like elements of A.*

1.3 Diagonalisable group schemes

sec:diag_gp_sch

Let M be an abelian group, k a ring, and $k[M]$ the group algebra. We construct a Hopf algebra structure on $k[M]$ by making the group elements group-like, i.e., we set

$$\Delta(m) = m \otimes m, \quad \epsilon(m) = 1, \quad S(m) = m^{-1},$$

and we extend by linearity to all $k[M]$. This is a Hopf algebra (easily checked on elements)

def:diag_gp_sch **Definition 1.3.1.** A group scheme constructed in this way is called a *diagonalisable group scheme.*

The name is not immediately a good one, but we will later see that most affine group schemes can be embedded into $\mathbf{GL}_n$ for some n , and that the diagonalisable group schemes will be precisely those corresponding to groups of diagonalisable matrices. ref!

We can describe these schemes completely:

hm:struct_of_diag_gps

Theorem 1.3.2. *Let G be a diagonalisable group scheme, represented by A, and suppose that A is a finitely generated k-algebra. Then*

$$G = \left(\prod_{i=1}^{n} \mathbb{G}_m \right) \times \prod_{j=1}^{n'} \boldsymbol{\mu}_{m_j}$$

for some integers n, n', m_j.

Proof. Take a finite set of generators of A. These are finite linear combinations of elements of M, so we have a finite number of elements of M generating A, Then these elements must also generate M. So M is a finitely generated abelian group. But such groups are direct sums of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, so we need only check two cases.

Case 1: $M = \mathbb{Z}$. Then $k[\mathbb{Z}]$ has basis $\{e_n | n \in \mathbb{Z}\}$, with multiplication $e_i e_j = e_{i+j}$. So $k[\mathbb{Z}] \xrightarrow{\sim} k[X, X^{-1}]$, the affine coordinate ring of $\mathbb{G}_m$.

Case 2: $M = \mathbb{Z}/n\mathbb{Z}$. The basis is now $\{1 = e_0, e_1, \dots, e_{n-1} = e^{n-1}\}$. So $e_1^n = 1$, and the Hopf algebra is $k[X]/(X^n - 1)$, which represents $\boldsymbol{\mu}_n$. ■

We will now really exploit the Hopf algebra structure. First, we have

lem:gplikes_lin_indep

Lemma 1.3.3. *The group-like elements in a Hopf algebra over a field k are linearly independent.*

Proof. Take group-like elements $b, \{b_i\}$ such that

$$b = \sum \lambda_i b_i \qquad \text{(a finite sum)},$$

and assume the b_i to be independent. Then

$$1 = \epsilon(b) = \sum \lambda_i \epsilon(b_i) = \sum \lambda_i.$$

But

$$\begin{aligned} \Delta(b) = b \otimes b &= \sum \lambda_i \lambda_j b_i \otimes b_j \\ &= \sum \lambda_i \Delta(b_i) = \sum \lambda_i b_i \otimes b_i \end{aligned}$$

so all $\lambda_i \lambda_j = 0$ for $j \neq i$, $\lambda_i^2 = \lambda_i$, so all λ_i are either 0 or 1. Since their sum is 1, exactly one $\lambda_i = 1$, and so $b = b_i$. ■

Furthermore:

_implies_cring_is_span_of_gplikes

Theorem 1.3.4. *An affine group scheme over a field is diagonalisable iff its representing algebra (coordinate ring!) is spanned by group-like elements. There is a natural coequivalence of categories*

$$\{\textit{Diagonalisable group schemes}/k\} \xrightarrow{\sim} \mathsf{AbGrp}.$$

Proof. By the lemma, if A is spanned by group-like elements, they are a basis of A. The group formed by them is the character group X_G. So $k[X_G] \xrightarrow{\sim} A$, at least as sets. Checking on basis elements shows that this is an isomorphism of Hopf algebras. If M is any abelian group, its elements are the only group-like elements of $k[M]$. Thus M is the character group of $\operatorname{Spec} k[M]$ (the group scheme rep. by $k[M]$). Also, if $G \to H$ is a homomorphism of affine algebraic groups, we have an induced map $X_H \to X_G$. This determines the Hopf algebra structure, since X_H spans $k[X_H]$. In the opposite direction, a map $X_H \to X_G$ clearly induces a map $k[X_H] \to k[X_G]$. ∎

We will develop this correspondence further in Section 1.5.

1.4 Finite constant groups

sec:fin_const_gps

Let Γ be a finite group, and define a functor

$$\overline{\Gamma}\colon \mathsf{k\text{-}alg} \to \mathsf{Grp}$$

sending every ring to Γ, every map to the identity map. This functor is unrepresentable (not sure why), but we'll do something almost as good.

Let $A := Mor(\Gamma \to k)$. Let e_σ be the indicator function for $\sigma \in \Gamma$, so $\{e_\sigma\}$ is a basis for A. We define a ring structure on A by $e_\sigma^2 = e_\sigma, e_\sigma e_\tau = 0$ for $\sigma \neq \tau$, $\sum e_\sigma = 1$.

Let R be a k-algebra where the only idempotents are 0 and 1. Then a homomorphism $\phi\colon A \to R$ is completely determined by which σ is sent to 1, so $\operatorname{Hom}(A, R) \xrightarrow{\sim} \Gamma$ as sets. If we define a comultiplication on A by $\Delta(e_\rho) = \sum_{\rho=\sigma\tau} e_\sigma \otimes e_\tau$, this induces the same group structure on $\operatorname{Hom}(A, R)$ as that of Γ. If we define $S(e_\sigma) = e_{\sigma^{-1}}$, $\epsilon(e_\sigma) = 1$ if σ is the unit of Γ, otherwise 0, A becomes a Hopf algebra. We say that it represents the *constant group scheme* Γ.

1.4.1 Scheme-theoretic aside

As a scheme, the construction is simpler: Just take $\operatorname{Spec}(\prod_{g\in\Gamma}) = \coprod_{g\in\Gamma} \operatorname{Spec} k$, and define the group structure by $G \times G \ni (\operatorname{Spec} k)_{g,h} \mapsto (\operatorname{Spec} k)_{gh} \in G$. Chasing through the definitions and various categorical equivalences shows that this really is the same construction.

1.5 Cartier Duality

sec:Cartier_duality

Let k be a ring, N a finite rank free k-module. Recall that then: $N^\vee$ is also free; if M is also finite rank free, $(M \otimes N)^\vee \xrightarrow{\sim} M^\vee \otimes N^\vee$; $\mathrm{Hom}_k(M^\vee, N^\vee) \xrightarrow{\sim} \mathrm{Hom}_k(M,N)^\vee$; and Hom, $\otimes$ commute with finite direct sums. Hence the same identities hold for finitely generated projective modules, as they are summands of free modules.

f:finite_group_scheme

Definition 1.5.1. Call a group scheme *finite* if it is represented a k-algebra A which is a finitely generated projective[1] module.

Note that being finitely generated as a module is a much stronger condition than being finitely generated as a k-algebra.

Remark 1.5.2. The scheme-theoretic statement would be: G is a group object in the category of schemes over k with a finite structure morphism.

So take such a group scheme G. Then we have five Hopf structure maps:

$$\begin{aligned} \Delta\colon A &\to A \otimes A && \text{comultiplication} \\ \epsilon\colon A &\to k && \text{counit} \\ S\colon A &\to A && \text{coinverse} \\ m\colon A \times A &\to A && \text{multiplication} \\ u\colon k &\to A && \text{unit.} \end{aligned}$$

If we dualise these, we end up with something very similar:

$$\begin{aligned} m^\vee\colon A^\vee &\to A^\vee \otimes A^\vee \\ u^\vee\colon A^\vee &\to k \\ S^\vee\colon A^\vee &\to A^\vee \\ \Delta^\vee\colon A^\vee \times A^\vee &\to A^\vee \\ \epsilon^\vee\colon k &\to A^\vee \end{aligned}$$

and we are immediately led to

thm:Cartier_duality

Theorem 1.5.3 (Cartier duality 1)**.** *Let G be a finite abelian group scheme represented by A. Then $A^\vee$ represents another finite abelian group scheme $G^\vee$, and $(G^\vee)^\vee \xrightarrow{\sim} G$.*

Sketch of proof. This is reasonably straightforward - just check that all the axioms are satisfied, dualise diagrams if needed. The only tricky bit is to show that $S^\vee$ is a homomorphism: We must show that

$$\Delta^\vee(S^\vee \otimes S^\vee) = S^\vee \Delta^\vee,$$

[1]The projectivity hypothesis is probably not part of the definition found elsewhere. But we will need it in these notes.

which is equivalent to $\Delta S = (S \otimes S)\Delta$. As group functors, this means that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "S"] & A \otimes A \arrow[d, "S\otimes S"] \\ A \arrow[r, "\Delta"] & A \otimes A \end{tikzcd} \quad \text{or equivalently} \quad \begin{tikzcd} G & G \times G \arrow[l, "mult."'] \\ G \arrow[u, "inv"] & G \times G \arrow[l, "mult."'] \arrow[u, "inv\times inv"] \end{tikzcd} \quad \text{which commutes}$$

iff $a^{-1}b^{-1} = (ab)^{-1}$ – that is, iff G is abelian. ■

How do we compute $G^\vee(k)$? This is the set of maps $\phi\colon A^\vee \to k$, which by duality are all of the form $\phi_b\colon f \mapsto f(b)$. We have

$$\phi_b(fg) = \phi_b\Delta^\vee(f \otimes g) = \Delta(f \otimes g)(b) = (f \otimes g)(\Delta b),$$

and

$$\phi_b(f)\phi_b(g) = f(b)g(b) = (f \otimes g)(b \otimes b)$$

so ϕ_b preserves products iff $\Delta b = b \otimes b$. A similar argument for ϵ leads us to conclude that

$$G^\vee(k) = \{\text{group-like elements of } A\}$$

which we know from before (Theorem 1.2.12) is the character group of G.

We can evaluate $G^\vee(R)$ by base change. G_R is the functor represented by $A \otimes R$, and so $G_R{}^\vee$ is represented by $(A \otimes R)^\vee = A^\vee \otimes R$, which also represents $(G^\vee)_R$. Hence

$$G^\vee(R) = (G^\vee)_R(R) = (G_R)^\vee(R) = \{\text{group-like elements of } A \otimes R\}.$$

We have

cor:cartier_duality_2

Corollary 1.5.4. *Forming the dual group scheme commutes with base change, and* $G^\vee(R) = \mathrm{Hom}(G_R, \mathbb{G}_{m\,R})$.

Note also that in this case, the group functor $\mathrm{Hom}(G, H)$ is representable. This is not in general true.

CHAPTER 2

Representations of algebraic groups

2.1 Actions & linear representations

Let G be a group functor, X a set functor. An *action* of G on X (written $G \curvearrowright X$) is the only sensible thing: a natural transformation $G \times X \to X$ such that all the maps $G(R) \times X(R) \to X(R)$ are group actions.

For now, we will take X of the form $R \mapsto V \times R$, for some k-module R. If the induced action of $G(R)$ is linear, we say that we have a *linear representation of G on V*.

Examples

no examples yet. write some!

Our first theorem characterizes these:

racterize_linear_reps **Theorem 2.1.1.** *Let G be an affine group scheme, represented by A. Then linear representations of G on V correspond to k-linear maps $V \to V \otimes A$ such that*

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes A \arrow[d, "\mathrm{id} \otimes \Delta"] \\ V \otimes A \arrow[r, "\rho \otimes \mathrm{id}"] & V \otimes A \otimes A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & V \arrow[dl, "\rho"'] \arrow[dr, hook] & \\ V \otimes A \arrow[rr, "\mathrm{id} \otimes \epsilon"] & & V \otimes k \end{tikzcd}$$

commute.

Proof. Omitted (it's super boring). You write out all the required diagrams and find out that you need exactly these to commute. ∎

Patterned on this, we make the following

def:comodule **Definition 2.1.2.** A k-module V, with a k-linear map $\rho\colon V \to V \otimes A$ satisfying

$$(\mathrm{id} \otimes \epsilon)\rho = \mathrm{id}, \quad (\mathrm{id} \otimes \Delta)\rho = (\rho \otimes \mathrm{id})\rho$$

is an *A-comodule.*

Example 2.1.3. If $V = A, \rho = \Delta$, this gives the *regular representation* of G.

Standard constructions on modules also work for comodules - tensor products and direct sums make sense, and we can speak of subcomodules and quotient comodules.[1]

2.2 Finiteness

sec:finiteness

prop:findim_cosubmodules

Proposition 2.2.1. *Let k be a field, A/k a Hopf algebra. Every A-comodule V is a direct limit of finite-dimensional subcomodules.*

Proof. We show that every $v \in V$ is contained in some finite-dimensional subcomodule. Let $\{a_i\}$ be a basis for A, so that $\rho(v) = \sum v_i \otimes a_i$, where only finitely many $v_i \neq 0$. Write $\Delta(a_i) = \sum r_{ijk} a_j \otimes a_k$. Then

$$\sum \rho(v_i) \otimes a_i = (\rho \otimes \mathrm{id})\rho(v) = (\mathrm{id} \otimes \Delta)(\rho(v)) = \sum v_i \otimes r_{ijk} a_j \otimes a_k.$$

We compare the coefficients of a_k to see that $\rho(v_k) = \sum v_i \otimes r_{ijk} a_j$. Hence $\mathrm{Span}\langle v, \{v_i\}\rangle$ is a finite-dimensional subcomodule. ■

We can extend this further to show:

thm:hopfalgs_are_direct_limits

Theorem 2.2.2. *Any Hopf algebra A over a field k equals* $\lim_{\to} A_\alpha$*, where the A_α are finitely generated k-Hopf-subalgebras.*

Proof sketch. We show that every finite subset of A is in some A_α. From Proposition 2.2.1, we know that the finite subset is in a finite-dimensional V, with $\Delta(V) \subseteq V \otimes A$. Let then $\{v_j\}$ be a basis for V. Then $\Delta(v_j) = \sum v_i \otimes a_{ij}$. Then $U := \mathrm{Span}\langle \{v_j\}, \{a_{ij}\}\rangle$ satisfies $\Delta(U) \subset U \otimes U$. Also, $L := \mathrm{Span}\langle U, S(U)\rangle$ satisfies $\Delta(L) \subset L \otimes L$, $S(L) \subseteq L$. Hence $A_\alpha = k[L]$ will work. ■

To make full use of this, we first need a

def:finite_type

Definition 2.2.3. An affine group scheme is *of finite type* if its Hopf algebra is finitely generated. [2]

Note that being *of finite type* is a weaker condition than being *finite*. Then we have:

cor:finite_type_is_inverse_limit

Corollary 2.2.4. *Every affine group scheme G over a field is an inverse limit of affine group schemes G_α of finite type.*

Proof. This is immediate, simply let G_α be the group schemes corresponding to the Hopf algebras A_α. (Or just $\mathrm{Spec}\, A_\alpha$.) ■

We conclude with today's final theorem:

[1] The category of comodules is not necessarily abelian, since kernels do not always exist. They will apparently exist if the Hopf algebra is flat. (see this MathOverflow discussion.)

[2] Waterhouse calls this *algebraic*, but *finite type* is more in line with standard scheme-theoretic terminology.

thm:everything_in_GL **Theorem 2.2.5.** *Every affine group scheme G of finite type over a field is isomorphic to a closed subgroup of some $\mathbf{GL}_n$.*

Proof. Let A be the Hopf algebra (the coordinate ring) of G. By Proposition 2.2.1, there is a finite-dimensional subcomodule $V \subseteq A$ containing the algebra generators. Let $\{v_j\}$ be a basis for V, such that $\Delta v_j = \sum v_i \otimes a_{ij}$.

Note that $\Delta|V : V \to V \otimes A$, so A acts on V. The corresponding map of Hopf algebras is

$$k[\{x_{ij}\}_{i,j\leq n}, 1/\det] \to A,$$

defined by $x_{ij} \mapsto a_{ij}$. But $v_j = (\epsilon \otimes \mathrm{id})\Delta(v_j) = \sum \epsilon(v_i) a_{ij}$, so the image contains V, thus all the algebra generators, so the image is all of A. Then we have a surjective map from the Hopf algebra of $\mathbf{GL}_{\dim V}$ onto A, which means that $G \hookrightarrow \mathbf{GL}_{\dim V}$ and is closed. ■