

Algebraic groups

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January 29, 2019

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CHAPTER 1

Affine group schemes

1.1 Affine schemes

Definition 1.1.1. Let k be a commutative ring. An *affine scheme over k* is a representable functor $F : \mathbf{Alg}_k \rightarrow \mathbf{Set}$.

Let us try to motivate this definition a bit. Let k be a commutative ring and $\{f_\alpha\} \subseteq k[x_1, \dots, x_n]$. For any k -algebra R we can consider the set of points in R^n satisfying $f_\alpha = 0$ for all α . Let us call this set $I(R)$. It is clear that if we have a k -algebra homomorphism $f : R \rightarrow S$ we obtain a map $I(f) : I(R) \rightarrow I(S)$, and that this turns I into a functor $I : \mathbf{Alg}_k \rightarrow \mathbf{Set}$.

Theorem 1.1.2. Let J be the ideal of $k[x_1, \dots, x_n]$ generated by the f_α and $A = k[x_1, \dots, x_n]/J$. Then I is a representable functor with representative A .

Proof. Let $a = (a_1, \dots, a_n) \in I(R)$ and let $f_a : k[x_1, \dots, x_n] \rightarrow R$ be the k -algebra homomorphism given by $x_i \mapsto a_i$. By definition of the point a , the ideal J must map to 0 under f_a . Thus f_a factors through A and we obtain a map $\bar{f}_a : A \rightarrow R$.

Conversely, given a map $f : A \rightarrow R$, let $a = (f(\bar{x}_1), \dots, f(\bar{x}_n)) \in R^n$ where \bar{x}_i is the image of x_i in A . Since f is a homomorphism it is clear that a lies in $I(R)$.

These two maps give a bijection between the sets $I(R)$ and $\mathbf{Hom}_k(A, R)$. It is straightforward to check this is a natural bijection and so I is naturally isomorphic to the functor $\mathbf{Hom}(A, -)$. ■

This theorem says that the affine schemes are exactly what we expect them to be.

1.1.1 The Yoneda lemma

Recall from category theory the following result on representable functors.

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Theorem 1.1.3. (*Yoneda's lemma*). *Let $F : \mathcal{C} \rightarrow \mathbf{Set}$ be a functor, \mathcal{C} be locally small and $c \in \mathcal{C}$. Then there is a natural bijection*

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}(c, -), F) &\leftrightarrow Fc \\ \eta &\rightarrow \eta_c(\mathrm{id}_c) \\ \eta_x : f &\mapsto F(f)(y) \leftarrow y. \end{aligned} \tag{1.1}$$

Corollary 1.1.4. (*Yoneda embedding*). *The functor*

$$\mathcal{C}(\bullet, -) : \mathcal{C}^{op} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathbf{Set}) \tag{1.2}$$

is full and faithful.

Proof. Let $c, d \in \mathcal{C}$. From the Yoneda lemma we have a bijection

$$\mathrm{Hom}(\mathcal{C}(d, -), \mathcal{C}(c, -)) \leftrightarrow \mathcal{C}(c, d). \tag{1.3}$$

Let $f \in \mathcal{C}(c, d)$. Under the bijection this gets sent to the natural transformation $\eta_x : g \mapsto \mathcal{C}(c, -)(g)(f) = g \circ f$. Thus the backwards map in the Yoneda bijection is just $\mathcal{C}(\bullet, -)$ and so the result follows. ■

Remark 1.1.5. It follows that \mathcal{C}^{op} is equivalent to the category of representable functors from \mathcal{C} . In fact there exists a functor $P : \mathbf{Fun}^{rep}(\mathcal{C}, \mathbf{Set}) \rightarrow \mathcal{C}^{op}$ such that $P \circ \mathcal{C}(\bullet, -) = \mathrm{id}_{\mathcal{C}^{op}}$, $\mathcal{C}(\bullet, -) \circ P \cong \mathrm{id}_{\mathbf{Fun}^{rep}(\mathcal{C}, \mathbf{Set})}$ and $\mathcal{C}(\bullet, -) \circ P|_{\mathrm{im}(\mathcal{C}(\bullet, -))} = \mathrm{id}_{\mathrm{im}(\mathcal{C}(\bullet, -))}$.

Corollary 1.1.6. *Let $c, d \in \mathcal{C}$. Then $c \cong d$ if and only if $\mathcal{C}(c, -) \cong \mathcal{C}(d, -)$.*

Proof. (\Rightarrow) This follows by functoriality.

(\Leftarrow) Let $\alpha : \mathcal{C}(c, -) \Rightarrow \mathcal{C}(d, -)$ be an isomorphism and β its inverse. Let $a : d \rightarrow c$ and $b : c \rightarrow d$ be the corresponding maps. By naturality of the Yoneda bijection

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}(c, -), \mathcal{C}(d, -)) & \longrightarrow & \mathcal{C}(d, c) \\ \downarrow \beta \circ & & \downarrow \beta_c \\ \mathrm{Hom}(\mathcal{C}(c, -), \mathcal{C}(c, -)) & \longrightarrow & \mathcal{C}(c, c) \end{array} \tag{1.4}$$

commutes. So $\beta_c(\alpha_c(\mathrm{id}_c)) = \mathrm{id}_c$. But $\beta_c(\alpha_c(\mathrm{id}_c)) = a \circ b$ and so $a \circ b = \mathrm{id}_c$. Similarly $b \circ a = \mathrm{id}_d$ and so the result follows.

Alternatively note use the remark. ■

This last corollary implies that two k -algebras are isomorphic if and only if the corresponding affine schemes are.

1.2 Affine group schemes

Definition 1.2.1. An *affine group scheme over k* is a functor $F : \text{Alg}_k \rightarrow \text{Grp}$ such that F composed with the forgetful functor $\text{Grp} \rightarrow \text{Set}$ is representable.

Example 1.2.2. Let $k = \mathbb{Z}$ (so $\text{Alg}_k = \text{Ring}$). Then $\mathbf{SL}_n : \text{Ring} \rightarrow \text{Grp}$, $R \mapsto \mathbf{SL}_n(R)$ is an affine group scheme over \mathbb{Z} .

Proposition 1.2.3. Let E, F, G be affine group schemes over k represented by A, B, C respectively. If we have morphisms $E \rightarrow G$ and $F \rightarrow G$ then the pullback exists and is represented by $A \otimes_C B$.

Proof. Existence follows from the fact that we can compute the pullback point-wise and Grp has pullbacks. Explicitly

$$(E \times_G F)(R) = \{(e, f) : e \text{ and } f \text{ have same image in } G(R)\}. \quad (1.5)$$

For the second part of the proposition note that the pullback in Alg_k^{op} is of the required form and that we have an equivalence of categories between Alg_k^{op} and affine schemes over k . ■

Definition 1.2.4. Let F be an affine group scheme over k and let $\phi : k \rightarrow k'$ be a ring homomorphism. Any k' -algebra can be turned into a k by composing by ϕ and so we can turn F into a functor on k' -algebras. Call this new functor $F_{k'}$.

Proposition 1.2.5. If F is represented by A then $F_{k'}$ then is represented by $A \otimes_k k'$.

Proof. There is a natural bijection

$$\text{Hom}_{k'}(A \otimes_k k', S) \leftrightarrow \text{Hom}_k(A, S). \quad (1.6)$$

■

1.2.1 Hopf algebras

Another way to define an affine group scheme is that it is a representable functor $F : \text{Alg}_k \rightarrow \text{Set}$ together with natural transformations $\mu : F \times F \rightarrow F$, $i : F \rightarrow F$ and $u : e \rightarrow F$ where e is the functor $\text{Hom}_k(k, -)$ such that

$$\begin{array}{ccc} F \times F \times F & \xrightarrow{\text{id} \times \mu} & F \times F \\ \downarrow \mu \times \text{id} & & \downarrow \mu \\ F \times F & \xrightarrow{\mu} & F \end{array} \quad (1.7)$$

$$\begin{array}{ccc} e \times F & \xrightarrow{u \times \text{id}} & F \times F \\ \searrow \cong & & \downarrow \mu \\ & & F \end{array} \quad \begin{array}{ccc} F \times e & \xrightarrow{\text{id} \times u} & F \times F \\ \searrow \cong & & \downarrow \mu \\ & & F \end{array} \quad (1.8)$$

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$$\begin{array}{ccc}
 F & \xrightleftharpoons[i \times \text{id}]{\text{id} \times i} & F \times F \\
 \downarrow & & \downarrow \mu \\
 e & \xrightarrow{u} & F
 \end{array} \tag{1.9}$$

all commute. If F is represented by A then translating back to Alg_k we obtain maps $\Delta : A \rightarrow A \otimes_k A$, $S : A \rightarrow A$ and $\varepsilon : A \rightarrow k$. Satisfying the same commutative diagrams, but with arrows reversed and μ replaced with Δ etc. The algebra A together with these maps is what we call a Hopf algebra.

Definition 1.2.6. Let $\psi : H \rightarrow G$ be a morphism of affine groups schemes. We say ψ is a *closed embedding* if the corresponding map on algebras is surjective. H is then isomorphic to a closed subgroup of G represented by the corresponding quotient of the algebra of A .

Theorem 1.2.7. *Affine group schemes over k correspond to Hopf algebras over k .*

Definition 1.2.8. Let A be a k -algebra. We call an ideal I of A a *Hopf ideal* if A/I inherits the structure of a Hopf algebra.

Proposition 1.2.9. *Let A be a k -algebra and $I \triangleleft A$. Then I is a Hopf ideal if and only if $\Delta(I) \subseteq I \otimes A + A \otimes I$, $S(I) \subseteq I$ and $\varepsilon(I) = 0$.*

Proof. Todo. ■

Definition 1.2.10. Let $\Phi : G \rightarrow H$ be a morphism of affine groups schemes. Then $\ker \Phi(R) = \ker(G(R) \rightarrow H(R))$ or alternatively $\ker \Phi = G \times_H \{e\}$. It follows that if G, H are represented by A, B respectively then $\ker \Phi$ is represented by $A \otimes_B k \cong A/AI_B$.

1.2.2 Characters

Definition 1.2.11. A homomorphism $G \rightarrow G_m$ is called a character.

Theorem 1.2.12. *The characters of an affine group scheme G represented by A correspond to the group-like elements of A .*

1.3 Diagonalisable group schemes

sec:diag_gp_sch

Let M be an abelian group, k a ring, and $k[M]$ the group algebra. We construct a Hopf algebra structure on $k[M]$ by making the group elements group-like, i.e., we set

$$\Delta(m) = m \otimes m, \quad \varepsilon(m) = 1, \quad S(m) = m^{-1},$$

and we extend by linearity to all $k[M]$. This is a Hopf algebra (easily checked on elements)

def:diag_gp_sch

Definition 1.3.1. A group scheme constructed in this way is called a *diagonalisable group scheme*.

The name is not immediately a good one, but we will later see that most affine group schemes can be embedded into \mathbf{GL}_n for some n (Theorem 2.2.5), and that the diagonalisable group schemes will be precisely those corresponding to groups of diagonalisable matrices.

We can describe these schemes completely:

hm:struct_of_diag_gps

Theorem 1.3.2. *Let G be a diagonalisable group scheme, represented by A , and suppose that A is a finitely generated k -algebra. Then*

$$G = \left(\prod_{i=1}^n \mathbb{G}_m \right) \times \prod_{j=1}^{n'} \mu_{m_j}$$

for some integers n, n', m_j .

Proof. Take a finite set of generators of A . These are finite linear combinations of elements of M , so we have a finite number of elements of M generating A . Then these elements must also generate M . So M is a finitely generated abelian group. But such groups are direct sums of copies of \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$, so we need only check two cases.

Case 1: $M = \mathbb{Z}$. Then $k[\mathbb{Z}]$ has basis $\{e_n | n \in \mathbb{Z}\}$, with multiplication $e_i e_j = e_{i+j}$. So $k[\mathbb{Z}] \xrightarrow{\sim} k[X, X^{-1}]$, the affine coordinate ring of \mathbb{G}_m .

Case 2: $M = \mathbb{Z}/n\mathbb{Z}$. The basis is now $\{1 = e_0, e_1, \dots, e_{n-1} = e^{n-1}\}$. So $e_1^n = 1$, and the Hopf algebra is $k[X]/(X^n - 1)$, which represents μ_n . ■

We will now really exploit the Hopf algebra structure. First, we have

lem:gplikes_lin_indep

Lemma 1.3.3. *The group-like elements in a Hopf algebra over a field k are linearly independent.*

Proof. Take group-like elements $b, \{b_i\}$ such that

$$b = \sum \lambda_i b_i \quad (\text{a finite sum}),$$

and assume the b_i to be independent. Then

$$1 = \varepsilon(b) = \sum \lambda_i \varepsilon(b_i) = \sum \lambda_i.$$

But

$$\begin{aligned} \Delta(b) &= b \otimes b = \sum \lambda_i \lambda_j b_i \otimes b_j \\ &= \sum \lambda_i \Delta(b_i) = \sum \lambda_i b_i \otimes b_i \end{aligned}$$

so all $\lambda_i \lambda_j = 0$ for $j \neq i$, $\lambda_i^2 = \lambda_i$, so all λ_i are either 0 or 1. Since their sum is 1, exactly one $\lambda_i = 1$, and so $b = b_i$. ■

Furthermore:

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implies_cring_is_span_of_gplikes

Theorem 1.3.4. *An affine group scheme over a field is diagonalisable iff its representing algebra (coordinate ring!) is spanned by group-like elements. There is a natural coequivalence of categories*

$$\{\text{Diagonalisable group schemes}/k\} \xrightarrow{\sim} \mathbf{AbGrp}.$$

Proof. By the lemma, if A is spanned by group-like elements, they are a basis of A . The group formed by them is the character group X_G . So $k[X_G] \xrightarrow{\sim} A$, at least as sets. Checking on basis elements shows that this is an isomorphism of Hopf algebras. If M is any abelian group, its elements are the only group-like elements of $k[M]$. Thus M is the character group of $\text{Spec } k[M]$ (the group scheme rep. by $k[M]$). Also, if $G \rightarrow H$ is a homomorphism of affine algebraic groups, we have an induced map $X_H \rightarrow X_G$. This determines the Hopf algebra structure, since X_H spans $k[X_H]$. In the opposite direction, a map $X_H \rightarrow X_G$ clearly induces a map $k[X_H] \rightarrow k[X_G]$. ■

We will develop this correspondence further in Section 1.5.

1.4 Finite constant groups

sec:fin_const_gps

Let Γ be a finite group, and define a functor

$$\bar{\Gamma}: k\text{-alg} \rightarrow \mathbf{Grp}$$

sending every ring to Γ , every map to the identity map. This functor is unrepresentable (not sure why), but we'll do something almost as good.

Let $A := \text{Mor}(\Gamma \rightarrow k)$. Let e_σ be the indicator function for $\sigma \in \Gamma$, so $\{e_\sigma\}$ is a basis for A . We define a ring structure on A by $e_\sigma^2 = e_\sigma$, $e_\sigma e_\tau = 0$ for $\sigma \neq \tau$, $\sum e_\sigma = 1$.

Let R be a k -algebra where the only idempotents are 0 and 1. Then a homomorphism $\phi: A \rightarrow R$ is completely determined by which σ is sent to 1, so $\text{Hom}(A, R) \xrightarrow{\sim} \Gamma$ as sets. If we define a comultiplication on A by $\Delta(e_\rho) = \sum_{\rho=\sigma\tau} e_\sigma \otimes e_\tau$, this induces the same group structure on $\text{Hom}(A, R)$ as that of Γ . If we define $S(e_\sigma) = e_{\sigma^{-1}}$, and $\varepsilon(e_\sigma) = 1$ if σ is the unit of Γ , otherwise 0, A becomes a Hopf algebra. We say that it represents the *constant group scheme* Γ .

1.4.1 Scheme-theoretic aside

As a scheme, the construction is simpler: Just take $\text{Spec}(\prod_{g \in \Gamma} k) = \coprod_{g \in \Gamma} \text{Spec } k$, and define the group structure by $G \times G \ni (\text{Spec } k)_{g,h} \mapsto (\text{Spec } k)_{gh} \in G$. Chasing through the definitions and various categorical equivalences shows that this really is the same construction.

1.5 Cartier Duality

sec:Cartier_duality

Let k be a ring, N a finite rank free k -module. Recall that then: N^\vee is also free; if M is also finite rank free, $(M \otimes N)^\vee \xrightarrow{\sim} M^\vee \otimes N^\vee$; $\text{Hom}_k(M^\vee, N^\vee) \xrightarrow{\sim} \text{Hom}_k(M, N)^\vee$; and Hom, \otimes commute with finite direct sums. Hence the same identities hold for finitely generated projective modules, as they are summands of free modules.

f:finite_group_scheme

Definition 1.5.1. Call a group scheme *finite* if it is represented a k -algebra A which is a finitely generated projective¹ module.

Note that being finitely generated as a module is a much stronger condition than being finitely generated as a k -algebra.

Remark 1.5.2. The scheme-theoretic statement would be: G is a group object in the category of schemes over k with a finite structure morphism.

So take such a group scheme G . Then we have five Hopf structure maps:

$$\begin{array}{ll} \Delta: A \rightarrow A \otimes A & \text{comultiplication} \\ \varepsilon: A \rightarrow k & \text{counit} \\ S: A \rightarrow A & \text{coinverse} \\ m: A \times A \rightarrow A & \text{multiplication} \\ u: k \rightarrow A & \text{unit.} \end{array}$$

If we dualise these, we end up with something very similar:

$$\begin{array}{l} m^\vee: A^\vee \rightarrow A^\vee \otimes A^\vee \\ u^\vee: A^\vee \rightarrow k \\ S^\vee: A^\vee \rightarrow A^\vee \\ \Delta^\vee: A^\vee \times A^\vee \rightarrow A^\vee \\ \varepsilon^\vee: k \rightarrow A^\vee \end{array}$$

and we are immediately led to

thm:Cartier_duality

Theorem 1.5.3 (Cartier duality 1). *Let G be a finite abelian group scheme represented by A . Then A^\vee represents another finite abelian group scheme G^\vee , and $(G^\vee)^\vee \xrightarrow{\sim} G$.*

Sketch of proof. This is reasonably straightforward - just check that all the axioms are satisfied, dualise diagrams if needed. The only tricky bit is to show that S^\vee is a homomorphism: We must show that

$$\Delta^\vee(S^\vee \otimes S^\vee) = S^\vee \Delta^\vee,$$

¹The projectivity hypothesis is probably not part of the definition found elsewhere. But we will need it in these notes.

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which is equivalent to $\Delta S = (S \otimes S)\Delta$. As group functors, this means that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow S & & \downarrow S \otimes S \\ A & \xrightarrow{\Delta} & A \otimes A \end{array} \quad \text{or equivalently} \quad \begin{array}{ccc} G & \xleftarrow{\text{mult.}} & G \times G \\ \uparrow \text{inv} & & \uparrow \text{inv} \times \text{inv} \\ G & \xleftarrow{\text{mult.}} & G \times G \end{array} \quad \text{which commutes}$$

iff $a^{-1}b^{-1} = (ab)^{-1}$ – that is, iff G is abelian. ■

How do we compute $G^\vee(k)$? This is the set of maps $\phi: A^\vee \rightarrow k$, which by duality are all of the form $\phi_b: f \mapsto f(b)$. We have

$$\phi_b(fg) = \phi_b \Delta^\vee(f \otimes g) = \Delta(f \otimes g)(b) = (f \otimes g)(\Delta b),$$

and

$$\phi_b(f)\phi_b(g) = f(b)g(b) = (f \otimes g)(b \otimes b)$$

so ϕ_b preserves products iff $\Delta b = b \otimes b$. A similar argument for ε leads us to conclude that

$$G^\vee(k) = \{\text{group-like elements of } A\}$$

which we know from before (Theorem 1.2.12) is the character group of G .

We can evaluate $G^\vee(R)$ by base change. G_R is the functor represented by $A \otimes R$, and so G_R^\vee is represented by $(A \otimes R)^\vee = A^\vee \otimes R$, which also represents $(G^\vee)_R$. Hence

$$G^\vee(R) = (G^\vee)_R(R) = (G_R)^\vee(R) = \{\text{group-like elements of } A \otimes R\}.$$

We have

cor:cartier_duality_2

Corollary 1.5.4. *Forming the dual group scheme commutes with base change, and $G^\vee(R) = \text{Hom}(G_R, \mathbb{G}_{mR})$.*

Note also that in this case, the group functor $\text{Hom}(G, H)$ is representable. This is not in general true.

CHAPTER 2

Representations of algebraic groups

2.1 Actions & linear representations

Let G be a group functor, X a set functor. An *action* of G on X (written $G \curvearrowright X$) is the only sensible thing: a natural transformation $G \times X \rightarrow X$ such that all the maps $G(R) \times X(R) \rightarrow X(R)$ are group actions.

For now, we will take X of the form $R \mapsto V \times R$, for some k -module R . If the induced action of $G(R)$ is linear, we say that we have a *linear representation* of G on V .

Examples

no examples yet.

write some!

Our first theorem characterizes these:

Theorem 2.1.1. *Let G be an affine group scheme, represented by A . Then linear representations of G on V correspond to k -linear maps $V \rightarrow V \otimes A$ such*

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow \rho & & \downarrow \text{id} \otimes \Delta \\ V \otimes A & \xrightarrow{\rho \otimes \text{id}} & V \otimes A \otimes A \end{array} \quad \text{and} \quad \begin{array}{ccc} & V & \\ \swarrow \rho & & \searrow \text{id} \otimes \varepsilon \\ V \otimes A & \xrightarrow{\text{id} \otimes \varepsilon} & V \otimes k \end{array} \quad \text{commute.}$$

Proof. Omitted (it's super boring). You write out all the required diagrams and find out that you need exactly these to commute. ■

Patterned on this, we make the following

`def:comodule`

Definition 2.1.2. A k -module V , with a k -linear map $\rho: V \rightarrow V \otimes A$ satisfying

$$(\text{id} \otimes \varepsilon)\rho = \text{id}, \quad (\text{id} \otimes \Delta)\rho = (\rho \otimes \text{id})\rho$$

is an A -comodule.

Example 2.1.3. If $V = A$, $\rho = \Delta$, this gives the *regular representation* of G .

2. Representations of algebraic groups

Standard constructions on modules also work for comodules - tensor products and direct sums make sense, and we can speak of subcomodules and quotient comodules.¹

2.2 Finiteness

sec:finiteness

prop:findim_cosubmodules

Proposition 2.2.1. *Let k be a field, A/k a Hopf algebra. Every A -comodule V is a direct limit of finite-dimensional subcomodules.*

Proof. We show that every $v \in V$ is contained in some finite-dimensional subcomodule. Let $\{a_i\}$ be a basis for A , so that $\rho(v) = \sum v_i \otimes a_i$, where only finitely many $v_i \neq 0$. Write $\Delta(a_i) = \sum r_{ijk} a_j \otimes a_k$. Then

$$\sum \rho(v_i) \otimes a_i = (\rho \otimes \text{id})\rho(v) = (\text{id} \otimes \Delta)(\rho(v)) = \sum v_i \otimes r_{ijk} a_j \otimes a_k.$$

We compare the coefficients of a_k to see that $\rho(v_k) = \sum v_i \otimes r_{ijk} a_j$. Hence $\text{Span}\langle v, \{v_i\} \rangle$ is a finite-dimensional subcomodule. ■

We can extend this further to show:

thm:hopfalgs_are_direct_limits

Theorem 2.2.2. *Any Hopf algebra A over a field k equals $\lim_{\rightarrow} A_\alpha$, where the A_α are finitely generated k -Hopf-subalgebras.*

Proof sketch. We show that every finite subset of A is in some A_α . From Proposition 2.2.1, we know that the finite subset is in a finite-dimensional V , with $\Delta(V) \subseteq V \otimes A$. Let then $\{v_j\}$ be a basis for V . Then $\Delta(v_j) = \sum v_i \otimes a_{ij}$. Then $U := \text{Span}\langle \{v_j\}, \{a_{ij}\} \rangle$ satisfies $\Delta(U) \subset U \otimes U$. Also, $L := \text{Span}\langle U, S(U) \rangle$ satisfies $\Delta(L) \subset L \otimes L$, $S(L) \subseteq L$. Hence $A_\alpha = k[L]$ will work. ■

To make full use of this, we first need a

def:finite_type

Definition 2.2.3. An affine group scheme is *of finite type* if its Hopf algebra is finitely generated.²

Note that being *of finite type* is a weaker condition than being *finite*. Then we have:

cor:finite_type_is_inverse_limit

Corollary 2.2.4. *Every affine group scheme G over a field is an inverse limit of affine group schemes G_α of finite type.*

Proof. This is immediate, simply let G_α be the group schemes corresponding to the Hopf algebras A_α . (Or just $\text{Spec } A_\alpha$.) ■

We conclude with today's final theorem:

¹The category of comodules is not necessarily abelian, since kernels do not always exist. They will apparently exist if the Hopf algebra is flat. (see [this MathOverflow discussion](#).)

²Waterhouse calls this *algebraic*, but *finite type* is more in line with standard scheme-theoretic terminology.

thm:everything_in_GL

Theorem 2.2.5. *Every affine group scheme G of finite type over a field is isomorphic to a closed subgroup of some \mathbf{GL}_n .*

Proof. Let A be the Hopf algebra (the coordinate ring) of G . By Proposition 2.2.1, there is a finite-dimensional subcomodule $V \subseteq A$ containing the algebra generators. Let $\{v_j\}$ be a basis for V , such that $\Delta v_j = \sum v_i \otimes a_{ij}$.

Note that $\Delta|_V: V \rightarrow V \otimes A$, so A acts on V . The corresponding map of Hopf algebras is

$$k[\{x_{ij}\}_{i,j \leq n}, 1/\det] \rightarrow A,$$

defined by $x_{ij} \mapsto a_{ij}$. But $v_j = (\varepsilon \otimes \text{id})\Delta(v_j) = \sum \varepsilon(v_i)a_{ij}$, so the image contains V , thus all the algebra generators, so the image is all of A . Then we have a surjective map from the Hopf algebra of $\mathbf{GL}_{\dim V}$ onto A , which means that $G \hookrightarrow \mathbf{GL}_{\dim V}$ and is closed. ■