Algebraic groups

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Contents

CHAPTER 1

Affine group schemes

1.1 Affine schemes

Definition 1.1.1. Let k be a commutative ring. An *affine scheme over* k is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$.

Let us try to motivate this definition a bit. Let k be a commutative ring and $\{f_\alpha\} \subseteq k[x_1, \ldots, x_n]$. For any k-algebra R we can consider the set of points in R^n satisfying $f_\alpha = 0$ for all α. Let us call this set $I(R)$. It is clear that if we have a k-algebra homomorphism $f : R \to S$ we obtain a map $I(f) : I(R) \to I(S)$, and that this turns I into a functor $I : \mathsf{Alg}_k \to \mathsf{Set}$.

Thm 1.1.2. *Let J be the ideal of $k[x_1, \ldots, x_n]$ generated by the f_α and $A = k[x_1, \ldots, x_n]/J$. Then I is a representable functor with representative A.*

Proof. Let $a = (a_1, \ldots, a_n) \in I(R)$ and let $f_a : k[x_1, \ldots, x_n] \to R$ be the k-algebra homomorphism given by $x_i \mapsto a_i$. By definition of the point a, the ideal J must map to 0 under f_a. Thus f_a factors through A and we obtain a map $\bar{f}_a : A \to R$.

Conversely, given a map $f : A \to R$, let $a = (f(\bar{x}_1), \ldots, f(\bar{x}_n)) \in R^n$ where $\bar{x}_i$ is the image of x_i in A. Since f is a homomorphism it is clear that a lies in $I(R)$.

These two maps give a bijection between the sets $I(R)$ and $\mathrm{Hom}_k(A, R)$. It is straightforward to check this is a natural bijection and so I is naturally isomorphic to the functor $\mathrm{Hom}(A, -)$. ∎

This theorem says that the affine schemes are exactly what we expect them to be.

1.1.1 The Yoneda lemma

Recall from category theory the following result on representable functors.

Thm 1.1.3. *(Yoneda's lemma). Let $F : \mathcal{C} \to \mathsf{Set}$ be a functor, $\mathcal{C}$ be locally small and $c \in \mathcal{C}$. Then there is a natrual bijection*

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}(c,-),F) &\leftrightarrow Fc \\ \eta \quad &\to \quad \eta_c(\mathrm{id}_c) \\ \eta_x : f \mapsto F(f)(y) \quad &\leftarrow \quad y. \end{aligned} \tag{1.1}$$

Corollary 1.1.4. *(Yoneda embedding). The functor*

$$\mathcal{C}(\bullet,-) : \mathcal{C}^{op} \to \mathsf{Fun}(\mathcal{C},\mathsf{Set}) \tag{1.2}$$

is full and faithful.

Proof. Let $c, d \in \mathcal{C}$. From the Yoneda lemma we have a bijection

$$\mathrm{Hom}(\mathcal{C}(d,-),\mathcal{C}(c,-)) \leftrightarrow \mathcal{C}(c,d). \tag{1.3}$$

Let $f \in \mathcal{C}(c,d)$. Under the bijection this gets sent to the natural transformation $\eta_x : g \mapsto \mathcal{C}(c,-)(g)(f) = g \circ f$. Thus the backwards map in the Yoneda bijection is just $\mathcal{C}(\bullet,-)$ and so the result follows. ∎

Remark 1.1.5. It follows that $\mathcal{C}^{op}$ is equivalent to the category of representable functors from $\mathcal{C}$. In fact there exists a functor $P : \mathsf{Fun}^{rep}(\mathcal{C},\mathsf{Set}) \to \mathcal{C}^{op}$ such that $P \circ \mathcal{C}(\bullet,-) = \mathrm{id}_{\mathcal{C}^{op}}$, $\mathcal{C}(\bullet,-) \circ P \cong \mathrm{id}_{\mathsf{Fun}^{rep}(\mathcal{C},\mathsf{Set})}$ and $\mathcal{C}(\bullet,-) \circ P|_{\mathrm{im}(\mathcal{C}(\bullet,-)} = \mathrm{id}_{\mathrm{im}(\mathcal{C}(\bullet,-)}$.

Corollary 1.1.6. *Let $c, d \in \mathcal{C}$. Then $c \cong d$ if and only if $\mathcal{C}(c,-) \cong \mathcal{C}(d,-)$.*

Proof. ($\Rightarrow$) This follows by functoriality.

($\Leftarrow$) Let $\alpha : \mathcal{C}(c,-) \Rightarrow \mathcal{C}(d,-)$ be an isomorphism and β its inverse. Let $a : d \to c$ and $b : c \to d$ be the corresponding maps. By naturality of the Yoneda bijection

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(d,-)) & \longrightarrow & \mathcal{C}(d,c) \\ \big\downarrow{\scriptstyle \beta\circ} & & \big\downarrow{\scriptstyle \beta_c} \\ \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(c,-)) & \longrightarrow & \mathcal{C}(c,c) \end{array} \tag{1.4}$$

commutes. So $\beta_c(\alpha_c(\mathrm{id}_c)) = \mathrm{id}_c$. But $\beta_c(\alpha_c(\mathrm{id}_c)) = a \circ b$ and so $a \circ b = \mathrm{id}_c$. Similarly $b \circ a = \mathrm{id}_b$ and so the result follows.

Alternatively note use the remark. ∎

This last corollary implies that two k-algebras are isomorphic if and only if the corresponding affine schemes are.

1.2 Affine group schemes

Definition 1.2.1. An *affine group scheme over k* is a functor $F : \mathsf{Alg}_k \to \mathsf{Grp}$ such that F composed with the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is representable.

Example 1.2.2. Let $k = \mathbb{Z}$ (so $\mathsf{Alg}_k = \mathsf{Ring}$). Then $\mathbf{SL}_n : \mathsf{Ring} \to \mathsf{Grp}$, $R \mapsto \mathbf{SL}_n(R)$ is an affine group scheme over $\mathbb{Z}$.

Proposition 1.2.3. *Let E, F, G be affine group schemes over k represented by A, B, C respectively. If we have morphisms $E \to G$ and $F \to G$ then the pullback exists and is represented by $A \otimes_C B$.*

Proof. Existence follows from the fact that we can compute the pullback pointwise and Grp has pullbacks. Explicitly

$$(E \times_G F)(R) = \{(e, f) : \text{ } e \text{ and } f \text{ have same image in } G(R)\}. \tag{1.5}$$

For the second part of the proposition note that the pullback in Alg_k^{op} is of the required form and that we have an equivalence of categories between Alg_k^{op} and affine schemes over k. ■

Definition 1.2.4. Let F be an affine group scheme over k and let $\phi : k \to k'$ be a ring homomorphism. Any k'-algebra can be turned into a k by composing by ϕ and so we can turn F into a functor on k'-algebras. Call this new functor $F_{k'}$.

Proposition 1.2.5. *If F is represented by A then $F_{k'}$ then is represented by $A \otimes_k k'$.*

Proof. There is a natural bijection

$$\mathrm{Hom}_{k'}(A \otimes_k k', S) \leftrightarrow \mathrm{Hom}_k(A, S). \tag{1.6}$$

■

1.2.1 Hopf algebras

Another way to define an affine group scheme is that it is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$ together with natural transformations $\mu : F \times F \to F$, $i : F \to F$ and $u : e \to G$ where e is the functor $\mathrm{Hom}_k(k, -)$ such that

$$\begin{tikzcd} F \times F \times F \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"] & F \times F \arrow[d, "\mu"] \\ F \times F \arrow[r, "\mu"] & F \end{tikzcd} \tag{1.7}$$

$$\begin{tikzcd} e \times F \arrow[r, "u \times \mathrm{id}"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} F \times e \arrow[r, "\mathrm{id} \times u"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \tag{1.8}$$

$$
\begin{tikzcd}
F \arrow[r, shift left, "\mathrm{id}\times i"] \arrow[r, shift right, "i\times\mathrm{id}"'] \arrow[d] & F \times F \arrow[d, "\mu"] \\
e \arrow[r, "u"] & F
\end{tikzcd}
\tag{1.9}
$$

all commute. If F is represented by A then translating back to Alg_k we obtain maps $\Delta : A \to A \otimes_k A$, $S : A \to A$ and $\epsilon : A \to k$. Satisfying the same commutative diagrams, but with arrows reversed and μ replaced with Δ etc. The algebra A together with these maps is what we call a Hopf algebra.

Definition 1.2.6. Let $\psi : H \to G$ be a morphism of affine groups schemes. We say ψ is a *closed embedding* if the corresponding map on algebras is surjective. H is then isomorphic to a closed subgroup of G represented by the corresponding quotient of the algebra of A.

Thm 1.2.7. *Affine group schemes over k correspond to Hopf algebras over k.*

Definition 1.2.8. Let A be a k-algebra. We call an ideal I of A a *Hopf ideal* if A/I inherits the structure of a Hopf algebra.

Proposition 1.2.9. *Let A be a k-algebra and $I \lhd A$. Then I is a hopf ideal if and only if $\Delta(I) \subseteq I \otimes A + A \otimes I$, $S(I) \subseteq I$ and $\epsilon(I) = 0$.*

Proof. Todo. ■

Definition 1.2.10. Let $\Phi : G \to H$ be a morphism of affine groups schemes. Then $\ker \Phi(R) = \ker(G(R) \to H(R))$ or alternatively $\ker \Phi = G \times_H \{e\}$. It follows that if G, H are represented by A, B respectively then $\ker \Phi$ is represented by $A \otimes_B k \cong A/AI_B$.

1.2.2 Characters

Definition 1.2.11. A homomorphism $G \to G_m$ is called a character.

Thm 1.2.12. *The characters of an affine group scheme G represented by A correspond to the group like elements of A.*