Algebraic groups

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CHAPTER 1

Affine group schemes

1.1 Affine schemes

Definition 1.1.1. Let k be a commutative ring. An *affine scheme over k* is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$.

Let us try to motivate this definition a bit. Let k be a commutative ring and $\{f_\alpha\} \subseteq k[x_1, \ldots, x_n]$. For any k-algebra R we can consider the set of points in R^n satisfying $f_\alpha = 0$ for all α. Let us call this set $I(R)$. It is clear that if we have a k-algebra homomorphism $f : R \to S$ we obtain a map $I(f) : I(R) \to I(S)$, and that this turns I into a functor $I : \mathsf{Alg}_k \to \mathsf{Set}$.

Thm 1.1.2. *Let J be the ideal of $k[x_1, \ldots, x_n]$ generated by the f_α and $A = k[x_1, \ldots, x_n]/J$. Then I is a representable functor with representative A.*

Proof. Let $a = (a_1, \ldots, a_n) \in I(R)$ and let $f_a : k[x_1, \ldots, x_n] \to R$ be the k-algebra homomorphism given by $x_i \mapsto a_i$. By definition of the point a, the ideal J must map to 0 under f_a. Thus f_a factors through A and we obtain a map $\bar{f}_a : A \to R$.

Conversely, given a map $f : A \to R$, let $a = (f(\bar{x}_1), \ldots, f(\bar{x}_n)) \in R^n$ where $\bar{x}_i$ is the image of x_i in A. Since f is a homomorphism it is clear that a lies in $I(R)$.

These two maps give a bijection between the sets $I(R)$ and $\mathrm{Hom}_k(A, R)$. It is straightforward to check this is a natural bijection and so I is naturally isomorphic to the functor $\mathrm{Hom}(A, -)$. ■

This theorem says that the affine schemes are exactly what we expect them to be.

1.1.1 The Yoneda lemma

Recall from category theory the following result on representable functors.

Thm 1.1.3. *(Yoneda's lemma). Let $F : \mathcal{C} \to \mathsf{Set}$ be a functor, $\mathcal{C}$ be locally small and $c \in \mathcal{C}$. Then there is a natrual bijection*

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}(c,-),F) &\leftrightarrow Fc \\ \eta \quad &\to \quad \eta_c(\mathrm{id}_c) \\ \eta_x : f \mapsto F(f)(y) \quad &\leftarrow \quad y. \end{aligned} \tag{1.1}$$

Corollary 1.1.4. *(Yoneda embedding). The functor*

$$\mathcal{C}(\bullet,-) : \mathcal{C}^{op} \to \mathsf{Fun}(\mathcal{C},\mathsf{Set}) \tag{1.2}$$

is full and faithful.

Proof. Let $c, d \in \mathcal{C}$. From the Yoneda lemma we have a bijection

$$\mathrm{Hom}(\mathcal{C}(d,-),\mathcal{C}(c,-)) \leftrightarrow \mathcal{C}(c,d). \tag{1.3}$$

Let $f \in \mathcal{C}(c,d)$. Under the bijection this gets sent to the natural transformation $\eta_x : g \mapsto \mathcal{C}(c,-)(g)(f) = g \circ f$. Thus the backwards map in the Yoneda bijection is just $\mathcal{C}(\bullet,-)$ and so the result follows. ■

Remark 1.1.5. It follows that $\mathcal{C}^{op}$ is equivalent to the category of representable functors from $\mathcal{C}$.

Corollary 1.1.6. *Let $c, d \in \mathcal{C}$. Then $c \cong d$ if and only if $\mathcal{C}(c,-) \cong \mathcal{C}(d,-)$.*

Proof. ($\Rightarrow$) This follows by functoriality.

($\Leftarrow$) Let $\alpha : \mathcal{C}(c,-) \Rightarrow \mathcal{C}(d,-)$ be an isomorphism and β its inverse. Let $a : d \to c$ and $b : c \to d$ be the corresponding maps. By naturality of the Yoneda bijection

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(d,-)) \arrow[r] \arrow[d, "\beta\circ"] & \mathcal{C}(d,c) \arrow[d, "\beta_c"] \\ \mathrm{Hom}(\mathcal{C}(c,-),\mathcal{C}(c,-)) \arrow[r] & \mathcal{C}(c,c) \end{tikzcd} \tag{1.4}$$

commutes. So $\beta_c(\alpha_c(\mathrm{id}_c)) = \mathrm{id}_c$. But $\beta_c(\alpha_c(\mathrm{id}_c)) = a \circ b$ and so $a \circ b = \mathrm{id}_c$. Similarly $b \circ a = \mathrm{id}_b$ and so the result follows.

Alternatively note that we can define an inverse of $\mathcal{C}(\bullet,-)$ from the image of the Yoneda embedding to $\mathcal{C}$ and it automatically is a functor. The result must then follow. ■

This last corollary implies that two k-algebras are isomorphic if and only if the corresponding affine schemes are.

1.2 Affine group schemes

Definition 1.2.1. An *affine group scheme over* k is a functor $F : \mathsf{Alg}_k \to \mathsf{Grp}$ such that F composed with the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is representable.

Example 1.2.2. Let $k = \mathbb{Z}$ (so $\mathsf{Alg}_k = \mathsf{Ring}$). Then $\mathbf{SL}_n : \mathsf{Ring} \to \mathsf{Grp}$, $R \mapsto \mathbf{SL}_n(R)$ is an affine group scheme over $\mathbb{Z}$.

Proposition 1.2.3. *Let E, F, G be affine group schemes over k represented by A, B, C respectively. If we have morphisms $E \to G$ and $F \to G$ then the pullback exists and is represented by $A \otimes_C B$.*

Proof. Existence follows from the fact that we can compute the pullback pointwise and Grp has pullbacks. Explicitly

$$(E \times_G F)(R) = \{(e, f) : \ e \text{ and } f \text{ have same image in } G(R)\}. \tag{1.5}$$

For the second part of the proposition note that the pullback in Alg_k^{op} is of the required form and that we have an equivalence of categories between Alg_k^{op} and affine schemes over k. ■

1.2.1 Hopf algebras

Another way to define an affine group scheme is that it is a representable functor $F : \mathsf{Alg}_k \to \mathsf{Set}$ together with natural transformations $\mu : F \times F \to F$, $i : F \to F$ and $u : e \to G$ where e is the functor $\mathrm{Hom}_k(k, -)$ such that

$$\begin{tikzcd} F \times F \times F \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"] & F \times F \arrow[d, "\mu"] \\ F \times F \arrow[r, "\mu"] & F \end{tikzcd} \tag{1.6}$$

$$\begin{tikzcd} e \times F \arrow[r, "u \times \mathrm{id}"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} F \times e \arrow[r, "\mathrm{id} \times u"] \arrow[dr, "\cong"] & F \times F \arrow[d, "\mu"] \\ & F \end{tikzcd} \tag{1.7}$$

$$\begin{tikzcd} F \arrow[r, shift left, "\mathrm{id} \times i"] \arrow[r, shift right, "i \times \mathrm{id}"'] \arrow[d] & F \times F \arrow[d, "\mu"] \\ e \arrow[r, "u"] & F \end{tikzcd} \tag{1.8}$$

all commute. If F is represented by A then translating back to Alg_k we obtain maps $\Delta : A \to A \otimes_k A$, $S : A \to A$ and $\epsilon : A \to k$. Satisfying the same commutative diagrams, but with arrows reversed and μ replaced with Δ etc. The algebra A together with these maps is what we call a Hopf algebra.

Thm 1.2.4. *Affine group schemes over k correspond to Hopf algebras over k.*