Homological algebra

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Contents

CHAPTER 1

Abelian Categories

1.1 Additive categories

Let $\mathcal{C}$ be a category such that the hom-sets carry the structure of an abelian group and composition is bilinear. We call such a category Ab-enriched. An additive category is an Ab-enriched category which has finite coproducts.

1.2 Abelian categories

1.3 Exact sequences

sec:es

$$\begin{tikzcd}
 & & \mathrm{coker}(\theta) \\
\mathcal{F} \arrow[r, "\theta"] \arrow[d] & \mathcal{G} \arrow[ur] \arrow[r, "\phi"] & \mathcal{H} \\
\mathrm{im}(\theta) \arrow[r, "\alpha"] \arrow[ur] & \ker(\phi) \arrow[u] &
\end{tikzcd} \tag{1.1}$$

1.3.1 Split sequences

1.3.2 Homology

1.4 Adjoint functors

Let $L : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. If L admits a right adjoint $R : \mathcal{B} \to \mathcal{A}$ then it turns out L has a lot of useful properties. In this section we explore these properties.

Proposition 1.4.1. *Suppose $L \dashv R$. Then L is right exact and R is left exact.*

Proof. Consider the short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$. For every $A \in \mathcal{A}$ we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(L(A), B_1) & \longrightarrow & \mathrm{Hom}(L(A), B_2) & \longrightarrow & \mathrm{Hom}(L(A), B_3) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Hom}(A, R(B_1)) & \longrightarrow & \mathrm{Hom}(A, R(B_2)) & \longrightarrow & \mathrm{Hom}(A, R(B_3)) \end{array} \tag{1.2}$$

where the top row is exact. It follows that the bottom row is exact for all A and so the bottow row is too. It follows that

$$0 \longrightarrow R(B_1) \longrightarrow R(B_2) \longrightarrow R(B_3) \tag{1.3}$$

is exact and so R is left exact. By a similar argument L is right exact. ■

Proposition 1.4.2. *Suppose $L \dashv R$. Then*

1. *if L is exact then R preserves injectives*
2. *if R is exact then L preserves projectives.*

Proof. Suppose L is exact and I is an injective object in $\mathcal{B}$. We need to show that $\mathrm{Hom}(-, R(I))$ is exact. To do this it suffices to show that given $f : A \to B$ injective, the map $f^* : \mathrm{Hom}(B, R(I)) \to \mathrm{Hom}(A, R(I))$ is surjective. But L is exact so Lf is injective and so $(Lf)^* : \mathrm{Hom}(LB, I) \to \mathrm{Hom}(LA, I)$ is surjective. We also have that $L \dashv R$ and so

$$\begin{array}{ccc} \mathrm{Hom}(L(B), I) & \xrightarrow{(Lf)^*} & \mathrm{Hom}(L(A), I) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(B, R(I)) & \xrightarrow{f^*} & \mathrm{Hom}(A, R(I)) \end{array} \tag{1.4}$$

commutes. It follows that f^* is surjective as required.

The corresponding result for R follows similarly. ■

CHAPTER 2

Sheaf Theory

2.1 Presheaves

Let $\mathcal{C}$ be any category, $\mathcal{A}$ be an abelian category and define $\mathsf{PreSh}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{A})$ to be the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$. The functor sending all objects to 0 is certainly both initial and terminal, direct sums can be defined pointwise, and the hom-sets in $\mathsf{PreSh}(\mathcal{C})$ inherit an additive structure from $\mathcal{A}$ so $\mathsf{PreSh}(\mathcal{C})$ is naturally an additive category. Moreover kernels and cokernels can be contructed in the obvious way and it is clear that they satisfy the axioms for an abelian category and so $\mathsf{PreSh}(C)$ is abelian.

2.2 Sheaves

To define sheaves we restrict to the case when X be a topological space, $\mathcal{U}$ the poset of open sets of X, and $\mathcal{A}$ be an abelian category. We write $\mathsf{PreSh}(X)$ for $\mathsf{PreSh}(\mathcal{U})$. The category of sheaves on X with values in $\mathcal{A}$, $\mathsf{Sh}(X)$, is defined to be the full subcategory of $\mathsf{PreSh}(X)$ with objects given by presheaves $\mathscr{F}$ for which the following diagram is an equalizer for all open coverings $U = \cup_i U_i$

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.1}$$

Since $\mathcal{A}$ is an abelian category this is equivalent to the following diagram being exact

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \xrightarrow{\text{diff}} \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.2}$$

Note that since $\varnothing$ admits the empty covering and the empty product is 0 this forces $\mathscr{F}(\varnothing) = 0$.

As in the case of $\mathsf{PreSh}(\mathcal{C})$, $\mathsf{Sh}(X)$ is an additive category. However, the cokernel of a morphism between sheaves need not be a sheaf and so we must do some more work to show that $\mathsf{Sh}(X)$ is abelian.

Fix $x \in X$. For a (pre)sheaf $\mathscr{F}$ define the stalk of $\mathscr{F}$ at x to be

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F} \tag{2.3}$$

when this limit exists. Note that this is a functor since morphisms between (pre)sheaves are natural transformations.

Thm 2.2.1. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

1. *If ϕ_x is injective for all $x \in X$ then ϕ is injective on sections.*
2. *If ϕ_x is an isomorphism for all $x \in X$ then ϕ is an isomorphism.*

Proof. Exercise. ■

Aside

Although we do not need this right away, given an $A \in \mathcal{A}$ we can define the (pre)sheaf x_*A by

$$(x_*A)(U) = \begin{cases} A & \textit{if } x \in U \\ 0 & \textit{otherwise} \end{cases} \tag{2.4}$$

Proposition 2.2.2. *When it exists, the functor $(-)_x : \mathsf{Sh}(X) \to \mathcal{A}$ is left adjoint to $x_* : \mathcal{A} \to \mathsf{Sh}(X)$.*

Proof. To see this simply note that morphisms between $\mathscr{F}$ and $x_*(A)$ correspond naturally to natrual transformations between $\mathscr{F}$ restricted to $U \ni x$ and $\Delta(A)$. ■

Remark 2.2.3. The result also holds in $\mathsf{PreSh}(X)$.

2.3 Étalé space of a presheaf and sheafification

For a presheaf $\mathscr{F}$ we are now in the position to define its étalé space. The étalé space of $\mathscr{F}$, denoted Spé($\mathscr{F}$) is the topological space with underlying set $\amalg_{x\in X}\mathscr{F}_x$ and topology generated by the basis of sets given by $\{s_x | x \in U\}$ for $s \in \mathscr{F}(U)$ where $U \subset X$ is open. Together with this space there is also a natural continuous map $\pi : \text{Spé}(\mathscr{F}) \to X$ sending an element s_x to x. The sheafification of $\mathscr{F}$, denoted $\mathscr{F}^+$, is then defined to be the sheaf of sections of $\pi : \text{Spé}(\mathscr{F}) \to X$. By unwrapping the definitions we see that the sections can be characterised as

$$\begin{aligned} \mathscr{F}^+(U) = \{s : U \to \amalg_{x\in U}\mathscr{F}_x : &\forall x \in U, \exists V \subset U \text{ open containing } x \text{ and} \\ & t \in \mathscr{F}(V) \text{ s.t. } s(y) = t_y \forall y \in V\} \end{aligned} \tag{2.5}$$

In particular there is a natural morphism $\mathscr{F} \to \mathscr{F}^+$ sending $s \in \mathscr{F}(U)$ to the section $x \mapsto s_x$ which is an isomorphism on stalks. From the characterisation of sections it clear that if $\mathscr{F}$ is a presheaf of $\mathsf{AbGrp}, \mathsf{Ring}, ...$ then $\mathscr{F}^+$ is a sheaf with values in the corresponding abelian category.

We have defined Spé and $(-)^+$ on objects but they can also be turned into functors. If we have a morphism $\phi : \mathscr{F} \to \mathscr{G}$ between presheaves, this induces a continuous map $\text{Spé}(\phi) : \text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ given by $s_x \mapsto \phi_x(s_x)$ so that

$$\begin{tikzcd} \text{Spé}(\mathscr{F}) \arrow[rr, "\text{Spé}(\phi)"] \arrow[dr, "\pi"'] & & \text{Spé}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.6}$$

commutes. This construction is functorial and turns Spé into a functor from presheaves to topological bundles over X. It follows that we also obtain a map of sheaves $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ by composing sections with $\text{Spé}(\phi)$. Thus we have a functor $(-)^+ : \mathsf{PreSh}(X) \to \mathsf{Sh}(X)$ and in fact the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\phi^+"] & \mathscr{G}^+ \\ \mathscr{F} \arrow[u] \arrow[r, "\phi"] & \mathscr{G} \arrow[u] \end{tikzcd} \tag{2.7}$$ eq:sheafif

Note that since the morphism $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism when $\mathscr{F}$ is a sheaf, this says that the functor $(-)^+$ restricted to $\mathsf{Sh}(X)$ is natrually isomorphism to the identity functor.

Thm 2.3.1. *Let $\theta : \mathscr{F} \to \mathscr{F}^+$ be the natural morphism. Then for any morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, there exists a unique morphism of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$ so that*

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\psi"] & \mathscr{G} \\ \mathscr{F} \arrow[u, "\theta"] \arrow[ur, "\phi"] & \end{tikzcd} \tag{2.8}$$

commutes.

Proof. This just follows from equation 2.7, the fact that $\theta : \mathscr{G} \to \mathscr{G}^+$ is an isomorphism when $\mathscr{G}$ is a sheaf, and by taking stalks. ■

Corollary 2.3.2. *The sheafification functor is left adjoint to the inclusion functor $\iota : \mathsf{Sh}(X) \to \mathsf{PreSh}(X)$.*

Proof. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. Given a morphism $\phi : \mathscr{F}^+ \to \mathscr{G}$ we can precompose it with $\theta : \mathscr{F} \to \mathscr{F}^+$ to obtain a map $\mathscr{F} \to \iota\mathscr{G}$. Conversely, given $\psi : \mathscr{F} \to \iota\mathscr{G}$, we obtain a map $\mathscr{F}^+ \to \mathscr{G}$ from the theorem. Then the theorem says these operations are inverse so we have a bijection

$$\text{Hom}(\mathscr{F}^+, \mathscr{G}) \cong \text{Hom}(\mathscr{F}, \iota\mathscr{G}). \tag{2.9}$$

Naturality is then an easy check. ■

Corollary 2.3.3. *The sheafification functor is exact.*

Proof. It is a left adjoint so it is right exact. It thus suffices to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is injective then so is ϕ^+. For this it suffices to show that ϕ_x is injective for all x. But this is obvious. ■

We can now define the cokernel of a morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathsf{Sh}(X)$. We simply define it to be the sheafification of the cokernel in $\mathsf{PreSh}(X)$ and it is an easy to check to see that this is indeed a cokernel object in $\mathsf{Sh}(X)$. It is then easy to see that $\ker \operatorname{coker} = \operatorname{coker} \ker$ by looking at stalks and so $\mathsf{Sh}(X)$ is an abelian category.

Remark 2.3.4. While $\mathsf{Sh}(X)$ is a full subcategory of $\mathsf{PreSh}(X)$ that is abelian, it is not a full abelian subcategory.

2.4 Exact sequences

Now that we know that we are working in an abelian category we can talk about exact sequences in $\mathsf{Sh}(X)$. Recall from section 1.3 that $\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ is exact at $\mathscr{G}$ if $\phi \circ \theta = 0$ and the map induced map $\operatorname{im}(\theta) \to \ker(\phi)$ is an isomorphism. But the map $\operatorname{im}(\theta) \to \ker(\phi)$ is an isomorphism iff it is an isomorphism at the level of stalks iff $\mathscr{F}_x \xrightarrow{\theta_x} \mathscr{G}_x \xrightarrow{\phi_x} \mathscr{H}_x$ is exact for all $x \in X$. Thus exactness in $\mathsf{Sh}(X)$ can be verified by checking exactness at all the stalks.

2.5 Sheaves over different spaces

2.5.1 Direct image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on X. We define the direct image of $\mathscr{F}$ under f to be the sheaf $f_*\mathscr{F}$ on Y defined by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. If we define f_* on morphisms in the obvious way then it is clear that we obtain a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. In fact we also obtain a functor $f_* : \mathsf{PreSh}(X) \to \mathsf{PreSh}(Y)$ and it turns out this functor has nice left adjoint.

Define $\lim_f : \mathsf{PreSh}(Y) \to \mathsf{PreSh}(X)$ to be the functor that sends $\mathscr{F} \in \mathsf{PreSh}(Y)$ to the presheaf $\lim_f(\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ on X, and does the obvious things to morphisms.

Thm 2.5.1. $\lim_f \dashv f_*$ *as functors between* $\mathsf{PreSh}(X)$ *and* $\mathsf{PreSh}(Y)$.

Proof. Let $\phi : \lim_f \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For V open in Y, $f^{-1}(V)$ is open in X and so we have maps

$$\mathscr{F}(V) \to \varinjlim_{W \supset f(U)} \mathscr{F}(W) \to \mathscr{G}(U) \tag{2.10}$$

where $U = f^{-1}(V)$. If $V' \subset V$, $U = f^{-1}(V)$ and $U' = f^{-1}(V')$ then

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r] \arrow[d] \arrow[dr] & \varinjlim_{W \supset f(U)} \mathscr{F}(W) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(V') \arrow[r] & \varinjlim_{W \supset f(U')} \mathscr{F}(W) \arrow[r] & \mathscr{G}(U')
\end{tikzcd} \tag{2.11}$$

commutes and so these maps in fact define a morphism $\mathscr{F} \to f_*\mathscr{G}$.

Conversely suppose we are given a morphism $\mathscr{F} \to f_*\mathscr{G}$. Let U be open in X. For $V \supset f(U)$ we have maps

$$\mathscr{F}(V) \to \mathscr{G}(f^{-1}(V)) \to \mathscr{G}(U). \tag{2.12}$$

Moreover if $V \supset V' \supset f(U)$ then

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(f^{-1}(V)) \arrow[d] \arrow[dr] & \\
 & & \mathscr{G}(U) \\
\mathscr{F}(V') \arrow[r] & \mathscr{G}(f^{-1}(V')) \arrow[ur] &
\end{tikzcd} \tag{2.13}$$

commutes so we obtain maps $\varinjlim_{V \supset f(U)} \mathscr{F}(V) \to \mathscr{G}(U)$. If $U \supset U'$ we have maps

$$\begin{tikzcd}
\varinjlim_{V \supset f(U)} \mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\varinjlim_{V \supset f(U')} \mathscr{F}(V) \arrow[r] & \mathscr{G}(U').
\end{tikzcd} \tag{2.14}$$

A straighforward calculation shows that this commutes and so we obatin a morphism $\lim_f \mathscr{F} \to \mathscr{G}$.

These operations are clearly inverse to each other. A straightforward calculation shows that the bijection is natural. ■

Corollary 2.5.2. $\lim_f$ *is an exact functor.*

Proof. It is a left adjoint so it is right exact. Thus it suffices to show that it sends injective maps to injective maps. But this is obvious. ■

2.5.2 Inverse image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on Y. Let $f^{-1}\text{Spé}(\mathscr{F})$ be the pullback

$$\begin{tikzcd} f^{-1}\text{Spé}(\mathscr{F}) \arrow[r, dashed] \arrow[d, dashed, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \text{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.15}$$

We define the inverse image sheaf $f^{-1}\mathscr{F}$ to be the sheaf of sections of $\pi : f^{-1}\text{Spé}(\mathscr{F}) \to X$. Equivalently, it is the sheaf

$$f^{-1}\mathscr{F}(U) = \left\{ s : U \to \text{Spé}(\mathscr{F}) : \begin{tikzcd} & \text{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "f|_U"] & Y \end{tikzcd} \text{ commutes} \right\} \tag{2.16}$$

eq:invimg

or also equivalently, the sheaf

$$f^{-1}\mathscr{F}(U) = \{ s : U \to \amalg_{x \in U} \mathscr{F}_{f(x)} : \forall x \in U, \exists W \subset Y, V \subset f^{-1}(W) \cap U \text{ open and } t \in \mathscr{F}(W) \text{ s.t. } x \in V \wedge s(y) = t_{f(y)} \forall y \in V \}. \tag{2.17}$$

It is clear from the construction that we obtain a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.

Remark 2.5.3. A direct calculation shows that $f^{-1}\mathscr{F}_x$ and $\mathscr{F}_{f(x)}$ are naturally isomorphic and so there is a natrual bijection between $f^{-1}\text{Spé}(\mathscr{F})$ and $\text{Spé}(f^{-1}\mathscr{F})$. It is then a straightforward exercise to check that this bijection is in fact a homeomorphism i.e. $f^{-1}\text{Spé}(\mathscr{F}) \cong \text{Spé}(f^{-1}\mathscr{F})$.

Thm 2.5.4. *f^{-1} is naturally isomorphic to $(-)^+ \circ \lim_f$.*

Proof. Let U be an open subset of X and $s \in \lim_f \mathscr{F}(U)$. There is a natural map $\phi_x : (\lim_f \mathscr{F})_x \to \mathscr{F}_{f(x)}$ so we can define a map $U \to \text{Spé}(\mathscr{F})$ by $x \mapsto \phi_x(s_x)$. It is clear that this gives an element of $f^{-1}\mathscr{F}(U)$ as characterised by equation 2.16. Thus we obtain a morphism $\lim_f \mathscr{F} \to f^{-1}\mathscr{F}$. On stalks this map is given by ϕ_x. A direct calculation shows that ϕ_x is an isomorphism for all $x \in X$ and so the induced map $(\lim_f \mathscr{F})^+ \to f^{-1}\mathscr{F}$ must be an isomorphism. It is straightforward to see that this defines a natural transformation. ∎

Corollary 2.5.5. *$f^{-1} \dashv f_*$ as functors between $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$.*

Proof. f^{-1} is naturally isomophic to $(-)^+ \circ \lim_f$ and so for $\mathscr{F} \in \mathsf{Sh}(Y), \mathscr{G} \in \mathsf{Sh}(X)$ we have natural bijections

$$\begin{aligned} \text{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) &\cong \text{Hom}_{\mathsf{Sh}(X)}\left((\lim_f \mathscr{F})^+, \mathscr{G} \right) \cong \text{Hom}_{\mathsf{PreSh}(X)}\left(\lim_f \mathscr{F}, \mathscr{G} \right) \\ &\cong \text{Hom}_{\mathsf{PreSh}(Y)}\left(\mathscr{F}, f_*\mathscr{G} \right) \cong \text{Hom}_{\mathsf{Sh}(Y)}\left(\mathscr{F}, f_*\mathscr{G} \right). \end{aligned} \tag{2.18}$$

■

Corollary 2.5.6. $(-)_x \circ f^{-1} = (-)_{f(x)}$.

Proof. $(-)_x \circ f^{-1} = (-)_x \circ (-)^+ \circ \lim_f = (-)_x \circ \lim_f = (-)_{f(x)}$. ■

Corollary 2.5.7. *f^{-1} is an exact functor.*

Proof. It is the composition of two exact functors. Alternatively take stalks. ■

2.6 The $\mathscr{H}\!om$ sheaf

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves and $f : \text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ be a continuous map so that

$$\begin{tikzcd} \text{Spé}(\mathscr{F}) \arrow[rr, "\text{Spé}(\phi)"] \arrow[dr, "\pi"'] & & \text{Spé}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.19}$$

commutes. Then we obtain a morphism $\mathscr{F}^+ \to \mathscr{G}^+$ by postcomposing sections with f. Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves we in fact obtain a morphism $\mathscr{F} \to \mathscr{G}$. But we also know that morphisms $\mathscr{F} \to \mathscr{G}$ give continuous maps $\text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ making the above diagram commute.

2.7 Injective sheaves

There are enough injectives.

CHAPTER 3

Spectral sequences

CHAPTER 4

Group cohomology