Homological algebra and schemes

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March 26, 2019

Contents

CHAPTER 1

Abelian Categories

1.1 Additive categories

Let $\mathcal{A}$ be a category such that the hom-sets carry the structure of an abelian group and composition is bilinear. We call such a category Ab-enriched. An additive category is an Ab-enriched category which has finite coproducts and a zero object.

thm:atosa **Thm 1.1.1.** *Let $\mathcal{A}$ be an additive category. Then finite coproducts in $\mathcal{A}$ are in fact finite biproducts.*

Proof. It is easy to see that initial objects are isomorphic to terminal objects (and they both exist) and so it suffices to show the result for binary coproducts. Let $A, B \in \mathcal{A}$. Define $p_A : A \coprod B \to A$ and $p_B : A \coprod B \to B$ as the maps making the following diagrams commute.

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\
& A \coprod B \arrow[r, dashed, "p_A"] & A \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "0"] & & \\
& A \coprod B \arrow[r, dashed, "p_B"] & B \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "\mathrm{id}_B"'] & &
\end{tikzcd}
\tag{1.1}
$$

Let $f = i_A \circ p_A + i_B \circ p_B$. Then

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "i_A"] & & \\
& A \coprod B \arrow[r, "f"] & A \coprod B \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "i_B"'] & &
\end{tikzcd}
\tag{1.2}
$$

commutes and so by universality we must have $f = \mathrm{id}_{A \coprod B}$. Now suppose we have maps $f : C \to A$ and $g : C \to B$. Let $h : C \to A \coprod B$ be the map $i_A \circ f + i_B \circ g$. Then $p_A \circ h = f$ and $p_B \circ h = g$. Moreover, if $h' : C \to A \coprod B$ is any other map satisfying $p_A \circ h' = f$ and $p_B \circ h' = g$ then $h' = id_{A \coprod B} \circ h' = i_A \circ f + i_B \circ g = h$ and so $A \coprod B$ is a biproduct. ∎

A functor between additive categories is called additive if it is a homomorphism on hom-sets.

1.2 Semiadditive categories

The above definition of an additive category includes the additive structure on the hom-sets as data. In this section we provide a definition where the additive structure arises as a property instead.

Let $\mathcal{A}$ be a category with a zero object. Recall that in such a category there always exists a morphism between to any two objects $A, B \in \mathcal{A}$ given by $A \to 0 \to B$. We call this the 0 morphism. Moreover if finite coproducts and finite products exists there is a canonical map $A \coprod B \to A \prod B$ arising from the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "0"'] & A \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "0"] & B. \end{tikzcd} \tag{1.3}$$

We call a category $\mathcal{A}$ *semiadditive* if it has a zero object, finite products, finite coproducts and the canonical map $A \coprod B \to A \prod B$ is an isomorphism for all $A, B \in \mathcal{A}$. In such a category we write $A \oplus B$ for the biproduct.

Thm 1.2.1. *Let $\mathcal{A}$ be a semiadditive category then it is naturally enriched over the monoidal category of commutative monoids.*

Proof. Let $\Delta_A : A \oplus A \to A$ and $\nabla_A : A \to A \oplus A$ be the maps that make

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\ & A \oplus A \arrow[r, dashed, "\Delta_A"] & A \\ A \arrow[ur, "i'_A"] \arrow[urr, bend right, "\mathrm{id}_A"'] & & \end{tikzcd} \qquad \begin{tikzcd} & & A \\ A \arrow[r, dashed, "\nabla_A"] \arrow[urr, bend left, "\mathrm{id}_A"] \arrow[drr, bend right, "\mathrm{id}_A"'] & A \oplus A \arrow[ur, "p_A"] \arrow[dr, "p'_A"'] & \\ & & A \end{tikzcd} \tag{1.4}$$

commute. Given $f, g : A \to B$ we can construct a map $f \oplus g : A \oplus A \to B \oplus B$ in the obvious way. We can then define $f + g : A \to B$ to be the composite

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.5}$$

Note that there is a map $t_A : A \oplus A \to A \oplus A$ arising from the diagram

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[dr, "\mathrm{id}_A" near start] & A \\ A \arrow[ur, "\mathrm{id}_A" near start] \arrow[r, "0"'] & A. \end{tikzcd} \tag{1.6}$$

It is then an easy check to see that $\Delta_A \circ t_A = \Delta_A$ and $t_A \circ \nabla_A = \nabla_A$, from which it follows that $+$ is commutative. Straightforward calculations also show that $+$ is associative, distributes over compositions and has the zero map as identity. The result follows. ∎

A functor between semiadditive categories is called semiadditive if it preserves zero objects and biproducts i.e. there are isomorphisms $F(A \oplus B) \cong F(A) \oplus F(B)$ such that

$$\begin{tikzcd} F(A) \arrow[dr, "F(i_A)"] \arrow[drr, bend left, "i_{F(A)}"] & & \\ & F(A \oplus B) \arrow[r, "\cong"] & F(A) \oplus F(B) \\ F(B) \arrow[ur, "F(i_B)"] \arrow[urr, bend right, "i_{F(B)}"'] & & \end{tikzcd} \tag{1.7}$$

commutes, and similarly for the projection maps.

prop:sa **Proposition 1.2.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a semiadditive functor and $f, g : A \to B$ for $A, B \in \mathcal{A}$. Then $F(f + g) = F(f) + F(g)$.*

Proof. Obvious. ∎

We now define an additive category to be a semiadditive category where the enriched hom-sets are in fact groups.

thm:as **Thm 1.2.3.** *Let $\mathcal{A}$ be an additive category according to the first definition. By theorem 1.1.1, $\mathcal{A}$ is semiadditive and so the hom-sets naturally carry the structure of a commutative monoid. This monoidal structure agrees with the original group structure.*

Proof. Let $A, B \in \mathcal{A}$ and $f, g : A \to B$. Then the addition arising from the semiadditive structure comes from the composition

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.8}$$

But $\nabla_A = i_A^L + i_A^R, \Delta_B = p_B^L + p_B^R$ and $f \oplus g = i_B^L \circ f \circ p_A^L + i_B^R \circ g \circ p_A^R$ and so their composition is just $f + g$. ∎

Corollary 1.2.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categoires. Then F is additive iff F it is semiadditive.*

Proof. Semiadditive $\implies$ additive follows from proposition 1.2.2 and theorem 1.2.3. Additive $\implies$ semiadditive is a straigtforward exercise. ∎

Corollary 1.2.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories which is a left adjoint. Then F is additive.*

Proof. F preserves colimits and so is semiadditive. ∎

Corollary 1.2.6. *If $\mathcal{A}$ is an additive category then $\mathcal{A}^{op}$ is also additive.*

Proof. The oppositive category of a semiadditive category is clearly also semiadditive. The resulting monoidal structure on the hom-sets are also clearly the same and so the result follows. ∎

1.3 Abelian categories

Abelian categories are additive categories with more strucure. Before we state exactly what we mean by this we give some definitions.

Definition 1.3.1. Let $\mathcal{A}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{A}$.

1. A kernel of f is an equaliser of $A \overset{f}{\underset{0}{\rightrightarrows}} B$.
2. A cokernel of f is a coequaliser of the same diagram.
3. f is called monic if $f \circ g = 0$ implies $g = 0$ for all g.
4. f is called epi if $g \circ f = 0$ implies $g = 0$ for all g.

Remark 1.3.2. It is easy to see that all kernels are monic, all cokernels are epi, a map is monic iff its kernel is 0, and a map is epi iff its cokernel is 0.

We call an additive category $\mathcal{A}$ pre-abelian if all morphisms have kernels and cokernels. In such a category, given any morphism $f : A \to B$ we can form

$$\begin{tikzcd}
 & & \ker(\operatorname{coker}(f)) \arrow[dr, "i"] & & \\
K \arrow[r, "\ker(f)"] & A \arrow[ur, dashed, "\alpha"] \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[r, "\operatorname{coker}(f)"] & C \\
 & & \operatorname{coker}(\ker(f)) \arrow[ur, dashed, "\beta"'] & &
\end{tikzcd} \tag{1.9}$$

eq:canon-decomp

where α and β exist from the universal property of kernels and cokernerls respectively. Since p is epi and $0 = \operatorname{coker}(f) \circ f = \operatorname{coker}(f) \circ \beta \circ p$ it follows that $\operatorname{coker}(f) \circ \beta = 0$ and so there is a map $\gamma : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ such that $i \circ \gamma = \beta$. Similarly there is a map $\gamma' : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ such that $\gamma' \circ p = \alpha$. Using that p is epi one can see that $\gamma' = \gamma$ and so for any morphism f there is a canonical decomposition

$$A \xrightarrow{p} \operatorname{coker}(\ker(f)) \xrightarrow{\gamma_f} \ker(\operatorname{coker}(f)) \xrightarrow{i} B. \tag{1.10}$$

An abelian category is a pre-abelian category in which γ_f is an isomorphism for every f.

`thm:abcat` **Thm 1.3.3.** *Let $\mathcal{A}$ be a pre-abelian category. Then γ_f is an isomorphism for all morphism f iff every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.*

Proof. ($\Rightarrow$) The kernel of a monic is the 0 object with the 0 map, and the cokernel of this is just A together with the identity. Thus, if γ_f is an isomorphism the canonical decomposition of f just becomes

$$A \xrightarrow{\text{id}} A \xrightarrow{\cong} \ker(\operatorname{coker}(f)) \xrightarrow{i} B \tag{1.11}$$

and so f is the kernel of its cokernel. Similarly one obtains that if f is epi it is the cokernel of its kernel.

($\Leftarrow$) First note that if a kernel is epi then it must be an isomorphism so all epic monics must be isomorphisms (since all monics are kernels). Thus, it suffices to show that the maps α and β in equation 1.9 are epi and monic respectively. To see that β is monic let $x : X \to \operatorname{coker}(\ker(f))$ be a map such that $\beta \circ x = 0$. Then let $q : \operatorname{coker}(\ker(f)) \to \operatorname{coker}(x)$ be the coker of x, and $j : \operatorname{coker}(x) \to B$ the map such that $j \circ q = \beta$. Finally let $l : \ker(q \circ p) \to A$ be the kernel of $q \circ p$. Then we have the following diagram

$$\begin{tikzcd}
\ker(q \circ p) \arrow[dd, dashed] \arrow[dr, "l"] & & & \\
& A \arrow[rr, "f"] \arrow[dr, "p"] & & B \\
\ker(f) \arrow[ur, "k"] & & \operatorname{coker}(\ker(f)) \arrow[ur, "\beta"] \arrow[dr, "q"] & \\
& X \arrow[ur, "x"] & & \operatorname{coker}(x). \arrow[uu, "j"']
\end{tikzcd} \tag{1.12}$$

Since $q \circ p$ is epi it is the coker of l. But also $f \circ l = j \circ q \circ p \circ l = 0$, so l factors through $\ker(f)$ and so $p \circ l = 0$. Thus there exists $p' : \operatorname{coker}(x) \to \operatorname{coker}(\ker(f))$ such that

$$\begin{tikzcd}
\ker(q \circ p) \arrow[r, "l"] & A \arrow[r, "p"] \arrow[d, "q \circ p"] & \operatorname{coker}(\ker(f)) \\
& \operatorname{coker}(x) \arrow[ur, dashed, "p'"'] &
\end{tikzcd} \tag{1.13}$$

commutes. Since p is epi, it must follow that $p' \circ q = \mathrm{id}$. Thus q is monic and so $x = 0$. It follows that β is monic. Similarly one can show that α is epi. ∎

It follows that an abelian category is equvalently a pre-abelian category in which every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.

Thm 1.3.4. *If $\mathcal{A}$ is an abelian category then $\mathcal{A}^{op}$ is also an abelian category.*

Proof. It is certainly additive. Moreover, kernels and cokernels simply swap roles. γ_f is then still an isomorphism for all f and so $\mathcal{A}^{op}$ is abelian. ∎

From now on we write $\mathrm{im}(f) := \ker(\mathrm{coker}(f))$ and $\mathrm{coim}(f) := \mathrm{coker}(\ker(f))$.

1.4 Exact sequences

sec:es

Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ be the category with objects given by $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$, and morphisms given by chain maps. Recall from earlier that f can be factored as

$$A \xrightarrow{p_f} \mathrm{im}(f) \xhookrightarrow{i_f} B. \tag{1.14}$$

Since p_f is epi, we must have $g \circ i_f = 0$. Thus we can factor f further through $\ker(g)$ to obtain $f : A \to \mathrm{im}(f) \to \ker(g) \to B$. Let $H(A \xrightarrow{f} B \xrightarrow{g} C)$ be the cokernel of the morphism $\mathrm{im}(f) \to \ker(g)$. If we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd} \tag{1.15}$$

then there exists maps so that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{im}(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ A' \arrow[r] & \mathrm{im}(f') \arrow[r] & \ker(g') \arrow[r] & B' \arrow[r] & C' \end{tikzcd} \tag{1.16}$$

commutes. In particular there is a morphism

$$\mathrm{coker}(\mathrm{im}(f) \to \ker(g)) \to \mathrm{coker}(\mathrm{im}(f') \to \ker(g')). \tag{1.17}$$

It is easy to check that this construction is functorial and so we obtain a functor $H : \mathcal{S} \to \mathcal{A}$.

One can similarly construct a functor $H' : \mathcal{S} \to \mathcal{A}$ by considering

$$\ker(\mathrm{coker}(f) \to \mathrm{coim}(g)) \tag{1.18}$$

instead.

Remark 1.4.1. We may also form a functor by looking simply at the fact that f factors through $\ker(g)$ and then looking at the coker of the resulting morphism $A \to \ker(g)$. It is an easy check to see that this yields a functor naturally isomorphic to H. Similarly for H'.

Lemma 1.4.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. Recall that we have the factorisation*

$$A \to \mathrm{im}(f) \to \ker(g) \xrightarrow{i_g} B \xrightarrow{p_f} \mathrm{coker}(f) \to \mathrm{coim}(g) \to C. \tag{1.19}$$

Let h be the composition $\ker(g) \to B \to \mathrm{coker}(f)$. Then

1. $\ker(h) = \mathrm{im}(f) \to \ker(g)$
2. $\mathrm{coker}(h) = \mathrm{coker}(f) \to \mathrm{coim}(g)$.

Proof. Let $l : C \to \ker(g)$ be such that $h \circ l = 0$. Then $p_f \circ i_g \circ l = 0$ and so $i_g \circ l$ factors through $\mathrm{im}(f)$. Since i_g is monic it follows that l factors through $\mathrm{im}(f)$. Uniqueness follows automatically. Thus the result follows. The second part follows similarly. ■

Thm 1.4.3. *The functors $H, H' : \mathcal{S} \to \mathcal{A}$ are naturally isomorphic.*

Proof. Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$ and h be as in the lemma. Then $H(S) = \mathrm{coker}(\ker(h))$ and $H'(S) = \ker(\mathrm{coker}(h))$ so we obtain the factorisation

$$\ker(g) \to H(S) \xrightarrow{\cong} H'(S) \to \mathrm{coker}(f). \tag{1.20}$$

Naturality of the isomophism then follows from naturality of this factorisation. ■

Remark 1.4.4. In a pre-abelian category we still have a natrual transformation $H \Rightarrow H'$, but it might not be an isomorphism.

Definition 1.4.5. Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. We say that S is exact at B if $H(S) = 0$.

Proposition 1.4.6. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a short exact sequence iff* $A = \ker(g)$ *and* $C = \mathrm{coker}(f)$.

Proof. ($\Rightarrow$) We have $\ker(g) \cong \mathrm{im}(f) \cong A$ and $\mathrm{coker}(f) \cong \mathrm{coim}(g) \cong C$.

($\Leftarrow$) Certainly have exactness at A and C. Exactness at B also holds. ■

1.4.1 Split sequences

Thm 1.4.7. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. The following are equivalent*

1. *there exists $q : B \to A$ such that $q \circ f = \mathrm{id}_A$*
2. *there exists $p : C \to B$ such that $g \circ p = \mathrm{id}_C$*
3. *there is an isomorphism $h : B \to A \oplus C$ such that $h \circ f$ and $g \circ h^{-1}$ are the natural inclusion and projection respectively.*

Proof. (3) certainly implies both (1) and (2).

(2) $\Rightarrow$ (3) Let $q : B \to A$ be the unique map making the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C. \\ B \arrow[u, dashed, "q"] \arrow[ur, "\mathrm{id}_B - p\circ g"'] & & \end{tikzcd} \tag{1.21}$$

Then $\mathrm{id}_B = p \circ g + f \circ q$. It follows that $p = f \circ q \circ p + p$. Since f is monic we have $q \circ p = 0$. Thus $q = q \circ f \circ q$ and so since q is epi, $q \circ f = \mathrm{id}_A$. The result follows. (1) $\Rightarrow$ (3) follows similarly. ∎

Corollary 1.4.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Then F applied to a split short exact sequence is also split exact.*

Proposition 1.4.9. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. If either*

1. *A is injective or*
2. *C is projective*

then the sequence is split.

1.5 Adjoint functors

Let $L : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. If L admits a right adjoint $R : \mathcal{B} \to \mathcal{A}$ then it turns out L has a lot of useful properties. In this section we explore these properties.

Proposition 1.5.1. *Suppose $L \dashv R$. Then L is right exact and R is left exact.*

Proof. Consider the short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$. For every $A \in \mathcal{A}$ we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(L(A), B_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(L(A), B_2) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(L(A), B_3) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{Hom}(A, R(B_1)) \arrow[r] & \mathrm{Hom}(A, R(B_2)) \arrow[r] & \mathrm{Hom}(A, R(B_3)) \end{tikzcd} \tag{1.22}$$

where the top row is exact. It follows that the bottom row is exact for all A and so the bottow row is too. It follows that

$$0 \longrightarrow R(B_1) \longrightarrow R(B_2) \longrightarrow R(B_3) \tag{1.23}$$

is exact and so R is left exact. By a similar argument L is right exact. ∎

Proposition 1.5.2. *Suppose $L \dashv R$. Then*

1. *if L is exact then R preserves injectives*
2. *if R is exact then L preserves projectives.*

Proof. Suppose L is exact and I is an injective object in $\mathcal{B}$. We need to show that $\mathrm{Hom}(-, R(I))$ is exact. To do this it suffices to show that given $f : A \to B$ injective, the map $f^* : \mathrm{Hom}(B, R(I)) \to \mathrm{Hom}(A, R(I))$ is surjective. But L is exact so Lf is injective and so $(Lf)^* : \mathrm{Hom}(LB, I) \to \mathrm{Hom}(LA, I)$ is surjective. We also have that $L \dashv R$ and so

$$\begin{array}{ccc} \mathrm{Hom}(L(B), I) & \xrightarrow{(Lf)^*} & \mathrm{Hom}(L(A), I) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(B, R(I)) & \xrightarrow{f^*} & \mathrm{Hom}(A, R(I)) \end{array} \tag{1.24}$$

commutes. It follows that f^* is surjective as required.

The corresponding result for R follows similarly. ∎

1.6 Limits and derived functors

Proposition 1.6.1. *An abelian category $\mathcal{A}$ is cocomplete iff it has all direct sums.*

Proof. We already have kernels and hence equalisers so the statement follows. ∎

Remark 1.6.2. The same result holds if we replace direct sums with product and cocomplete with complete.

Thm 1.6.3. *Let $\mathcal{A}$ be a cocomplete abelian category with enough projectives. If $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every set $\{A_i\}$ of objects in $\mathcal{A}$ we have*

$$L_*F\left(\bigoplus_{i\in I} A_i\right) \cong \bigoplus_{i\in I} L_*F(A_i). \tag{1.25}$$

Proof. Let $P_i \to A_i$ be projective resolutions. Then $\oplus_i P_i \to \oplus_i A_i$ is also a projective resolution. Hence

$$L_*F(\oplus_i A_i) = H_*(F(\oplus_i P_i)) \cong H_*(\oplus_i F(P_i)) \cong \oplus_i H_*(F(P_i)) = \oplus_i L_*F(A_i). \tag{1.26}$$

∎

1.6.1 Filtered colimits

Definition 1.6.4. A category I is called filtered if it has coproduct and coequaliser diagrams. A filtered colimit is the colimit of a functor from a filtered category.

Lemma 1.6.5. *Let I be a filtered category, and $A : I \to \mathsf{Mod} - R$. Then*

1. *Every element $a \in colim_I A$ is the image of some element $a_i \in A_i$ for some $i \in I$ under the canonical map $A_i \to colim_I A$.*
2. *For every i, the kernel of the canonical map $A_i \to colim_I A$ is the union of the kernels of the maps $A(\phi) : A_i \to A_j$ for $\phi : i \to j$ in I.*

Proof. Use the explicit construction of the colimit as the cokernel of

$$\bigoplus_{i \to j} A_i \to \bigoplus_i A_i. \tag{1.27}$$

■

Thm 1.6.6. *Filtered colimits of R-modules are exact considered as functors from $\mathsf{Fun}(I, \mathsf{Mod} - R)$ to $\mathsf{Mod} - R$.*

Proof. We know that colim is a left adjoint and so is right exact. It thus suffices to show that if $t : A \to B$ is monic then $\mathrm{colim}_I A \to \mathrm{colim}_I B$ is too. But this follows immediately from the previous proposition. ■

Definition 1.6.7. We say an abelian category $\mathcal{A}$ satisfies axiom (AB5) if it is cocomplete and filtered colimits are exact.

Thm 1.6.8. *Let $\mathcal{A}$ be an abelian category satisfying axiom (AB5). Then for $F : \mathcal{A} \to \mathcal{B}$ a left adjoint, we have that for all filtered I,*

$$L_* F(colim_I A) \cong colim_I L_* F(A_i). \tag{1.28}$$

Proof. colim_I is exact so commutes with H_i. The rest of the proof is similar to the direct sum proof. ■

CHAPTER 2

Sheaf Theory

ch:sheafs

2.1 Presheaves

Let $\mathcal{C}$ be any category, $\mathcal{A}$ be an abelian category and define $\mathsf{PreSh}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{A})$ to be the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$. The functor sending all objects to 0 is certainly both initial and terminal, direct sums can be defined pointwise, and the hom-sets in $\mathsf{PreSh}(\mathcal{C})$ inherit an additive structure from $\mathcal{A}$ so $\mathsf{PreSh}(\mathcal{C})$ is naturally an additive category. Moreover kernels and cokernels can be contructed in the obvious way and it is clear that they satisfy the axioms for an abelian category and so $\mathsf{PreSh}(C)$ is abelian.

2.2 Sheaves

To define sheaves we restrict to the case when X be a topological space, $\mathcal{U}$ the poset of open sets of X, and $\mathcal{A}$ be an abelian category. We write $\mathsf{PreSh}(X)$ for $\mathsf{PreSh}(\mathcal{U})$. The category of sheaves on X with values in $\mathcal{A}$, $\mathsf{Sh}(X)$, is defined to be the full subcategory of $\mathsf{PreSh}(X)$ with objects given by presheaves $\mathscr{F}$ for which the following diagram is an equalizer for all open coverings $U = \cup_i U_i$

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.1}$$

Since $\mathcal{A}$ is an abelian category this is equivalent to the following diagram being exact

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \xrightarrow{\text{diff}} \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.2}$$

Note that since $\varnothing$ admits the empty covering and the empty product is 0 this forces $\mathscr{F}(\varnothing) = 0$.

As in the case of $\mathsf{PreSh}(\mathcal{C})$, $\mathsf{Sh}(X)$ is an additive category. However, the cokernel of a morphism between sheaves need not be a sheaf and so we must do some more work to show that $\mathsf{Sh}(X)$ is abelian.

Fix $x \in X$. For a (pre)sheaf $\mathscr{F}$ define the stalk of $\mathscr{F}$ at x to be

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F} \tag{2.3}$$

when this limit exists. Note that this is a functor since morphisms between (pre)sheaves are natural transformations.

Thm 2.2.1. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

1. *If ϕ_x is injective for all $x \in X$ then ϕ is injective on sections.*
2. *If ϕ_x is an isomorphism for all $x \in X$ then ϕ is an isomorphism.*

Proof. Exercise. ■

Proposition 2.2.2. *Let $\mathscr{F}, \mathscr{G}$ be presheaves and $\phi, \psi : \mathscr{F} \to \mathscr{G}$ be morphisms that are equal on stalks. If $\mathscr{G}$ satisfies sheaf condition (A) then $\phi = \psi$.*

Proof. Consider $\phi - \psi$. ■

Aside

Although we do not need this right away, given an $A \in \mathcal{A}$ we can define the (pre)sheaf x_*A by

$$(x_*A)(U) = \begin{cases} A & \textit{if } x \in U \\ 0 & \textit{otherwise} \end{cases} \tag{2.4}$$

Proposition 2.2.3. *When it exists, the functor $(-)_x : \mathsf{Sh}(X) \to \mathcal{A}$ is left adjoint to $x_* : \mathcal{A} \to \mathsf{Sh}(X)$.*

Proof. To see this simply note that morphisms between $\mathscr{F}$ and $x_*(A)$ correspond naturally to natrual transformations between $\mathscr{F}$ restricted to $U \ni x$ and $\Delta(A)$. ■

Remark 2.2.4. The result also holds in $\mathsf{PreSh}(X)$.

2.3 Étalé space of a presheaf and sheafification

For a presheaf $\mathscr{F}$ we are now in the position to define its étalé space. The étalé space of $\mathscr{F}$, denoted Spé($\mathscr{F}$) is the topological space with underlying set $\amalg_{x\in X}\mathscr{F}_x$ and topology generated by the basis of sets given by $\{s_x | x \in U\}$ for $s \in \mathscr{F}(U)$ where $U \subset X$ is open. Together with this space there is also a natural continuous map $\pi : \text{Spé}(\mathscr{F}) \to X$ sending an element s_x to x. The sheafification of $\mathscr{F}$, denoted $\mathscr{F}^+$, is then defined to be the sheaf of sections of $\pi : \text{Spé}(\mathscr{F}) \to X$. By unwrapping the definitions we see that the sections can be characterised as

$$\begin{aligned} \mathscr{F}^+(U) = \{s : U \to \amalg_{x\in U}\mathscr{F}_x : &\forall x \in U, \exists V \subset U \text{ open containing } x \text{ and} \\ &t \in \mathscr{F}(V) \text{ s.t. } s(y) = t_y \forall y \in V\} \end{aligned} \tag{2.5}$$

In particular there is a natural morphism $\mathscr{F} \to \mathscr{F}^+$ sending $s \in \mathscr{F}(U)$ to the section $x \mapsto s_x$ which is an isomorphism on stalks. From the characterisation of sections it clear that if $\mathscr{F}$ is a presheaf of $\mathsf{AbGrp}, \mathsf{Ring}, \ldots$ then $\mathscr{F}^+$ is a sheaf with values in the corresponding abelian category.

We have defined Spé and $(-)^+$ on objects but they can also be turned into functors. If we have a morphism $\phi : \mathscr{F} \to \mathscr{G}$ between presheaves, this induces a continuous map $\text{Spé}(\phi) : \text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ given by $s_x \mapsto \phi_x(s_x)$ so that

$$\begin{tikzcd} \text{Spé}(\mathscr{F}) \arrow[rr, "\text{Spé}(\phi)"] \arrow[dr, "\pi"] & & \text{Spé}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.6}$$

commutes. This construction is functorial and turns Spé into a functor from presheaves to topological bundles over X.

Remark 2.3.1. The natural map $\mathscr{F} \to \mathscr{F}^+$ induces a homeomorphism $\text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{F}^+)$.

It follows that we also obtain a map of sheaves $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ by composing sections with $\text{Spé}(\phi)$. Thus we have a functor $(-)^+ : \mathsf{PreSh}(X) \to \mathsf{Sh}(X)$ and in fact the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\phi^+"] & \mathscr{G}^+ \\ \mathscr{F} \arrow[u] \arrow[r, "\phi"] & \mathscr{G} \arrow[u] \end{tikzcd} \tag{2.7}$$ eq:sheafif

Note that since the morphism $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism when $\mathscr{F}$ is a sheaf, this says that the functor $(-)^+$ restricted to $\mathsf{Sh}(X)$ is natrually isomorphic to the identity functor.

Thm 2.3.2. *Let $\theta : \mathscr{F} \to \mathscr{F}^+$ be the natural morphism. Then for any morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, there exists a unique morphism of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$ so that*

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\psi"] & \mathscr{G} \\ \mathscr{F} \arrow[u, "\theta"] \arrow[ur, "\phi"] & \end{tikzcd} \tag{2.8}$$

commutes.

Proof. This just follows from equation 2.7, the fact that $\theta : \mathscr{G} \to \mathscr{G}^+$ is an isomorphism when $\mathscr{G}$ is a sheaf, and by taking stalks. ∎

Corollary 2.3.3. *The sheafification functor is left adjoint to the inclusion functor $\iota : \mathsf{Sh}(X) \to \mathsf{PreSh}(X)$.*

Proof. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. Given a morphism $\phi : \mathscr{F}^+ \to \mathscr{G}$ we can precompose it with $\theta : \mathscr{F} \to \mathscr{F}^+$ to obtain a map $\mathscr{F} \to \iota\mathscr{G}$. Conversely, given $\psi : \mathscr{F} \to \iota\mathscr{G}$, we obtain a map $\mathscr{F}^+ \to \mathscr{G}$ from the theorem. Then the theorem says these operations are inverse so we have a bijection

$$\mathrm{Hom}(\mathscr{F}^+, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \iota\mathscr{G}). \tag{2.9}$$

Naturality is then an easy check. ∎

Corollary 2.3.4. *The sheafification functor is exact.*

Proof. It is a left adjoint so it is right exact. It thus suffices to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is injective then so is ϕ^+. For this it suffices to show that ϕ_x is injective for all x. But this is obvious. ∎

We can now define the cokernel of a morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathsf{Sh}(X)$. We simply define it to be the sheafification of the cokernel in $\mathsf{PreSh}(X)$ and it is an easy to check to see that this is indeed a cokernel object in $\mathsf{Sh}(X)$. It is then easy to see that $\ker \mathrm{coker} = \mathrm{coker} \ker$ by looking at stalks and so $\mathsf{Sh}(X)$ is an abelian category.

Remark 2.3.5. While $\mathsf{Sh}(X)$ is a full subcategory of $\mathsf{PreSh}(X)$ that is abelian, it is not a full abelian subcategory.

2.4 Maps defined on a basis

Thm 2.4.1. *Let $\mathscr{F}, \mathscr{G}$ be sheafs on X and let $\mathcal{B}$ be a basis for the topology on X. Then any morphism $\phi|_{\mathcal{B}} : \mathscr{F}|_{\mathcal{B}} \to \mathscr{G}|_{\mathcal{B}}$ extends uniquely to a morphism $\phi : \mathscr{F} \to \mathscr{G}$. Moreover this procedure is functorial.*

Proof. There is a natural isomorphism between $\varinjlim_{U \ni x} \mathscr{F}$ and $\varinjlim_{B \ni x} \mathscr{F}$. Thus we obtain a map $\phi := \phi|_{\mathcal{B}}^+ : \mathscr{F} \to \mathscr{G}$. It is clear that this is a morphism of sheaves. Moreover for $U \in \mathcal{B}$ and $s \in \mathscr{F}(U)$ it is clear that $\phi(U)(s)$ and $\phi|_{\mathcal{B}}(U)(s)$ have the same stalks and so must be equal. Thus ϕ extends $\phi|_{\mathcal{B}}$. Finally, if a morphism extends $\phi|_{\mathcal{B}}$ then it is determined on stalks and hence must equal to ϕ, which gives us uniqueness. Functoriality is clear. ∎

2.5 Exact sequences

Now that we know that we are working in an abelian category we can talk about exact sequences in $\mathsf{Sh}(X)$. Recall from section 1.4 that $\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ is exact at $\mathscr{G}$ if $\phi \circ \theta = 0$ and the map induced map $\mathrm{im}(\theta) \to \ker(\phi)$ is an isomorphism. But the map $\mathrm{im}(\theta) \to \ker(\phi)$ is an isomorphism iff it is an isomorphism at the level of stalks iff $\mathscr{F}_x \xrightarrow{\theta_x} \mathscr{G}_x \xrightarrow{\phi_x} \mathscr{H}_x$ is exact for all $x \in X$. Thus $(-)_x$ is an exact functor and exactness in $\mathsf{Sh}(X)$ can be verified by checking exactness at all the stalks.

2.6 Direct sums of sheaves

If $\mathcal{A}$ has direct sums, then so does $\mathsf{PreSh}(X)$ since we can compute the direct sum pointwise. It follows that $\mathsf{PreSh}(X)$ is cocomplete. The sheafification of the direct sum in $\mathsf{PreSh}(X)$ gives us a direct sum in $\mathsf{Sh}(X)$ and hence $\mathsf{Sh}(X)$ is also cocomplete.

We also have products in both $\mathsf{PreSh}(X)$ and $\mathsf{Sh}(X)$ (computed pointwise) and so they are also both complete.

2.7 Sheaves over different spaces

2.7.1 Direct image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on X. We define the direct image of $\mathscr{F}$ under f to be the sheaf $f_*\mathscr{F}$ on Y defined by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. If we define f_* on morphisms in the obvious way then it is clear that we obtain a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. In fact we also obtain a functor $f_* : \mathsf{PreSh}(X) \to \mathsf{PreSh}(Y)$ and it turns out this functor has nice left adjoint.

Define $\lim_f : \mathsf{PreSh}(Y) \to \mathsf{PreSh}(X)$ to be the functor that sends $\mathscr{F} \in \mathsf{PreSh}(Y)$ to the presheaf $\lim_f(\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ on X, and does the obvious things to morphisms.

Thm 2.7.1. $\lim_f \dashv f_*$ *as functors between* $\mathsf{PreSh}(X)$ *and* $\mathsf{PreSh}(Y)$*.*

Proof. Let $\phi : \lim_f \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For V open in Y, $f^{-1}(V)$ is open in X and so we have maps

$$\mathscr{F}(V) \to \varinjlim_{W \supset f(U)} \mathscr{F}(W) \to \mathscr{G}(U) \tag{2.10}$$

where $U = f^{-1}(V)$. If $V' \subset V$, $U = f^{-1}(V)$ and $U' = f^{-1}(V')$ then

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r] \arrow[d] \arrow[dr] & \varinjlim_{W \supset f(U)} \mathscr{F}(W) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(V') \arrow[r] & \varinjlim_{W \supset f(U')} \mathscr{F}(W) \arrow[r] & \mathscr{G}(U')
\end{tikzcd} \tag{2.11}$$

commutes and so these maps in fact define a morphism $\mathscr{F} \to f_*\mathscr{G}$.

Conversely suppose we are given a morphism $\mathscr{F} \to f_*\mathscr{G}$. Let U be open in X. For $V \supset f(U)$ we have maps

$$\mathscr{F}(V) \to \mathscr{G}(f^{-1}(V)) \to \mathscr{G}(U). \tag{2.12}$$

Moreover if $V \supset V' \supset f(U)$ then

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(f^{-1}(V)) \arrow[d] \arrow[dr] & \\ & & \mathscr{G}(U) \\ \mathscr{F}(V') \arrow[r] & \mathscr{G}(f^{-1}(V')) \arrow[ur] & \end{tikzcd} \tag{2.13}$$

commutes so we obtain maps $\varinjlim_{V \supset f(U)} \mathscr{F}(V) \to \mathscr{G}(U)$. If $U \supset U'$ we have maps

$$\begin{tikzcd} \varinjlim\limits_{V \supset f(U)} \mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \varinjlim\limits_{V \supset f(U')} \mathscr{F}(V) \arrow[r] & \mathscr{G}(U'). \end{tikzcd} \tag{2.14}$$

A straighforward calculation shows that this commutes and so we obatin a morphism $\lim_f \mathscr{F} \to \mathscr{G}$.

These operations are clearly inverse to each other. A straightforward calculation shows that the bijection is natural. ■

Corollary 2.7.2. $\lim_f$ *is an exact functor.*

Proof. It is a left adjoint so it is right exact. Thus it suffices to show that it sends injective maps to injective maps. But this is obvious. ■

Stalks

Proposition 2.7.3. *Let $\mathscr{F}$ be a sheaf on X and $f : X \to Y$ a continuous map. Then there is a natural map $(f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$ in the sense that if $\mathscr{G}$ is another sheaf on X and $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism then*

$$\begin{tikzcd} (f_*\mathscr{F})_{f(p)} \arrow[r, "(f_*\phi)_{f(p)}"] \arrow[d] & (f_*\mathscr{G})_{f(p)} \arrow[d] \\ \mathscr{F}_p \arrow[r, "\phi_p"] & \mathscr{G}_p \end{tikzcd} \tag{2.15}$$

commutes.

Proof. We have

$$(f_*\mathscr{F})_{f(p)} = \varinjlim_{U \ni f(p)} f_*\mathscr{F}(U) = \varinjlim_{U : f^{-1}(U) \ni p} \mathscr{F}(f^{-1}(U)). \tag{2.16}$$

But $\{U : f^{-1}(U) \ni p\} \subseteq \{V : V \ni p\}$ and so there is map

$$(f_*\mathscr{F})_{f(p)} = \varinjlim_{U : f^{-1}(U) \ni p} \mathscr{F}(f^{-1}(U)) \to \varinjlim_{V \ni p} \mathscr{F}(V) = \mathscr{F}_p. \tag{2.17}$$

Naturality is an easy exercise. ■

2.7.2 Inverse image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on Y. Let $f^{-1}\mathrm{Spé}(\mathscr{F})$ be the pullback

$$\begin{tikzcd} f^{-1}\mathrm{Spé}(\mathscr{F}) \arrow[r, dashed] \arrow[d, dashed, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.18}$$

We define the inverse image sheaf $f^{-1}\mathscr{F}$ to be the sheaf of sections of $\pi : f^{-1}\mathrm{Spé}(\mathscr{F}) \to X$. Equivalently, it is the sheaf

$$f^{-1}\mathscr{F}(U) = \left\{ s : U \to \mathrm{Spé}(\mathscr{F}) : \begin{tikzcd} & \mathrm{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "f|_U"'] & Y \end{tikzcd} \text{ commutes} \right\} \tag{2.19}$$

eq:invimg

or also equivalently, the sheaf

$$f^{-1}\mathscr{F}(U) = \{s : U \to \amalg_{x\in U}\mathscr{F}_{f(x)} : \forall x \in U, \exists W \subset Y, V \subset f^{-1}(W) \cap U \text{ open and } t \in \mathscr{F}(W) \text{ s.t. } x \in V \wedge s(y) = t_{f(y)} \forall y \in V\}. \tag{2.20}$$

It is clear from the construction that we obtain a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.

Remark 2.7.4. A direct calculation shows that $f^{-1}\mathscr{F}_x$ and $\mathscr{F}_{f(x)}$ are naturally isomorphic and so there is a natrual bijection between $f^{-1}\mathrm{Spé}(\mathscr{F})$ and $\mathrm{Spé}(f^{-1}\mathscr{F})$. It is then a straightforward exercise to check that this bijection is in fact a homeomorphism i.e. $f^{-1}\mathrm{Spé}(\mathscr{F}) \cong \mathrm{Spé}(f^{-1}\mathscr{F})$.

Thm 2.7.5. *f^{-1} is naturally isomorphic to $(-)^+ \circ \lim_f$ as functors $\mathsf{PreSh}(Y) \to \mathsf{Sh}(X)$.*

Proof. Let U be an open subset of X and $s \in \lim_f \mathscr{F}(U)$. There is a natural map $\phi_x : (\lim_f \mathscr{F})_x \to \mathscr{F}_{f(x)}$ so we can define a map $U \to \mathrm{Spé}(\mathscr{F})$ by $x \mapsto \phi_x(s_x)$. It is clear that this gives an element of $f^{-1}\mathscr{F}(U)$ as characterised by equation 2.19. Thus we obtain a morphism $\lim_f \mathscr{F} \to f^{-1}\mathscr{F}$. On stalks this map is given by ϕ_x. A direct calculation shows that ϕ_x is an isomorphism for all $x \in X$ and so the induced map $(\lim_f \mathscr{F})^+ \to f^{-1}\mathscr{F}$ must be an isomorphism. It is straightforward to see that this defines a natural transformation. ■

Remark 2.7.6. In fact $f^{-1} \circ (-)^+$, f^{-1} and $(-)^+ \circ \lim_f$ are all naturally isomorphic.

Corollary 2.7.7. *$f^{-1} \dashv f_*$ as functors between $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$.*

Proof. f^{-1} is naturally isomophic to $(-)^+ \circ \lim_f$ and so for $\mathscr{F} \in \mathsf{Sh}(Y), \mathscr{G} \in \mathsf{Sh}(X)$ we have natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathsf{Sh}(X)}\left((\lim_f \mathscr{F})^+, \mathscr{G}\right) \cong \mathrm{Hom}_{\mathsf{PreSh}(X)}\left(\lim_f \mathscr{F}, \mathscr{G}\right) \\ \cong \mathrm{Hom}_{\mathsf{PreSh}(Y)}\left(\mathscr{F}, f_*\mathscr{G}\right) \cong \mathrm{Hom}_{\mathsf{Sh}(Y)}\left(\mathscr{F}, f_*\mathscr{G}\right). \end{aligned} \tag{2.21}$$

■

Corollary 2.7.8. $(-)_x \circ f^{-1} = (-)_{f(x)}$.

Proof. $(-)_x \circ f^{-1} = (-)_x \circ (-)^+ \circ \lim_f = (-)_x \circ \lim_f = (-)_{f(x)}$. ■

Corollary 2.7.9. *f^{-1} is an exact functor.*

Proof. It is the composition of two exact functors. Alternatively take stalks. ■

Corollary 2.7.10. *There are natural transformations $e : \mathrm{id} \Rightarrow f_* f^{-1}$ and $\epsilon : f^{-1} f_* \to \mathrm{id}$ such that*

$$f^{-1} \xRightarrow{f^{-1}e} f^{-1} f_* f^{-1} \xRightarrow{\epsilon f^{-1}} f^{-1} \tag{2.22}$$

$$f_* \xRightarrow{e f_*} f_* f^{-1} f_* \xRightarrow{f_* \epsilon} f_* \tag{2.23}$$

both compose to the identity natural transformation.

2.8 The $\mathscr{H}\!om$ sheaf

Lemma 2.8.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves and $f : Sp\acute{e}(\mathscr{F}) \to Sp\acute{e}(\mathscr{G})$ be a continuous map so that*

$$\begin{tikzcd} Sp\acute{e}(\mathscr{F}) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Sp\acute{e}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.24}$$

commutes. Let $\widetilde{f} : \mathscr{F}^+ \to \mathscr{G}^+$ be the morphism obtained by postcomposing sections with f. Then $\widetilde{f}_x = f|_x$.

Proof. This follows from the fact that if $s \in \mathscr{F}^+(U)$ then for $x \in U$, $s_x = s(x)$. ■

Thm 2.8.2. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves. Then there is a bijection between continuous maps $Sp\acute{e}(\mathscr{F}) \to Sp\acute{e}(\mathscr{G})$ and morphisms of sheaves $\mathscr{F} \to \mathscr{G}$.*

Proof. For sheaves we have $\mathscr{F} \cong \mathscr{F}^+$ and so the results follows from the lemma. ■

Corollary 2.8.3. *The presheaf* $\mathscr{H}\!om(\mathscr{F},\mathscr{G})$ *defined by*

$$\mathscr{H}\!om(\mathscr{F},\mathscr{G})(U) = \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \tag{2.25}$$

is in fact a sheaf.

2.9 Sheaves of modules

Definition 2.9.1. Let $\mathscr{A}$ be a presheaf of rings and $\mathscr{F}$ a presheaf of groups. We say that $\mathscr{F}$ is an $\mathscr{A}$-module if there is a morphism $\mathscr{A} \times \mathscr{F} \to \mathscr{F}$ satisfying the usual commutative diagrams. Write $\mathsf{PreMod}(\mathscr{A})$ for the category of presheaf $\mathscr{A}$-modules, and $\mathsf{Mod}(\mathscr{A})$ for the category of $\mathscr{A}$-modules.

Proposition 2.9.2. *Let* $\mathscr{A}$ *be a presheaf of rings. There is an isomorphism of categories between* $\mathsf{Mod}(\mathscr{A})$ *and* $\mathsf{Mod}(\mathscr{A}^+)$.

Proof. Given $\mathscr{A} \times \mathscr{F} \to \mathscr{F}$ we obtain $\mathscr{A}^+ \times \mathscr{F} \to \mathscr{F}$ via sheafification. Conversely, given $\mathscr{A}^+ \times \mathscr{F} \to \mathscr{F}$ we obtain $\mathscr{A} \times \mathscr{F} \to \mathscr{F}$ by composing with $\mathscr{A} \times \mathscr{F} \to \mathscr{A}^+ \times \mathscr{F}$. These operations are clearly inverse and respect morphisms. ∎

Proposition 2.9.3. *Let* $\mathscr{A}$ *be a sheaf of rings and* $\mathscr{F} \in \mathsf{PreMod}(\mathscr{A})$. *Then* $\mathscr{F}^+ \in \mathsf{Mod}(\mathscr{A}^+)$ *and the canonical morphism* $\mathscr{F} \to \mathscr{F}^+$ *is an* $\mathscr{A}$*-module map.*

2.10 Tensors

Definition 2.10.1. Let $\mathscr{A}$ be a presheaf of rings, and $\mathscr{F},\mathscr{G} \in \mathsf{PreMod}(\mathscr{A})$. Define $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ to be the sheafification of the presheaf tensor product. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ is a $\mathscr{A}^+$-module. Write $\mathscr{F} \otimes'_{\mathscr{A}} \mathscr{G}$ for the presheaf tensor.

Thm 2.10.2. *Let* $\mathscr{A}$ *be a sheaf of rings. Then any* $\mathscr{A}$*-bilinear morphism* $\mathscr{F} \times \mathscr{G} \to \mathscr{H}$ *where* $\mathscr{H} \in \mathsf{Mod}(\mathscr{A})$ *factors uniquely through* $\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$.

Thm 2.10.3. $- \otimes_{\mathscr{A}} - : \mathsf{PreMod}(\mathscr{A}) \times \mathsf{PreMod}(\mathscr{A}) \to \mathsf{Mod}(\mathscr{A}^+)$ *is a functor.*

Proposition 2.10.4. *Let* $\mathscr{A}$ *be a presheaf of rings on* Y, $\mathscr{F},\mathscr{G} \in \mathsf{PreMod}(\mathscr{A})$, *and* $f : X \to Y, g : Y \to Z$ *be continuous maps. Then there are natural isomorphisms*

1. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{A} \cong \mathscr{F}$
2. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G} \cong \mathscr{F}^+ \otimes_{\mathscr{A}^+} \mathscr{G}^+$
3. $f^{-1}(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}) \cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{A}} f^{-1}\mathscr{G}$

Proof. 1. Obvious.

2. Use universal property.

3. It is straightforward to check that $\lim_f(\mathscr{F} \otimes'_{\mathscr{A}} \mathscr{G}) \cong \lim_f \mathscr{F} \otimes'_{\lim_f \mathscr{A}} \lim_f \mathscr{G}$. But then

$$\begin{aligned} f^{-1}(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}) \cong f^{-1}(\mathscr{F} \otimes'_{\mathscr{A}} \mathscr{G}) &\cong \left(\lim_f (\mathscr{F} \otimes'_{\mathscr{A}} \mathscr{G})\right)^+ \\ &\cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{A}} f^{-1}\mathscr{G} \end{aligned} \tag{2.26}$$

■

Proposition 2.10.5. *Let $\mathscr{A}$ be a presheaf of rings on X and $\mathscr{F}, \mathscr{G} \in \mathsf{Mod}(X)$. Then $(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})_p \cong \mathscr{F}_p \otimes_{\mathscr{A}_p} \mathscr{G}_p$ for all $p \in X$.*

2.11 Injective sheaves

Definition 2.11.1. Let $\mathscr{F}$ be a sheaf. Define $D(\mathscr{F})$ to be the sheaf of all (not necessarily continuous) sections of $\text{Spé}(\mathscr{F}) \to X$.

Lemma 2.11.2. $D(\mathscr{F}) = \prod_{x \in X} x_*(\mathscr{F}_x)$.

Proof. Obvious. ■

Thm 2.11.3. $\mathsf{Sh}(X)$ *over the abelian category* $\mathsf{AbGrp}/\mathsf{Ring}/\mathsf{Mod}_R$ *has enough injectives.*

Proof. Let $\mathcal{A}$ denote the abelian category. Recall that $x_* : \mathcal{A} \to \mathsf{Sh}(X)$ is the right adjoint of an exact functor. Thus it is left exact and preserves injectives. Let $x \in X$. $\mathcal{A}$ has enough injectives, so there is some injective object I_x such that $0 \to \mathscr{F}_x \to I_x$ is exact. It follows that $0 \to x_*(\mathscr{F}_x) \to x_*(I_x)$ is also exact. We can then form the exact sequence $0 \to \prod_{x \in X} x_*(\mathscr{F}_x) \to \prod_{x \in X} x_*(I_x)$. The last term is injective since it is a product of injective objects. Composing this with the canonical map $\mathscr{F} \to D(\mathscr{F})$ gives the required injection into an injective object. ■

CHAPTER 3

Scheme Theory

3.1 Locally ringed spaces

A locally ringed space is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf $\mathscr{O}_X$ of rings on X such that the stalks are local rings. A morphism of between the locally ringed spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ is a pair $(f, f^\#)$ consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ which induces morphisms of local rings on stalks $f_p^\# : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$.

Given morphisms $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\#) : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ we define their composition $(h, h^\#) : (X, \mathscr{O}_X) \to (Z, \mathscr{O}_Z)$ by $h = g \circ f$ and

$$h^\# = \mathscr{O}_Z \to g_*\mathscr{O}_Y \to g_*(f_*\mathscr{O}_X) = h_*\mathscr{O}_X. \tag{3.1}$$

Note that

$$\begin{tikzcd}
\mathscr{O}_{Z,h(p)} \arrow[r] \arrow[rr, bend left, "(h^\#)_{h(p)}"] \arrow[dr, "g^\#_{f(p)}"'] & (g_*\mathscr{O}_Y)_{g\circ f(p)} \arrow[r] \arrow[d] & (g_*f_*\mathscr{O}_X)_{g\circ f(p)} \arrow[d] \arrow[dd, bend left] \\
& \mathscr{O}_{Y,f(p)} \arrow[r] \arrow[dr, "f_p^\#"'] & (f_*\mathscr{O}_X)_{f(p)} \arrow[d] \\
& & \mathscr{O}_{X,p}
\end{tikzcd} \tag{3.2}$$

commutes and so $h_p^\# = f_p^\# \circ g^\#_{f(p)}$ is a morphism of local rings and so $(h, h^\#)$ is indeed a morphism of locally ringed spaces.

`prop:factor` **Proposition 3.1.1.** *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. If $f(X) \subseteq U$ for some open subset $U \subseteq Y$ then $(f, f^\#)$ factors through $(U, \mathscr{O}_Y|_U)$.*

Proof. Let $\bar{f} : X \to U$ denote the map f viewed as having codomain U, and $i : U \to Y$. Then $f = i \circ \bar{f}$. Moreover, there is a natural morphism

$i^\# : \mathscr{O}_Y \to i_*(\mathscr{O}_Y|_U)$ given by the restriction maps. Since $\bar{f}^{-1}(V) = f^{-1}(V)$ for $V \subseteq U$, there is also a natural map $\bar{f}^\# : \mathscr{O}_Y|_U \to \bar{f}_*\mathscr{O}_X$ given by the restriction of $f^\#$. It is straightforward to see that $f^\# = i^\# \circ \bar{f}^\#$. ∎

Thm 3.1.2. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. $(f, f^\#)$ is an isomorphism iff f is a homeomorphism and $f^\#$ is an isomorphism.*

Proof. The forwards direction is obvious. Now suppose f is a homeomorphism and $f^\#$ is an isomorphism. Let $g = f^{-1} : Y \to X$ and $g^\# = (g_* f^\#)^{-1}$. Then $(g, g^\#) \circ (f, f^\#) = \mathrm{id}$ and $(f, f^\#) \circ (g, g^\#) = \mathrm{id}$. ∎

Corollary 3.1.3. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. If*

1. *$U := f(X)$ is an open subset of Y,*
2. *f is a homeomorphism onto its image,*
3. *$f_p^\#$ is an isomorphism for all $p \in X$*

then $(X, \mathscr{O}_X) \cong (U, \mathscr{O}_Y|_U)$.

Proof. By proposition 3.1.1, $(f, f^\#)$ factors through $(\bar{f}, \bar{f}^\#) : (X, \mathscr{O}_X) \to (U, \mathscr{O}_Y|_U)$. By the theorem it suffices to check that $\bar{f}_p^\#$ is an isomorphism for all $p \in X$. But this follows from the fact that $i^\#$ is an isomorphism on stalks. ∎

Thm 3.1.4. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces. The presheaf $U \mapsto \mathrm{Hom}((U, \mathscr{O}_X|_U), (Y, \mathscr{O}_Y))$ on $\mathsf{Top}(X)$ is a sheaf of sets.*

3.2 Morphisms

3.2.1 Quasi-compact

Definition 3.2.1. Let $(f, f^\#) : X \to Y$ be a morphism of schemes. We say $(f, f^\#)$ is quasi-compact if there is an affine covering $\{V_i\}_i$ of Y such that $f^{-1}(V_i)$ is quasi-compact for all i.

Lemma 3.2.2. *Let X be a topological space. If X is a finite union of quasi-compact open sets then X is quasi-compact.*

Thm 3.2.3. *Let $(f, f^\#) : X \to Y$ be quasi-compact. Then for any affine $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

Proof. Let us say that an open affine subset $V \subseteq Y$ has the property (P) if $f^{-1}(V)$ is quasi-compact. We show that (P) is an affine-local property.

1. If V has property (P), then certainly V_g for $g \in \mathscr{O}_Y(V)$ does too.

2. Suppose that $(g_1, \dots, g_k) = \mathscr{O}_Y(V)$ and that V_{g_i} has property (P) for all i. Then $f^{-1}(V) = \cup_i f^{-1}(V_{f_i})$ is a finite union of quasi-compact open sets and so is quasi compact.

The result follows from the affine communication lemma. ∎

Remark 3.2.4. It follows easily from the theorem that $f^{-1}(V)$ is quasi-compact for all open quasi-compact subsets V of Y.

Remark 3.2.5. If $\phi : A \to B$ is ring homomorphism, then $\operatorname{Spec}(\phi)$ is always quasi-compact (choose the trivial covers for both spaces).

Proposition 3.2.6. *Being quasi-compact is invariant under base change.*

3.2.2 Quasi-separated

Definition 3.2.7. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is quasi-separated if there is an affine covering $\{V_i\}_i$ of Y such that $f^{-1}(V_i)$ is quasi-separated for all i.

Thm 3.2.8. *Let $(f, f^{\#}) : X \to Y$ be quasi-separated. Then for any affine $V \subseteq Y$, $f^{-1}(V)$ is quasi-separated.*

Proof. Let us say that an open affine subset $V \subseteq Y$ has the property (P) if $f^{-1}(V)$ is quasi-separated. We show that (P) is an affine-local property.

1. If V has property (P), then certainly V_g for $g \in \mathscr{O}_Y(V)$ does too.

2. Suppose that $(g_1, \dots, g_k) = \mathscr{O}_Y(V)$ and that V_{g_i} has property (P) for all i. Then for each i, $f^{-1}(V_{g_i})$ has an open cover $\{W_{ij}\}_j$ such that the intersection between any two elements of the cover is quasi-compact. It is clear that $\{W_{ij}\}_{i,j}$ cover $f^{-1}(V)$ and so it suffices to show that $W_{ij} \cap W_{kl}$ is quasi-compact for $i \neq k$. But by part (1), the $V_{g_i g_j}$ also have property (P). Moreover, $W_{ij} \cap W_{kl} \subseteq f^{-1}(V_{g_i g_k})$ and so $W_{ij} \cap W_{kl} = W_{ij} \cap f^{-1}(V_{g_i g_k}) \cap W_{kl} \cap f^{-1}(V_{g_i g_k})$. But $W_{ij} \cap f^{-1}(V_{g_i g_k}) = (W_{ij})_{f^{\#}(V)(g_k)}$ is affine. Thus $W_{ij} \cap W_{kl}$ is the intersection of two affines in the quasi-separated space $f^{-1}(V_{g_i g_k})$ and is thus quasi-compact.

The result follows from the affine communication lemma. ∎

Remark 3.2.9. If $\phi : A \to B$ is ring homomorphism, then $\operatorname{Spec}(\phi)$ is always quasi-separated (choose the trivial covers for both spaces).

Proposition 3.2.10. *Being quasi-separated is invariant under base change.*

3.2.3 Locally of finite type

Definition 3.2.11. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is locally of finite type if there is an affine covering $\{V_i\}_i$ of Y, and for each i, and affine covering $\{U_{ij}\}_j$ of $f^{-1}(V_i)$ such that $\mathscr{O}_X(U_{ij})$ is as finitely generated $\mathscr{O}_Y(V_i)$-algebra.

prop:fg_aff_local

Proposition 3.2.12. *Let $\phi : B \to A$ be a ring homomorphism and $(f_1, \dots, f_n) = A$. Then A is a finitely generated B-algebra iff A_{f_i} is a finitely generated B-algebra for all i.*

Thm 3.2.13. *Let $(f, f^{\#}) : X \to Y$ be locally of finite type. Then for any affine $V \subseteq Y$ and affine $U \subseteq f^{-1}(V)$, $\mathscr{O}_X(U)$ is a finitely generated $\mathscr{O}_Y(V)$-algebra.*

Proof. Let us say that an open affine subset $V \subseteq Y$ has the property (P) if for all open affine $U \subseteq f^{-1}(V)$, $\mathscr{O}_X(U)$ is finitely generated as a $\mathscr{O}_Y(V)$-algebra. We show that (P) is an affine-local property.

1. It is clear that if V has property (P) then so does V_g for $g \in \mathscr{O}_Y(V)$.
2. Suppose that $(g_1, \dots, g_k) = \mathscr{O}_Y(V)$ and that V_{g_i} has property (P) for all i. Let $U \subseteq f^{-1}(V)$ be affine and open and let $\psi : A \to C$ denote the corresponding morphism where $A = \mathscr{O}_Y(V)$ and $C = \mathscr{O}_X(U)$. We have that f restricts to a morphism $U_{\psi(g_i)} \to V_{f_i}$ for all i. Since $U_{\psi(g_i)}$ is affine, $C_{\psi(g_i)}$ is a finitely generated A_{g_i}-algebra and hence a finitely generated A-algebra. Since $(g_1, \dots, g_k) = A$ it follows that $(\psi(g_1), \dots, \psi(g_k)) = C$ and so C is a finitely generated A-algebra.

It follows that the property (P) is an affine-local property. But by proposition 3.2.12 and the affine communication lemma, each of the V_i have property (P). Since the V_i cover Y, the result follows from the affine communication lemma. ∎

Remark 3.2.14. If $\phi : A \to B$ is ring homomorphism, then $\mathrm{Spec}(\phi)$ is locally of finite type iff B is a finitely generated A-algebra via ϕ.

Proposition 3.2.15. *Being locally of finite type is invariant under base change.*

3.2.4 Finite type

Definition 3.2.16. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is of finite type if it is quasi-compact and locally of finite type.

Remark 3.2.17. If $\phi : A \to B$ is ring homomorphism, then $\mathrm{Spec}(\phi)$ is of finite type iff B is a finitely generated A-algebra via ϕ.

3.2.5 Finite

Definition 3.2.18. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is finite if it is affine and locally of finite type.

3.2.6 Closed immersion

Definition 3.2.19. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is a closed immersion if $f(X)$ is closed in Y, f is a homeomorphism onto its image, and the morphism $f^{\#}$ is surjective.

Remark 3.2.20. If $\phi : A \to B$ is ring homomorphism, then $\mathrm{Spec}(\phi)$ is a closed immersion iff ϕ is a surjection.

Proposition 3.2.21. *Let $f : X \to Y$ be a morphism and suppose there is an open cover $\{U_i\}_i$ of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a closed immersion for all i. Then f is a closed immersion.*

Lemma 3.2.22. *Let $Y \to \mathrm{Spec}(A)$ be a closed immersion. Then Y is affine.*

Proof. See Hartshorne II.3.11 b). ∎

Corollary 3.2.23. *Let $f : X \to Y$ be a closed immersion. Then f is an affine morphism.*

Proof. Let U be an affine susbet of Y. Then f restricted to $f^{-1}(U)$ is a closed immersion. The result then follows from the lemma. ∎

Proposition 3.2.24. *Being a closed immersion is invariant under base change.*

Proof. Let $f : X \to S$ be a closed immersion and $g : S' \to S$ any morphism of schemes. It suffices to check for X, S, S' affine. But then the result follows from the fact that tensoring is right exact. ∎

3.2.7 Open immersion

Definition 3.2.25. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is an open immersion if $f(X)$ is open in Y, f is a homeomorphism onto its image, and $f_p^{\#}$ is an isomorphism for all $p \in X$.

Proposition 3.2.26. *Being an open immersion is invariant under base change.*

Proof. Clear if you view $i : U \to X$ as a subset of X (then $V \times_S S'$ is an open subset of $S \times_S S' = S'$). ∎

3.2.8 Affine

Definition 3.2.27. Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is affine if there is an affine covering $\{V_i\}_i$ of Y such that $f^{-1}(V_i)$ is affine for all i.

Thm 3.2.28. *Let $(f, f^{\#}) : X \to Y$ be affine. Then for any affine $V \subseteq Y$, $f^{-1}(V)$ is affine.*

Proof. Let us say that an open affine subset $V \subseteq Y$ has the property (P) if $f^{-1}(V)$ is affine. We show that (P) is an affine-local property.

1. If V has property (P), then clearly V_g for $g \in \mathscr{O}_Y(V)$ does too.
2. Suppose that $(g_1, \dots, g_k) = \mathscr{O}_Y(V)$ and that V_{g_i} has property (P) for all i. Then f restricted to $f^{-1}(V) \to V$ is quasi-compact and quasi-separated. It follows that $f^{-1}(V)$ is quasi-compact and quasi-separated. Moreover, $f^{-1}(V)_{f^\#(V)(g_i)}$ is affine for all i, and $(f^\#(V)(g_1), \dots, f^\#(V)(g_k)) = \Gamma(f^{-1}(V), \mathscr{O}_X)$. It follows from proposition 9.2.5 that $f^{-1}(V)$ is affine.

The result follows from the affine communication lemma. ■

Remark 3.2.29. If $\phi : A \to B$ is a ring homomorphism, then $\operatorname{Spec}(\phi)$ is always affine (choose the trivial cover for both spaces).

Proposition 3.2.30. *Affineness is invariant under base-change.*

Proof. Let $f : X \to S$ be an affine morphism and $g : S' \to S$ any morphism of schemes. We wish to show that $f' : X_{S'} = X \times_S S' \to S'$ is affine as well. Let $\{U_i\}_i$ be a cover of S by affine open subsets and for each i let $\{V_{ji}\}_{ji}$ be a cover of $g^{-1}(U_i)$ by affine open subsets. Then

$$f'^{-1}(V_{ji}) = X \times_S V_{ji} = f^{-1}(U_i) \times_{U_i} V_{ji} \tag{3.3}$$

is affine since $f^{-1}(U_i)$ is. Thus f' is affine. ■

3.3 $\mathscr{O}_X$-Modules

Definition 3.3.1. Let $(X, \mathscr{O}_X)$ be a locally ringed space. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ of abelian groups with a compatible $\mathscr{O}_X$ action. Morphisms of $\mathscr{O}_X$-modules are morphisms of sheaves of abelian groups that respect the $\mathscr{O}_X$-module structure.

Thm 3.3.2. *The category of $\mathscr{O}_X$-modules is an abelian category.*

Proof. Additive structure on hom-sets is obvious. Kernels are the same as the kernels in $\mathsf{Ab}(X)$, with the obvious $\mathscr{O}_X$-module structure. Similarly for cokernels (if a presheaf has an $\mathscr{O}_X$-module structure, then so does its sheafification by acting on the stalks). The rest then follows. ■

Definition 3.3.3. (Tensor product). Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. Define the tensor product of $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ to be the sheafification of the presheaf tensor product with the obvious $\mathscr{O}_X$-module structure.

Definition 3.3.4. (Pullback). Let $f : X \to Y$ be a continuous map, and $\mathscr{F}$ an $\mathscr{O}_Y$. Then $f^{-1}\mathscr{F}$ is naturally a $f^{-1}\mathscr{O}_Y$-module. Moreover from the inverse image - direct image adjunction we obtain a map $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ from $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$. We can thus form the sheaf $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$. This sheaf is naturally an $\mathscr{O}_X$-module and we call it $f^*\mathscr{F}$.

Proposition 3.3.5. *The maps $X \mapsto \mathsf{Mod}(X)$, $f \mapsto f^*$ give rise to a functor $\mathsf{Sch}^{op} \to \mathsf{Mod}$.*

Proof. Clear that the identity maps to the identity. Now suppose we have maps $f : X \to Y, g : Y \to Z$ and let $\mathscr{F} \in \mathsf{Mod}(Z)$. Then

$$\begin{aligned} f^*(g^*(\mathscr{F})) &= f^*(g^{-1}\mathscr{F} \otimes_{g^{-1}\mathscr{O}_Z} \mathscr{O}_Y) \\ &= f^{-1}\left(g^{-1}\mathscr{F} \otimes_{g^{-1}\mathscr{O}_Z} \mathscr{O}_Y\right) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\ &\cong (g \circ f)^{-1}\mathscr{F} \otimes_{(g\circ f)^{-1}\mathscr{O}_Z} \mathscr{O}_X \\ &= (g \circ f)^*(\mathscr{F}). \end{aligned} \tag{3.4}$$

■

Thm 3.3.6. *Let $\mathscr{F}, \mathscr{G} \in \mathsf{Mod}(Y)$ and $f : X \to Y$. Then*

$$f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}. \tag{3.5}$$

Remark 3.3.7. If $i : U \hookrightarrow X$ is the inclusion map for an open subset $U \subseteq X$, then $i^* = i^{-1}$.

Definition 3.3.8. (Direct image). Let $f : X \to Y$ be a continuous map, and $\mathscr{F}$ an $\mathscr{O}_X$. Then $f_*\mathscr{F}$ is naturally a $f_*\mathscr{O}_X$ module, and hence a $\mathscr{O}_Y$-module via $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$.

thm:tensor-hom **Thm 3.3.9.** *Let $\mathscr{F}$ be a $(\mathscr{A}, \mathscr{B})$-bimodule. Then $- \otimes_{\mathscr{A}} \mathscr{F} \dashv \mathscr{H}om_{\mathscr{B}}(\mathscr{F}, -)$ as functors between $\mathsf{Mod}(\mathscr{A})$ and $\mathsf{Mod}(\mathscr{B})$.*

Proof. Follows from the corresponding tensor-hom adjunction for modules. ■

Lemma 3.3.10. *Let $f : X \to Y$, $\mathscr{F} \in \mathsf{Mod}(Y)$ and $\mathscr{G} \in \mathsf{Mod}(X)$. Then under the natural bijection*

$$\mathrm{Hom}_{\mathsf{Ab}}(f^{-1}\mathscr{F}, \mathscr{G}) \leftrightarrow \mathrm{Hom}_{\mathsf{Ab}}(\mathscr{F}, f_*\mathscr{G}) \tag{3.6}$$

$f^{-1}\mathscr{O}_Y$-module morphisms biject with $\mathscr{O}_Y$-module morphisms.

Thm 3.3.11. *Let $f : X \to Y$ be a continuous map. Then $f^* \dashv f_*$ as functors between $\mathsf{Mod}(X)$ and $\mathsf{Mod}(Y)$.*

Proof. Let $\mathscr{F} \in \mathsf{Mod}(Y)$ and $\mathscr{G} \in \mathsf{Mod}(X)$. Note that $\mathscr{O}_X$ is an $(f^{-1}\mathscr{O}_Y, \mathscr{O}_X)$-bimodule. We thus have the following chain of natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, \mathscr{G}) &\leftrightarrow \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{G})) \\ &\leftrightarrow \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \mathscr{G}) \\ &\leftrightarrow \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_*\mathscr{G}) \end{aligned} \tag{3.7}$$

where the last bijection follows from the lemma. ■

Remark 3.3.12. Given an $\mathscr{O}_Y$-module $\mathscr{F}$ we have a morphism

$$f^{-1}\mathscr{F} \to f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X = f^*\mathscr{F}. \tag{3.8}$$

Thus we have a morphism $\mathscr{F} \to f_*f^{-1}\mathscr{F} \to f_*f^*\mathscr{F}$. This morphism is the same as the one arising from the $f^* \dashv f_*$ adjunction.

Definition 3.3.13. Let $\mathscr{F}$ be an $\mathscr{O}_Y$-moudle, and $\sigma \in \mathscr{F}(U)$. Write $f^*\sigma$ for the element in $(f^*\mathscr{F})(f^{-1}(U))$ under the morphism $\mathscr{F} \to f_*f^*\mathscr{F}$.

Remark 3.3.14. When $\mathscr{F} = \mathscr{O}_Y$, $f^* = f^\#$.

Proposition 3.3.15. *Let $\mathscr{F}$ be an $\mathscr{O}_Y$-module, $\sigma \in \mathscr{F}(Y)$ and $\phi : \mathscr{O}_Y \to \mathscr{F}$ the corresponding map. Then $f^*\phi : \mathscr{O}_X \to f^*\mathscr{F}$ is mulitplication by $f^*\sigma$.*

Proof. id $\to f_*f^*$ a natural transformation and so

$$\begin{tikzcd} \mathscr{O}_Y \arrow[r] \arrow[d] & \mathscr{F} \arrow[d] \\ f_*\mathscr{O}_X \arrow[r] & f_*f^*\mathscr{F} \end{tikzcd} \tag{3.9}$$

commutes. ■

Definition 3.3.16. Let R be a ring and M an R-module. Define $\widetilde{M}$ to be the $\mathscr{O}_{\mathrm{Spec}(R)}$-module which is locally M_r.

Thm 3.3.17. *$\widetilde{\bullet}$ is a fully faithful exact functor from Mod_R to $\mathsf{Mod}(\mathrm{Spec}(R))$.*

Proof. Localisation is exact. ■

Corollary 3.3.18. *$\widetilde{\bullet}$ and Γ form part of an adjoint equivalence of categories between Mod_R and $\mathsf{Mod}(\mathrm{Spec}(R))$.*

3.4 Locally free $\mathscr{O}_X$-modules

Definition 3.4.1. Let $(X, \mathscr{O}_X)$ be a scheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. We say $\mathscr{F}$ is locally free of rank n if there exists an open cover $\{U_i\}_i$ of X such that $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$ for all i.

Remark 3.4.2. Given a locally free sheaf $\mathscr{F}$ and an open cover we obtain transition functions $\psi_{ji} \in \mathbf{GL}_n(\mathscr{O}_{U_{ij}})$. Conversely, given such data we obtain a sheaf isomorphic to the original one.

`lem:pullback` **Lemma 3.4.3.** *Let $f : X \to Y$ be a morphism of schemes and $\phi : \mathscr{O}_Y \to \mathscr{O}_Y$ the $\mathscr{O}_Y$-module homomorphism given by multiplication by $\alpha \in \mathscr{O}_Y(Y)$. Then $f^*\phi : \mathscr{O}_X \to \mathscr{O}_X$ is given by multiplication by $f^\#(Y)(\alpha)$.*

Proof. We check that they are equal on stalks.

$$\begin{tikzcd} \mathscr{O}_{Y,f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p} \arrow[r] & \mathscr{O}_{Y,f(p)} \otimes_{\mathscr{O}_{Y,f(p)}} \mathscr{O}_{X,p} \arrow[d] \\ \mathscr{O}_{X,p} \arrow[u] \arrow[r, "f^*\phi"] & \mathscr{O}_{X,p}. \end{tikzcd} \tag{3.10}$$

Following this diagram we get that

$$1 \mapsto 1 \otimes 1 \mapsto \alpha_p \otimes 1 = 1 \otimes f_p^{\#}(\alpha_p) \mapsto f_p^{\#}(\alpha_p). \tag{3.11}$$

Thus $f^*\phi$ is given by multiplication by $f^{\#}(Y)(\alpha)$. ■

Thm 3.4.4. *Let $f : X \to Y$ be a morphism of schemes, and $\mathscr{F}$ a locally free $\mathscr{O}_Y$-module. Then $f^*\mathscr{F}$ is locally free of the same rank. Moreover, if $\{\psi_{ji}\}$ denote the transition functions for an open cover $\{U_i\}_i$ for Y, then $f^{\#}(\psi_{ji})$ are transition functions for $f^*\mathscr{F}$ on the open cover $\{f^{-1}(U_i)\}_i$.*

Proof. There is an isomorphism $\mathscr{F}|_{U_i} \to \mathscr{O}_{U_i}^{\oplus n}$. Thus

$$(f^*\mathscr{F})|_{f^{-1}(U_i)} \cong \left(f|_{f^{-1}(U_i)}\right)^* (\mathscr{F}|_{U_i}) \cong \mathscr{O}_{f^{-1}(U_i)}^{\oplus n} \tag{3.12}$$

and so $f^*\mathscr{F}$ is also locally free of rank n. The result on the transition functions follows from lemma 3.4.3 ■

Proposition 3.4.5. *Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module. Then $- \otimes_{\mathscr{O}_X} \mathscr{F}$ is exact.*

3.5 Line bundles

Definition 3.5.1. Let S be a scheme and $\mathscr{L}, \mathscr{L}'$ be locally free sheaves on S. We say the morphisms $\phi : \bigoplus \mathscr{O}_S \to \mathscr{L}$, $\psi : \bigoplus \mathscr{O}_S \to \mathscr{L}'$ are isomorphic if there is an isomorphism $i : \mathscr{L} \to \mathscr{L}'$ such that $\psi = i \circ \phi$.

Definition 3.5.2. Let $r \geq 0$. Define $\underline{\mathbb{P}}^r$ to be the functor from Sch^{op} to Set which associates with the scheme S the set of isomorphism classes of sujective morphisms $\phi : \bigoplus_{k=0}^r \mathscr{O}_S \to \mathscr{L}$ where $\mathscr{L}$ is a locally free sheaf of rank 1. Given $f : T \to S$, $\underline{\mathbb{P}}^r(f)$ sends ϕ to $f^*\phi$.

Definition 3.5.3. $\mathbb{P}^r_{\mathbb{Z}} := \mathrm{Proj}(\mathbb{Z}[x_0, \ldots, x_r])$.

Lemma 3.5.4. *Let $\mathscr{L}$ be a locally free sheaf of rank 1 on a scheme S and $\sigma \in \mathscr{L}(S)$. Then $S_\sigma := \{s \in S : \sigma \notin \mathfrak{m}_s\mathscr{L}_s\}$ is an open subset of S and trivialises $\mathscr{L}$. Moreover, if $\xi : T \to S$ is a morphism of schemes then $T_{\xi^*\sigma} = \xi^{-1}(S_\sigma)$.*

Proof. Let $\{U_i\}_i$ be a trivialising open cover for $\mathscr{L}$. Then $\psi_i : \mathscr{L}|_{U_i} \xrightarrow{\sim} \mathscr{O}_S|_{U_i}$ and under this isomorphism $\mathfrak{m}_s\mathscr{L}_s$ corresponds to $\mathfrak{m}_s$. Thus $S_\sigma \cap U_i$ is open for all i and so S_σ is open. Now define the map $\phi : \mathscr{O}_X \to \mathscr{L}$ by $1 \mapsto \sigma$. On $S_\sigma \cap U_i$ the composition $\mathscr{O}_X|_{S_\sigma \cap U_i} \xrightarrow{\phi} \mathscr{L}|_{S_\sigma \cap U_i} \xrightarrow{\psi_i} \mathscr{O}_X|_{S_\sigma \cap U_i}$ must be multiplication by some $\alpha \in \Gamma(S_\sigma \cap U_i, \mathscr{O}_X)$. But looking at stalks, this α must be invertible and so the composition must be an isomorphism. But then $\phi|_{S_\sigma \cap U_i}$ must be too and hence $\phi|_{S_\sigma}$ is an isomorphism.

For the last part note that we have the following commutative diagram

$$\begin{tikzcd} \mathscr{L}|_{U_i} \arrow[r] \arrow[d] & \mathscr{O}_S|_{U_i} \arrow[d] \\ (\xi|_{\xi^{-1}(U_i)})_*((\xi^*\mathscr{L})|_{\xi^{-1}(U_i)}) \arrow[r] & (\xi|_{\xi^{-1}(U_i)})_*(\mathscr{O}_T|_{\xi^{-1}(U_i)}). \end{tikzcd} \tag{3.13}$$

It follows that $\xi^*(\sigma|_{U_i})$ maps to $\xi^\#(\psi_i(\sigma|_{U_i}))$ under $\xi^*\psi_i$. Thus

$$T_{\xi^*\sigma} \cap \xi^{-1}(U_i) = \xi^{-1}(U_i)_{\xi^\#(\psi_i(\sigma|_{U_i}))} = \xi^{-1}\left((U_i)_{\psi_i(\sigma|_{U_i})}\right) = \xi^{-1}(U_i \cap S_\sigma) \tag{3.14}$$

and so the result follows. ■

Lemma 3.5.5. *Let S be a scheme, $\mathscr{L}$ a locally free sheaf of rank 1 and*

$$\phi : \bigoplus_{k=0}^{r} \mathscr{O}_S \to \mathscr{L} \tag{3.15}$$

a surjective morphism of $\mathscr{O}_S$-modules. Then there is a morphism $\eta_\phi : S \to \mathbb{P}^r_{\mathbb{Z}}$. Moreover, this construction only depends on the isomorphism class of ϕ and is functorial in S i.e. given $\gamma : T \to S$ we have $\eta_{\gamma^*(\phi)} = \eta_\phi \circ \gamma$.

Remark 3.5.6. Morally such a morphism gives $r+1$ sections $\sigma_0, \ldots, \sigma_r$. We then obtain the morphism $S \to \mathbb{P}^r_{\mathbb{Z}}$ by $s \mapsto (\sigma_0 : \cdots : \sigma_r)$.

Proof. Let $e_i = \delta_{ij}$ and let $\sigma_0, \ldots, \sigma_r$ be the images of $e_0, \ldots, e_r$ respectively. Since ϕ is surjective, $S = \cup_i S_{\sigma_i}$. Let $\psi_i : \mathscr{O}_S|_{S_{\sigma_i}} \to \mathscr{L}|_{S_{\sigma_i}}$ denote the trivialising isomorphisms from the previous lemma and let ξ_i be the inverse of ψ_i. Restricting ϕ to S_{σ_i} we obtain the composition

$$\bigoplus_{k=0}^{r} \mathscr{O}_{S_{\sigma_i}} \xrightarrow{\phi|_{S_{\sigma_i}}} \mathscr{L}|_{S_{\sigma_i}} \xrightarrow{\xi_i} \mathscr{O}_{S_{\sigma_i}}. \tag{3.16}$$ eq:aff

Let f_{ji} denote the image of e_j under this composition. Then $f_{ji} \cdot \sigma_i|_{S_{\sigma_i}} = \sigma_j|_{S_{\sigma_i}}$. It follows that

$$f_{ji}|_{S_{\sigma_i} \cap S_{\sigma_j}} \cdot f_{ij}|_{S_{\sigma_i} \cap S_{\sigma_j}} \cdot \sigma_j|_{S_{\sigma_i} \cap S_{\sigma_j}} = \sigma_j|_{S_{\sigma_i} \cap S_{\sigma_j}}. \tag{3.17}$$

Applying $\xi_i|_{S_{\sigma_i} \cap S_{\sigma_j}}$ to both sides yields that

$$f_{ji}|_{S_{\sigma_i} \cap S_{\sigma_j}} = f_{ij}|^{-1}_{S_{\sigma_i} \cap S_{\sigma_j}}. \tag{3.18}$$

Similarly we get

$$f_{kj}|_{S_{\sigma_i} \cap S_{\sigma_j}} f_{ji}|_{S_{\sigma_i} \cap S_{\sigma_j}} = f_{ki}|_{S_{\sigma_i} \cap S_{\sigma_j}}, \quad f_{ii} = 1. \tag{3.19}$$

Now, equation 3.16 gives rise to a morphism $(\eta_\phi)_i : S_{\sigma_i} \to U_i$ where U_i is the standard affine patch $D_+(x_i) \subseteq \mathbb{P}^r_{\mathbb{Z}}$. This morphism sends $x_j/x_i \in \mathscr{O}_{\mathbb{P}^r_{\mathbb{Z}}}(U_i)$ to f_{ji} (and is uniquely defined by this fact). It follows that

$$(\eta_\phi)_i^{-1}(U_i \cap U_j) = (S_{\sigma_i})_{f_{ji}} = S_{\sigma_i} \cap S_{\sigma_j}. \tag{3.20}$$

Thus the map $(\eta_\phi)_i|_{S_{\sigma_i} \cap S_{\sigma_j}}$ factors

$$\begin{tikzcd} S_{\sigma_i} \cap S_{\sigma_j} \arrow[r] \arrow[dr] & \operatorname{Spec}(\mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]) \\ & \operatorname{Spec}(\mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]_{x_j/x_i}) \arrow[u] \end{tikzcd} \tag{3.21}$$

and similarly for $(\eta_\phi)_j|_{S_{\sigma_i} \cap S_{\sigma_j}}$. We thus have maps

$$\begin{tikzcd} & \operatorname{Spec}(\mathbb{Z}[x_0/x_i, \ldots, x_0/x_i]_{x_j/x_i}) \\ S_{\sigma_i} \cap S_{\sigma_j} \arrow[ur, "(\eta_\phi)_{ji}"] \arrow[dr, "(\eta_\phi)_{ij}"'] & \\ & \operatorname{Spec}(\mathbb{Z}[x_0/x_j, \ldots, x_0/x_j]_{x_i/x_j}) \arrow[uu, "\zeta"'] \end{tikzcd} \tag{3.22}$$

where the vertical map is induced from $x_k/x_i \mapsto (x_k/x_j) \cdot (x_i/x_j)^{-1}$. To see that this diagram commutes note that $(\eta_\phi)_{ji}(x_k/x_i) = f_{ki}|_{S_{\sigma_i} \cap S_{\sigma_j}}$ while

$$\begin{aligned} (\eta_\phi)_{ij} \circ \zeta(x_k/x_i) &= (\eta_\phi)_{ij}\left((x_k/x_j) \cdot (x_i/x_j)^{-1}\right) \\ &= f_{kj}|_{S_{\sigma_i} \cap S_{\sigma_j}} \cdot f_{ij}|^{-1}_{S_{\sigma_i} \cap S_{\sigma_j}} \\ &= f_{kj}|_{S_{\sigma_i} \cap S_{\sigma_j}} \cdot f_{ji}|_{S_{\sigma_i} \cap S_{\sigma_j}} = f_{ki}|_{S_{\sigma_i} \cap S_{\sigma_j}}. \end{aligned} \tag{3.23}$$

It follows that $(\eta_\phi)_i|_{S_{\sigma_i} \cap S_{\sigma_j}} = (\eta_\phi)_j|_{S_{\sigma_i} \cap S_{\sigma_j}}$ and hence we obatin a map $\eta_\phi : S \to \mathbb{P}^r_{\mathbb{Z}}$. This construction clearly only depends on the isomorphism class of ϕ (the f_{ji}'s are independent of isomorphism class).

Finally we show functoriality. Let $\gamma : T \to S$ be a morphism of schemes. Then $e_1, \ldots, e_r$ get sent to $\gamma^*\sigma_1, \ldots, \gamma^*\sigma_r$ under $\gamma^*(\phi)$. The corresponding open sets are $T_{\gamma^*\sigma_i} = \gamma^{-1}(S_{\sigma_i})$ and the corresponding f_{ji}'s are $\gamma^* f_{ji} = \gamma^{\#} f_{ji}$. It follows that for each i, $(\eta_{\gamma^*(\phi)})_i = (\eta_\phi)_i \circ \gamma|_{T_{\gamma^*\sigma_i}}$. Therefore we get that $\eta_{\gamma^*(\phi)} = \eta_\phi \circ \gamma$. ∎

rem:trans_fn *Remark* 3.5.7. The $f_{ji}|_{S_{\sigma_i} \cap S_{\sigma_j}}$ are the transition functions with respect to the open cover S_{σ_i}.

Definition 3.5.8. Let $\mathscr{O}(1)$ denote the sheaf on $\mathbb{P}^r_{\mathbb{Z}}$ with transition functions $\psi_{ji} = x_i/x_j$. Write χ_k for the section corresponding to x_k/x_i on U_i for $i = 0, \ldots, r$. Then we obtain a surjective morphism $\phi_{\mathbb{P}^r_{\mathbb{Z}}} : \bigoplus_{k=0}^r \mathscr{O}_{\mathbb{P}^r_{\mathbb{Z}}} \to \mathscr{O}(1)$. Note that $\eta_{\phi_{\mathbb{P}^r_{\mathbb{Z}}}} = \mathrm{id}_{\mathbb{P}^r_{\mathbb{Z}}}$.

Thm 3.5.9. $\underline{\mathbb{P}}^r$ *is a representable functor with representative* $\mathbb{P}^r_{\mathbb{Z}}$.

Proof. From the lemma we have a natural transformation $\underline{\mathbb{P}}^r \Rightarrow \mathrm{Hom}(-, \mathbb{P}^r_{\mathbb{Z}})$. Now consider that map $\mathrm{Hom}(S, \mathbb{P}^r_{\mathbb{Z}}) \to \underline{\mathbb{P}}^r(S)$ given by $\xi \to \xi^*(\phi_{\mathbb{P}^r_{\mathbb{Z}}})$. Then

$$\eta_{\xi^*(\phi_{\mathbb{P}^r_{\mathbb{Z}}})} = \eta_{\phi_{\mathbb{P}^r_{\mathbb{Z}}}} \circ \xi = \xi \tag{3.24}$$

so the composition one way is the identity. Conversely, consider a $\phi : \bigoplus_{k=0}^r \mathscr{O}_S \to \mathscr{L} \in \underline{\mathbb{P}}^r(S)$. We wish to show that $(\eta_\phi)^*\mathscr{O}(1) \cong \mathscr{L}$ and under this isomorphism, $(\eta_\phi)^*(\phi_{\mathbb{P}^r_{\mathbb{Z}}})) = \phi$. First note that $(\eta_\phi)^{-1}(U_i) = S_{\sigma_i}$. Moreover, the corresponding transition functions for $(\eta_\phi)^*\mathscr{O}(1)$ are $(\eta_\phi)^\#(x_i/x_j) = f_{ij}|_{S_{\sigma_i} \cap S_{\sigma_j}}$. It follows from remark 3.5.7 that $(\eta_\phi)^*\mathscr{O}(1) \cong \mathscr{L}$. Finally $(\eta_\phi)^*(\chi_k)$ maps to f_{ki} on the trivialisation on S_{σ_i}. The result follows. ∎

3.6 Ample line bundles

lem:ext

Lemma 3.6.1. *Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, $\mathscr{L}$ a line bundle and $f \in \Gamma(X, \mathscr{L})$.*

1. *If X is quasi-compact, $s \in \mathscr{F}(X)$ and $s|_{X_f} = 0$ then there is an $n \geq 1$ such that $s \otimes f^{\otimes n} = 0$ in $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.*

2. *If X is qsqc and $s \in \mathscr{F}(X_f)$ then there is an $n \geq 1$ such that $s \otimes f^{\otimes n} = t|_{X_f}$ for some $t \in (\mathscr{F} \otimes \mathscr{L}^{\otimes n})(X)$.*

Proof. (1) Let $\{U_i\}$ be a finite trivialising cover for $\mathscr{L}$ consisting of affines and let $g_i \in \mathscr{O}_X(U_i)$ be the section corresponding to $f|_{U_i}$ under $\mathscr{L}|_{U_i} \cong \mathscr{O}_X|_{U_i}$. Then $U_i \cap X_f = (U_i)_{g_i}$ so there is an $n \geq 1$ (independent of i) such that $g_i^n s|_{U_i} = 0$. But under the isomorphism $\mathscr{F}|_{U_i} \cong \mathscr{F}|_{U_i} \otimes \mathscr{O}_X|_{U_i}^{\otimes n} \cong \mathscr{F}|_{U_i} \otimes \mathscr{L}|_{U_i}^{\otimes n}$, $g_i^n s|_{U_i}$ maps to $s|_{U_i} \otimes f|_{U_i}^{\otimes n}$. It follows that $s \otimes f^{\otimes n} = 0$. (2) Let $\{U_i\}$ and $\{g_i\}$ be as before and let $\{\psi_{ji}\}$ be the transition functions. There exists an n (independent of the i) such that $g_i^n s|_{(U_i)_{g_i}} = t_i|_{(U_i)_{g_i}}$ for some $t_i \in \mathscr{F}(U_i)$. Let $\mathscr{G} = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ and t_i' be the element in $\mathscr{G}(U_i)$ corresponding to t_i. Then $t_i'|_{U_i \cap X_f} = s \otimes f^{\otimes n}|_{U_i \cap X_f}$. Thus we have $t_i'|_{U_i \cap U_j \cap X_f} = t_j'|_{U_i \cap U_j \cap X_f}$. But $U_i \cap U_j$ is quasi-compact since X is quasi-serparated and so by (1) there is a m (independent of i, j) such that $t_i' \otimes f^{\otimes m}|_{U_i \cap U_j} = t_j' \otimes f^{\otimes m}|_{U_i \cap U_j}$. It follows that there is a $t \in \mathscr{G} \otimes \mathscr{L}^{\otimes m}(X)$ that restricts to the $t_i' \otimes f^{\otimes m}$ and hence to $s \otimes f^{\otimes (m+n)}$ on X_f. ∎

Definition 3.6.2. A line bundle $\mathscr{L}$ is ample if for any coherent sheaf $\mathscr{F}$ on S, there is a $n_0 \in \mathbb{N}$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections for all $n \geq n_0$.

Thm 3.6.3. *Let S be a noetherian scheme. The line bundle $\mathscr{L}$ is very ample iff there is $n \in N$ and $\sigma_1, \ldots, \sigma_k \in \Gamma(S, \mathscr{L}^{\otimes n})$ such that S_{σ_i} is affine for all i and $S = \cup_i S_{\sigma_i}$.*

Proof. ($\Leftarrow$) Let $\mathscr{F}$ be a coherent sheaf on S. Then $\mathscr{F}|_{S_{\sigma_i}}$ is globally generated, say by $t_{ji} \in \mathscr{F}(S_{\sigma_i})$, since S_{σ_i} is affine. By lemma 3.6.1, there is an $n \geq 1$ independent of i, and $\lambda_{ji} \in (\mathscr{F} \otimes \mathscr{L}^{\otimes n})(X)$ such that $\lambda_{ji}|_{S_{\sigma_i}} = t_{ji} \otimes \sigma_i^{\otimes n}$. It is clear that these λ_{ji} globally generate $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$. It is straighfroward to see that $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is also globally generated for $m \geq n$.

($\Rightarrow$) Let $\{U_i\}_i$ be an open cover of S consisting of affines which trivialise $\mathscr{L}$. Let $Y_i = S \backslash U_i$ and $\mathscr{J}_{Y_i}$ be the ideal sheaf associated to Y_i. Since $\mathscr{L}$ is ample there is an n independent of i, such that $\mathscr{J}_{Y_i} \otimes \mathscr{L}^{\otimes n}$ is globally generated for all i. Let $\widetilde{\sigma}_{ji} \in \mathscr{J}_{Y_i} \otimes \mathscr{L}^{\otimes n}$ be such that for all $x \in U$ there is a j such that $\widetilde{\sigma}_{ji} \notin \mathfrak{m}_x$ (note that $\mathscr{J}|_{U_i} \cong \mathscr{O}_S|_{U_i}$). We have the exact sequence $0 \to \mathscr{J}_{U_i} \to \mathscr{O}_S$ and hence the exact sequence $0 \to \mathscr{J}|_{U_i} \otimes \mathscr{L}^{\otimes n} \to \mathscr{L}^{\otimes n}$. Let σ_{ji} be the image of $\widetilde{\sigma}_{ji}$ in $\mathscr{L}^{\otimes n}$. Then by construction we have $S_{\sigma_{ji}} \subseteq U_i$ and in fact $U_i = \cup_j S_{\sigma_{ji}}$. Since U_i is affine and trivialises $\mathscr{L}$ we also have that $S_{\sigma_{ji}}$ is also affine. The result follows. ■

Corollary 3.6.4. *$\mathscr{O}(1)$ is ample.*

Lemma 3.6.5. *Let S be a Noetherian scheme, $\mathscr{L}$ a line bundle on S, $\phi : \bigoplus_{k=0}^r \mathscr{O}_S \to \mathscr{L}$ a surjective $\mathscr{O}_S$-module homomorphism and $\sigma_0, \ldots, \sigma_r$ the distinguished images of ϕ in $\mathscr{L}$. If S_{σ_i} is affine for all i, then the resulting morphism $S \to \mathbb{P}^r_{\mathbb{Z}}$ is a closed immersion.*

Thm 3.6.6. *Let $f : S \to \operatorname{Spec}(R)$ be a morphism of finite type, R a Noetherian ring and $\mathscr{L}$ an ample line bundle on S. Then there is an $n \geq 1$ and $\sigma_0, \ldots, \sigma_r \in \Gamma(S, \mathscr{L}^{\otimes n})$ such that the corresponding morphism $S \to \mathbb{P}^r_R$ is a closed immersion into an open subset of $\mathbb{P}^r_R$.*

Proof. Since f is finite, S is Noetherian. Replacing $\mathscr{L}^{\otimes n}$ with $\mathscr{L}$, we get that there are $\sigma_1, \ldots, \sigma_k \in \Gamma(S, \mathscr{L})$ such that $S = \cup_i S_{\sigma_i}$ and the S_{σ_i} are affine. Each $\Gamma(S_{\sigma_i}, \mathscr{O}_S)$ is a finitely generated R-algebra. Let $\{\sigma_{ji}\}$ be the image of a generating set in $\Gamma(S_{\sigma_i}, \mathscr{L})$. There exists an n independent of i and $t_{ji} \in \Gamma(S, \mathscr{L}^{\otimes n})$ such that $\sigma_{ji} \otimes \sigma_i^{\otimes n} = t_{ji}|_{S_{\sigma_i}}$. Let Σ be the set of t_{ji}'s and σ_i's and $\psi : \{0, \ldots, r\} \to \Sigma$ an arbitrary enumeration of Σ. The resulting morphism $S \to \mathbb{P}^r_R$ is obtained by gluing together the morphisms

$$S_{\sigma_i} \to \operatorname{Spec}(R[x_0/x_{\psi^{-1}(\sigma_i)}, \ldots, x_r/x_{\psi^{-1}(\sigma_i)}]). \tag{3.25}$$

These maps are closed immersions (the corresponding maps on global sections are surjective) and so the result follows. ■

3.7 Cohomological results

Lemma 3.7.1. *Let $f : X \to Y$ be an affine morphism of schemes and suppose that X is noetherian. Then for all quasi-coherent $\mathscr{O}_X$-modules $\mathscr{F}$, we have $R^k f_*(\mathscr{F}) = 0$ for all $k > 0$.*

Lemma 3.7.2. *Let X be a Noetherian scheme and $i : X \to Y$ be a closed immersion. Then i is an affine morphism.*

Proof. WLOG Y is affine. Note that $i_* : \mathsf{Ab}(X) \to \mathsf{Ab}(Y)$ is an exact functor. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ a flasque quasi-coherent resolution. Then $0 \to i_*\mathscr{F} \to i_*\mathscr{I}^\bullet$ is a flasque quasi-coherent resolution of $i_*\mathscr{F}$. Thus, as Y is affine,

$$0 = H^k(\Gamma(Y, i_*\mathscr{I}^\bullet)) = H^k(\Gamma(X, \mathscr{I}^\bullet)) = H^k(X, \mathscr{F}) \tag{3.26}$$

for $k > 0$. Thus X must be affine. ■

3.8 Flatness

Definition 3.8.1. Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is flat over Y at $x \in X$ if $\mathscr{F}_x$ is flat as an $\mathscr{O}_{Y,f(x)}$-module. We sat that $\mathscr{F}$ is flat over Y if $\mathscr{F}$ is flat at every $x \in X$.

Proposition 3.8.2. *Let $\phi : A \to B$ be a ring homomorphism and M a B-module. $\widetilde{M}$ is flat over* $\operatorname{Spec}(A)$ *iff M is flat as an A-module.*

Proof. $\widetilde{M}$ is flat over $\operatorname{Spec}(A)$ iff $M_{\mathfrak{p}}$ is flat as an $A_{\phi^{-1}(\mathfrak{p})}$ module for all $\mathfrak{p} \in \operatorname{Spec}(B)$. ■

CHAPTER 4

Spectral sequences

4.1 Grothendieck spectral sequence

Thm 4.1.1. *(Grothendieck spectral sequence). Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors and suppose that F sends injective objects to G-acylic objects. Then for A an object in $\mathcal{A}$ there is a spectral sequence $\{E_r(A)\}$ such that*

$$E_2^{p,q}(A) = R^pG(R^qF(A)) \Rightarrow R^{p+q}(GF)(A). \tag{4.1}$$

Corollary 4.1.2. *(Leray spectral sequence). Let $f : X \to Y, g : Y \to Z$ be continuous maps. Then for a sheaf $\mathscr{F}$, there is a E_2 cohomological spectral sequence*

$$R^pg_*(R^qf_*(\mathscr{F})) \Rightarrow R^{p+q}(g \circ f)_*(\mathscr{F}) \tag{4.2}$$

which is functorial in $\mathscr{F}$.

Proof. f_* sends injective sheaves to flabby sheaves, which are g_*-acyclic. ■

CHAPTER 5

Group cohomology

CHAPTER 6

Appendix - Categorical results

6.1 Category theory results

prop:cat_factor **Proposition 6.1.1.** *Let $F \dashv G$ and G be full. Let e be the unit of the adjunction. Then every morphism $x \to Gy$ factors uniquely through $e_x : x \to GFx$.*

Proof. Let α and β denote the forward and backward maps in

$$\mathrm{Hom}(Fx, y) \leftrightarrow \mathrm{Hom}(x, Gy) \tag{6.1}$$

respectively. Let $f : x \to Gy$. Then $f = \alpha(\beta(f))$. But $\alpha(\beta(f)) = G\beta(f) \circ e_x$ so we get existence of a factorisation. For uniqueness, suppose $f = h \circ e_x$. Since G is full there is a $l : Fx \to y$ such that $h = Gl$. So $\alpha(l) = \alpha(\beta(f))$. But α is a bijection so $l = \beta(f)$ and hence $h = G\beta(f)$ which gives uniqueness. ∎

CHAPTER 7

Appendix - Sheaf theoretic results

7.1 Properties of sheaves of rings

Thm 7.1.1. *Let $\mathscr{F}$ be a sheaf of rings on X and $s \in \mathscr{F}(X)$. The following are equivalent:*

1. *s is invertible,*
2. *There exists an open cover $\{U_i\}_i$ of X such that $s|_{U_i}$ is invertible for all i,*
3. *s_x is invertible for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2) Trivial. (2) $\Rightarrow$ (1) Suppose $s|_{U_i}$ is invertible for all i. Then there are $t_i \in \mathscr{F}(U_i)$ such that $t_i s|_{U_i} = 1$. But then, since inverses are unique we must have $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$ since they are both the inverse of $s|_{U_i \cap U_j}$. Thus there is a section $t \in \mathscr{F}(U)$ that restricts to the t_i. Checking locally it follows that $ts = 1$ and so s is invertible. (2) $\Leftrightarrow$ (3) Trivial. ∎

7.2 Locally ringed spaces

Lemma 7.2.1. *Let $(f, f^{\#}), (g, g^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be morphisms of locally ringed spaces. Let $\mathcal{U} = \{U_i\}$ be an open covering of X. If the morphisms agree on the restrictions to the U_i then they are equal.*

Proof. We certainly have $f = g$. The result then follows from sheaf condition (A). ∎

Proposition 7.2.2. *Let X, Y and $\{Z_i\}_i$ be locally ringed spaces together with open immersions $f_i : Z_i \to X, g_i : Z_i \to Y$. Let $\alpha : X \to Y$ be a morphism such that $\alpha \circ f_i = g_i$ for all i and $\alpha : f_i(Z_i) \cap f_j(Z_j) \to g_i(Z_i) \cap g_j(Z_j)$ is an isomorphism for all i, j. Then α is an isomorphism.*

Proof. We have that $\alpha : f_i(Z_i) \to g_i(Z_i)$ is an isomorphism for all i. So we can define inverses $\beta_i : g_i(Z_i) \to f_i(Z_i)$. They agree on overlaps and so they glue to give a global inverse β. ∎

Proposition 7.2.3. *Let $f : X \to Y$ be a morphism of schemes and let $\{U_i\}$ be an open cover of Y such that the restriction of f to a morphism $f^{-1}(U_i) \to U_i$ is an isomorphism for all i. Then f is an isomorphism.*

7.3 Restriction

Remark 7.3.1. Recall from chapter 2 that given $f : X \to Y$ we obtain functors $f_*, \lim_f, f^{-1}$ between $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$. These constructions were themselves functorial and give rise to contra/co-variant functors $\mathsf{Top} \to \mathsf{Set}$. The same also holds for f_*, f^* as functors between $\mathsf{Mod}(X)$ and $\mathsf{Mod}(Y)$.

Thm 7.3.2. *Let $f : X \to Y$ be a continuous map and $U \subseteq X$, $V \subseteq Y$ be open subsets such that $f(U) \subseteq V$. Moreover, let $f|_{U,V}$ denote the map $U \to V$ arising from $f|_U$. Then for $\mathscr{F} \in \mathsf{Sh}(X)$ and $\mathscr{G} \in \mathsf{Sh}(Y)$ we have*

1. $(f^{-1}\mathscr{G})|_U \cong f|_U^{-1}\mathscr{G} \cong f|_{U,V}^{-1}(\mathscr{G}|_V)$
2. $(f_*\mathscr{F})|_V \cong (f|_{U,V})_*(\mathscr{F}|_U)$ *when* $U = f^{-1}(V)$

where are isomorphisms are natural.

Proof.
1. $f|_U = f \circ i_U$ and so we obtain the first isomorphism. $f|_U = i_V \circ f|_{U,V}$ and so we obtain the second isomorphism.
2. Straightforward calculation.

■

7.4 Miscellaneous

Proposition 7.4.1. *Let X be a topological space and $\mathscr{F}$ a sheaf on X. If U, V are disjoint open subsets of X, the $\mathscr{F}(U \cup V) \cong \mathscr{F}(U) \times \mathscr{F}(V)$.*

Proof. Obvious. ■

CHAPTER 8

Appendix - Graded rings and Proj

8.1 Graded rings

8.1.1 General results

Definition 8.1.1. A ($\mathbb{Z}$-)graded ring is a ring S together with a decomposition $S = \bigoplus_{d\in\mathbb{Z}} S_d$ as abelian groups, such that $S_d \cdot S_e \subseteq S_{d+e}$. An ideal $\mathfrak{a}$ of S is called homogeneous if $\mathfrak{a} = \bigoplus_{d\in\mathbb{Z}} \mathfrak{a} \cap S_d$. A map $\phi : S \to T$ between graded rings is called a graded morphism if it is a ring homomorphism and $\phi(S_d) \subseteq T_d$ for all d.

Proposition 8.1.2. *Let S be a graded ring and $\mathfrak{a}$ an ideal of R. The following are equivalent:*

1. *$\mathfrak{a}$ is a homogeneous ideal*
2. *$\mathfrak{a}$ is generated by homogeneous elements*
3. *$a \in \mathfrak{a} \Rightarrow$ the homogeneous parts of a lie in $\mathfrak{a}$.*

Proof. (1) $\Rightarrow$ (2) $\mathfrak{a}$ is generated by $\cup_{d\in\mathbb{Z}}\mathfrak{a} \cap S_d$. (2) $\Rightarrow$ (3) Trivial. (3) $\Rightarrow$ (1) We always have $\mathfrak{a} \supseteq \bigoplus_{d\in\mathbb{Z}} \mathfrak{a} \cap S_d$. Now suppose $a \in \mathfrak{a}$. Then clearly a lies in $\bigoplus_{d\in\mathbb{Z}} \mathfrak{a} \cap S_d$. ■

Proposition 8.1.3. *Let $\phi : S \to T$ be a graded homomorphism and $\mathfrak{a}$ a homogeneous ideal of T. Then $\phi^{-1}(\mathfrak{a})$ is a homogeneous ideal of S.*

Proof. Suppose $\phi(s) \in \mathfrak{a}$. Then $\sum_n \phi(s_n) \in \mathfrak{a}$. But $\mathfrak{a}$ is homogeneous and so $\phi(s_n) \in \mathfrak{a}$ for all n. Thus $s_n \in \phi^{-1}(\mathfrak{a})$ for all n and so $\phi^{-1}(\mathfrak{a})$ is a homogeneous ideal. ■

Proposition 8.1.4. *The sum, product, intersection and radical of homogeneous ideals are homogeneous.*

Proof. The only nontrivial case is showing that if $\mathfrak{a}$ is homogeneous then so is $\mathrm{rad}(\mathfrak{a})$. Let $r = \sum_d r_d \in \mathrm{rad}(\mathfrak{a})$. Then there exists an $n \geq 1$ such that $r^n \in \mathfrak{a}$. Let d_0 be minimal such that $r_{d_0} \neq 0$. Then $r_{d_0}^n$ is the degree nd_0 homogeneous part of r^d. Since $\mathfrak{a}$ is homogeneous, we must have $r_{d_0}^n \in \mathfrak{a}$ and so $r_{d_0} \in \mathrm{rad}(\mathfrak{a})$. Now repeat with $r - r_{d_0}$ to get that the homogeneous parts of r must lie in $\mathrm{rad}(\mathfrak{a})$. ■

Proposition 8.1.5. *Let S be a graded ring and $\mathfrak{a}$ a homogeneous ideal of S. Then $\mathfrak{a}$ is prime iff $fg \in \mathfrak{a} \Rightarrow f \in \mathfrak{a}$ or $g \in \mathfrak{a}$ for all homogeneous $f, g \in S$.*

Proof. ($\Rightarrow$) Trivial. ($\Leftarrow$) Let $f = \sum_d f_d, g = \sum_d g_d \in S$ and suppose $fg \in \mathfrak{a}$. Let d_0 be the minimal d such that $f_d \neq 0$, and similarly define d_0' in terms of g. Then $f_{d_0} g_{d_0'}$ is the $d_0 + d_0'$ homogeneous part of fg and so must lie in $\mathfrak{a}$. Thus either $f_{d_0} \in \mathfrak{a}$ or $g_{d_0'} \in \mathfrak{a}$. If both are in $\mathfrak{a}$, then repeat to $(f - f_{d_0})(g - g_{d_0'}) \in \mathfrak{a}$. Otherwise, one must lie in $\mathfrak{a}$ and the other does not. If $f_{d_0} \in \mathfrak{a}$ and $g_{d_0'} \notin \mathfrak{a}$ then $(f - f_{d_0})g \in \mathfrak{a}$. Let d_1 denote the degree of the homogeneous part of $f - f_{d_0}$ of minimal degree. Then $f_{d_1} g_{d_0'} \in \mathfrak{a}$. But $g_{d_0'} \notin \mathfrak{a}$ and so $f_{d_1} \in \mathfrak{a}$. In the other case argue similarly. Repeating in the manner we obtain that either f of g must lie in $\mathfrak{a}$. ■

Proposition 8.1.6. *Let S be a graded ring and $\mathfrak{a}$ a homogeneous ideal of S. Then $\mathfrak{a}$ is a radical ideal iff $f^n \in \mathfrak{a} \Rightarrow f \in \mathfrak{a}$ for all homogeneous $f \in S$.*

Proof. ($\Rightarrow$) Trivial. ($\Leftarrow$) Let $f \in S$ be such that $f^n \in \mathfrak{a}$. Let d_0 be the minimal d such that $f_d \neq 0$. Then $(f^n)_{dn} = f_{d_0}^n$. Since $\mathfrak{a}$ is homogeneous, this implies that $f_{d_0}^n \in \mathfrak{a}$ and so $f_{d_0} \in \mathfrak{a}$. Repeat to $f - f_{d_0}$. ■

Thm 8.1.7. *(Graded localisation). Let S be a $\mathbb{Z}$-graded ring and T a multiplicative subset of S consisting of homogeneous elements. Then $T^{-1}S$ is a $\mathbb{Z}$-graded ring and the canonical morphism $\phi : S \to T^{-1}S$ is a graded morphism.*

Proof. $T^{-1}S$ is a certainly a ring so we only need to define a grading on it. Let T_k denote the elements of T with grading k and define $(T^{-1}S)_n$ to be the set of elements in $T^{-1}S$ that can be written as s/t for some $s \in S_i, t \in T_j$ with $i - j = n$. We claim that $T^{-1}S = \bigoplus_{n\in\mathbb{Z}}(T^{-1}S)_n$. It is clear that $\sum_n (T^{-1}S)_n = T^{-1}S$. Now suppose there are distinct $n_1, \ldots, n_k \in \mathbb{Z}$ and $x_1 \in (T^{-1}S)_{n_1}, \ldots, x_k \in (T^{-1}S)_{n_k}$ such that $\sum_i x_i = 0$. We can write each x_i as s_i/t_i where $s_i \in S, t_i \in T$ are both homogeneous and $\deg(s_i) - \deg(t_i) = n_i$. WLOG all the t_i are equal to some $t \in T$. Then there is some $u \in T$ such that $u \sum_i s_i = 0$. But each s_i has a different degree and so $us_i = 0$ for all i. It follows that each $x_i = 0$ too. Thus $T^{-1}S = \bigoplus_{n\in\mathbb{Z}}(T^{-1}S)_n$. Finally, the fact that $(T^{-1}S)_n(T^{-1}S)_m \subseteq (T^{-1}S)_{n+m}$ is obvious, and so is the fact that ϕ is a graded morphism. ■

Proposition 8.1.8. *Let $\phi : S \to T^{-1}S$ be the canonical homomorphism.*

1. *Let I be a homogeneous ideal of S. Then $T^{-1}I$ is a homogeneous ideal of $T^{-1}S$.*

2. *Let J be a homogeneous ideal of $T^{-1}S$. Then $\phi^{-1}(J)$ is a homogeneous ideal of S.*

Proof. (1) $T^{-1}I$ is generated by the homogeneous elements of I. (2) ϕ is a graded morphism. ■

Definition 8.1.9. We say a $\mathbb{Z}$-graded ring S is $\mathbb{Z}_{\geq 0}$-graded if $S_d = 0$ for all $d < 0$. For an element $s \in S$ we write $\deg(s)$ for the grading of the homogeneous part of largest grading amongst the non-zero homogeneous parts of s. We write S_+ for the ideal $\bigoplus_{d>0} S_d$. We call S_+ the irrelevant ideal.

Remark 8.1.10. From now on graded ring will refer to a $\mathbb{Z}_{\geq 0}$-graded ring.

Definition 8.1.11. Fix a ring A and let S be a graded ring. If $S_0 = A$ we say that S is a *graded ring over* A. If S_+ is a finitely generated ideal of S, we say that S is a *finitely generated graded ring over* A. If S is generated by S_1 as an A-algebra, we say that S is *generated in degree 1.*

Proposition 8.1.12. *S is a finitely generated graded ring over A iff S is a finitely generated graded A-algebra.*

Proof. ($\Rightarrow$) Suppose $S_+ = (r_1, \dots, r_k)$. WLOG the r_i are all homogeneous. We prove by induction on $\deg(r)$ that r lies in $A[r_1, \dots, r_n]$. If $\deg(r) = 0$ the result is clear. Now suppose r has $\deg(r) > 0$. Write $r = r_a + r_+$ where $r_a \in A$ and $r_+ \in S_+$. Since $r_+ \in S_+$, we can write $r_+ = \sum_i s_i r_i$ for some $s_i \in S$. Since $\deg(r_i) > 0$, $\deg(s_i) < \deg(r_+)$ for all i. But then each $s_i \in A[r_1, \dots, r_n]$ by the induction hypothesis. It is then clear that $r \in A[r_1, \dots, r_n]$.

($\Leftarrow$) Suppose $S = A[r_1, \dots, r_n]$. WLOG $r_i \in S_+$ for all i. Then $(r_1, \dots, r_n) \subseteq S_+$. But $S = A \oplus (r_1, \dots, r_n)$. Thus $(r_1, \dots, r_n) = S_+$. ■

Corollary 8.1.13. *S is Noetherian iff A is Noetherian and S is a finitely generated graded ring.*

Proof. ($\Rightarrow$) If S is Noetherian, then S_+ is finitely generated, so S is a finitely generated graded ring. To see that A must be Noetherian note that $A \cong S/S_+$ is a quotient of a Noetherian ring and so must be Noetherian itself. ($\Leftarrow$) If A is Noetherian and S is a finitely generated graded ring, then S is a finitely generated A-algebra and so must be Noetherian too (by Hilbert's basis theorem + quotients). ■

8.1.2 Graded localisation

Definition 8.1.14. Let S be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal of S. Write $S_{(\mathfrak{p})}$ for the 0^{th} graded component of $T^{-1}S$ where T is the set of homogeneous elements in S not in $\mathfrak{p}$.

For $f \in S_+$ homogeneous write $S_{(f)}$ for the 0^{th} graded component of S_f.

Proposition 8.1.15. *Let S be a graded ring and write $S^{(d)}$ for the subring $\bigoplus_{k \geq 0} S_{kd}$. If $f \in S$ is a homogeneous element of degree d then $S_{(f)} \cong S^{(d)}/(1-f)$.*

Definition 8.1.16. Let $\mathfrak{a} \lhd S_{(f)}$ be a radical ideal. Define $\Psi(\mathfrak{a})_n = \{x \in S_n : x^d/f^n \in \mathfrak{a}\}$ and $\Psi(\mathfrak{a}) = \bigoplus_{n\geq 0} \Psi(\mathfrak{a})_n$.

Lemma 8.1.17. *$\Psi(\mathfrak{a})$ is a homogeneous radical ideal of S. If $\mathfrak{a}$ is additionally prime, then so is $\Psi(\mathfrak{a})$.*

Proof. We first check that $\Psi(\mathfrak{a})$ is an ideal. Let $x, y \in \Psi(\mathfrak{a})_n$. Then x^d/f^n and y^d/f^n lie in $\mathfrak{a}$. Thus $((x+y)^d/f^n)^2 \in \mathfrak{a}$ and so $(x+y)^d/f^n \in \mathfrak{a}$. Let $x \in \Psi(\mathfrak{a})_n$ and $s \in S_k$. Then $x^d/f^n \in \mathfrak{a}$ and so $(sx)^d/f^{k+n} \in \mathfrak{a}$. Thus $\mathfrak{a}$ is an ideal. By definition it must be homogeneous. To check that $\Psi(\mathfrak{a})$ is a radical ideal suppose $x \in S_k$ and $x^n \in \Psi(\mathfrak{a})_{kn}$ for some n. Then $x^{dn}/f^{kn} = (x^d/f^k)^n \in \mathfrak{a}$. But $\mathfrak{a}$ is a radical ideal and so $x^d/f^k \in \mathfrak{a}$ as required.

Finally, suppose $\mathfrak{a}$ is prime. Let $x \in S_m, y \in S_n$ be such that $xy \in \Psi(\mathfrak{a})_{m+n}$. Then $(xy)^d/f^{m+n} = (x^d/f^m)(y^d/f^n) \in \mathfrak{a}$. Thus either $x \in \Psi(\mathfrak{a})_m$ of $y \in \Psi(\mathfrak{a})_n$. ∎

thm:f_loc **Thm 8.1.18.** *Let S be a graded ring and let $f \in S_+$ be a homogeneous element of degree d. Then there are maps Φ and Ψ,*

$$\left\{\mathfrak{b} \lhd S : \ \mathfrak{b} \textit{ is homog. and radical } \right\} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \{\mathfrak{a} \lhd S_{(f)} : \mathfrak{a} \textit{ radical}\} \tag{8.1}$$

such that $\Phi \circ \Psi = \mathrm{id}$ and $\mathfrak{b} \subseteq \Psi \circ \Phi(\mathfrak{b})$. Moreover, these maps restict to a bijection

$$\left\{\mathfrak{q} \lhd S : \begin{array}{c} \mathfrak{q} \textit{ is homog. and} \\ \textit{prime and } \mathfrak{q} \not\ni f \end{array} \right\} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \{\mathfrak{p} \lhd S_{(f)} : \mathfrak{p} \textit{ prime}\}. \tag{8.2}$$

Proof. Define Φ by $\mathfrak{b} \mapsto \mathfrak{b}S_f \cap S_{(f)}$. It is clear that $\Phi(\mathfrak{b})$ is radical (resp. prime) if $\mathfrak{b}$ is radical (resp. prime not containing f). Moreover we have the explicit description

$$\Phi(\mathfrak{b}) = \{x \in S_f : x = b/f^k, b \in \mathfrak{b}_{kd}\} \tag{8.3}$$

(this holds for any ideal $\mathfrak{b}$). Let Ψ be as defined earlier. By the lemma it sends radical (resp. prime) ideals to radical (resp. prime not containing f) ideals.

It is straightforward to check that $\Phi(\Psi(\mathfrak{a})) = \mathfrak{a}$ whenever $\mathfrak{a}$ is a radical ideal. It is also easy to see that $\mathfrak{b} \subseteq \Psi(\Phi(\mathfrak{b}))$ for any ideal $\mathfrak{b}$. It remains to check that $\mathfrak{q} \supseteq \Psi(\Phi(\mathfrak{q}))$ when $\mathfrak{q}$ is a homogeneous prime not containing f. But this is obvious. ∎

Corollary 8.1.19. *We have a bijection*

$$\{\mathfrak{p} \lhd S_{(f)} : \mathfrak{p} \textit{ prime}\} \leftrightarrow \{\mathfrak{q} \lhd S_f : \mathfrak{q} \textit{ homog. and prime}\}. \tag{8.4}$$

Proposition 8.1.20. *Let S be a graded ring, $f \in S_+$ be a homogeneous element of degree d and Φ be as in the theorem. Then for $\mathfrak{q} \lhd S$ a homogeneous prime not containing f we have*

$$S_{(\mathfrak{q})} \cong \left(S_{(f)}\right)_{\Phi(\mathfrak{q})}. \tag{8.5}$$

Proof. Let T be the multiplicative set of homogeneous elements in S not in $\mathfrak{q}$. There is a canonical homomorphism $S \to T^{-1}S$. Since $f \in T$, this induces a homomorphism $S_f \to T^{-1}S$ which preserves the grading. We thus obtain a map $\phi : S_{(f)} \to S_{(\mathfrak{q})}$. Moreover,

$$\begin{aligned} S_{(f)} \backslash \Phi(\mathfrak{q}) &= \{x \in S_f : \exists a \in S_{dk} \backslash q_{dk} \text{ s.t. } x = a/f^k\} \\ &= \{x \in S_{(f)} : x = a/f^k, a \in S_{dk} \Rightarrow a \notin \mathfrak{q}_{dk}\}. \end{aligned} \tag{8.6}$$

and so it follows that $S_{(f)} \backslash \Phi(\mathfrak{q})$ maps into $(S_\mathfrak{q})^\times$ under ϕ. We now wish to show that $S_{(\mathfrak{q})}$ has the required universal property. Let $\psi : S_{(f)} \to R$ be any ring homomorphism such that $S_{(f)} \backslash \Phi(\mathfrak{q})$ gets sent into $R^\times$ and let $x \in S_{(\mathfrak{q})}$. We can write $x = a/t$ where $a \in S_m, t \in T$ where $m := \deg(t)$. Since $T \cap \mathfrak{q} = \emptyset$, $t^d \notin \mathfrak{q}$ and so $t^d/f^m \in S_{(f)} \backslash \Phi(\mathfrak{q})$. We thus define $\eta : S_{(\mathfrak{q})} \to R$ by

$$\eta(x) = \phi(at^{d-1}/f^m) \cdot \phi(t^d/f^m)^{-1}. \tag{8.7}$$

This is well defined: if $x = a'/t'$ write m' for the degree of t'. There exists $u \in T$ such that $u(t'a - ta') = 0$. Write k for the degree of u. Then

$$\frac{u^d t^{d-1} t'^d a}{f^{k+m+m'}} = \frac{u^d t'^{d-1} t^d a'}{f^{k+m+m'}} \tag{8.8}$$

and so since $u^d/f^k \in S_{(f)} \backslash \Phi(\mathfrak{q})$,

$$\phi\left(\frac{t^{d-1}a}{f^m}\right) \phi\left(\frac{t^d}{f^m}\right)^{-1} = \phi\left(\frac{t^{d'-1}a'}{f^{m'}}\right) \phi\left(\frac{t'^d}{f^{m'}}\right)^{-1} \tag{8.9}$$

as required. It is then straightforward to check that η is a ring homomorphism and that $\phi = \eta \circ \psi$. Uniqueness follows from that fact that $x = (at^{d-1}/f^m) \cdot (t^d/f^m)^{-1}$ in $S_{(\mathfrak{q})}$. ■

Corollary 8.1.21. *$S_{(\mathfrak{p})}$ is a local ring with maximal ideal $\mathfrak{q} \cdot (T^{-1}S) \cap S_{(\mathfrak{q})}$.*

8.1.3 Miscellaneous

Definition 8.1.22. Let S be a graded ring and let I be an ideal of S. Write I^h for the ideal generated by the homogeneous elements of I.

Proposition 8.1.23. *If $\mathfrak{p}$ is a prime ideal of S. Then $\mathfrak{p}^h$ is also prime.*

Proof. Let $a, b \in S$ be homogeneous. If $ab \in \mathfrak{p}^h \subseteq \mathfrak{p}$ then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. But since a, b are homogeneous either $a \in \mathfrak{p}^h$ or $b \in \mathfrak{p}^h$. ■

Corollary 8.1.24. *Let S be a graded ring. Then for a homogeneous ideal I, we have*

$$rad(I) = \bigcap_{\substack{\mathfrak{p} \supseteq I, \\ homog}} \mathfrak{p}. \tag{8.10}$$

Proof. We certainly have $(\subseteq)$. $(\supseteq)$ follows from the fact that if $\mathfrak{a} \subseteq \mathfrak{p}$ then $\mathfrak{a} \subseteq \mathfrak{p}^h \subseteq \mathfrak{p}$. ■

8.2 The Proj construction

Definition 8.2.1. Let S be a graded ring. Define $\mathrm{Proj}(S)$ to be the set of all homogeneous prime ideals of S which do not contain S_+. If $\mathfrak{a}$ is a homogeneous ideal of S we define $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) : \mathfrak{p} \supseteq \mathfrak{a}\}$. As before these sets define a topology on $\mathrm{Proj}(S)$.

Remark 8.2.2. Note that we still have $V(I) = V(\mathrm{rad}(I))$.

Definition 8.2.3. (Basic open sets). Let $f \in S_+$ be a homogeneous element of S. Define $D_+(f) = \mathrm{Proj}(S) \backslash V((f))$.

Proposition 8.2.4. *The basic open sets form a base of the topology on* $\mathrm{Proj}(S)$.

Proof. Clearly $D_+(f) \cap D_+(g) = D_+(fg)$. Finally note that

$$V(I) = V(I \cap S_+) = \bigcap_{\substack{f \in I \cap S_+ \\ \text{homog}}} V((f)) \tag{8.11}$$

and so $\mathrm{Proj}(S) \backslash V(I) = \bigcup_{\substack{f \in I \cap S_+ \\ \text{homog}}} D_+(f)$. ∎

Proposition 8.2.5. *Let S be a graded ring and I a homogeneous ideal of S. Then $V(I) = \emptyset$ iff rad$(I) \supseteq S_+$.*

Proof. ($\Leftarrow$) Follows from $V(I) = V(\mathrm{rad}(I))$. ($\Rightarrow$) $V(I) = \emptyset$. Thus $I \subseteq \mathfrak{p}$ implies $S_+ \subseteq \mathfrak{p}$. But then $S_+ \subseteq \mathrm{rad}(I)$. ∎

Lemma 8.2.6. *Let $f \in S_+$ be a homogeneous element and let $\Phi : D_+(f) \to$* $\mathrm{Spec}(S_{(f)})$ *be the map from theorem 8.1.18. Then Φ is a homeomorphism.*

Proof. Let C be a closed subset of $\mathrm{Spec}(S_{(f)})$. Then $C = V(\mathfrak{a})$ for some radical ideal $\mathfrak{a} \lhd S_{(f)}$. It is easy to see from theorem 8.1.18 that $\Phi^{-1}(C) = V(\Psi(\mathfrak{a})) \cap D_+(f)$. Similarly, any closed subset of $D_+(f)$ is of the form $D_+(f) \cap V(\mathfrak{b})$ for some homogeneous radical ideal $\mathfrak{b}$ and $\Phi(D_+(f) \cap V(\mathfrak{b})) = V(\Phi(\mathfrak{b}))$. ∎

Remark 8.2.7. Let $g = s/f^k \in S_{(f)}$. Then $\mathfrak{q} \in \Phi^{-1}(V(g))$ iff $\Phi(\mathfrak{q}) \supseteq (g)$ iff $s \in \mathfrak{q}$ iff $\mathfrak{q} \in D_+(f) \cap V((s))$. Thus $\Phi^{-1}(D(g)) = D_+(fs)$.

Definition 8.2.8. We can turn $\mathrm{Proj}(S)$ into a locally ringed space by defining the structure sheaf to be

$$\mathscr{O}_{\mathrm{Proj}(S)}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in \mathrm{Proj}(S)} S_{(\mathfrak{p})} : \begin{array}{c} \text{for each } \mathfrak{p} \in U \text{ there exists an} \\ \text{open neighbourhood } V \text{ of } \mathfrak{p} \text{ in } U, \\ \text{and homogeneous elements } a, f \in S \\ \text{of the same degree such that for all} \\ \mathfrak{q} \in V,\ f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \text{ in } S_{(\mathfrak{q})}. \end{array} \right\}. \tag{8.12}$$

Thm 8.2.9. *Let S be a graded ring.*

1. *For any* $\mathfrak{p} \in \operatorname{Proj}(S)$, $\mathscr{O}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$.

2. *For any homogeneous* $f \in S_+$, $(D_+(f), \mathscr{O}|_{D_+(f)}) \cong \operatorname{Spec}(S_{(f)})$.

3. $\operatorname{Proj}(S)$ *is a scheme.*

Proof. (1) Trivial. Note that this means that $\operatorname{Proj}(S)$ is a locally ringed space. (2) We have from the previous lemma that the underlying topological spaces are homeomorphic. We have also established that $S_{(\mathfrak{q})} \cong (S_{(f)})_{\Phi(\mathfrak{q})}$. Using this isomorphism we can construct a map $D_+(f) \to \operatorname{Spec}(S_{(f)})$ which is an isomorphism on stalks. It follows that they must be isomorphisms as locally ringed spaces. (3) then follows. ■

CHAPTER 9

Appendix - Scheme theoretic results

9.1 Basic open sets

Proposition 9.1.1. *Let X be a scheme and $f \in \Gamma(X, \mathscr{O}_X)$. Then for affine $U \subseteq X$, $X_f \cap U = U_{f|_U}$ is a basic open set.*

Proof. It suffice to prove that for $X = \operatorname{Spec}(R)$ and $r \in R$, $X_r = D_r(R)$. But

$$X_r = \{\mathfrak{p} \lhd R : r/1 \notin \mathfrak{p}_\mathfrak{p}\} = \{\mathfrak{p} \lhd R : r \notin \mathfrak{p}\} = D_r(R). \tag{9.1}$$

■

Proposition 9.1.2. *Let $(f, f^\#) : X \to Y$ be a morphism of schemes and $r \in \Gamma(Y, \mathscr{O}_Y)$. Then $f^{-1}(Y_r) = X_{f^\#(Y)(r)}$.*

Proof. Recall that $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a morphism of local rings. Thus $r_{f(x)} \in \mathfrak{m}_{f(x)}$ iff $f_x^\#(r_{f(x)}) \in \mathfrak{m}_x$. But $f_x^\#(r_{f(x)}) = f^\#(Y)(r)_x$ and so

$$\begin{aligned} f^{-1}(Y_r) &= \{x \in X : r_{f(x)} \notin \mathfrak{m}_{f(x)}\} \\ &= \{x \in X : f^\#(Y)(r)_x \notin \mathfrak{m}_x\} = X_{f^\#(Y)(r)}. \end{aligned} \tag{9.2}$$

■

Proposition 9.1.3. *Let X be a scheme and U, V be open affine subsets. Then there exists a cover of $U \cap V$ consisting of sets which are basic with respect to both U and V.*

Proof. Let $x \in U \cap V$. Then there is a $f \in \mathscr{O}_X(U)$ such that $x \in U_f \subseteq U \cap V$. Let $g \in \mathscr{O}_X(V)$ be such that $x \in V_g \subseteq U_f$. Then $(U_f)_{g|_{U_f}} = V_g$, both of which are basic with respect to U and V respectively. ■

Lemma 9.1.4. *(The Affine Communication Lemma). Let P be some property enjoyed by some affine open subsets of a scheme X such that*

1. *if an affine open subset* $\operatorname{Spec}(A) \hookrightarrow X$ *has property* P*, then for any* $f \in A$*,* $\operatorname{Spec}(A_f) \hookrightarrow X$ *does too*

2. *if* $(f_1, \dots, f_n) = A$ *and* $\operatorname{Spec}(A_{f_i}) \hookrightarrow X$ *has* P *for all* i*, then so does* $\operatorname{Spec}(A) \hookrightarrow X$*.*

Suppose that $X = \cup_{i \in I} \operatorname{Spec}(A_i)$ *where* $\operatorname{Spec}(A_i)$ *has property* P*. Then every affine open subset of* X *has* P *too.*

Definition 9.1.5. We call such a property an affine-local property.

9.2 Quasi-separated schemes

Definition 9.2.1. We say a topological space X is quasi-separated if the intersection of any two quasi-compact open subsets is quasi-compact.

Thm 9.2.2. *Let* X *be a scheme. The following are equivalent:*

1. X *is quasi-separated*

2. *The intersection of any two affine open subsets is a finite union of affine open subsets*

3. *There exists an open cover* $\{V_i\}_i$ *such that* $V_i \cap V_j$ *is a finite union of affine open subsets for any* i, j*.*

Proof. (1) $\Rightarrow$ (2) Obvious. (2) $\Rightarrow$ (3) Obvious. (3) $\Rightarrow$ (1) Let U, V be quasi-compact open subsets of X. The inclusion maps $V_i \hookrightarrow X$ are all quasi-compact and so $V_i \cap U$ is quasi-compact for all i. Thus $U \hookrightarrow X$ is quasi-compact and so $U \cap V$ is quasi-compact. ∎

Corollary 9.2.3. *Affine schemes are quasi-separated.*

Corollary 9.2.4. *A scheme* X *is quasi-compact and quasi-separated iff* X *can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.*

prop:aff_cover_aff

Proposition 9.2.5. *Let* X *be a quasi-compact and quasi-separated scheme and* $f_1, \dots, f_k \in \Gamma(X, \mathscr{O}_X)$ *be such that* $(f_1, \dots, f_k) = \Gamma(X, \mathscr{O}_X)$*. If* X_{f_i} *is affine for all* i*, then* X *is affine.*

Proof. Identical to the noetherian case. ∎

9.3 Spec adjunction

Thm 9.3.1. *Let $(X, \mathscr{O}_X)$ be a scheme and A a ring. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Sch}}(X, \mathrm{Spec}(A)) \leftrightarrow \mathrm{Hom}_{\mathsf{Ring}}(A, \Gamma(X, \mathscr{O}_X)). \tag{9.3}$$

In other words $\Gamma \dashv \mathrm{Spec}$ *as functors between* Sch *and* Ring^{op}.

Proof. Given a morphism $(f, f^\#) : (X, \mathscr{O}_X) \to (\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ we obtain map $A \to \Gamma(X, \mathscr{O}_X)$ from $f^\#(\mathrm{Spec}(A))$.

Conversely, suppose we have $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For an affine $U \subseteq X$, we have the map $A \to \Gamma(X, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$, and we thus obtain a map $U \to \mathrm{Spec}(A)$. Let $U, V \subseteq X$ be affine and $W \subseteq U \cap V$ also be affine. The following diagram commutes

$$\begin{tikzcd} & & \Gamma(U, \mathscr{O}_X) \arrow[dr] & \\ A \arrow[r] & \Gamma(X, \mathscr{O}_X) \arrow[ur] \arrow[rr] \arrow[dr] & & \Gamma(W, \mathscr{O}_X) \\ & & \Gamma(V, \mathscr{O}_X) \arrow[ur] & \end{tikzcd} \tag{9.4}$$

and so

$$\begin{tikzcd} & U \arrow[dl] & \\ \mathrm{Spec}(A) & & W \arrow[ll] \arrow[ul] \arrow[dl] \\ & V \arrow[ul] & \end{tikzcd} \tag{9.5}$$

also commutes. So the morphisms agree on overlaps and so can be glued to get a morphism $X \to \mathrm{Spec}(A)$.

It is straightforward to check that this defines a bijection. ■

Corollary 9.3.2. *Let $(X, \mathscr{O}_X)$ be a scheme. There is a canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathscr{O}_X)$ such that every morphism from X to an affine scheme factors through this map uniquely.*

Proof. This follows from proposition 6.1.1. ■

9.4 Sheaf of ideals

Definition 9.4.1. Let $\mathscr{F}$ be a sheaf on X. Then $\mathrm{supp}(\mathscr{F}) = \{x \in X : \mathscr{F}_x \neq 0\}$.

prop:supp **Proposition 9.4.2.** *If $\mathscr{F}$ is a finitely generated $\mathscr{O}_X$-module then supp(X) is a closed subset of X.*

Definition 9.4.3. A subsheaf of $\mathscr{O}_X$ is called a *sheaf of ideals* on X.

Definition 9.4.4. Let $\mathscr{J}$ be a sheaf of ideals on X. Let $Z = \operatorname{supp}(\mathscr{O}_X/\mathscr{J})$. By proposition 9.4.2, Z is a closed subset of X. Let $i : Z \to X$ be the inclusion map. Then we define the structure sheaf on X to be $\mathscr{O}_Z = i^{-1}(\mathscr{O}_X/\mathscr{J})$. This turns Z into a locally ringed space.

Proposition 9.4.5. *$i_*\mathscr{O}_Z \cong \mathscr{O}_X/\mathscr{J}$.*

Proof. There is a natural map $\mathscr{O}_X/\mathscr{J} \to i_*\mathscr{O}_Z = i_*i^{-1}(\mathscr{O}_X/\mathscr{J})$ arising from the inverse image-direct image adjunction. Looking at stalks shows that this is an isomorphism. ∎

Remark 9.4.6. In particular there is a natural map $i^\# : \mathscr{O}_X \to i_*\mathscr{O}_Z$ given by the composition $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J} \to i_*\mathscr{O}_Z$ inducing a morphism $(i, i^\#)$ of locally ringed spaces.

Corollary 9.4.7. *The map $(i, i^\#) : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ is a closed immersion and $\mathscr{J} = \ker(i^\#)$.*

Lemma 9.4.8. *Let A be a ring and $I \lhd A$ be an ideal. Then the sheaf $(A/I)^\sim$ on $\operatorname{Spec}(A)$ has support $V(I)$.*

Proof. Consider the following exact sequence of A-modules

$$0 \to I \to A \to A/I \to 0. \tag{9.6}$$

If $I \not\subseteq \mathfrak{p}$ then $IA_\mathfrak{p} = A_\mathfrak{p}$ and so $(A/I)_\mathfrak{p} = 0$. If $I \subseteq \mathfrak{p}$ then $(A/I)_\mathfrak{p} \cong (A/I)_\mathfrak{q}$ where $\mathfrak{q} = \mathfrak{p}/I$ and so is in particular not 0. ∎

thm:sheaf_of_ideals **Thm 9.4.9.** *If $\mathscr{J}$ is quasi-coherent then $(Z, \mathscr{O}_Z)$ is a scheme and for any affine piece $(U, \mathscr{O}_X|_U) \cong (\operatorname{Spec}(A), \mathscr{O}_{\operatorname{Spec}(A)})$ of X, $(Z \cap U, \mathscr{O}_Z|_{Z\cap U})$ is isomorphic to $(\operatorname{Spec}(A/I), \mathscr{O}_{\operatorname{Spec}(A/I)})$ where I is the ideal of A corresponding to $J(U)$.*

Proof. It suffices to show the second part of the theorem. Let $(U, \mathscr{O}_X|_U) \cong (\operatorname{Spec}(A), \mathscr{O}_{\operatorname{Spec}(A)})$ be an affine piece of X. Restricting the short exact sequence $0 \to \mathscr{J} \to \mathscr{O}_X \to i_*\mathscr{O}_Z \to 0$ to U we get

$$0 \to \mathscr{J}|_U \to \mathscr{O}_X|_U \to (i_{U\cap Z})_*(\mathscr{O}_Z|_{U\cap Z}) \to 0. \tag{9.7}$$

It follows that

$$(i_{U\cap Z})_*(\mathscr{O}_Z|_{U\cap Z}) \cong (A/I)^\sim \cong \operatorname{Spec}(\phi)_*\mathscr{O}_{\operatorname{Spec}(A/I)} \tag{9.8}$$ eq:sh_isoms

where $\phi : A \to A/I$ is the quotient map. By the lemma $U \cap Z = V(I)$ and so there is a homeomorphism $\psi : \operatorname{Spec}(A/I) \to U \cap Z$. Since both $i_{U\cap Z}$ and $\operatorname{Spec}(\phi)$ are homeomorphisms onto their images, the isomorphisms in equation 9.8 induce isomorphisms of sheaves. Taking stalks moreover shows that we get an isomorphism of locally ringed spaces as required. ∎

thm:ideal-subsch **Thm 9.4.10.** *There is a bijection*

$$\{\textit{quasi-coherent ideal sheaves of } \mathscr{O}_X\} \leftrightarrow \{\textit{closed subschemes of } X\}. \tag{9.9}$$

Proof. Given a quasi-coherent ideal sheaf $\mathscr{I}$ of $\mathscr{O}_X$ we obatin a closed embedding $Z \to X$. Conversely, given a closed embedding $i : Z \to X$ we obatin an ideal sheaf from the kernel of $\mathscr{O}_X \to i_*\mathscr{O}_Z$. We need only show that any such kernel is quasi-coherent since we have already showed that the maps are mutually inverse. Let $\mathscr{I}$ denote the kernel of $i^\#$ and let $U = \mathrm{Spec}(A)$ be an affine piece of X. i is an affine morphism so $W = i^{-1}(U)$ is also affine and so $i|_W : W \to U$ is given by $\mathrm{Spec}(A \to A/I)$ for some ideal I of A. Thus the exact sequence

$$0 \to \mathscr{I}|_U \to \mathscr{O}_X|_U \to i|_{W*}\mathscr{O}_Z|_W \to 0 \tag{9.10}$$

becomes

$$0 \to \mathscr{I}|_U \to \mathscr{O}_{\mathrm{Spec}(A)} \to (A/I)^\sim \to 0 \tag{9.11}$$

and so $\mathscr{I}|_U = \widetilde{I}$ and so $\mathscr{I}$ is quasi-coherent. ■

Remark 9.4.11. Let Z be a closed subset of X. Then the bijection specialises to

$$\left\{ \begin{array}{c} \text{q-c ideal sheafs } \mathscr{I} \text{ such} \\ \text{that } \mathrm{supp}(\mathscr{O}_X/\mathscr{I}) = Z \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{closed subschemes w/ underlying} \\ \text{topological space } Z \end{array} \right\}. \tag{9.12}$$

9.5 Reduced schemes

Definition 9.5.1. A scheme $(X, \mathscr{O}_X)$ is reduced if $\mathscr{O}_X(U)$ is reduced for all $U \subseteq X$ open.

Lemma 9.5.2. *$(X, \mathscr{O}_X)$ is reduced iff $\mathscr{O}_{X,p}$ is reduced for all $p \in X$.*

Lemma 9.5.3. *Let $\mathscr{J}$ be the ideal sheaf of $\mathscr{O}_X$ given by $U \mapsto N(\mathscr{O}_X)$. Then $\mathscr{J}$ is quasi-coherent.*

Proof. It suffices to show that $\mathscr{J} \cong N(\mathscr{O}_X(X))^\sim$ when X is affine. But we have an isomorphism on the basis and hence between sheaves. ■

Definition 9.5.4. Let $(X, \mathscr{O}_X)$ be a scheme. We define $(X_{red}, (\mathscr{O}_X)_{red})$ to be the scheme associated with the sheaf of ideals $\mathscr{J}$ given by $\mathscr{J}(U) = N(\mathscr{O}_X(U))$. Let $(z, z^\#) : (X_{red}, (\mathscr{O}_X)_{red}) \to (X, \mathscr{O}_X)$ be the associated closed embedding.

Remark 9.5.5. X_{red} is reduced since it is reduced on affine pieces.

Proposition 9.5.6. *z is a homeomorphism.*

Proof. It suffices to check that $\mathrm{supp}(\mathscr{O}_X/\mathscr{J}) = X$ for affine X. Let $\phi : R \to R/N(R)$ be the quotient map. Then $\mathrm{Spec}(\phi)$ is a homeomorphism. It follows that $\mathrm{supp}(\mathscr{O}_X/\mathscr{J}) = X$ and so z is the identity map. ■

Thm 9.5.7. *Let $f : X \to Y$ be a morphism of schemes and suppose X is reduced. Then f factors through Y_{red}.*

Proof. Universal property of cokernels. ■

Definition 9.5.8. For an affine scheme X, let $I(Z)$ be the radical ideal corresponding to a closed set $Z \subset X$. For a general scheme X and a closed subset $Z \subseteq X$, let $\mathscr{I}_Z$ be the sheaf

$$\mathscr{I}_Z(U) = \{f \in \mathscr{O}_X(U) : f_x \in m_x, \forall x \in U \cap Z\}. \tag{9.13}$$

Lemma 9.5.9. *Let X be an affine scheme and $Z \subseteq X$ a closed subset. Then $\mathscr{I}_Z \cong \widetilde{I(Z)}$.*

Proof. This holds on global sections and rad commutes with localisation. ■

Thm 9.5.10. *Let X be a scheme and $Z \subseteq X$ a closed subset. Then there is a unique quasi-coherent ideal $\mathscr{I}$ such that the associated closed immersion $Z' \to X$ has image Z and Z' reduced.*

Proof. $\mathscr{I} = \mathscr{I}_Z$ is quasi-coherent and the associated embedding has image Z. It is clear that Z' is reduced (check on affine pieces). It thus remains to check the uniqueness of $\mathscr{I}$. For this it suffices to consider the affine case. Let $X = \operatorname{Spec}(A)$ and $\mathscr{I} = \widetilde{I}$. Then $Z' = \operatorname{Spec}(A/I)$ and $V(I) = Z$. But Z' is reduced iff $I = I(Z)$. Thus $\mathscr{I} = \mathscr{I}_Z$. ■

Remark 9.5.11. If we take $Z = X$ then $Z' = X_{red}$.

Remark 9.5.12. If X is a Noetherian scheme then for any affine $U = \operatorname{Spec}(R)$, we have that $\mathscr{I}_Z(U)$ is an ideal of R and so is finitely generated. It follows that all ideal sheaves on Noetherian schemes are coherent.

Proposition 9.5.13. *Let Z be a closed subset of X and $i : Z' \to X$ the corresponding reduced scheme. If $j : Z'' \to X$ is any closed immersion with image Z, then i factors through j.*

Proof. By theorem 9.4.10 it suffices to show that $\mathscr{I}_Z$ is the largest ideal with $\operatorname{supp}(\mathscr{O}_X/\mathscr{I}) = Z$. But any such sheaf $\mathscr{I}$ must have $\mathscr{I}_x \subseteq \mathfrak{m}_x$ for $x \in Z$. It follows that $\mathscr{I} \subseteq \mathscr{I}_Z$ as required. ■

Proposition 9.5.14. *Let $f : Y \to X$ be a morphism. Then there exists a unique smallest closed subscheme which f factors through.*

9.6 Fibre products

Proposition 9.6.1. *Let $f : X \to S$ and $g : S' \to S$ be morphisms, $U \subseteq S$ and $V \subseteq g^{-1}(U)$. Then $X \times_S V = f^{-1}(U) \times_U V$.*

9.7 Presheaves on the category of schemes

Definition 9.7.1. Let $F : \mathsf{Sch}^{op} \to \mathsf{Set}$ be a functor. We call F locally sheafy if for any scheme X, $F|_{\mathsf{Top}(X)}$ is a sheaf of sets.

Thm 9.7.2. *Let $F, G : \mathsf{Sch}^{op} \to \mathsf{Set}$ be locally sheafy functors and suppose there is a natural transformation $\eta : F|_{\mathsf{Aff}^{op}} \Rightarrow G|_{\mathsf{Aff}^{op}}$. Then there is a unique natural transformation $\zeta : F \Rightarrow G$ such that $\zeta|_{\mathsf{Aff}} = \eta$.*

Proof. Let X be a scheme and $s \in F(X)$. We wish to define $\zeta_X(s) \in G(X)$. For each affine piece U of X, define $t_U = \eta_U(s|_U) \in G(U)$. Given any two affine pieces U and V we have $t_U|_{U\cap V} = \eta_{U\cap V}(s|_{U\cap V}) = t_V|_{U\cap V}$. Since the union of all affine pieces of X is X we obtain an element $t \in G(X)$ such that $t|_U = t_U$ for all affine $U \subseteq X$. Define $\zeta_X(s) = t$. Note that if X was already affine then $\zeta_X = \eta_X$. We claim that ζ is a natural transformation.

Let X, Y be schemes and $f : X \to Y$ a morphism (in Sch). Let $U \subseteq Y$ and $V \subseteq f^{-1}(U) \subseteq X$ be affine pieces and $f|_{V,U} : V \to U$ denote the map such that $f \circ i_V = i_U \circ f|_{V,U}$. Then we know that

$$
\begin{tikzcd}
F(U) \arrow[rrr, "\eta_U"] \arrow[ddd, "Ff_{V,U}"'] & & & G(U) \arrow[ddd, "Ff_{V,U}"] \\
& F(Y) \arrow[ul] \arrow[r, "\zeta_Y"] \arrow[d, "Ff"] & G(Y) \arrow[ur] \arrow[d, "Gf"] & \\
& F(X) \arrow[dl] \arrow[r, "\zeta_X"] & G(X) \arrow[dr] & \\
F(V) \arrow[rrr, "\eta_V"] & & & G(V)
\end{tikzcd}
\tag{9.14}
$$

commutes except for the middle square. Thus $G(i_V) \circ (Gf \circ \zeta_Y) = G(i_V) \circ (\zeta_X \circ Ff)$. But we can vary the U and V so that the V cover X. It follows that $Gf \circ \zeta_Y = \zeta_X \circ Gf$. Thus ζ is a natural transformation.

To see that ζ is unique, suppose $\xi : F \Rightarrow G$ is another natural transformation extending η. Then let $s \in F(X)$ and $U \subseteq X$ be an affine piece. We must have $G(i_U) \circ \zeta_X(s) = \eta_U \circ F(i_U) = G(i_U) \circ \xi_X(s)$. But we can vary U to cover X and so we must have $\zeta_X(s) = \xi_X(s)$ for all $s \in F(X)$ and hence $\zeta_X = \xi_X$ for all X and hence $\zeta = \xi$. ∎

Corollary 9.7.3. *Let $F, G : \mathsf{Sch}^{op} \to \mathsf{Set}$ be locally sheafy functors such that $F|_{\mathsf{Aff}^{op}} \cong G|_{\mathsf{Aff}^{op}}$. Then $F \cong G$.*

Conjecture 9.7.4. *There is an equivalence of categories between locally sheafy presheafs on Sch and locally sheafy presheafs on Aff.*

Proof. Given $F : \mathsf{Aff}^{op} \to \mathsf{Set}$ define $\widetilde{F} : \mathsf{Sch}^{op} \to \mathsf{Set}$ by $X \mapsto \varprojlim_{U\subseteq X} F(U)$ where U ranges over affine subsets of X and send morphisms to the obvious things. ∎

CHAPTER 10

Appendix - Vector Bundles

Proposition 10.0.1. *Let $\pi : E \to X$ be a vector bundle of rank n with trivialisation $\{U_i\}_i$ and transition functions $\{\psi_{ji}\}_{ji}$.*

1. *If $C \subseteq X$ is a closed subset of X then $\pi : \pi^{-1}(C) \to C$ is a vector bundle of rank n with trivialisation $\{U_i \cap C\}_i$ and transition functions $\{\psi_{ji}|_{U_{ij} \cap C}\}$.*
2. *If Z is a topological space then $\pi' = \mathrm{id} \times \pi : Z \times E \to Z \times X$ is a vector bundle with trivialisation $\{Z \times U_i\}_i$ and transition functions $\{\psi'_{ji}(z, u) = \psi_{ji}(u)\}$.*

Corollary 10.0.2. *Let $\pi : E \to Y$ be a vector bundle of rank n with trivialisation $\{U_i\}$ and transition functions $\{\psi_{ji}(u)\}_{ji}$. If $f : X \to Y$ is a continuous map then $\pi' : f^*E \to X$ is a vector bundle of rank n with trivialisation $\{f^{-1}(U_i)\}_i$ and transition functions $\{\psi_{ji}(f(v))\}_{ji}$.*

Proof. f^*E is the vector bundle arising from the closed subset of $X \times E \to X \times Y$ given by $G = \{(x, f(x)) : x \in X\}$. But there is a homeomorphism $X \leftrightarrow G$ which descends to $f^{-1}(U_i) \leftrightarrow (X \times U_i) \cap G$. This gives the required trivialisations. It also follows that the transition functions are of the required form. ∎