Homological algebra

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CHAPTER 1

Abelian Categories

1.1 Additive categories

Let $\mathcal{A}$ be a category such that the hom-sets carry the structure of an abelian group and composition is bilinear. We call such a category Ab-enriched. An additive category is an Ab-enriched category which has finite coproducts and a zero object.

thm:atosa **Thm 1.1.1.** *Let $\mathcal{A}$ be an additive category. Then finite coproducts in $\mathcal{A}$ are in fact finite biproducts.*

Proof. It is easy to see that initial objects are isomorphic to terminal objects (and they both exist) and so it suffices to show the result for binary coproducts. Let $A, B \in \mathcal{A}$. Define $p_A : A \coprod B \to A$ and $p_B : A \coprod B \to B$ as the maps making the following diagrams commute.

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\
& A \coprod B \arrow[r, "p_A"] & A \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "0"] & & \\
& A \coprod B \arrow[r] & B \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "\mathrm{id}_B"'] & &
\end{tikzcd}
\tag{1.1}
$$

Let $f = i_A \circ p_A + i_B \circ p_B$. Then

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "i_A"] & & \\
& A \coprod B \arrow[r, "f"] & A \coprod B \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "i_B"'] & &
\end{tikzcd}
\tag{1.2}
$$

commutes and so by universality we must have $f = \mathrm{id}_{A \coprod B}$. Now suppose we have maps $f : C \to A$ and $g : C \to B$. Let $h : C \to A \coprod B$ be the map $i_A \circ f + i_B \circ g$. Then $p_A \circ h = f$ and $p_B \circ h = g$. Moreover, if $h' : C \to A \coprod B$ is any other map satisfying $p_A \circ h' = f$ and $p_B \circ h' = g$ then $h' = id_{A \coprod B} \circ h' = i_A \circ f + i_B \circ g = h$ and so $A \coprod B$ is a biproduct. ■

A functor between additive categories is called additive if it is a homomorphism on hom-sets.

1.2 Semiadditive categories

The above definition of an additive category includes the additive structure on the hom-sets as data. In this section we provide a definition where the additive structure arises as a property instead.

Let $\mathcal{A}$ be a category with a zero object. Recall that in such a category there always exists a morphism between to any two objects $A, B \in \mathcal{A}$ given by $A \to 0 \to B$. We call this the 0 morphism. Moreover if finite coproducts and finite products exists there is a canonical map $A \coprod B \to A \prod B$ arising from the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "0" near start] & A \\ B \arrow[r, "0", "\mathrm{id}_B"'] \arrow[ur] & B. \end{tikzcd} \tag{1.3}$$

We call a category $\mathcal{A}$ *semiadditive* if it has a zero object, finite products, finite coproducts and the canonical map $A \coprod B \to A \prod B$ is an isomorphism for all $A, B \in \mathcal{A}$. In such a category we write $A \oplus B$ for the biproduct.

Thm 1.2.1. *Let $\mathcal{A}$ be a semiadditive category then it is naturally enriched over the monoidal category of commutative monoids.*

Proof. Let $\Delta_A : A \oplus A \to A$ and $\nabla_A : A \to A \oplus A$ be the maps that make

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\ & A \oplus A \arrow[r, "\Delta_A"] & A \\ A \arrow[ur, "i'_A"] \arrow[urr, bend right, "\mathrm{id}_A"'] & & \end{tikzcd} \qquad \begin{tikzcd} & & A \\ A \arrow[r, "\nabla_A"] \arrow[urr, bend left, "\mathrm{id}_A"] \arrow[drr, bend right, "\mathrm{id}_A"'] & A \oplus A \arrow[ur, "p_A"'] \arrow[dr, "p'_A"] & \\ & & A \end{tikzcd} \tag{1.4}$$

commute. Given $f, g : A \to B$ we can construct a map $f \oplus g : A \oplus A \to B \oplus B$ in the obvious way. We can then define $f + g : A \to B$ to be the composite

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.5}$$

Note that there is a map $t_A : A \oplus A \to A \oplus A$ arising from the diagram

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[dr, "\mathrm{id}_A" near start] & A \\ A \arrow[r, "0"'] \arrow[ur, "\mathrm{id}_A"' near start] & A. \end{tikzcd} \tag{1.6}$$

It is then an easy check to see that $\Delta_A \circ t_A = \Delta_A$ and $t_A \circ \nabla_A = \nabla_A$, from which it follows that $+$ is commutative. Straightforward calculations also show that $+$ is associative, distributes over compositions and has the zero map as identity. The result follows. ■

A functor between semiadditive categories is called semiadditive if it preserves zero objects and biproducts i.e. there are isomorphisms $F(A \oplus B) \cong F(A) \oplus F(B)$ such that

$$\begin{tikzcd} F(A) \arrow[dr, "F(i_A)"] \arrow[drr, bend left, "i_{F(A)}"] & & \\ & F(A \oplus B) \arrow[r, "\cong"] & F(A) \oplus F(B) \\ F(B) \arrow[ur, "F(i_B)"] \arrow[urr, bend right, "i_{F(B)}"'] & & \end{tikzcd} \tag{1.7}$$

commutes, and similarly for the projection maps.

prop:sa **Proposition 1.2.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a semiadditive functor and $f, g : A \to B$ for $A, B \in \mathcal{A}$. Then $F(f + g) = F(f) + F(g)$.*

Proof. Obvious. ■

We now define an additive category to be a semiadditive category where the enriched hom-sets are in fact groups.

thm:as **Thm 1.2.3.** *Let $\mathcal{A}$ be an additive category according to the first definition. By theorem 1.1.1, $\mathcal{A}$ is semiadditive and so the hom-sets naturally carry the structure of a commutative monoid. This monoidal structure agrees with the original group structure.*

Proof. Let $A, B \in \mathcal{A}$ and $f, g : A \to B$. Then the addition arising from the semiadditive structure comes from the composition

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.8}$$

But $\nabla_A = i_A^L + i_A^R, \Delta_B = p_B^L + p_B^R$ and $f \oplus g = i_B^L \circ f \circ p_A^L + i_B^R \circ g \circ p_A^R$ and so their composition is just $f + g$. ■

Corollary 1.2.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categoires. Then F is additive iff F it is semiadditive.*

Proof. Semiadditive $\implies$ additive follows from proposition 1.2.2 and theorem 1.2.3. Additive $\implies$ semiadditive is a straigtforward exercise. ■

Corollary 1.2.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories which is a left adjoint. Then F is additive.*

Proof. F preserves colimits and so is semiadditive. ■

Corollary 1.2.6. *If $\mathcal{A}$ is an additive category then $\mathcal{A}^{op}$ is also additive.*

Proof. The oppositive category of a semiadditive category is clearly also semiadditive. The resulting monoidal structure on the hom-sets are also clearly the same and so the result follows. ■

1.3 Abelian categories

Abelian categories are additive categories with more strucure. Before we state exactly what we mean by this we give some definitions.

Definition 1.3.1. Let $\mathcal{A}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{A}$.

1. A kernel of f is an equaliser of $A \underset{0}{\overset{f}{\rightrightarrows}} B$.
2. A cokernel of f is a coequaliser of the same diagram.
3. f is called monic if $f \circ g = 0$ implies $g = 0$ for all g.
4. f is called epi if $g \circ f = 0$ implies $g = 0$ for all g.

Remark 1.3.2. It is easy to see that all kernels are monic, all cokernels are epi, a map is monic iff its kernel is 0, and a map is epi iff its cokernel is 0.

We call an additive category $\mathcal{A}$ pre-abelian if all morphisms have kernels and cokernels. In such a category, given any morphism $f : A \to B$ we can form

$$\begin{tikzcd}
 & & \ker(\operatorname{coker}(f)) \arrow[dr, "i"] & & \\
K \arrow[r, "\ker(f)"] & A \arrow[ur, dashed, "\alpha"] \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[r, "\operatorname{coker}(f)"] & C \\
 & & \operatorname{coker}(\ker(f)) \arrow[ur, dashed, "\beta"'] & &
\end{tikzcd} \tag{1.9}$$

eq:canon-decomp

where α and β exist from the universal property of kernels and cokernerls respectively. Since p is epi and $0 = \operatorname{coker}(f) \circ f = \operatorname{coker}(f) \circ \beta \circ p$ it follows that $\operatorname{coker}(f) \circ \beta = 0$ and so there is a map $\gamma : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ such that $i \circ \gamma = \beta$. Similarly there is a map $\gamma' : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ such that $\gamma' \circ p = \alpha$. Using that p is epi one can see that $\gamma' = \gamma$ and so for any morphism f there is a canonical decomposition

$$A \xrightarrow{p} \operatorname{coker}(\ker(f)) \xrightarrow{\gamma_f} \ker(\operatorname{coker}(f)) \xrightarrow{i} B. \tag{1.10}$$

An abelian category is a pre-abelian category in which γ_f is an isomorphism for every f.

thm:abcat

Thm 1.3.3. *Let $\mathcal{A}$ be a pre-abelian category. Then γ_f is an isomorphism for all morphism f iff every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.*

Proof. ($\Rightarrow$) The kernel of a monic is the 0 object with the 0 map, and the cokernel of this is just A together with the identity. Thus, if γ_f is an isomorphism the canonical decomposition of f just becomes

$$A \xrightarrow{\mathrm{id}} A \xrightarrow{\cong} \ker(\operatorname{coker}(f)) \xrightarrow{i} B \tag{1.11}$$

and so f is the kernel of its cokernel. Similarly one obtains that if f is epi it is the cokernel of its kernel.

($\Leftarrow$) First note that if a kernel is epi then it must be an isomorphism so all epic monics must be isomorphisms (since all monics are kernels). Thus, it suffices to show that the maps α and β in equation 1.9 are epi and monic respectively. To see that β is monic let $x : X \to \operatorname{coker}(\ker(f))$ be a map such that $\beta \circ x = 0$. Then let $q : \operatorname{coker}(\ker(f)) \to \operatorname{coker}(x)$ be the coker of x, and $j : \operatorname{coker}(x) \to B$ the map such that $j \circ q = \beta$. Finally let $l : \ker(q \circ p) \to A$ be the kernel of $q \circ p$. Then we have the following diagram

$$\begin{tikzcd}
\ker(q\circ p) \arrow[dr, "l"] \arrow[dd, dashed] & & & \\
& A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\
\ker(f) \arrow[ur, "k"] & & \operatorname{coker}(\ker(f)) \arrow[ur, "\beta"'] \arrow[dr, "q"] & \\
& X \arrow[ur, "x"] & & \operatorname{coker}(x). \arrow[uu, "j"']
\end{tikzcd} \tag{1.12}$$

Since $q \circ p$ is epi it is the coker of l. But also $f \circ l = j \circ q \circ p \circ l = 0$, so l factors through $\ker(f)$ and so $p \circ l = 0$. Thus there exists $p' : \operatorname{coker}(x) \to \operatorname{coker}(\ker(f))$ such that

$$\begin{tikzcd}
\ker(q\circ p) \arrow[r, "l"] & A \arrow[r, "p"] \arrow[d, "q\circ p"] & \operatorname{coker}(\ker(f)) \\
& \operatorname{coker}(x) \arrow[ur, dashed, "p'"'] &
\end{tikzcd} \tag{1.13}$$

commutes. Since p is epi, it must follow that $p' \circ q = \mathrm{id}$. Thus q is monic and so $x = 0$. It follows that β is monic. Similarly one can show that α is epi. ∎

It follows that an abelian category is equvalently a pre-abelian category in which every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.

Thm 1.3.4. *If $\mathcal{A}$ is an abelian category then $\mathcal{A}^{op}$ is also an abelian category.*

Proof. It is certainly additive. Moreover, kernels and cokernels simply swap roles. γ_f is then still an isomorphism for all f and so $\mathcal{A}^{op}$ is abelian. ∎

From now on we write $\mathrm{im}(f) := \ker(\mathrm{coker}(f))$ and $\mathrm{coim}(f) := \mathrm{coker}(\ker(f))$.

1.4 Exact sequences

sec:es

Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ be the category with objects given by $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$, and morphisms given by chain maps. Recall from earlier that f can be factored as

$$A \xrightarrow{p_f} \mathrm{im}(f) \xhookrightarrow{i_f} B. \tag{1.14}$$

Since p_f is epi, we must have $g \circ i_f = 0$. Thus we can factor f further through $\ker(g)$ to obtain $f : A \to \mathrm{im}(f) \to \ker(g) \to B$. Let $H(A \xrightarrow{f} B \xrightarrow{g} C)$ be the cokernel of the morphism $\mathrm{im}(f) \to \ker(g)$. If we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd} \tag{1.15}$$

then there exists maps so that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{im}(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ A' \arrow[r] & \mathrm{im}(f') \arrow[r] & \ker(g') \arrow[r] & B' \arrow[r] & C' \end{tikzcd} \tag{1.16}$$

commutes. In particular there is a morphism

$$\mathrm{coker}(\mathrm{im}(f) \to \ker(g)) \to \mathrm{coker}(\mathrm{im}(f') \to \ker(g')). \tag{1.17}$$

It is easy to check that this construction is functorial so we obtain a functor $H : \mathcal{S} \to \mathcal{A}$.

One can similarly construct a functor $H' : \mathcal{S} \to \mathcal{A}$ by considering

$$\ker(\mathrm{coker}(f) \to \mathrm{coim}(g)) \tag{1.18}$$

instead.

Remark 1.4.1. We may also form a functor by looking simply at the fact that f factors through $\ker(g)$ and then looking at the coker of the resulting morphism $A \to \ker(g)$. It is an easy check to see that this yields a functor naturally isomorphic to H. Similarly for H'.

Lemma 1.4.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. Recall that we have the factorisation*

$$A \to \operatorname{im}(f) \to \ker(g) \to B \to \operatorname{coker}(f) \to \operatorname{coim}(g) \to C. \tag{1.19}$$

Let h be the composition $\ker(g) \to B \to \operatorname{coker}(f)$. Then

1. $\ker(h) = \operatorname{im}(f) \to \ker(g)$
2. $\operatorname{coker}(h) = \operatorname{coker}(f) \to \operatorname{coim}(g)$.

Proof. Straightforward. ■

Thm 1.4.3. *The functors $H, H' : \mathcal{S} \to \mathcal{A}$ are naturally isomorphic.*

Proof. Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$ and h be as in the lemma. Then $H(S) = \operatorname{coker}(\ker(h))$ and $H'(S) = \ker(\operatorname{coker}(h))$ so we obtain the factorisation

$$\ker(g) \to H(S) \xrightarrow{\cong} H'(S) \to \operatorname{coker}(f). \tag{1.20}$$

Naturality of the isomophism then follows from naturality of this factorisation. ■

Remark 1.4.4. In a pre-abelian category we still have a natrual transformation $H \Rightarrow H'$, but it might not be an isomorphism.

Definition 1.4.5. Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. We say that S is exact at B if $H(S) = 0$.

1.4.1 Split sequences

1.5 Adjoint functors

Let $L : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. If L admits a right adjoint $R : \mathcal{B} \to \mathcal{A}$ then it turns out L has a lot of useful properties. In this section we explore these properties.

Proposition 1.5.1. *Suppose $L \dashv R$. Then L is right exact and R is left exact.*

Proof. Consider the short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$. For every $A \in \mathcal{A}$ we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(L(A), B_1) & \longrightarrow & \operatorname{Hom}(L(A), B_2) & \longrightarrow & \operatorname{Hom}(L(A), B_3) \\ & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\ 0 & \longrightarrow & \operatorname{Hom}(A, R(B_1)) & \longrightarrow & \operatorname{Hom}(A, R(B_2)) & \longrightarrow & \operatorname{Hom}(A, R(B_3)) \end{array} \tag{1.21}$$

where the top row is exact. It follows that the bottom row is exact for all A and so the bottow row is too. It follows that

$$0 \longrightarrow R(B_1) \longrightarrow R(B_2) \longrightarrow R(B_3) \tag{1.22}$$

is exact and so R is left exact. By a similar argument L is right exact. ∎

Proposition 1.5.2. *Suppose $L \dashv R$. Then*

1. *if L is exact then R preserves injectives*
2. *if R is exact then L preserves projectives.*

Proof. Suppose L is exact and I is an injective object in $\mathcal{B}$. We need to show that $\mathrm{Hom}(-, R(I))$ is exact. To do this it suffices to show that given $f : A \to B$ injective, the map $f^* : \mathrm{Hom}(B, R(I)) \to \mathrm{Hom}(A, R(I))$ is surjective. But L is exact so Lf is injective and so $(Lf)^* : \mathrm{Hom}(LB, I) \to \mathrm{Hom}(LA, I)$ is surjective. We also have that $L \dashv R$ and so

$$\begin{array}{ccc} \mathrm{Hom}(L(B), I) & \xrightarrow{(Lf)^*} & \mathrm{Hom}(L(A), I) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{Hom}(B, R(I)) & \xrightarrow{f^*} & \mathrm{Hom}(A, R(I)) \end{array} \tag{1.23}$$

commutes. It follows that f^* is surjective as required.

The corresponding result for R follows similarly. ∎

CHAPTER 2

Sheaf Theory

2.1 Presheaves

Let $\mathcal{C}$ be any category, $\mathcal{A}$ be an abelian category and define $\mathsf{PreSh}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{A})$ to be the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$. The functor sending all objects to 0 is certainly both initial and terminal, direct sums can be defined pointwise, and the hom-sets in $\mathsf{PreSh}(\mathcal{C})$ inherit an additive structure from $\mathcal{A}$ so $\mathsf{PreSh}(\mathcal{C})$ is naturally an additive category. Moreover kernels and cokernels can be contructed in the obvious way and it is clear that they satisfy the axioms for an abelian category and so $\mathsf{PreSh}(C)$ is abelian.

2.2 Sheaves

To define sheaves we restrict to the case when X be a topological space, $\mathcal{U}$ the poset of open sets of X, and $\mathcal{A}$ be an abelian category. We write $\mathsf{PreSh}(X)$ for $\mathsf{PreSh}(\mathcal{U})$. The category of sheaves on X with values in $\mathcal{A}$, $\mathsf{Sh}(X)$, is defined to be the full subcategory of $\mathsf{PreSh}(X)$ with objects given by presheaves $\mathscr{F}$ for which the following diagram is an equalizer for all open coverings $U = \cup_i U_i$

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.1}$$

Since $\mathcal{A}$ is an abelian category this is equivalent to the following diagram being exact

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \xrightarrow{\text{diff}} \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.2}$$

Note that since $\varnothing$ admits the empty covering and the empty product is 0 this forces $\mathscr{F}(\varnothing) = 0$.

As in the case of $\mathsf{PreSh}(\mathcal{C})$, $\mathsf{Sh}(X)$ is an additive category. However, the cokernel of a morphism between sheaves need not be a sheaf and so we must do some more work to show that $\mathsf{Sh}(X)$ is abelian.

Fix $x \in X$. For a (pre)sheaf $\mathscr{F}$ define the stalk of $\mathscr{F}$ at x to be

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F} \tag{2.3}$$

when this limit exists. Note that this is a functor since morphisms between (pre)sheaves are natural transformations.

Thm 2.2.1. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

1. *If ϕ_x is injective for all $x \in X$ then ϕ is injective on sections.*
2. *If ϕ_x is an isomorphism for all $x \in X$ then ϕ is an isomorphism.*

Proof. Exercise. ■

Aside

Although we do not need this right away, given an $A \in \mathcal{A}$ we can define the (pre)sheaf x_*A by

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

Proposition 2.2.2. *When it exists, the functor $(-)_x : \mathsf{Sh}(X) \to \mathcal{A}$ is left adjoint to $x_* : \mathcal{A} \to \mathsf{Sh}(X)$.*

Proof. To see this simply note that morphisms between $\mathscr{F}$ and $x_*(A)$ correspond naturally to natrual transformations between $\mathscr{F}$ restricted to $U \ni x$ and $\Delta(A)$. ■

Remark 2.2.3. The result also holds in $\mathsf{PreSh}(X)$.

2.3 Étalé space of a presheaf and sheafification

For a presheaf $\mathscr{F}$ we are now in the position to define its étalé space. The étalé space of $\mathscr{F}$, denoted Spé($\mathscr{F}$) is the topological space with underlying set $\amalg_{x \in X} \mathscr{F}_x$ and topology generated by the basis of sets given by $\{s_x | x \in U\}$ for $s \in \mathscr{F}(U)$ where $U \subset X$ is open. Together with this space there is also a natural continuous map $\pi : \text{Spé}(\mathscr{F}) \to X$ sending an element s_x to x. The sheafification of $\mathscr{F}$, denoted $\mathscr{F}^+$, is then defined to be the sheaf of sections of $\pi : \text{Spé}(\mathscr{F}) \to X$. By unwrapping the definitions we see that the sections can be characterised as

$$\begin{aligned} \mathscr{F}^+(U) = \{s : U \to \amalg_{x \in U} \mathscr{F}_x : & \forall x \in U, \exists V \subset U \text{ open containing } x \text{ and} \\ & t \in \mathscr{F}(V) \text{ s.t. } s(y) = t_y \forall y \in V\} \end{aligned} \tag{2.5}$$

In particular there is a natural morphism $\mathscr{F} \to \mathscr{F}^+$ sending $s \in \mathscr{F}(U)$ to the section $x \mapsto s_x$ which is an isomorphism on stalks. From the characterisation of sections it clear that if $\mathscr{F}$ is a presheaf of $\mathsf{AbGrp}, \mathsf{Ring}, ...$ then $\mathscr{F}^+$ is a sheaf with values in the corresponding abelian category.

We have defined Spé and $(-)^+$ on objects but they can also be turned into functors. If we have a morphism $\phi : \mathscr{F} \to \mathscr{G}$ between presheaves, this induces a continuous map $\text{Spé}(\phi) : \text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ given by $s_x \mapsto \phi_x(s_x)$ so that

$$\begin{tikzcd} \text{Spé}(\mathscr{F}) \arrow[rr, "\text{Spé}(\phi)"] \arrow[dr, "\pi"'] & & \text{Spé}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.6}$$

commutes. This construction is functorial and turns Spé into a functor from presheaves to topological bundles over X. It follows that we also obtain a map of sheaves $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ by composing sections with $\text{Spé}(\phi)$. Thus we have a functor $(-)^+ : \mathsf{PreSh}(X) \to \mathsf{Sh}(X)$ and in fact the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\phi^+"] & \mathscr{G}^+ \\ \mathscr{F} \arrow[u] \arrow[r, "\phi"] & \mathscr{G} \arrow[u] \end{tikzcd} \tag{2.7}$$

eq:sheafif

Note that since the morphism $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism when $\mathscr{F}$ is a sheaf, this says that the functor $(-)^+$ restricted to $\mathsf{Sh}(X)$ is natrually isomorphism to the identity functor.

Thm 2.3.1. *Let $\theta : \mathscr{F} \to \mathscr{F}^+$ be the natural morphism. Then for any morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, there exists a unique morphism of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$ so that*

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\psi"] & \mathscr{G} \\ \mathscr{F} \arrow[u, "\theta"] \arrow[ur, "\phi"] & \end{tikzcd} \tag{2.8}$$

commutes.

Proof. This just follows from equation 2.7, the fact that $\theta : \mathscr{G} \to \mathscr{G}^+$ is an isomorphism when $\mathscr{G}$ is a sheaf, and by taking stalks. ■

Corollary 2.3.2. *The sheafification functor is left adjoint to the inclusion functor $\iota : \mathsf{Sh}(X) \to \mathsf{PreSh}(X)$.*

Proof. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. Given a morphism $\phi : \mathscr{F}^+ \to \mathscr{G}$ we can precompose it with $\theta : \mathscr{F} \to \mathscr{F}^+$ to obtain a map $\mathscr{F} \to \iota\mathscr{G}$. Conversely, given $\psi : \mathscr{F} \to \iota\mathscr{G}$, we obtain a map $\mathscr{F}^+ \to \mathscr{G}$ from the theorem. Then the theorem says these operations are inverse so we have a bijection

$$\text{Hom}(\mathscr{F}^+, \mathscr{G}) \cong \text{Hom}(\mathscr{F}, \iota\mathscr{G}). \tag{2.9}$$

Naturality is then an easy check. ■

Corollary 2.3.3. *The sheafification functor is exact.*

Proof. It is a left adjoint so it is right exact. It thus suffices to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is injective then so is ϕ^+. For this it suffices to show that ϕ_x is injective for all x. But this is obvious. ∎

We can now define the cokernel of a morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathsf{Sh}(X)$. We simply define it to be the sheafification of the cokernel in $\mathsf{PreSh}(X)$ and it is an easy to check to see that this is indeed a cokernel object in $\mathsf{Sh}(X)$. It is then easy to see that $\ker \operatorname{coker} = \operatorname{coker} \ker$ by looking at stalks and so $\mathsf{Sh}(X)$ is an abelian category.

Remark 2.3.4. While $\mathsf{Sh}(X)$ is a full subcategory of $\mathsf{PreSh}(X)$ that is abelian, it is not a full abelian subcategory.

2.4 Exact sequences

Now that we know that we are working in an abelian category we can talk about exact sequences in $\mathsf{Sh}(X)$. Recall from section 1.4 that $\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ is exact at $\mathscr{G}$ if $\phi \circ \theta = 0$ and the map induced map $\operatorname{im}(\theta) \to \ker(\phi)$ is an isomorphism. But the map $\operatorname{im}(\theta) \to \ker(\phi)$ is an isomorphism iff it is an isomorphism at the level of stalks iff $\mathscr{F}_x \xrightarrow{\theta_x} \mathscr{G}_x \xrightarrow{\phi_x} \mathscr{H}_x$ is exact for all $x \in X$. Thus exactness in $\mathsf{Sh}(X)$ can be verified by checking exactness at all the stalks.

2.5 Sheaves over different spaces

2.5.1 Direct image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on X. We define the direct image of $\mathscr{F}$ under f to be the sheaf $f_*\mathscr{F}$ on Y defined by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. If we define f_* on morphisms in the obvious way then it is clear that we obtain a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. In fact we also obtain a functor $f_* : \mathsf{PreSh}(X) \to \mathsf{PreSh}(Y)$ and it turns out this functor has nice left adjoint.

Define $\lim_f : \mathsf{PreSh}(Y) \to \mathsf{PreSh}(X)$ to be the functor that sends $\mathscr{F} \in \mathsf{PreSh}(Y)$ to the presheaf $\lim_f(\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ on X, and does the obvious things to morphisms.

Thm 2.5.1. $\lim_f \dashv f_*$ *as functors between* $\mathsf{PreSh}(X)$ *and* $\mathsf{PreSh}(Y)$.

Proof. Let $\phi : \lim_f \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For V open in Y, $f^{-1}(V)$ is open in X and so we have maps

$$\mathscr{F}(V) \to \varinjlim_{W \supset f(U)} \mathscr{F}(W) \to \mathscr{G}(U) \tag{2.10}$$

where $U = f^{-1}(V)$. If $V' \subset V$, $U = f^{-1}(V)$ and $U' = f^{-1}(V')$ then

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r] \arrow[d] \arrow[dr] & \varinjlim_{W \supset f(U)} \mathscr{F}(W) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}(V') \arrow[r] & \varinjlim_{W \supset f(U')} \mathscr{F}(W) \arrow[r] & \mathscr{G}(U') \end{tikzcd} \tag{2.11}$$

commutes and so these maps in fact define a morphism $\mathscr{F} \to f_*\mathscr{G}$.

Conversely suppose we are given a morphism $\mathscr{F} \to f_*\mathscr{G}$. Let U be open in X. For $V \supset f(U)$ we have maps

$$\mathscr{F}(V) \to \mathscr{G}(f^{-1}(V)) \to \mathscr{G}(U). \tag{2.12}$$

Moreover if $V \supset V' \supset f(U)$ then

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(f^{-1}(V)) \arrow[d] \arrow[dr] & \\ & & \mathscr{G}(U) \\ \mathscr{F}(V') \arrow[r] & \mathscr{G}(f^{-1}(V')) \arrow[ur] & \end{tikzcd} \tag{2.13}$$

commutes so we obtain maps $\varinjlim_{V \supset f(U)} \mathscr{F}(V) \to \mathscr{G}(U)$. If $U \supset U'$ we have maps

$$\begin{tikzcd} \varinjlim_{V \supset f(U)} \mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \varinjlim_{V \supset f(U')} \mathscr{F}(V) \arrow[r] & \mathscr{G}(U'). \end{tikzcd} \tag{2.14}$$

A straighforward calculation shows that this commutes and so we obatin a morphism $\lim_f \mathscr{F} \to \mathscr{G}$.

These operations are clearly inverse to each other. A straightforward calculation shows that the bijection is natural. ■

Corollary 2.5.2. $\lim_f$ *is an exact functor.*

Proof. It is a left adjoint so it is right exact. Thus it suffices to show that it sends injective maps to injective maps. But this is obvious. ■

2.5.2 Inverse image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on Y. Let $f^{-1}\text{Spé}(\mathscr{F})$ be the pullback

$$\begin{tikzcd} f^{-1}\text{Spé}(\mathscr{F}) \arrow[r, dashed] \arrow[d, dashed, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \text{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.15}$$

We define the inverse image sheaf $f^{-1}\mathscr{F}$ to be the sheaf of sections of $\pi : f^{-1}\text{Spé}(\mathscr{F}) \to X$. Equivalently, it is the sheaf

$$f^{-1}\mathscr{F}(U) = \left\{ s : U \to \text{Spé}(\mathscr{F}) : \begin{tikzcd} & \text{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "f|_U"] & Y \end{tikzcd} \text{ commutes} \right\} \tag{2.16}$$ eq:invimg

or also equivalently, the sheaf

$$f^{-1}\mathscr{F}(U) = \{s : U \to \amalg_{x\in U}\mathscr{F}_{f(x)} : \forall x \in U, \exists W \subset Y, V \subset f^{-1}(W) \cap U \text{ open and } t \in \mathscr{F}(W) \text{ s.t. } x \in V \wedge s(y) = t_{f(y)} \forall y \in V\}. \tag{2.17}$$

It is clear from the construction that we obtain a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.

Remark 2.5.3. A direct calculation shows that $f^{-1}\mathscr{F}_x$ and $\mathscr{F}_{f(x)}$ are naturally isomorphic and so there is a natrual bijection between $f^{-1}\text{Spé}(\mathscr{F})$ and $\text{Spé}(f^{-1}\mathscr{F})$. It is then a straightforward exercise to check that this bijection is in fact a homeomorphism i.e. $f^{-1}\text{Spé}(\mathscr{F}) \cong \text{Spé}(f^{-1}\mathscr{F})$.

Thm 2.5.4. *f^{-1} is naturally isomorphic to $(-)^+ \circ \lim_f$.*

Proof. Let U be an open subset of X and $s \in \lim_f \mathscr{F}(U)$. There is a natural map $\phi_x : (\lim_f \mathscr{F})_x \to \mathscr{F}_{f(x)}$ so we can define a map $U \to \text{Spé}(\mathscr{F})$ by $x \mapsto \phi_x(s_x)$. It is clear that this gives an element of $f^{-1}\mathscr{F}(U)$ as characterised by equation 2.16. Thus we obtain a morphism $\lim_f \mathscr{F} \to f^{-1}\mathscr{F}$. On stalks this map is given by ϕ_x. A direct calculation shows that ϕ_x is an isomorphism for all $x \in X$ and so the induced map $(\lim_f \mathscr{F})^+ \to f^{-1}\mathscr{F}$ must be an isomorphism. It is straightforward to see that this defines a natural transformation. ∎

Corollary 2.5.5. *$f^{-1} \dashv f_*$ as functors between $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$.*

Proof. f^{-1} is naturally isomophic to $(-)^+ \circ \lim_f$ and so for $\mathscr{F} \in \mathsf{Sh}(Y), \mathscr{G} \in \mathsf{Sh}(X)$ we have natural bijections

$$\begin{aligned} \text{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) &\cong \text{Hom}_{\mathsf{Sh}(X)}\left((\lim_f \mathscr{F})^+, \mathscr{G}\right) \cong \text{Hom}_{\mathsf{PreSh}(X)}\left(\lim_f \mathscr{F}, \mathscr{G}\right) \\ &\cong \text{Hom}_{\mathsf{PreSh}(Y)}\left(\mathscr{F}, f_*\mathscr{G}\right) \cong \text{Hom}_{\mathsf{Sh}(Y)}\left(\mathscr{F}, f_*\mathscr{G}\right). \end{aligned} \tag{2.18}$$

■

Corollary 2.5.6. $(-)_x \circ f^{-1} = (-)_{f(x)}$.

Proof. $(-)_x \circ f^{-1} = (-)_x \circ (-)^+ \circ \lim_f = (-)_x \circ \lim_f = (-)_{f(x)}$. ■

Corollary 2.5.7. *f^{-1} is an exact functor.*

Proof. It is the composition of two exact functors. Alternatively take stalks. ■

2.6 The $\mathscr{H}\!om$ sheaf

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves and $f : \text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ be a continuous map so that

$$\begin{tikzcd} \text{Spé}(\mathscr{F}) \arrow[rr, "\text{Spé}(\phi)"] \arrow[dr, "\pi"'] & & \text{Spé}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.19}$$

commutes. Then we obtain a morphism $\mathscr{F}^+ \to \mathscr{G}^+$ by postcomposing sections with f. Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves we in fact obtain a morphism $\mathscr{F} \to \mathscr{G}$. But we also know that morphisms $\mathscr{F} \to \mathscr{G}$ give continuous maps $\text{Spé}(\mathscr{F}) \to \text{Spé}(\mathscr{G})$ making the above diagram commute.

2.7 Injective sheaves

There are enough injectives.

CHAPTER 3

Spectral sequences

CHAPTER 4

Group cohomology