Homological algebra

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Abelian Categories

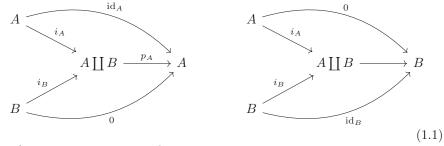
1.1 Additive categories

Let \mathcal{A} be a category such that the hom-sets carry the structure of an abelian group and composition is bilinear. We call such a category Ab-enriched. An additive category is an Ab-enriched category which has finite coproducts and a zero object.

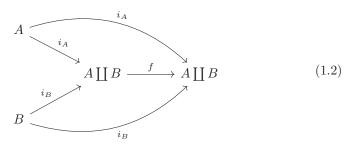
thm:atosa

Thm 1.1.1. Let A be an additive category. Then finite coproducts in A are in fact finite biproducts.

Proof. It is easy to see that initial objects are isomorphic to terminal objects (and they both exist) and so it suffices to show the result for binary coproducts. Let $A, B \in \mathcal{A}$. Define $p_A : A \coprod B \to A$ and $p_B : A \coprod B \to B$ as the maps making the following diagrams commute.



Let $f = i_A \circ p_A + i_B \circ p_B$. Then



commutes and so by universality we must have $f = \mathrm{id}_{A \coprod B}$. Now suppose we have maps $f: C \to A$ and $g: C \to B$. Let $h: C \to A \coprod B$ be the map $i_A \circ f + i_B \circ g$. Then $p_A \circ h = f$ and $p_B \circ h = g$. Moreover, if $h': C \to A \coprod B$ is any other map satisfying $p_A \circ h' = f$ and $p_B \circ h' = g$ then $h' = id_{A \coprod B} \circ h' = i_A \circ f + i_B \circ g = h$ and so $A \coprod B$ is a biproduct.

A functor between additive categories is called additive if it is a homomorphism on hom-sets.

1.2 Semiadditive categories

The above definition of an additive category includes the additive structure on the hom-sets as data. In this section we provide a definition where the additive structure arises as a property instead.

Let \mathcal{A} be a category with a zero object. Recall that in such a category there always exists a morphism between to any two objects $A, B \in \mathcal{A}$ given by $A \to 0 \to B$. We call this the 0 morphism. Moreover if finite coproducts and finite products exists there is a canonical map $A \coprod B \to A \coprod B$ arising from the diagram

$$A \xrightarrow{\operatorname{id}_{A}} A$$

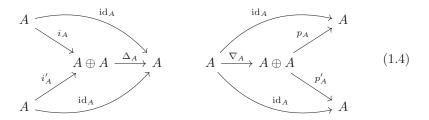
$$B \xrightarrow{\operatorname{id}_{B}} B.$$

$$(1.3)$$

We call a category \mathcal{A} semiadditive if it has a zero object, finite products, finite coproducts and the canonical map $A \coprod B \to A \coprod B$ is an isomorphism for all $A, B \in \mathcal{A}$. In such a category we write $A \oplus B$ for the biproduct.

Thm 1.2.1. Let A be a semiadditive category then it is naturally enriched over the monoidal category of commutative monoids.

Proof. Let $\Delta_A: A \oplus A \to A$ and $\nabla_A: A \to A \oplus A$ be the maps that make



commute. Given $f, g: A \to B$ we can construct a map $f \oplus g: A \oplus A \to B \oplus B$ in the obvious way. We can then define $f + g: A \to B$ to be the composite

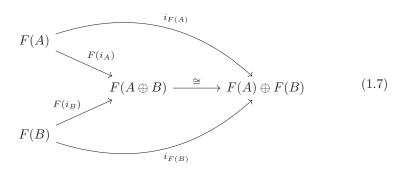
$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.5}$$

Note that there is a map $t_A: A \oplus A \to A \oplus A$ arising from the diagram

$$\begin{array}{ccc}
A & \xrightarrow{0} & A \\
& & \downarrow & \downarrow \\
A & \xrightarrow{\mathrm{id}_A} & A.
\end{array} \tag{1.6}$$

It is then an easy check to see that $\Delta_A \circ t_A = \Delta_A$ and $t_A \circ \nabla_A = \nabla_A$, from which it follows that + is commutative. Straightforward calculations also show that + is associative, distributes over compositions and has the zero map as identity. The result follows.

A functor between semiadditive categories is called semiadditive if it preserves zero objects and biproducts i.e. there are isomorphisms $F(A \oplus B) \cong F(A) \oplus F(B)$ such that



commutes, and similarly for the projection maps.

prop:sa

Proposition 1.2.2. Let $F: A \to B$ be a semiadditive functor and $f, g: A \to B$ for $A, B \in A$. Then F(f+g) = F(f) + F(g).

We now define an additive category to be a semiadditive category where the enriched hom-sets are in fact groups.

thm:as

Thm 1.2.3. Let A be an additive category according to the first definition. By theorem 1.1.1, A is semiadditive and so the hom-sets naturally carry the structure of a commutative monoid. This monoidal structure agrees with the original group structure.

Proof. Let $A, B \in \mathcal{A}$ and $f, g : A \to B$. Then the addition arising from the semiadditive structure comes from the composition

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.8}$$

But $\nabla_A = i_A^L + i_A^R$, $\Delta_B = p_B^L + p_B^R$ and $f \oplus g = i_B^L \circ f \circ p_A^L + i_B^R \circ g \circ p_A^R$ and so their composition is just f + g.

Corollary 1.2.4. Let $F: A \to B$ be a functor between additive categoires. Then F is additive iff F it is semiadditive.

Proof. Semiadditive \implies additive follows from proposition 1.2.2 and theorem 1.2.3. Additive \implies semiadditive is a straigtforward exercise.

Corollary 1.2.5. Let $F: \mathcal{A} \to \mathcal{B}$ be a functor between additive categories which is a left adjoint. Then F is additive.

Proof. F preserves colimits and so is semiadditive.

Corollary 1.2.6. If A is an additive category then A^{op} is also additive.

Proof. The oppositive category of a semiadditive category is clearly also semiadditive. The resulting monoidal structure on the hom-sets are also clearly the same and so the result follows.

1.3 Abelian categories

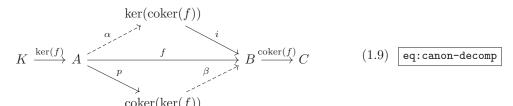
Abelian categories are additive categories with more strucure. Before we state exactly what we mean by this we give some definitions.

Definition 1.3.1. Let \mathcal{A} be an additive category and $f: A \to B$ a morphism in \mathcal{A} .

- 1. A kernel of f is an equaliser of $A \xrightarrow{f \atop 0} B$.
- 2. A cokernel of f is a coequaliser of the same diagram.
- 3. f is called monic if $f \circ g = 0$ implies g = 0 for all g.
- 4. f is called epi if $g \circ f = 0$ implies g = 0 for all g.

Remark 1.3.2. It is easy to see that all kernels are monic, all cokernels are epi, a map is monic iff its kernel is 0, and a map is epi iff its cokernel is 0.

We call an additive category \mathcal{A} pre-abelian if all morphisms have kernels and cokernels. In such a category, given any morphism $f: A \to B$ we can form



where α and β exist from the universal property of kernels and cokerners respectively. Since p is epi and $0 = \operatorname{coker}(f) \circ f = \operatorname{coker}(f) \circ \beta \circ p$ it follows

that $\operatorname{coker}(f) \circ \beta = 0$ and so there is a map $\gamma : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ such that $i \circ \gamma = \beta$. Similarly there is a map $\gamma' : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ such that $\gamma' \circ p = \alpha$. Using that p is epi one can see that $\gamma' = \gamma$ and so for any morphism f there is a canonical decomposition

$$A \xrightarrow{p} \operatorname{coker}(\ker(f)) \xrightarrow{\gamma_f} \ker(\operatorname{coker}(f)) \xrightarrow{i} B.$$
 (1.10)

An abelian category is a pre-abelian category in which γ_f is an isomorphism for every f.

thm:abcat

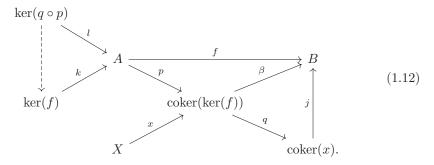
Thm 1.3.3. Let A be a pre-abelian category. Then γ_f is an isomorphism for all morphism f iff every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.

Proof. (\Rightarrow) The kernel of a monic is the 0 object with the 0 map, and the cokernel of this is just A together with the identity. Thus, if γ_f is an isomorphism the canonical decomposition of f just becomes

$$A \xrightarrow{\mathrm{id}} A \xrightarrow{\cong} \ker(\operatorname{coker}(f)) \xrightarrow{i} B$$
 (1.11)

and so f is the kernel of its cokernel. Similarly one obtains that if f is epi it is the cokernel of its kernel.

(\Leftarrow) First note that if a kernel is epi then it must be an isomorphism so all epic monics must be isomorphisms (since all monics are kernels). Thus, it suffices to show that the maps α and β in equation 1.9 are epi and monic respectively. To see that β is monic let $x:X\to \operatorname{coker}(\ker(f))$ be a map such that $\beta\circ x=0$. Then let $q:\operatorname{coker}(\ker(f))\to\operatorname{coker}(x)$ be the coker of x, and $j:\operatorname{coker}(x)\to B$ the map such that $j\circ q=\beta$. Finally let $l:\ker(q\circ p)\to A$ be the kernel of $q\circ p$. Then we have the following diagram



Since $q \circ p$ is epi it is the coker of l. But also $f \circ l = j \circ q \circ p \circ l = 0$, so l factors through $\ker(f)$ and so $p \circ l = 0$. Thus there exists $p' : \operatorname{coker}(x) \to \operatorname{coker}(\ker(f))$ such that

$$\ker(q \circ p) \xrightarrow{l} A \xrightarrow{p} \operatorname{coker}(\ker(f))$$

$$\downarrow^{q \circ p} \qquad (1.13)$$

$$\operatorname{coker}(x)$$

commutes. Since p is epi, it must follow that $p' \circ q = \text{id}$. Thus q is monic and so x = 0. It follows that β is monic. Similarly one can show that α is epi.

It follows that an abelian category is equivalently a pre-abelian category in which every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.

Thm 1.3.4. If A is an abelian category then A^{op} is also an abelian category.

Proof. It is certainly additive. Moreover, kernels and cokernels simply swap roles. γ_f is then still an isomorphism for all f and so \mathcal{A}^{op} is abelian.

From now on we write im(f) := ker(coker(f)) and coim(f) := coker(ker(f)).

1.4 Exact sequences

sec:es

Let \mathcal{A} be an abelian category and \mathcal{S} be the category with objects given by $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$, and morphisms given by chain maps. Recall from earlier that f can be factored as

$$A \xrightarrow{p_f} \operatorname{im}(f) \xrightarrow{i_f} B.$$
 (1.14)

Since p_f is epi, we must have $g \circ i_f = 0$. Thus we can factor f further through $\ker(g)$ to obtain $f: A \to \operatorname{im}(f) \to \ker(g) \to B$. Let $H(A \xrightarrow{f} B \xrightarrow{g} C)$ be the cokernel of the morphism $\operatorname{im}(f) \to \ker(g)$. If we have the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow & & \downarrow & \downarrow \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array}$$
(1.15)

then there exists maps so that

commutes. In particular there is a morphism

$$\operatorname{coker}(\operatorname{im}(f) \to \ker(g)) \to \operatorname{coker}(\operatorname{im}(f') \to \ker(g')).$$
 (1.17)

It is easy to check that this construction is functorial so we obtain a functor $H: \mathcal{S} \to \mathcal{A}$.

One can similarly construct a functor $H': \mathcal{S} \to \mathcal{A}$ by considering

$$\ker(\operatorname{coker}(f) \to \operatorname{coim}(g))$$
 (1.18)

instead.

Remark 1.4.1. We may also form a functor by looking simply at the fact that f factors through $\ker(g)$ and then looking at the coker of the resulting morphism $A \to \ker(g)$. It is an easy check to see that this yields a functor naturally isomorphic to H. Similarly for H'.

Lemma 1.4.2. Let $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. Recall that we have the factorisation

$$A \to \operatorname{im}(f) \to \ker(g) \to B \to \operatorname{coker}(f) \to \operatorname{coim}(g) \to C.$$
 (1.19)

Let h be the composition $\ker(g) \to B \to \operatorname{coker}(f)$. Then

1.
$$\ker(h) = \operatorname{im}(f) \to \ker(g)$$

2.
$$\operatorname{coker}(h) = \operatorname{coker}(f) \to \operatorname{coim}(g)$$
.

Proof. Straightforward.

Thm 1.4.3. The functors $H, H': \mathcal{S} \to \mathcal{A}$ are naturally isomorphic.

Proof. Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$ and h be as in the lemma. Then $H(S) = \operatorname{coker}(\ker(h))$ and $H'(S) = \ker(\operatorname{coker}(h))$ so we obtain the factorisation

$$\ker(g) \to H(S) \xrightarrow{\cong} H'(S) \to \operatorname{coker}(f).$$
 (1.20)

Naturality of the isomophism then follows from naturality of this factorisation.

Remark 1.4.4. In a pre-abelian category we still have a natural transformation $H \Rightarrow H'$, but it might not be an isomorphism.

Definition 1.4.5. Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. We say that S is exact at B if H(S) = 0.

1.4.1 Split sequences

1.5 Adjoint functors

Let $L: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. If L admits a right adjoint $R: \mathcal{B} \to \mathcal{A}$ then it turns out L has a lot of useful properties. In this section we explore these properties.

Proposition 1.5.1. Suppose $L \dashv R$. Then L is right exact and R is left exact.

Proof. Consider the short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$. For every $A \in \mathcal{A}$ we get the following commutative diagram

$$0 \longrightarrow \operatorname{Hom}(L(A), B_1) \longrightarrow \operatorname{Hom}(L(A), B_2) \longrightarrow \operatorname{Hom}(L(A), B_3)$$

$$\downarrow \cong \qquad \qquad \downarrow \cong \qquad \qquad \downarrow \cong$$

$$0 \longrightarrow \operatorname{Hom}(A, R(B_1)) \longrightarrow \operatorname{Hom}(A, R(B_2)) \longrightarrow \operatorname{Hom}(A, R(B_3))$$

$$(1.21)$$

where the top row is exact. It follows that the bottom row is exact for all A and so the bottow row is too. It follows that

$$0 \longrightarrow R(B_1) \longrightarrow R(B_2) \longrightarrow R(B_3) \tag{1.22}$$

is exact and so R is left exact. By a similar argument L is right exact.

Proposition 1.5.2. Suppose $L \dashv R$. Then

- 1. if L is exact then R preserves injectives
- 2. if R is exact then L preserves projectives.

Proof. Suppose L is exact and I is an injective object in \mathcal{B} . We need to show that $\operatorname{Hom}(-,R(I))$ is exact. To do this it suffices to show that given $f:A\to B$ injective, the map $f^*:\operatorname{Hom}(B,R(I))\to\operatorname{Hom}(A,R(I))$ is surjective. But L is exact so Lf is injective and so $(Lf)^*:\operatorname{Hom}(LB,I)\to\operatorname{Hom}(LA,I)$ is surjective. We also have that $L\dashv R$ and so

$$\operatorname{Hom}(L(B), I) \xrightarrow{(Lf)^*} \operatorname{Hom}(L(A), I)$$

$$\downarrow \cong \qquad \qquad \downarrow \cong \qquad \qquad (1.23)$$

$$\operatorname{Hom}(B, R(I)) \xrightarrow{f^*} \operatorname{Hom}(A, R(I))$$

commutes. It follows that f^* is surjective as required.

The corresponding result for R follows similarly.

Sheaf Theory

2.1 Presheaves

Let \mathcal{C} be any category, \mathcal{A} be an abelian category and define $\mathsf{PreSh}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{A})$ to be the category of presheaves on \mathcal{C} with values in \mathcal{A} . The functor sending all objects to 0 is certainly both initial and terminal, direct sums can be defined pointwise, and the hom-sets in $\mathsf{PreSh}(\mathcal{C})$ inherit an additive structure from \mathcal{A} so $\mathsf{PreSh}(\mathcal{C})$ is naturally an additive category. Moreover kernels and cokernels can be contructed in the obvious way and it is clear that they satisfy the axioms for an abelian category and so $\mathsf{PreSh}(\mathcal{C})$ is abelian.

2.2 Sheaves

To define sheaves we restrict to the case when X be a topological space, \mathcal{U} the poset of open sets of X, and \mathcal{A} be an abelian category. We write $\mathsf{PreSh}(X)$ for $\mathsf{PreSh}(\mathcal{U})$. The category of sheaves on X with values in \mathcal{A} , $\mathsf{Sh}(X)$, is defined to be the full subcategory of $\mathsf{PreSh}(X)$ with objects given by presheaves \mathscr{F} for which the following diagram is an equalizer for all open coverings $U = \cup_i U_i$

$$\mathscr{F}(U) \to \prod_{i} \mathscr{F}(U_i) \Longrightarrow \prod_{i,j} \mathscr{F}(U_i \cap U_j).$$
 (2.1)

Since \mathcal{A} is an abelian category this is equivalent to the following diagram being exact

$$0 \to \mathscr{F}(U) \to \prod_{i} \mathscr{F}(U_{i}) \xrightarrow{\text{diff}} \prod_{i,j} \mathscr{F}(U_{i} \cap U_{j}). \tag{2.2}$$

Note that since \emptyset admits the empty covering and the empty product is 0 this forces $\mathscr{F}(\emptyset) = 0$.

As in the case of $\mathsf{PreSh}(\mathcal{C})$, $\mathsf{Sh}(X)$ is an additive category. However, the cokernel of a morphism between sheaves need not be a sheaf and so we must do some more work to show that $\mathsf{Sh}(X)$ is abelian.

Fix $x \in X$. For a (pre)sheaf \mathscr{F} define the stalk of \mathscr{F} at x to be

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F} \tag{2.3}$$

when this limit exists. Note that this is a functor since morphisms between (pre)sheaves are natural transformations.

Thm 2.2.1. Let $\phi: \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

- 1. If ϕ_x is injective for all $x \in X$ then ϕ is injective on sections.
- 2. If ϕ_x is an isomorphism for all $x \in X$ then ϕ is an isomorphism.

Proof. Exercise.

Aside

Although we do not need this right away, given an $A \in \mathcal{A}$ we can define the (pre)sheaf x_*A by

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$
 (2.4)

Proposition 2.2.2. When it exists, the functor $(-)_x : Sh(X) \to \mathcal{A}$ is left adjoint to $x_* : \mathcal{A} \to Sh(X)$.

Proof. To see this simply note that morphisms between \mathscr{F} and $x_*(A)$ correspond naturally to natural transformations between \mathscr{F} restricted to $U \ni x$ and $\Delta(A)$.

Remark 2.2.3. The result also holds in PreSh(X).

2.3 Étalé space of a presheaf and sheafification

For a presheaf \mathscr{F} we are now in the position to define its étalé space. The étalé space of \mathscr{F} , denoted $\operatorname{Sp\'e}(\mathscr{F})$ is the topological space with underlying set $\coprod_{x\in X}\mathscr{F}_x$ and topology generated by the basis of sets given by $\{s_x|x\in U\}$ for $s\in\mathscr{F}(U)$ where $U\subset X$ is open. Together with this space there is also a natural continuous map $\pi:\operatorname{Sp\'e}(\mathscr{F})\to X$ sending an element s_x to x. The sheafification of \mathscr{F} , denoted \mathscr{F}^+ , is then defined to be the sheaf of sections of $\pi:\operatorname{Sp\'e}(\mathscr{F})\to X$. By unwrapping the definitions we see that the sections can be characterised as

$$\mathscr{F}^+(U) = \{s : U \to \coprod_{x \in U} \mathscr{F}_x : \forall x \in U, \exists V \subset U \text{ open containing } x \text{ and}$$

$$t \in \mathscr{F}(V) \text{ s.t. } s(y) = t_y \forall y \in V \}$$
 (2.5)

In particular there is a natural morphism $\mathscr{F} \to \mathscr{F}^+$ sending $s \in \mathscr{F}(U)$ to the section $x \mapsto s_x$ which is an isomorphism on stalks. From the characterisation of sections it clear that if \mathscr{F} is a presheaf of AbGrp, Ring, ... then \mathscr{F}^+ is a sheaf with values in the corresponding abelian category.

We have defined Spé and $(-)^+$ on objects but they can also be turned into functors. If we have a morphism $\phi: \mathscr{F} \to \mathscr{G}$ between presheaves, this induces a continuous map $\operatorname{Sp\acute{e}}(\phi): \operatorname{Sp\acute{e}}(\mathscr{F}) \to \operatorname{Sp\acute{e}}(\mathscr{G})$ given by $s_x \mapsto \phi_x(s_x)$ so that

$$\operatorname{Sp\acute{e}}(\mathscr{F}) \xrightarrow{\operatorname{Sp\acute{e}}(\phi)} \operatorname{Sp\acute{e}}(\mathscr{G}) \tag{2.6}$$

commutes. This construction is functorial and turns Spé into a functor from presheaves to topological bundles over X. It follows that we also obtain a map of sheaves $\phi^+: \mathscr{F}^+ \to \mathscr{G}^+$ by composing sections with $\operatorname{Sp\'e}(\phi)$. Thus we have a functor $(-)^+: \operatorname{PreSh}(X) \to \operatorname{Sh}(X)$ and in fact the following diagram commutes.

Note that since the morphism $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism when \mathscr{F} is a sheaf, this says that the functor $(-)^+$ restricted to $\mathsf{Sh}(X)$ is naturally isomorphism to the identity functor.

Thm 2.3.1. Let $\theta: \mathscr{F} \to \mathscr{F}^+$ be the natural morphism. Then for any morphism of presheaves $\phi: \mathscr{F} \to \mathscr{G}$ with \mathscr{G} a sheaf, there exists a unique morphism of sheaves $\psi: \mathscr{F}^+ \to \mathscr{G}$ so that

$$\begin{array}{ccc}
\mathscr{F}^{+} & \xrightarrow{\psi} \mathscr{G} \\
\emptyset & & & & \\
\mathscr{F} & & & & \\
\mathscr{F} & & & & \\
\end{array} (2.8)$$

commutes.

Proof. This just follows from equation 2.7, the fact that $\theta: \mathcal{G} \to \mathcal{G}^+$ is an isomorphism when \mathcal{G} is a sheaf, and by taking stalks.

Corollary 2.3.2. The sheafification functor is left adjoint to the inclusion functor $\iota : \mathsf{Sh}(X) \to \mathsf{PreSh}(X)$.

Proof. Let \mathscr{F} be a presheaf and \mathscr{G} be a sheaf. Given a morphism $\phi: \mathscr{F}^+ \to \mathscr{G}$ we can precompose it with $\theta: \mathscr{F} \to \mathscr{F}^+$ to obtain a map $\mathscr{F} \to \iota \mathscr{G}$. Conversely, given $\psi: \mathscr{F} \to \iota \mathscr{G}$, we obtain a map $\mathscr{F}^+ \to \mathscr{G}$ from the theorem. Then the theorem says these operations are inverse so we have a bijection

$$\operatorname{Hom}(\mathscr{F}^+,\mathscr{G}) \cong \operatorname{Hom}(\mathscr{F}, \iota\mathscr{G}).$$
 (2.9)

Naturality is then an easy check.

Corollary 2.3.3. The sheafification functor is exact.

Proof. It is a left adjoint so it is right exact. It thus suffices to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is injective then so is ϕ^+ . For this it suffices to show that ϕ_x is injective for all x. But this is obvious.

We can now define the cokernel of a morphism $\phi: \mathscr{F} \to \mathscr{G}$ in $\mathsf{Sh}(X)$. We simply define it to be the sheafification of the cokernel in $\mathsf{PreSh}(X)$ and it is an easy to check to see that this is indeed a cokernel object in $\mathsf{Sh}(X)$. It is then easy to see that ker coker = coker ker by looking at stalks and so $\mathsf{Sh}(X)$ is an abelian category.

Remark 2.3.4. While Sh(X) is a full subcategory of PreSh(X) that is abelian, it is not a full abelian subcategory.

2.4 Exact sequences

Now that we know that we are working in an abelian category we can talk about exact sequences in $\mathsf{Sh}(X)$. Recall from section 1.4 that $\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ is exact at \mathscr{G} if $\phi \circ \theta = 0$ and the map induced map $\mathsf{im}(\theta) \to \mathsf{ker}(\phi)$ is an isomorphism. But the map $\mathsf{im}(\theta) \to \mathsf{ker}(\phi)$ is an isomorphism iff it is an isomorphism at the level of stalks iff $\mathscr{F}_x \xrightarrow{\theta_x} \mathscr{G}_x \xrightarrow{\phi_x} \mathscr{H}_x$ is exact for all $x \in X$. Thus exactness in $\mathsf{Sh}(X)$ can be verified by checking exactness at all the stalks.

2.5 Sheaves over different spaces

2.5.1 Direct image sheaf

Let $f: X \to Y$ be a continuous map between topological spaces and \mathscr{F} a sheaf on X. We define the direct image of \mathscr{F} under f to be the sheaf $f_*\mathscr{F}$ on Y defined by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. If we define f_* on morphisms in the obvious way then it is clear that we obtain a functor $f_*: \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. In fact we also obtain a functor $f_*: \mathsf{PreSh}(X) \to \mathsf{PreSh}(Y)$ and it turns out this functor has nice left adjoint.

Define $\lim_f : \mathsf{PreSh}(Y) \to \mathsf{PreSh}(X)$ to be the functor that sends $\mathscr{F} \in \mathsf{PreSh}(Y)$ to the presheaf $\lim_f (\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ on X, and does the obvious things to morphisms.

Thm 2.5.1. $\lim_{f} \dashv f_*$ as functors between PreSh(X) and PreSh(Y).

Proof. Let $\phi: \lim_f \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For V open in Y, $f^{-1}(V)$ is open in X and so we have maps

$$\mathscr{F}(V) \to \varinjlim_{W \supset f(U)} \mathscr{F}(W) \to \mathscr{G}(U)$$
 (2.10)

where $U = f^{-1}(V)$. If $V' \subset V$, $U = f^{-1}(V)$ and $U' = f^{-1}(V')$ then

$$\mathscr{F}(V) \longrightarrow \varinjlim_{W \supset f(U)} \mathscr{F}(W) \longrightarrow \mathscr{G}(U)
\downarrow \qquad \qquad (2.11)$$

$$\mathscr{F}(V') \longrightarrow \varinjlim_{W \supset f(U')} \mathscr{F}(W) \longrightarrow \mathscr{G}(U')$$

commutes and so these maps in fact define a morphism $\mathscr{F} \to f_*\mathscr{G}$.

Conversely suppose we are given a morphism $\mathscr{F} \to f_*\mathscr{G}$. Let U be open in X. For $V \supset f(U)$ we have maps

$$\mathscr{F}(V) \to \mathscr{G}(f^{-1}(V)) \to \mathscr{G}(U).$$
 (2.12)

Moreover if $V \supset V' \supset f(U)$ then

commutes so we obtain maps $\varinjlim_{V\supset f(U)}\mathscr{F}(V)\to\mathscr{G}(U).$ If $U\supset U'$ we have maps

$$\lim_{\substack{V \supset f(U) \\ V \supset f(U')}} \mathscr{F}(V) \longrightarrow \mathscr{G}(U) \\
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad (2.14)$$

$$\lim_{\substack{V \supset f(U')}} \mathscr{F}(V) \longrightarrow \mathscr{G}(U').$$

A straighforward calculation shows that this commutes and so we obtain a morphism $\lim_f \mathscr{F} \to \mathscr{G}$.

These operations are clearly inverse to each other. A straightforward calculation shows that the bijection is natural.

Corollary 2.5.2. \lim_{f} is an exact functor.

Proof. It is a left adjoint so it is right exact. Thus it suffices to show that it sends injective maps to injective maps. But this is obvious.

2.5.2 Inverse image sheaf

Let $f:X\to Y$ be a continuous map between topological spaces and $\mathscr F$ a sheaf on Y. Let $f^{-1}\mathrm{Sp\acute{e}}(\mathscr F)$ be the pullback

$$f^{-1}\operatorname{Sp\acute{e}}(\mathscr{F}) \xrightarrow{f} Y. \tag{2.15}$$

We define the inverse image sheaf $f^{-1}\mathscr{F}$ to be the sheaf of sections of $\pi: f^{-1}\mathrm{Sp\acute{e}}(\mathscr{F}) \to X$. Equivalently, it is the sheaf

$$f^{-1}\mathscr{F}(U) = \left\{ s : U \to \operatorname{Sp\acute{e}}(\mathscr{F}) : \sup_{s \to f|_{U}} \downarrow_{\pi}^{\pi} \text{ commutes} \right\} \qquad (2.16) \quad \boxed{\operatorname{eq:invimg}}$$

or also equivalently, the sheaf

$$f^{-1}\mathscr{F}(U) = \{s: U \to \coprod_{x \in U} \mathscr{F}_{f(x)} : \forall x \in U, \exists W \subset Y, V \subset f^{-1}(W) \cap U \text{ open and } t \in \mathscr{F}(W) \text{ s.t. } x \in V \land s(y) = t_{f(y)} \forall y \in V \}.$$

$$(2.17)$$

It is clear from the construction that we obtain a functor $f^{-1}: \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.

Remark 2.5.3. A direct calculation shows that $f^{-1}\mathscr{F}_x$ and $\mathscr{F}_{f(x)}$ are naturally isomorphic and so there is a natrual bijection between $f^{-1}\operatorname{Sp\'e}(\mathscr{F})$ and $\operatorname{Sp\'e}(f^{-1}\mathscr{F})$. It is then a straightforward exercise to check that this bijection is in fact a homeomorphism i.e. $f^{-1}\operatorname{Sp\'e}(\mathscr{F})\cong\operatorname{Sp\'e}(f^{-1}\mathscr{F})$.

Thm 2.5.4. f^{-1} is naturally isomorphic to $(-)^+ \circ \lim_f$.

Proof. Let U be an open subset of X and $s \in \lim_f \mathscr{F}(U)$. There is a natural map $\phi_x : (\lim_f \mathscr{F})_x \to \mathscr{F}_{f(x)}$ so we can define a map $U \to \operatorname{Sp\'e}(\mathscr{F})$ by $x \mapsto \phi_x(s_x)$. It is clear that this gives an element of $f^{-1}\mathscr{F}(U)$ as characterised by equation 2.16. Thus we obtain a morphism $\lim_f \mathscr{F} \to f^{-1}\mathscr{F}$. On stalks this map is given by ϕ_x . A direct calculation shows that ϕ_x is an isomorphism for all $x \in X$ and so the induced map $(\lim_f \mathscr{F})^+ \to f^{-1}\mathscr{F}$ must be an isomorphism. It is straightforward to see that this defines a natural transformation.

Corollary 2.5.5. $f^{-1} \dashv f_*$ as functors between Sh(X) and Sh(Y).

Proof. f^{-1} is naturally isomorphic to $(-)^+ \circ \lim_f$ and so for $\mathscr{F} \in \mathsf{Sh}(Y), \mathscr{G} \in \mathsf{Sh}(X)$ we have natural bijections

$$\begin{aligned} \operatorname{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{F},\mathscr{G}) &\cong \operatorname{Hom}_{\mathsf{Sh}(X)}\left((\lim_{f}\mathscr{F})^{+},\mathscr{G}\right) \cong \operatorname{Hom}_{\mathsf{PreSh}(X)}\left(\lim_{f}\mathscr{F},\mathscr{G}\right) \\ &\cong \operatorname{Hom}_{\mathsf{PreSh}(Y)}\left(\mathscr{F},f_{*}\mathscr{G}\right) \cong \operatorname{Hom}_{\mathsf{Sh}(Y)}\left(\mathscr{F},f_{*}\mathscr{G}\right). \end{aligned} \tag{2.18}$$

Corollary 2.5.6. $(-)_x \circ f^{-1} = (-)_{f(x)}$.

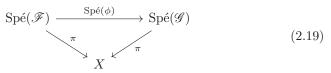
Proof.
$$(-)_x \circ f^{-1} = (-)_x \circ (-)^+ \circ \lim_f = (-)_x \circ \lim_f = (-)_{f(x)}.$$

Corollary 2.5.7. f^{-1} is an exact functor.

Proof. It is the composition of two exact functors. Alternatively take stalks.

2.6 The *Hom* sheaf

Let $\mathscr F$ and $\mathscr G$ be sheaves and $f:\mathrm{Sp\acute{e}}(\mathscr F)\to\mathrm{Sp\acute{e}}(\mathscr G)$ be a continuous map so that



commutes. Then we obtain a morphism $\mathscr{F}^+ \to \mathscr{G}^+$ by postcomposing sections with f. Since \mathscr{F} and \mathscr{G} are sheaves we in fact obtain a morphism $\mathscr{F} \to \mathscr{G}$. But we also know that morphisms $\mathscr{F} \to \mathscr{G}$ give continuous maps $\operatorname{Sp\'e}(\mathscr{F}) \to \operatorname{Sp\'e}(\mathscr{G})$ making the above diagram commute.

2.7 Injective sheaves

There are enough injectives.

Spectral sequences

Group cohomology