# Homological algebra and schemes

Emile T. Okada

February 21, 2019

# Contents

**Contents** 1

**1 Abelian Categories** 3
- 1.1 Additive categories . . . 3
- 1.2 Semiadditive categories . . . 4
- 1.3 Abelian categories . . . 6
- 1.4 Exact sequences . . . 8
- 1.5 Adjoint functors . . . 10
- 1.6 Limits and derived functors . . . 11

**2 Sheaf Theory** 13
- 2.1 Presheaves . . . 13
- 2.2 Sheaves . . . 13
- 2.3 Étalé space of a presheaf and sheafification . . . 14
- 2.4 Maps defined on a basis . . . 16
- 2.5 Exact sequences . . . 16
- 2.6 Direct sums of sheaves . . . 16
- 2.7 Sheaves over different spaces . . . 17
- 2.8 The $\mathscr{H}om$ sheaf . . . 20
- 2.9 Injective sheaves . . . 21

**3 Scheme Theory** **23**
3.1 Locally ringed spaces . . . . . 23
3.2 Morphisms . . . . . 24
3.3 $\mathscr{O}_X$-Modules . . . . . 26
3.4 Sheaf of ideals . . . . . 27
3.5 Reduced schemes . . . . . 28
3.6 Tangent space . . . . . 29

**4 Spectral sequences** **31**

**5 Group cohomology** **33**

**6 Appendix** **35**
6.1 Category theory results . . . . . 35
6.2 Properties of sheaves of rings . . . . . 35
6.3 Locally ringed spaces . . . . . 35
6.4 Restriction . . . . . 36
6.5 Results on schemes . . . . . 36

# CHAPTER 1

# Abelian Categories

## 1.1 Additive categories

Let $\mathcal{A}$ be a category such that the hom-sets carry the structure of an abelian group and composition is bilinear. We call such a category $\mathsf{Ab}$-enriched. An additive category is an $\mathsf{Ab}$-enriched category which has finite coproducts and a zero object.

thm:atosa **Thm 1.1.1.** *Let $\mathcal{A}$ be an additive category. Then finite coproducts in $\mathcal{A}$ are in fact finite biproducts.*

*Proof.* It is easy to see that initial objects are isomorphic to terminal objects (and they both exist) and so it suffices to show the result for binary coproducts. Let $A, B \in \mathcal{A}$. Define $p_A : A \coprod B \to A$ and $p_B : A \coprod B \to B$ as the maps making the following diagrams commute.

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\
& A \coprod B \arrow[r, dashed, "p_A"] & A \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "0"] & & \\
& A \coprod B \arrow[r, dashed, "p_B"] & B \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "\mathrm{id}_B"'] & &
\end{tikzcd}
\tag{1.1}
$$

Let $f = i_A \circ p_A + i_B \circ p_B$. Then

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "i_A"] & & \\
& A \coprod B \arrow[r, "f"] & A \coprod B \\
B \arrow[ur, "i_B"] \arrow[urr, bend right, "i_B"'] & &
\end{tikzcd}
\tag{1.2}
$$

commutes and so by universality we must have $f = \mathrm{id}_{A \coprod B}$. Now suppose we have maps $f : C \to A$ and $g : C \to B$. Let $h : C \to A \coprod B$ be the map $i_A \circ f + i_B \circ g$. Then $p_A \circ h = f$ and $p_B \circ h = g$. Moreover, if $h' : C \to A \coprod B$ is any other map satisfying $p_A \circ h' = f$ and $p_B \circ h' = g$ then $h' = id_{A \coprod B} \circ h' = i_A \circ f + i_B \circ g = h$ and so $A \coprod B$ is a biproduct. ■

A functor between additive categories is called additive if it is a homomorphism on hom-sets.

## 1.2 Semiadditive categories

The above definition of an additive category includes the additive structure on the hom-sets as data. In this section we provide a definition where the additive structure arises as a property instead.

Let $\mathcal{A}$ be a category with a zero object. Recall that in such a category there always exists a morphism between to any two objects $A, B \in \mathcal{A}$ given by $A \to 0 \to B$. We call this the 0 morphism. Moreover if finite coproducts and finite products exists there is a canonical map $A \coprod B \to A \prod B$ arising from the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "0" near start] & A \\ B \arrow[ur, "0"' near start] \arrow[r, "\mathrm{id}_B"'] & B. \end{tikzcd} \tag{1.3}$$

We call a category $\mathcal{A}$ *semiadditive* if it has a zero object, finite products, finite coproducts and the canonical map $A \coprod B \to A \prod B$ is an isomorphism for all $A, B \in \mathcal{A}$. In such a category we write $A \oplus B$ for the biproduct.

**Thm 1.2.1.** *Let $\mathcal{A}$ be a semiadditive category then it is naturally enriched over the monoidal category of commutative monoids.*

*Proof.* Let $\Delta_A : A \oplus A \to A$ and $\nabla_A : A \to A \oplus A$ be the maps that make

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[drr, bend left, "\mathrm{id}_A"] & & \\ & A \oplus A \arrow[r, dashed, "\Delta_A"] & A \\ A \arrow[ur, "i'_A"] \arrow[urr, bend right, "\mathrm{id}_A"'] & & \end{tikzcd} \qquad \begin{tikzcd} & & A \\ A \arrow[r, dashed, "\nabla_A"] \arrow[urr, bend left, "\mathrm{id}_A"] \arrow[drr, bend right, "\mathrm{id}_A"'] & A \oplus A \arrow[ur, "p_A"] \arrow[dr, "p'_A"] & \\ & & A \end{tikzcd} \tag{1.4}$$

commute. Given $f, g : A \to B$ we can construct a map $f \oplus g : A \oplus A \to B \oplus B$ in the obvious way. We can then define $f + g : A \to B$ to be the composite

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.5}$$

Note that there is a map $t_A : A \oplus A \to A \oplus A$ arising from the diagram

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[dr, "\mathrm{id}_A" near start] & A \\ A \arrow[r, "0"'] \arrow[ur, "\mathrm{id}_A"' near start] & A. \end{tikzcd} \tag{1.6}$$

It is then an easy check to see that $\Delta_A \circ t_A = \Delta_A$ and $t_A \circ \nabla_A = \nabla_A$, from which it follows that $+$ is commutative. Straightforward calculations also show that $+$ is associative, distributes over compositions and has the zero map as identity. The result follows. ■

A functor between semiadditive categories is called semiadditive if it preserves zero objects and biproducts i.e. there are isomorphisms $F(A \oplus B) \cong F(A) \oplus F(B)$ such that

$$\begin{tikzcd} F(A) \arrow[dr, "F(i_A)"] \arrow[drr, bend left, "i_{F(A)}"] & & \\ & F(A \oplus B) \arrow[r, "\cong"] & F(A) \oplus F(B) \\ F(B) \arrow[ur, "F(i_B)"] \arrow[urr, bend right, "i_{F(B)}"'] & & \end{tikzcd} \tag{1.7}$$

commutes, and similarly for the projection maps.

prop:sa **Proposition 1.2.2.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a semiadditive functor and $f, g : A \to B$ for $A, B \in \mathcal{A}$. Then $F(f+g) = F(f) + F(g)$.*

*Proof.* Obvious. ■

We now define an additive category to be a semiadditive category where the enriched hom-sets are in fact groups.

thm:as **Thm 1.2.3.** *Let $\mathcal{A}$ be an additive category according to the first definition. By theorem 1.1.1, $\mathcal{A}$ is semiadditive and so the hom-sets naturally carry the structure of a commutative monoid. This monoidal structure agrees with the original group structure.*

*Proof.* Let $A, B \in \mathcal{A}$ and $f, g : A \to B$. Then the addition arising from the semiadditive structure comes from the composition

$$A \xrightarrow{\nabla_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B. \tag{1.8}$$

But $\nabla_A = i_A^L + i_A^R, \Delta_B = p_B^L + p_B^R$ and $f \oplus g = i_B^L \circ f \circ p_A^L + i_B^R \circ g \circ p_A^R$ and so their composition is just $f + g$. ■

**Corollary 1.2.4.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categoires. Then $F$ is additive iff $F$ it is semiadditive.*

*Proof.* Semiadditive $\implies$ additive follows from proposition 1.2.2 and theorem 1.2.3. Additive $\implies$ semiadditive is a straigtforward exercise. ■

**Corollary 1.2.5.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories which is a left adjoint. Then $F$ is additive.*

*Proof.* $F$ preserves colimits and so is semiadditive. ■

**Corollary 1.2.6.** *If $\mathcal{A}$ is an additive category then $\mathcal{A}^{op}$ is also additive.*

*Proof.* The oppositive category of a semiadditive category is clearly also semi-additive. The resulting monoidal structure on the hom-sets are also clearly the same and so the result follows. ■

## 1.3 Abelian categories

Abelian categories are additive categories with more strucure. Before we state exactly what we mean by this we give some definitions.

**Definition 1.3.1.** Let $\mathcal{A}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{A}$.

1. A kernel of $f$ is an equaliser of $A \overset{f}{\underset{0}{\rightrightarrows}} B$ .
2. A cokernel of $f$ is a coequaliser of the same diagram.
3. $f$ is called monic if $f \circ g = 0$ implies $g = 0$ for all $g$.
4. $f$ is called epi if $g \circ f = 0$ implies $g = 0$ for all $g$.

*Remark* 1.3.2. It is easy to see that all kernels are monic, all cokernels are epi, a map is monic iff its kernel is 0, and a map is epi iff its cokernel is 0.

We call an additive category $\mathcal{A}$ pre-abelian if all morphisms have kernels and cokernels. In such a category, given any morphism $f : A \to B$ we can form

$$\begin{tikzcd}
 & & \ker(\mathrm{coker}(f)) \arrow[dr, "i"] & & \\
K \arrow[r, "\ker(f)"] & A \arrow[ur, dashed, "\alpha"] \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[r, "\mathrm{coker}(f)"] & C \\
 & & \mathrm{coker}(\ker(f)) \arrow[ur, dashed, "\beta"'] & &
\end{tikzcd} \tag{1.9}$$

eq:canon-decomp

where $\alpha$ and $\beta$ exist from the universal property of kernels and cokernerls respectively. Since $p$ is epi and $0 = \mathrm{coker}(f) \circ f = \mathrm{coker}(f) \circ \beta \circ p$ it follows that $\mathrm{coker}(f) \circ \beta = 0$ and so there is a map $\gamma : \mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$ such that $i \circ \gamma = \beta$. Similarly there is a map $\gamma' : \mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$ such that $\gamma' \circ p = \alpha$. Using that $p$ is epi one can see that $\gamma' = \gamma$ and so for any morphism $f$ there is a canonical decomposition

$$A \xrightarrow{p} \mathrm{coker}(\ker(f)) \xrightarrow{\gamma_f} \ker(\mathrm{coker}(f)) \xrightarrow{i} B. \tag{1.10}$$

An abelian category is a pre-abelian category in which $\gamma_f$ is an isomorphism for every $f$.

thm:abcat **Thm 1.3.3.** *Let $\mathcal{A}$ be a pre-abelian category. Then $\gamma_f$ is an isomorphism for all morphism $f$ iff every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.*

*Proof.* ($\Rightarrow$) The kernel of a monic is the 0 object with the 0 map, and the cokernel of this is just $A$ together with the identity. Thus, if $\gamma_f$ is an isomorphism the canonical decomposition of $f$ just becomes

$$A \xrightarrow{\mathrm{id}} A \xrightarrow{\cong} \ker(\mathrm{coker}(f)) \xrightarrow{i} B \tag{1.11}$$

and so $f$ is the kernel of its cokernel. Similarly one obtains that if $f$ is epi it is the cokernel of its kernel.

($\Leftarrow$) First note that if a kernel is epi then it must be an isomorphism so all epic monics must be isomorphisms (since all monics are kernels). Thus, it suffices to show that the maps $\alpha$ and $\beta$ in equation 1.9 are epi and monic respectively. To see that $\beta$ is monic let $x : X \to \mathrm{coker}(\ker(f))$ be a map such that $\beta \circ x = 0$. Then let $q : \mathrm{coker}(\ker(f)) \to \mathrm{coker}(x)$ be the coker of $x$, and $j : \mathrm{coker}(x) \to B$ the map such that $j \circ q = \beta$. Finally let $l : \ker(q \circ p) \to A$ be the kernel of $q \circ p$. Then we have the following diagram

$$\begin{tikzcd}
\ker(q\circ p) \arrow[dd, dashed] \arrow[dr, "l"] & & & & \\
& A \arrow[rrr, "f"] \arrow[dr, "p"'] & & & B \\
\ker(f) \arrow[ur, "k"] & & \mathrm{coker}(\ker(f)) \arrow[urr, "\beta"] \arrow[drr, "q"] & & \\
& X \arrow[ur, "x"] & & & \mathrm{coker}(x). \arrow[uu, "j"']
\end{tikzcd} \tag{1.12}$$

Since $q \circ p$ is epi it is the coker of $l$. But also $f \circ l = j \circ q \circ p \circ l = 0$, so $l$ factors through $\ker(f)$ and so $p \circ l = 0$. Thus there exists $p' : \mathrm{coker}(x) \to \mathrm{coker}(\ker(f))$ such that

$$\begin{tikzcd}
\ker(q\circ p) \arrow[r, "l"] & A \arrow[r, "p"] \arrow[d, "q\circ p"] & \mathrm{coker}(\ker(f)) \\
& \mathrm{coker}(x) \arrow[ur, dashed, "p'"'] &
\end{tikzcd} \tag{1.13}$$

commutes. Since $p$ is epi, it must follow that $p' \circ q = \mathrm{id}$. Thus $q$ is monic and so $x = 0$. It follows that $\beta$ is monic. Similarly one can show that $\alpha$ is epi. ■

It follows that an abelian category is equvalently a pre-abelian category in which every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.

**Thm 1.3.4.** *If $\mathcal{A}$ is an abelian category then $\mathcal{A}^{op}$ is also an abelian category.*

*Proof.* It is certainly additive. Moreover, kernels and cokernels simply swap roles. $\gamma_f$ is then still an isomorphism for all $f$ and so $\mathcal{A}^{op}$ is abelian. ■

From now on we write $\mathrm{im}(f) := \ker(\mathrm{coker}(f))$ and $\mathrm{coim}(f) := \mathrm{coker}(\ker(f))$.

## 1.4 Exact sequences

sec:es

Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ be the category with objects given by $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$, and morphisms given by chain maps. Recall from earlier that $f$ can be factored as

$$A \xrightarrow{p_f} \mathrm{im}(f) \xhookrightarrow{i_f} B. \tag{1.14}$$

Since $p_f$ is epi, we must have $g \circ i_f = 0$. Thus we can factor $f$ further through $\ker(g)$ to obtain $f : A \to \mathrm{im}(f) \to \ker(g) \to B$. Let $H(A \xrightarrow{f} B \xrightarrow{g} C)$ be the cokernel of the morphism $\mathrm{im}(f) \to \ker(g)$. If we have the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array} \tag{1.15}$$

then there exists maps so that

$$\begin{array}{ccccccccc} A & \longrightarrow & \mathrm{im}(f) & \longrightarrow & \ker(g) & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & \mathrm{im}(f') & \longrightarrow & \ker(g') & \longrightarrow & B' & \longrightarrow & C' \end{array} \tag{1.16}$$

commutes. In particular there is a morphism

$$\mathrm{coker}(\mathrm{im}(f) \to \ker(g)) \to \mathrm{coker}(\mathrm{im}(f') \to \ker(g')). \tag{1.17}$$

It is easy to check that this construction is functorial and so we obtain a functor $H : \mathcal{S} \to \mathcal{A}$.

One can similarly construct a functor $H' : \mathcal{S} \to \mathcal{A}$ by considering

$$\ker(\mathrm{coker}(f) \to \mathrm{coim}(g)) \tag{1.18}$$

instead.

*Remark* 1.4.1. We may also form a functor by looking simply at the fact that $f$ factors through $\ker(g)$ and then looking at the coker of the resulting morphism $A \to \ker(g)$. It is an easy check to see that this yields a functor naturally isomorphic to $H$. Similarly for $H'$.

**Lemma 1.4.2.** *Let $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. Recall that we have the factorisation*

$$A \to \mathrm{im}(f) \to \ker(g) \xrightarrow{i_g} B \xrightarrow{p_f} \mathrm{coker}(f) \to \mathrm{coim}(g) \to C. \tag{1.19}$$

*Let $h$ be the composition $\ker(g) \to B \to \mathrm{coker}(f)$. Then*

1. $\ker(h) = \mathrm{im}(f) \to \ker(g)$
2. $\mathrm{coker}(h) = \mathrm{coker}(f) \to \mathrm{coim}(g)$.

*Proof.* Let $l : C \to \ker(g)$ be such that $h \circ l = 0$. Then $p_f \circ i_g \circ l = 0$ and so $i_g \circ l$ factors through $\mathrm{im}(f)$. Since $i_g$ is monic it follows that $l$ factors through $\mathrm{im}(f)$. Uniqueness follows automatically. Thus the result follows. The second part follows similarly. ■

**Thm 1.4.3.** *The functors $H, H' : \mathcal{S} \to \mathcal{A}$ are naturally isomorphic.*

*Proof.* Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$ and $h$ be as in the lemma. Then $H(S) = \mathrm{coker}(\ker(h))$ and $H'(S) = \ker(\mathrm{coker}(h))$ so we obtain the factorisation

$$\ker(g) \to H(S) \xrightarrow{\cong} H'(S) \to \mathrm{coker}(f). \tag{1.20}$$

Naturality of the isomophism then follows from naturality of this factorisation. ■

*Remark* 1.4.4. In a pre-abelian category we still have a natrual transformation $H \Rightarrow H'$, but it might not be an isomorphism.

**Definition 1.4.5.** Let $S := A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{S}$. We say that $S$ is exact at $B$ if $H(S) = 0$.

**Proposition 1.4.6.** $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a short exact sequence iff* $A = \ker(g)$ *and* $C = \mathrm{coker}(f)$.

*Proof.* ($\Rightarrow$) We have $\ker(g) \cong \mathrm{im}(f) \cong A$ and $\mathrm{coker}(f) \cong \mathrm{coim}(g) \cong C$.
($\Leftarrow$) Certainly have exactness at $A$ and $C$. Exactness at $B$ also holds. ■

### 1.4.1 Split sequences

**Thm 1.4.7.** *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. The following are equivalent*

1. *there exists $q : B \to A$ such that $q \circ f = \mathrm{id}_A$*
2. *there exists $p : C \to B$ such that $g \circ p = \mathrm{id}_C$*
3. *there is an isomorphism $h : B \to A \oplus C$ such that $h \circ f$ and $g \circ h^{-1}$ are the natural inclusion and projection respectively.*

*Proof.* (3) certainly implies both (1) and (2).

(2) $\Rightarrow$ (3) Let $q : B \to A$ be the unique map making the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C. \\ B \arrow[u, dashed, "q"] \arrow[ur, "\mathrm{id}_B - p\circ g"'] & & \end{tikzcd} \tag{1.21}$$

Then $\mathrm{id}_B = p \circ g + f \circ q$. It follows that $p = f \circ q \circ p + p$. Since $f$ is monic we have $q \circ p = 0$. Thus $q = q \circ f \circ q$ and so since $q$ is epi, $q \circ f = \mathrm{id}_A$. The result follows. (1) $\Rightarrow$ (3) follows similarly. ■

**Corollary 1.4.8.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Then $F$ applied to a split short exact sequence is also split exact.*

**Proposition 1.4.9.** *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. If either*

1. *$A$ is injective or*
2. *$C$ is projective*

*then the sequence is split.*

## 1.5 Adjoint functors

Let $L : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. If $L$ admits a right adjoint $R : \mathcal{B} \to \mathcal{A}$ then it turns out $L$ has a lot of useful properties. In this section we explore these properties.

**Proposition 1.5.1.** *Suppose $L \dashv R$. Then $L$ is right exact and $R$ is left exact.*

*Proof.* Consider the short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$. For every $A \in \mathcal{A}$ we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(L(A), B_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(L(A), B_2) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(L(A), B_3) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{Hom}(A, R(B_1)) \arrow[r] & \mathrm{Hom}(A, R(B_2)) \arrow[r] & \mathrm{Hom}(A, R(B_3)) \end{tikzcd} \tag{1.22}$$

where the top row is exact. It follows that the bottom row is exact for all $A$ and so the bottow row is too. It follows that

$$0 \longrightarrow R(B_1) \longrightarrow R(B_2) \longrightarrow R(B_3) \tag{1.23}$$

is exact and so $R$ is left exact. By a similar argument $L$ is right exact. ■

**Proposition 1.5.2.** *Suppose $L \dashv R$. Then*

1. *if $L$ is exact then $R$ preserves injectives*
2. *if $R$ is exact then $L$ preserves projectives.*

*Proof.* Suppose $L$ is exact and $I$ is an injective object in $\mathcal{B}$. We need to show that $\mathrm{Hom}(-, R(I))$ is exact. To do this it suffices to show that given $f : A \to B$ injective, the map $f^* : \mathrm{Hom}(B, R(I)) \to \mathrm{Hom}(A, R(I))$ is surjective. But $L$ is exact so $Lf$ is injective and so $(Lf)^* : \mathrm{Hom}(LB, I) \to \mathrm{Hom}(LA, I)$ is surjective. We also have that $L \dashv R$ and so

$$\begin{array}{ccc} \mathrm{Hom}(L(B), I) & \xrightarrow{(Lf)^*} & \mathrm{Hom}(L(A), I) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{Hom}(B, R(I)) & \xrightarrow{f^*} & \mathrm{Hom}(A, R(I)) \end{array} \tag{1.24}$$

commutes. It follows that $f^*$ is surjective as required.

The corresponding result for $R$ follows similarly. ■

## 1.6 Limits and derived functors

**Proposition 1.6.1.** *An abelian category $\mathcal{A}$ is cocomplete iff it has all direct sums.*

*Proof.* We already have kernels and hence equalisers so the statement follows. ■

*Remark* 1.6.2. The same result holds if we replace direct sums with product and cocomplete with complete.

**Thm 1.6.3.** *Let $\mathcal{A}$ be a cocomplete abelian category with enough projectives. If $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every set $\{A_i\}$ of objects in $\mathcal{A}$ we have*

$$L_* F\left(\bigoplus_{i \in I} A_i\right) \cong \bigoplus_{i \in I} L_* F(A_i). \tag{1.25}$$

*Proof.* Let $P_i \to A_i$ be projective resolutions. Then $\oplus_i P_i \to \oplus_i A_i$ is also a projective resolution. Hence

$$L_* F(\oplus_i A_i) = H_*(F(\oplus_i P_i)) \cong H_*(\oplus_i F(P_i)) \cong \oplus_i H_*(F(P_i)) = \oplus_i L_* F(A_i). \tag{1.26}$$

■

### 1.6.1 Filtered colimits

**Definition 1.6.4.** A category $I$ is called filtered if it has coproduct and coequaliser diagrams. A filtered colimit is the colimit of a functor from a filtered category.

**Lemma 1.6.5.** *Let $I$ be a filtered category, and $A : I \to \mathsf{Mod} - R$. Then*

1. *Every element $a \in colim_I A$ is the image of some element $a_i \in A_i$ for some $i \in I$ under the canonical map $A_i \to colim_I A$.*

2. *For every $i$, the kernel of the canonical map $A_i \to colim_I A$ is the union of the kernels of the maps $A(\phi) : A_i \to A_j$ for $\phi : i \to j$ in $I$.*

*Proof.* Use the explicit construction of the colimit as the cokernel of

$$\bigoplus_{i \to j} A_i \to \bigoplus_i A_i. \tag{1.27}$$

■

**Thm 1.6.6.** *Filtered colimits of $R$-modules are exact considered as functors from $\mathsf{Fun}(I, \mathsf{Mod} - R)$ to $\mathsf{Mod} - R$.*

*Proof.* We know that colim is a left adjoint and so is right exact. It thus suffices to show that if $t : A \to B$ is monic then $\mathrm{colim}_I A \to \mathrm{colim}_I B$ is too. But this follows immediately from the previous proposition. ■

**Definition 1.6.7.** We say an abelian category $\mathcal{A}$ satisfies axiom (AB5) if it is cocomplete and filtered colimits are exact.

**Thm 1.6.8.** *Let $\mathcal{A}$ be an abelian category satisfying axiom (AB5). Then for $F : \mathcal{A} \to \mathcal{B}$ a left adjoint, we have that for all filtered $I$,*

$$L_* F(colim_I A) \cong colim_I L_* F(A_i). \tag{1.28}$$

*Proof.* $\mathrm{colim}_I$ is exact so commutes with $H_i$. The rest of the proof is similar to the direct sum proof. ■

# CHAPTER 2

# Sheaf Theory

ch:sheafs

## 2.1 Presheaves

Let $\mathcal{C}$ be any category, $\mathcal{A}$ be an abelian category and define $\mathsf{PreSh}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{A})$ to be the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$. The functor sending all objects to 0 is certainly both initial and terminal, direct sums can be defined pointwise, and the hom-sets in $\mathsf{PreSh}(\mathcal{C})$ inherit an additive structure from $\mathcal{A}$ so $\mathsf{PreSh}(\mathcal{C})$ is naturally an additive category. Moreover kernels and cokernels can be contructed in the obvious way and it is clear that they satisfy the axioms for an abelian category and so $\mathsf{PreSh}(C)$ is abelian.

## 2.2 Sheaves

To define sheaves we restrict to the case when $X$ be a topological space, $\mathcal{U}$ the poset of open sets of X, and $\mathcal{A}$ be an abelian category. We write $\mathsf{PreSh}(X)$ for $\mathsf{PreSh}(\mathcal{U})$. The category of sheaves on $X$ with values in $\mathcal{A}$, $\mathsf{Sh}(X)$, is defined to be the full subcategory of $\mathsf{PreSh}(X)$ with objects given by presheaves $\mathscr{F}$ for which the following diagram is an equalizer for all open coverings $U = \cup_i U_i$

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.1}$$

Since $\mathcal{A}$ is an abelian category this is equivalent to the following diagram being exact

$$0 \to \mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \xrightarrow{\text{diff}} \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{2.2}$$

Note that since $\varnothing$ admits the empty covering and the empty product is 0 this forces $\mathscr{F}(\varnothing) = 0$.

As in the case of $\mathsf{PreSh}(\mathcal{C})$, $\mathsf{Sh}(X)$ is an additive category. However, the cokernel of a morphism between sheaves need not be a sheaf and so we must do some more work to show that $\mathsf{Sh}(X)$ is abelian.

Fix $x \in X$. For a (pre)sheaf $\mathscr{F}$ define the stalk of $\mathscr{F}$ at $x$ to be

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F} \tag{2.3}$$

when this limit exists. Note that this is a functor since morphisms between (pre)sheaves are natural transformations.

**Thm 2.2.1.** *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

1. *If $\phi_x$ is injective for all $x \in X$ then $\phi$ is injective on sections.*
2. *If $\phi_x$ is an isomorphism for all $x \in X$ then $\phi$ is an isomorphism.*

*Proof.* Exercise. ∎

**Proposition 2.2.2.** *Let $\mathscr{F}, \mathscr{G}$ be presheaves and $\phi, \psi : \mathscr{F} \to \mathscr{G}$ be morphisms that are equal on stalks. If $\mathscr{G}$ satisfies sheaf condition (A) then $\phi = \psi$.*

*Proof.* Consider $\phi - \psi$. ∎

### Aside

Although we do not need this right away, given an $A \in \mathcal{A}$ we can define the (pre)sheaf $x_*A$ by

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

**Proposition 2.2.3.** *When it exists, the functor $(-)_x : \mathsf{Sh}(X) \to \mathcal{A}$ is left adjoint to $x_* : \mathcal{A} \to \mathsf{Sh}(X)$.*

*Proof.* To see this simply note that morphisms between $\mathscr{F}$ and $x_*(A)$ correspond naturally to natrual transformations between $\mathscr{F}$ restricted to $U \ni x$ and $\Delta(A)$. ∎

*Remark* 2.2.4. The result also holds in $\mathsf{PreSh}(X)$.

## 2.3 Étalé space of a presheaf and sheafification

For a presheaf $\mathscr{F}$ we are now in the position to define its étalé space. The étalé space of $\mathscr{F}$, denoted Spé($\mathscr{F}$) is the topological space with underlying set $\amalg_{x \in X} \mathscr{F}_x$ and topology generated by the basis of sets given by $\{s_x | x \in U\}$ for $s \in \mathscr{F}(U)$ where $U \subset X$ is open. Together with this space there is also a natural continuous map $\pi : \text{Spé}(\mathscr{F}) \to X$ sending an element $s_x$ to $x$. The sheafification of $\mathscr{F}$, denoted $\mathscr{F}^+$, is then defined to be the sheaf of sections of $\pi : \text{Spé}(\mathscr{F}) \to X$. By unwrapping the definitions we see that the sections can be characterised as

$$\begin{aligned} \mathscr{F}^+(U) = \{s : U \to \amalg_{x \in U} \mathscr{F}_x : &\forall x \in U, \exists V \subset U \text{ open containing } x \text{ and} \\ &t \in \mathscr{F}(V) \text{ s.t. } s(y) = t_y \forall y \in V\} \end{aligned} \tag{2.5}$$

In particular there is a natural morphism $\mathscr{F} \to \mathscr{F}^+$ sending $s \in \mathscr{F}(U)$ to the section $x \mapsto s_x$ which is an isomorphism on stalks. From the characterisation of sections it clear that if $\mathscr{F}$ is a presheaf of $\mathsf{AbGrp}, \mathsf{Ring}, \ldots$ then $\mathscr{F}^+$ is a sheaf with values in the corresponding abelian category.

We have defined Spé and $(-)^+$ on objects but they can also be turned into functors. If we have a morphism $\phi : \mathscr{F} \to \mathscr{G}$ between presheaves, this induces a continuous map $\mathrm{Spé}(\phi) : \mathrm{Spé}(\mathscr{F}) \to \mathrm{Spé}(\mathscr{G})$ given by $s_x \mapsto \phi_x(s_x)$ so that

$$\begin{tikzcd} \mathrm{Spé}(\mathscr{F}) \arrow[rr, "\mathrm{Spé}(\phi)"] \arrow[dr, "\pi"'] & & \mathrm{Spé}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.6}$$

commutes. This construction is functorial and turns Spé into a functor from presheaves to topological bundles over $X$. It follows that we also obtain a map of sheaves $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ by composing sections with $\mathrm{Spé}(\phi)$. Thus we have a functor $(-)^+ : \mathsf{PreSh}(X) \to \mathsf{Sh}(X)$ and in fact the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\phi^+"] & \mathscr{G}^+ \\ \mathscr{F} \arrow[u] \arrow[r, "\phi"] & \mathscr{G} \arrow[u] \end{tikzcd} \tag{2.7}$$

eq:sheafif

Note that since the morphism $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism when $\mathscr{F}$ is a sheaf, this says that the functor $(-)^+$ restricted to $\mathsf{Sh}(X)$ is natrually isomorphism to the identity functor.

**Thm 2.3.1.** *Let $\theta : \mathscr{F} \to \mathscr{F}^+$ be the natural morphism. Then for any morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ a sheaf, there exists a unique morphism of sheaves $\psi : \mathscr{F}^+ \to \mathscr{G}$ so that*

$$\begin{tikzcd} \mathscr{F}^+ \arrow[r, "\psi"] & \mathscr{G} \\ \mathscr{F} \arrow[u, "\theta"] \arrow[ur, "\phi"] & \end{tikzcd} \tag{2.8}$$

*commutes.*

*Proof.* This just follows from equation 2.7, the fact that $\theta : \mathscr{G} \to \mathscr{G}^+$ is an isomorphism when $\mathscr{G}$ is a sheaf, and by taking stalks. ∎

**Corollary 2.3.2.** *The sheafification functor is left adjoint to the inclusion functor $\iota : \mathsf{Sh}(X) \to \mathsf{PreSh}(X)$.*

*Proof.* Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. Given a morphism $\phi : \mathscr{F}^+ \to \mathscr{G}$ we can precompose it with $\theta : \mathscr{F} \to \mathscr{F}^+$ to obtain a map $\mathscr{F} \to \iota\mathscr{G}$. Conversely, given $\psi : \mathscr{F} \to \iota\mathscr{G}$, we obtain a map $\mathscr{F}^+ \to \mathscr{G}$ from the theorem. Then the theorem says these operations are inverse so we have a bijection

$$\mathrm{Hom}(\mathscr{F}^+, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \iota\mathscr{G}). \tag{2.9}$$

Naturality is then an easy check. ∎

**Corollary 2.3.3.** *The sheafification functor is exact.*

*Proof.* It is a left adjoint so it is right exact. It thus suffices to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is injective then so is $\phi^+$. For this it suffices to show that $\phi_x$ is injective for all $x$. But this is obvious. ■

We can now define the cokernel of a morphism $\phi : \mathscr{F} \to \mathscr{G}$ in $\mathsf{Sh}(X)$. We simply define it to be the sheafification of the cokernel in $\mathsf{PreSh}(X)$ and it is an easy to check to see that this is indeed a cokernel object in $\mathsf{Sh}(X)$. It is then easy to see that $\ker \operatorname{coker} = \operatorname{coker} \ker$ by looking at stalks and so $\mathsf{Sh}(X)$ is an abelian category.

*Remark* 2.3.4. While $\mathsf{Sh}(X)$ is a full subcategory of $\mathsf{PreSh}(X)$ that is abelian, it is not a full abelian subcategory.

## 2.4 Maps defined on a basis

**Thm 2.4.1.** *Let $\mathscr{F}, \mathscr{G}$ be sheafs on $X$ and let $\mathcal{B}$ be a basis for the topology on $X$. Then any morphism $\phi|_{\mathcal{B}} : \mathscr{F}|_{\mathcal{B}} \to \mathscr{G}|_{\mathcal{B}}$ extends uniquely to a morphism $\phi : \mathscr{F} \to \mathscr{G}$. Moreover this procedure is functorial.*

*Proof.* There is a natural isomorphism between $\varinjlim_{U \ni x} \mathscr{F}$ and $\varinjlim_{B \ni x} \mathscr{F}$. Thus we obtain a map $\phi := \phi|_{\mathcal{B}}^+ : \mathscr{F} \to \mathscr{G}$. It is clear that this is a morphism of sheaves. Moreover for $U \in \mathcal{B}$ and $s \in \mathscr{F}(U)$ it is clear that $\phi(U)(s)$ and $\phi|_{\mathcal{B}}(U)(s)$ have the same stalks and so must be equal. Thus $\phi$ extends $\phi|_{\mathcal{B}}$. Finally, if a morphism extends $\phi|_{\mathcal{B}}$ then it is determined on stalks and hence must equal to $\phi$, which gives us uniqueness. Functoriality is clear. ■

## 2.5 Exact sequences

Now that we know that we are working in an abelian category we can talk about exact sequences in $\mathsf{Sh}(X)$. Recall from section 1.4 that $\mathscr{F} \xrightarrow{\theta} \mathscr{G} \xrightarrow{\phi} \mathscr{H}$ is exact at $\mathscr{G}$ if $\phi \circ \theta = 0$ and the map induced map $\operatorname{im}(\theta) \to \ker(\phi)$ is an isomorphism. But the map $\operatorname{im}(\theta) \to \ker(\phi)$ is an isomorphism iff it is an isomorphism at the level of stalks iff $\mathscr{F}_x \xrightarrow{\theta_x} \mathscr{G}_x \xrightarrow{\phi_x} \mathscr{H}_x$ is exact for all $x \in X$. Thus $(-)_x$ is an exact functor and exactness in $\mathsf{Sh}(X)$ can be verified by checking exactness at all the stalks.

## 2.6 Direct sums of sheaves

If $\mathcal{A}$ has direct sums, then so does $\mathsf{PreSh}(X)$ since we can compute the direct sum pointwise. It follows that $\mathsf{PreSh}(X)$ is cocomplete. The sheafification of the direct sum in $\mathsf{PreSh}(X)$ gives us a direct sum in $\mathsf{Sh}(X)$ and hence $\mathsf{Sh}(X)$ is also cocomplete.

We also have products in both $\mathsf{PreSh}(X)$ and $\mathsf{Sh}(X)$ (computed pointwise) and so they are also both complete.

## 2.7 Sheaves over different spaces

### 2.7.1 Direct image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on $X$. We define the direct image of $\mathscr{F}$ under $f$ to be the sheaf $f_*\mathscr{F}$ on $Y$ defined by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. If we define $f_*$ on morphisms in the obvious way then it is clear that we obtain a functor $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. In fact we also obtain a functor $f_* : \mathsf{PreSh}(X) \to \mathsf{PreSh}(Y)$ and it turns out this functor has nice left adjoint.

Define $\lim_f : \mathsf{PreSh}(Y) \to \mathsf{PreSh}(X)$ to be the functor that sends $\mathscr{F} \in \mathsf{PreSh}(Y)$ to the presheaf $\lim_f(\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ on $X$, and does the obvious things to morphisms.

**Thm 2.7.1.** $\lim_f \dashv f_*$ *as functors between* $\mathsf{PreSh}(X)$ *and* $\mathsf{PreSh}(Y)$.

*Proof.* Let $\phi : \lim_f \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. For $V$ open in $Y$, $f^{-1}(V)$ is open in $X$ and so we have maps

$$\mathscr{F}(V) \to \varinjlim_{W \supset f(U)} \mathscr{F}(W) \to \mathscr{G}(U) \tag{2.10}$$

where $U = f^{-1}(V)$. If $V' \subset V$, $U = f^{-1}(V)$ and $U' = f^{-1}(V')$ then

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r] \arrow[d] \arrow[dr] & \varinjlim_{W \supset f(U)} \mathscr{F}(W) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(V') \arrow[r] & \varinjlim_{W \supset f(U')} \mathscr{F}(W) \arrow[r] & \mathscr{G}(U')
\end{tikzcd} \tag{2.11}$$

commutes and so these maps in fact define a morphism $\mathscr{F} \to f_*\mathscr{G}$.

Conversely suppose we are given a morphism $\mathscr{F} \to f_*\mathscr{G}$. Let $U$ be open in $X$. For $V \supset f(U)$ we have maps

$$\mathscr{F}(V) \to \mathscr{G}(f^{-1}(V)) \to \mathscr{G}(U). \tag{2.12}$$

Moreover if $V \supset V' \supset f(U)$ then

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r] \arrow[d] & \mathscr{G}(f^{-1}(V)) \arrow[d] \arrow[dr] & \\
 & & \mathscr{G}(U) \\
\mathscr{F}(V') \arrow[r] & \mathscr{G}(f^{-1}(V')) \arrow[ur] &
\end{tikzcd} \tag{2.13}$$

commutes so we obtain maps $\varinjlim_{V\supset f(U)} \mathscr{F}(V) \to \mathscr{G}(U)$. If $U \supset U'$ we have maps

$$\begin{array}{ccc} \varinjlim\limits_{V\supset f(U)} \mathscr{F}(V) & \longrightarrow & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \varinjlim\limits_{V\supset f(U')} \mathscr{F}(V) & \longrightarrow & \mathscr{G}(U'). \end{array} \tag{2.14}$$

A straighforward calculation shows that this commutes and so we obatin a morphism $\lim_f \mathscr{F} \to \mathscr{G}$.

These operations are clearly inverse to each other. A straightforward calculation shows that the bijection is natural. ∎

**Corollary 2.7.2.** $\lim_f$ *is an exact functor.*

*Proof.* It is a left adjoint so it is right exact. Thus it suffices to show that it sends injective maps to injective maps. But this is obvious. ∎

### Stalks

**Proposition 2.7.3.** *Let $\mathscr{F}$ be a sheaf on $X$ and $f : X \to Y$ a continuous map. Then there is a natural map $(f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$ in the sense that if $\mathscr{G}$ is another sheaf on $X$ and $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism then*

$$\begin{array}{ccc} (f_*\mathscr{F})_{f(p)} & \xrightarrow{(f_*\phi)_{f(p)}} & (f_*\mathscr{G})_{f(p)} \\ \downarrow & & \downarrow \\ \mathscr{F}_p & \xrightarrow{\phi_p} & \mathscr{G}_p \end{array} \tag{2.15}$$

*commutes.*

*Proof.* We have

$$(f_*\mathscr{F})_{f(p)} = \varinjlim_{U\ni f(p)} f_*\mathscr{F}(U) = \varinjlim_{U:f^{-1}(U)\ni p} \mathscr{F}(f^{-1}(U)). \tag{2.16}$$

But $\{U : f^{-1}(U) \ni p\} \subseteq \{V : V \ni p\}$ and so there is map

$$(f_*\mathscr{F})_{f(p)} = \varinjlim_{U:f^{-1}(U)\ni p} \mathscr{F}(f^{-1}(U)) \to \varinjlim_{V\ni p} \mathscr{F}(V) = \mathscr{F}_p. \tag{2.17}$$

Naturality is an easy exercise. ∎

### 2.7.2 Inverse image sheaf

Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ a sheaf on $Y$. Let $f^{-1}\text{Spé}(\mathscr{F})$ be the pullback

$$\begin{tikzcd} f^{-1}\text{Spé}(\mathscr{F}) \arrow[r, dashed] \arrow[d, dashed, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \text{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.18}$$

We define the inverse image sheaf $f^{-1}\mathscr{F}$ to be the sheaf of sections of $\pi : f^{-1}\text{Spé}(\mathscr{F}) \to X$. Equivalently, it is the sheaf

$$f^{-1}\mathscr{F}(U) = \left\{ s : U \to \text{Spé}(\mathscr{F}) : \begin{tikzcd} & \text{Spé}(\mathscr{F}) \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "f|_U"'] & Y \end{tikzcd} \text{ commutes} \right\} \tag{2.19}$$

eq:invimg

or also equivalently, the sheaf

$$f^{-1}\mathscr{F}(U) = \{s : U \to \amalg_{x\in U}\mathscr{F}_{f(x)} : \forall x \in U, \exists W \subset Y, V \subset f^{-1}(W) \cap U \text{ open and } t \in \mathscr{F}(W) \text{ s.t. } x \in V \wedge s(y) = t_{f(y)} \forall y \in V\}. \tag{2.20}$$

It is clear from the construction that we obtain a functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$.

*Remark* 2.7.4. A direct calculation shows that $f^{-1}\mathscr{F}_x$ and $\mathscr{F}_{f(x)}$ are naturally isomorphic and so there is a natrual bijection between $f^{-1}\text{Spé}(\mathscr{F})$ and $\text{Spé}(f^{-1}\mathscr{F})$. It is then a straightforward exercise to check that this bijection is in fact a homeomorphism i.e. $f^{-1}\text{Spé}(\mathscr{F}) \cong \text{Spé}(f^{-1}\mathscr{F})$.

**Thm 2.7.5.** *$f^{-1}$ is naturally isomorphic to $(-)^+ \circ \lim_f$.*

*Proof.* Let $U$ be an open subset of $X$ and $s \in \lim_f \mathscr{F}(U)$. There is a natural map $\phi_x : (\lim_f \mathscr{F})_x \to \mathscr{F}_{f(x)}$ so we can define a map $U \to \text{Spé}(\mathscr{F})$ by $x \mapsto \phi_x(s_x)$. It is clear that this gives an element of $f^{-1}\mathscr{F}(U)$ as characterised by equation 2.19. Thus we obtain a morphism $\lim_f \mathscr{F} \to f^{-1}\mathscr{F}$. On stalks this map is given by $\phi_x$. A direct calculation shows that $\phi_x$ is an isomorphism for all $x \in X$ and so the induced map $(\lim_f \mathscr{F})^+ \to f^{-1}\mathscr{F}$ must be an isomorphism. It is straightforward to see that this defines a natural transformation. ■

**Corollary 2.7.6.** *$f^{-1} \dashv f_*$ as functors between $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$.*

*Proof.* $f^{-1}$ is naturally isomophic to $(-)^+ \circ \lim_f$ and so for $\mathscr{F} \in \mathsf{Sh}(Y), \mathscr{G} \in \mathsf{Sh}(X)$ we have natural bijections

$$\begin{aligned} \text{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) &\cong \text{Hom}_{\mathsf{Sh}(X)}\left((\lim_f \mathscr{F})^+, \mathscr{G}\right) \cong \text{Hom}_{\mathsf{PreSh}(X)}\left(\lim_f \mathscr{F}, \mathscr{G}\right) \\ &\cong \text{Hom}_{\mathsf{PreSh}(Y)}\left(\mathscr{F}, f_*\mathscr{G}\right) \cong \text{Hom}_{\mathsf{Sh}(Y)}\left(\mathscr{F}, f_*\mathscr{G}\right). \end{aligned} \tag{2.21}$$

■

**Corollary 2.7.7.** $(-)_x \circ f^{-1} = (-)_{f(x)}$.

*Proof.* $(-)_x \circ f^{-1} = (-)_x \circ (-)^+ \circ \lim_f = (-)_x \circ \lim_f = (-)_{f(x)}$. ■

**Corollary 2.7.8.** *$f^{-1}$ is an exact functor.*

*Proof.* It is the composition of two exact functors. Alternatively take stalks. ■

**Corollary 2.7.9.** *There are natural transformations $e : \mathrm{id} \Rightarrow f_* f^{-1}$ and $\epsilon : f^{-1} f_* \to \mathrm{id}$ such that*

$$f^{-1} \xRightarrow{f^{-1}e} f^{-1} f_* f^{-1} \xRightarrow{\epsilon f^{-1}} f^{-1} \tag{2.22}$$

$$f_* \xRightarrow{e f_*} f_* f^{-1} f_* \xRightarrow{f_* \epsilon} f_* \tag{2.23}$$

*both compose to the identity natural transformation.*

## 2.8 The $\mathscr{H}om$ sheaf

**Lemma 2.8.1.** *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves and $f : Sp\acute{e}(\mathscr{F}) \to Sp\acute{e}(\mathscr{G})$ be a continuous map so that*

$$\begin{tikzcd} Sp\acute{e}(\mathscr{F}) \arrow[rr, "f"] \arrow[dr, "\pi"] & & Sp\acute{e}(\mathscr{G}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{2.24}$$

*commutes. Let $\widetilde{f} : \mathscr{F}^+ \to \mathscr{G}^+$ be the morphism obtained by postcomposing sections with $f$. Then $\widetilde{f}_x = f|_x$.*

*Proof.* This follows from the fact that if $s \in \mathscr{F}^+(U)$ then for $x \in U$, $s_x = s(x)$. ■

**Thm 2.8.2.** *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves. Then there is a bijection between continuous maps $Sp\acute{e}(\mathscr{F}) \to Sp\acute{e}(\mathscr{G})$ and morphisms of sheaves $\mathscr{F} \to \mathscr{G}$.*

*Proof.* For sheaves we have $\mathscr{F} \cong \mathscr{F}^+$ and so the results follows from the lemma. ■

**Corollary 2.8.3.** *The presheaf $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ defined by*

$$\mathscr{H}om(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \tag{2.25}$$

*is in fact a sheaf.*

## 2.9 Injective sheaves

**Definition 2.9.1.** Let $\mathscr{F}$ be a sheaf. Define $D(\mathscr{F})$ to be the sheaf of all (not necessarily continuous) sections of $\mathrm{Spé}(\mathscr{F}) \to X$.

**Lemma 2.9.2.** $D(\mathscr{F}) = \prod_{x \in X} x_*(\mathscr{F}_x)$.

*Proof.* Obvious. ∎

**Thm 2.9.3.** $\mathsf{Sh}(X)$ *over the abelian category* $\mathsf{AbGrp}/\mathsf{Ring}/\mathsf{Mod}_R$ *has enough injectives.*

*Proof.* Let $\mathcal{A}$ denote the abelian category. Recall that $x_* : \mathcal{A} \to \mathsf{Sh}(X)$ is the right adjoint of an exact functor. Thus it is left exact and preserves injectives. Let $x \in X$. $\mathcal{A}$ has enough injectives, so there is some injective object $I_x$ such that $0 \to \mathscr{F}_x \to I_x$ is exact. It follows that $0 \to x_*(\mathscr{F}_x) \to x_*(I_x)$ is also exact. We can then form the exact sequence $0 \to \prod_{x \in X} x_*(\mathscr{F}_x) \to \prod_{x \in X} x_*(I_x)$. The last term is injective since it is a product of injective objects. Composing this with the canonical map $\mathscr{F} \to D(\mathscr{F})$ gives the required injection into an injective object. ∎

# CHAPTER 3

# Scheme Theory

## 3.1 Locally ringed spaces

A locally ringed space is a pair $(X, \mathscr{O}_X)$ consisting of a topological space $X$ and a sheaf $\mathscr{O}_X$ of rings on $X$ such that the stalks are local rings. A morphism of between the locally ringed spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ is a pair $(f, f^\#)$ consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ which induces morphisms of local rings on stalks $f^\#_p : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$.

Given morphisms $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\#) : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ we define their composition $(h, h^\#) : (X, \mathscr{O}_X) \to (Z, \mathscr{O}_Z)$ by $h = g \circ f$ and

$$h^\# = \mathscr{O}_Z \to g_*\mathscr{O}_Y \to g_*(f_*\mathscr{O}_X) = h_*\mathscr{O}_X. \tag{3.1}$$

Note that

$$\begin{tikzcd}
\mathscr{O}_{Z,h(p)} \arrow[r] \arrow[rr, bend left, "(h^\#)_{h(p)}"] \arrow[dr, "g^\#_{f(p)}"'] & (g_*\mathscr{O}_Y)_{g\circ f(p)} \arrow[r] \arrow[d] & (g_*f_*\mathscr{O}_X)_{g\circ f(p)} \arrow[d] \arrow[dd, bend left] \\
& \mathscr{O}_{Y,f(p)} \arrow[r] \arrow[dr, "f^\#_p"'] & (f_*\mathscr{O}_X)_{f(p)} \arrow[d] \\
& & \mathscr{O}_{X,p}
\end{tikzcd} \tag{3.2}$$

commutes and so $h^\#_p = f^\#_p \circ g^\#_{f(p)}$ is a morphism of local rings and so $(h, h^\#)$ is indeed a morphism of locally ringed spaces.

`prop:factor` **Proposition 3.1.1.** *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. If $f(X) \subseteq U$ for some open subset $U \subseteq Y$ then $(f, f^\#)$ factors through $(U, \mathscr{O}_Y|_U)$.*

*Proof.* Let $\bar{f} : X \to U$ denote the map $f$ viewed as having codomain $U$, and $i : U \to Y$. Then $f = i \circ \bar{f}$. Moreover, there is a natural morphism

$i^\# : \mathscr{O}_Y \to i_*(\mathscr{O}_Y|_U)$ given by the restriction maps. Since $\bar{f}^{-1}(V) = f^{-1}(V)$ for $V \subseteq U$, there is also a natural map $\bar{f}^\# : \mathscr{O}_Y|_U \to \bar{f}_*\mathscr{O}_X$ given by the restriction of $f^\#$. It is straightforward to see that $f^\# = i^\# \circ \bar{f}^\#$. ■

**Thm 3.1.2.** *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. $(f, f^\#)$ is an isomorphism iff $f$ is a homeomorphism and $f^\#$ is an isomorphism.*

*Proof.* The forwards direction is obvious. Now suppose $f$ is a homeomorphism and $f^\#$ is an isomorphism. Let $g = f^{-1} : Y \to X$ and $g^\# = (g_* f^\#)^{-1}$. Then $(g, g^\#) \circ (f, f^\#) = \mathrm{id}$ and $(f, f^\#) \circ (g, g^\#) = \mathrm{id}$. ■

**Corollary 3.1.3.** *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. If*

1. *$U := f(X)$ is an open subset of $Y$,*
2. *$f$ is a homeomorphism onto its image,*
3. *$f_p^\#$ is an isomorphism for all $p \in X$*

*then $(X, \mathscr{O}_X) \cong (U, \mathscr{O}_Y|_U)$.*

*Proof.* By proposition 3.1.1, $(f, f^\#)$ factors through $(\bar{f}, \bar{f}^\#) : (X, \mathscr{O}_X) \to (U, \mathscr{O}_Y|_U)$. By the theorem it suffices to check that $\bar{f}_p^\#$ is an isomorphism for all $p \in X$. But this follows from the fact that $i^\#$ is an isomorphism on stalks. ■

**Thm 3.1.4.** *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces. The presheaf $U \mapsto \mathrm{Hom}((U, \mathscr{O}_X|_U), (Y, \mathscr{O}_Y))$ on $\mathsf{Top}(X)$ is a sheaf of sets.*

## 3.2 Morphisms

### 3.2.1 Quasi-compact

**Definition 3.2.1.** Let $(f, f^\#) : X \to Y$ be a morphism of schemes. We say $(f, f^\#)$ is quasi-compact if there is an affine covering $\{V_i\}_i$ of $Y$ such that $f^{-1}(V_i)$ is quasi-compact for all $i$.

**Lemma 3.2.2.** *Let $X$ be a topological space. If $X$ is a finite union of quasi-compact open sets then $X$ is quasi-compact.*

lemma:qc-aff **Lemma 3.2.3.** *Let $(f, f^\#) : X \to Y$ be a morphism of schemes, $X$ be quasi-compact and $Y$ affine. Then for any affine $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

*Proof.* $X$ is quasi-compact so has a finite affine covering $\{V_i\}_i$. The preimage of the basic open sets in $Y$ are of the form $X_r$. But $X_r$ admits a covering $\{(V_i)_r\}_i$ which are also affine. Thus $X_r$ is quasi-compact. But any affine in $Y$ can be covered by finitely many basic open sets, and so its preimage is covered by finitely many quasi-compact sets and hence is quasi-compact. ■

**Thm 3.2.4.** *Let $(f, f^{\#}) : X \to Y$ be quasi-compact. Then for any affine $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

*Proof.* Let $\{V_i\}_i$ be and affine covering of $Y$ such that $f^{-1}(V_i)$ is quasi-compact for all $i$. Then for any $W \subseteq Y$ affine, since affines are quasi-compact, there are $i_1, \ldots, i_k$ such that $W = \cup_j W \cap V_{i_j}$. Since $W$ is quasi-compact, we can find a finite collection of affines $\{Z_{jk}\}$ such that $W \cap V_{i_j} = \cup_k Z_{jk}$ (since $W = \cup_{j,k} Z_{jk}$). By lemma 3.2.3, $f^{-1}(Z_{jk})$ is quasi-compact for all $j, k$. Thus $f^{-1}(W)$ is a finite union of quasi-compact open sets and so is quasi-compact. ∎

### 3.2.2 Locally of finite type

**Definition 3.2.5.** Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is locally of finite type if there is an affine covering $\{V_i\}_i$ of $Y$, and for each $i$, and affine covering $\{U_{ij}\}_j$ of $f^{-1}(V_i)$ such that $\mathscr{O}_X(U_{ij})$ is as finitely generated $\mathscr{O}_Y(V_i)$-algebra.

**Proposition 3.2.6.** *Let $(f, f^{\#}) : X \to Y$ be locally of finite type. Then for any affine $V \subseteq Y$ and affine $U \subseteq f^{-1}(V)$, $\mathscr{O}_X(U)$ is a finitely generated $\mathscr{O}_Y(V)$-algebra.*

*Proof.* Affine schemes are quasi-compact and so $V$ can be covered by finitely many $V \cap V_i$. But. ∎

### 3.2.3 Finite type

**Definition 3.2.7.** Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is of finite type if it is quasi-compact and locally of finite type.

### 3.2.4 Closed immersion

**Definition 3.2.8.** Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is a closed immersion if $f(X)$ is closed in $Y$, $f$ is a homeomorphism onto its image, and the morphism $f^{\#}$ is surjective.

### 3.2.5 Open immersion

**Definition 3.2.9.** Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is an open immersion if $f(X)$ is open in $Y$, $f$ is a homeomorphism onto its image, and $f_p^{\#}$ is an isomorphism for all $p \in X$.

### 3.2.6 Affine

**Definition 3.2.10.** Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes. We say $(f, f^{\#})$ is affine if there is an affine covering $\{V_i\}_i$ of $Y$ such that $f^{-1}(V_i)$ is affine for all $i$.

**Proposition 3.2.11.** *Let $(f, f^{\#}) : X \to Y$ be affine. Then for any affine $V \subseteq Y$, $f^{-1}(V)$ is affine.*

*Proof.* ∎

## 3.3 $\mathscr{O}_X$-Modules

**Definition 3.3.1.** Let $(X, \mathscr{O}_X)$ be a locally ringed space. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ of abelian groups with a compatible $\mathscr{O}_X$ action. Morphisms of $\mathscr{O}_X$-modules are morphisms of sheaves of abelian groups that respect the $\mathscr{O}_X$-module structure.

**Thm 3.3.2.** *The category of $\mathscr{O}_X$-modules is an abelian category.*

*Proof.* Additive structure on hom-sets is obvious. Kernels are the same as the kernels in $\mathsf{Ab}(X)$, with the obvious $\mathscr{O}_X$-module structure. Similarly for cokernels (if a presheaf has an $\mathscr{O}_X$-module structure, then so does its sheafification by acting on the stalks). The rest then follows. ∎

**Definition 3.3.3.** (Tensor product). Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. Define the tensor product of $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ to be the sheafification of the presheaf tensor product.

**Definition 3.3.4.** (Pullback). Let $f : X \to Y$ be a continuous map, and $\mathscr{F}$ an $\mathscr{O}_Y$. Then $f^{-1}\mathscr{F}$ is naturally a $f^{-1}\mathscr{O}_Y$-module. Moreover from the inverse image - direct image adjunction we obtain a map $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ from $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$. We can thus form the sheaf $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$. This sheaf is naturally an $\mathscr{O}_X$-module and we call it $f^*\mathscr{F}$.

**Definition 3.3.5.** (Direct image). Let $f : X \to Y$ be a continuous map, and $\mathscr{F}$ an $\mathscr{O}_X$. Then $f_*\mathscr{F}$ is naturally a $f_*\mathscr{O}_X$ module, and hence a $\mathscr{O}_Y$-module via $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$.

thm:tensor-hom **Thm 3.3.6.** *Let $\mathscr{F}$ be a $(\mathscr{A}, \mathscr{B})$-bimodule. Then $- \otimes_{\mathscr{A}} \mathscr{F} \dashv \mathscr{H}om_{\mathscr{B}}(\mathscr{F}, -)$ as functors between $\mathsf{Mod}(\mathscr{A})$ and $\mathsf{Mod}(\mathscr{B})$.*

*Proof.* Follows from the corresponding tensor-hom adjunction for modules. ∎

**Lemma 3.3.7.** *Let $f : X \to Y$, $\mathscr{F} \in \mathsf{Mod}(Y)$ and $\mathscr{G} \in \mathsf{Mod}(X)$. Then under the natural bijection*

$$\mathrm{Hom}_{\mathsf{Ab}}(f^{-1}\mathscr{F}, \mathscr{G}) \leftrightarrow \mathrm{Hom}_{\mathsf{Ab}}(\mathscr{F}, f_*\mathscr{G}) \tag{3.3}$$

*$f^{-1}\mathscr{O}_Y$-module morphisms biject with $\mathscr{O}_Y$-module morphisms.*

**Thm 3.3.8.** *Let $f : X \to Y$ be a continuous map. Then $f^* \dashv f_*$ as functors between $\mathsf{Mod}(X)$ and $\mathsf{Mod}(Y)$.*

*Proof.* Let $\mathscr{F} \in \mathsf{Mod}(Y)$ and $\mathscr{G} \in \mathsf{Mod}(X)$. Note that $\mathscr{O}_X$ is an $(f^{-1}\mathscr{O}_Y, \mathscr{O}_X)$-bimodule. We thus have the following chain of natural bijections

$$\begin{aligned}\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, \mathscr{G}) &\leftrightarrow \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{G})) \\ &\leftrightarrow \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \mathscr{G}) \\ &\leftrightarrow \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_*\mathscr{G}) \qquad (3.4)\end{aligned}$$

where the last bijection follows from the lemma. ■

**Definition 3.3.9.** Let $R$ be a ring and $M$ an $R$-module. Define $\widetilde{M}$ to be the $\mathscr{O}_{\mathrm{Spec}(R)}$-module which is locally $M_r$.

**Thm 3.3.10.** *$\widetilde{\bullet}$ is a fully faithful exact functor from $\mathsf{Mod}_R$ to $\mathsf{Mod}(\mathrm{Spec}(R))$.*

*Proof.* Localisation is exact. ■

**Corollary 3.3.11.** *$\widetilde{\bullet}$ and $\Gamma$ form part of an adjoint equivalence of categories between $\mathsf{Mod}_R$ and $\mathsf{Mod}(\mathrm{Spec}(R))$.*

## 3.4 Sheaf of ideals

**Definition 3.4.1.** Let $\mathscr{F}$ be a sheaf on $X$. Then $\mathrm{supp}(\mathscr{F}) = \{x \in X : \mathscr{F}_x \neq 0\}$.

prop:supp **Proposition 3.4.2.** *If $\mathscr{F}$ is a finitely generated $\mathscr{O}_X$-module then supp$(X)$ is a closed subset of $X$.*

**Definition 3.4.3.** A subsheaf of $\mathscr{O}_X$ is called a *sheaf of ideals* on $X$.

**Definition 3.4.4.** Let $\mathscr{J}$ be a sheaf of ideals on $X$. Let $Z = \mathrm{supp}(\mathscr{O}_X/\mathscr{J})$. By proposition 3.4.2, $Z$ is a closed subset of $X$. Let $i : Z \to X$ be the inclusion map. Then we define the structure sheaf on $X$ to be $\mathscr{O}_Z = i^{-1}(\mathscr{O}_X/\mathscr{J})$. This turns $Z$ into a locally ringed space.

**Proposition 3.4.5.** *$i_*\mathscr{O}_Z \cong \mathscr{O}_X/\mathscr{J}$.*

*Proof.* There is a natural map $\mathscr{O}_X/\mathscr{J} \to i_*\mathscr{O}_Z = i_*i^{-1}(\mathscr{O}_X/\mathscr{J})$ arising from the inverse image-direct image adjunction. Looking at stalks shows that this is an isomorphism. ■

*Remark* 3.4.6. In particular there is a natural map $i^\# : \mathscr{O}_X \to i_*\mathscr{O}_Z$ given by the composition $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J} \to i_*\mathscr{O}_Z$ inducing a morphism $(i, i^\#)$ of locally ringed spaces.

**Corollary 3.4.7.** *The map $(i, i^\#) : (Z, \mathscr{O}_Z) \to (X, \mathscr{O}_X)$ is a closed immersion and $\mathscr{J} = \ker(i^\#)$.*

**Lemma 3.4.8.** *Let $A$ be a ring and $I \lhd A$ be an ideal. Then the sheaf $(A/I)^\sim$ on $\mathrm{Spec}(A)$ has support $V(I)$.*

*Proof.* Consider the following exact sequence of $A$-modules

$$0 \to I \to A \to A/I \to 0. \tag{3.5}$$

If $I \not\subseteq \mathfrak{p}$ then $IA_{\mathfrak{p}} = A_{\mathfrak{p}}$ and so $(A/I)_{\mathfrak{p}} = 0$. If $I \subseteq \mathfrak{p}$ then $(A/I)_{\mathfrak{p}} \cong (A/I)_{\mathfrak{q}}$ where $\mathfrak{q} = \mathfrak{p}/I$ and so is in particular not 0. ■

thm:sheaf_of_ideals **Thm 3.4.9.** *If $\mathscr{J}$ is quasi-coherent then $(Z, \mathscr{O}_Z)$ is a scheme and for any affine piece $(U, \mathscr{O}_X|_U) \cong (\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ of $X$, $(Z \cap U, \mathscr{O}_Z|_{Z\cap U})$ is isomorphic to $(\mathrm{Spec}(A/I), \mathscr{O}_{\mathrm{Spec}(A/I)})$ where $I$ is the ideal of $A$ corresponding to $J(U)$.*

*Proof.* It suffices to show the second part of the theorem. Let $(U, \mathscr{O}_X|_U) \cong (\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ be an affine piece of $X$. Restricting the short exact sequence $0 \to \mathscr{J} \to \mathscr{O}_X \to i_*\mathscr{O}_Z \to 0$ to $U$ we get

$$0 \to \mathscr{J}|_U \to \mathscr{O}_X|_U \to (i_{U\cap Z})_*(\mathscr{O}_Z|_{U\cap Z}) \to 0. \tag{3.6}$$

It follows that

$$(i_{U\cap Z})_*(\mathscr{O}_Z|_{U\cap Z}) \cong (A/I)^{\sim} \cong \mathrm{Spec}(\phi)_*\mathscr{O}_{\mathrm{Spec}(A/I)} \tag{3.7}$$ eq:sh_isoms

where $\phi : A \to A/I$ is the quotient map. By the lemma $U \cap Z = V(I)$ and so there is a homeomorphism $\psi : \mathrm{Spec}(A/I) \to U \cap Z$. Since both $i_{U\cap Z}$ and $\mathrm{Spec}(\phi)$ are homeomorphisms onto their images, the isomorphisms in equation 3.7 induce isomorphisms of sheaves. Taking stalks moreover shows that we get an isomorphism of locally ringed spaces as required. ■

## 3.5 Reduced schemes

**Definition 3.5.1.** A scheme $(X, \mathscr{O}_X)$ is reduced if $\mathscr{O}_X(U)$ is reduced for all $U \subseteq X$ open.

**Lemma 3.5.2.** *$(X, \mathscr{O}_X)$ is reduced iff $\mathscr{O}_{X,p}$ is reduced for all $p \in X$.*

**Lemma 3.5.3.** *Let $\mathscr{J}$ be the ideal sheaf of $\mathscr{O}_X$ given by $U \mapsto N(\mathscr{O}_X)$. Then $\mathscr{J}$ is quasi-coherent.*

*Proof.* It suffices to show that $\mathscr{J} \cong N(\mathscr{O}_X(X))^{\sim}$ when $X$ is affine. But we have an isomorphism on the basis and hence between sheaves. ■

**Definition 3.5.4.** Let $(X, \mathscr{O}_X)$ be a scheme. We define $(X_{red}, (\mathscr{O}_X)_{red})$ to be the scheme associated with the sheaf of ideals $\mathscr{J}$ given by $\mathscr{J}(U) = N(\mathscr{O}_X(U))$. Let $(z, z^{\#}) : (X_{red}, (\mathscr{O}_X)_{red}) \to (X, \mathscr{O}_X)$ be the associated closed embedding.

*Remark* 3.5.5. $X_{red}$ is reduced since it is reduced on affine pieces.

**Proposition 3.5.6.** *$z$ is a homeomorphism.*

*Proof.* It suffices to check that $\mathrm{supp}(\mathscr{O}_X/\mathscr{I}) = X$ for affine $X$. Let $\phi : R \to R/N(R)$ be the quotient map. Then $\mathrm{Spec}(\phi)$ is a homeomorphism. It follows that $\mathrm{supp}(\mathscr{O}_X/\mathscr{I}) = X$ and so $z$ is the identity map. ∎

**Thm 3.5.7.** *Let $f : X \to Y$ be a morphism of schemes and suppose $X$ is reduced. Then $f$ factors through $Y_{red}$.*

*Proof.* Universal property of cokernels. ∎

**Definition 3.5.8.** For an affine scheme $X$, let $I(Z)$ be the radical ideal corresponding to a closed set $Z \subset X$. For a general scheme $X$ and a closed subset $Z \subseteq X$, let $\mathscr{I}_Z$ be the sheaf

$$\mathscr{I}_Z(U) = \{f \in \mathscr{O}_X(U) : f_x \in m_x, \forall x \in U \cap Z\}. \tag{3.8}$$

**Lemma 3.5.9.** *Let $X$ be an affine scheme and $Z \subseteq X$ a closed subset. Then $\mathscr{I}_Z \cong \widetilde{I(Z)}$.*

*Proof.* This holds on global sections and rad commutes with localisation. ∎

**Thm 3.5.10.** *Let $X$ be a scheme and $Z \subseteq X$ a closed subset. Then there is a unique quasi-coherent ideal $\mathscr{I}$ such that the associated closed immersion $Z' \to X$ has image $Z$ and $Z'$ reduced.*

*Proof.* $\mathscr{I} = \mathscr{I}_Z$ is quasi-coherent and the associated embedding has image $Z$. It is clear that $Z'$ is reduced (check on affine pieces). It thus remains to check the uniqueness of $\mathscr{I}$. For this it suffices to consider the affine case. Let $X = \mathrm{Spec}(A)$ and $\mathscr{I} = \widetilde{I}$. Then $Z' = \mathrm{Spec}(A/I)$ and $V(I) = Z$. But $Z'$ is reduced iff $I = I(Z)$. Thus $\mathscr{I} = \mathscr{I}_Z$. ∎

*Remark* 3.5.11. If we take $Z = X$ then $Z' = X_{red}$.

## 3.6 Tangent space

# CHAPTER 4

# Spectral sequences

**Thm 4.0.1.** *(Grothendieck spectral sequence). Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors and suppose that $F$ sends injective objects to $G$-acylic objects. Then for $A$ an object in $\mathcal{A}$ there is a spectral sequence $\{E_r(A)\}$ such that*

$$E_2^{p,q}(A) = R^pG(R^qF(A)) \Rightarrow R^{p+q}(GF)(A). \tag{4.1}$$

**Corollary 4.0.2.** *(Leray spectral sequence). Let $f : X \to Y, g : Y \to Z$ be continuous maps. Then for a sheaf $\mathscr{F}$, there is a $E_2$ cohomological spectral sequence*

$$R^pg_*(R^qf_*(\mathscr{F})) \Rightarrow R^{p+q}(g \circ f)_*(\mathscr{F}) \tag{4.2}$$

*which is functorial in $\mathscr{F}$.*

*Proof.* $f_*$ sends injective sheaves to flabby sheaves, which are $g_*$-acyclic. ■

# CHAPTER 5

# Group cohomology

# CHAPTER 6

# Appendix

## 6.1 Category theory results

prop:cat_factor

**Proposition 6.1.1.** *Let $F \dashv G$ and $G$ be full. Let $e$ be the unit of the adjunction. Then every morphism $x \to Gy$ factors uniquely through $e_x : x \to GFx$.*

*Proof.* Let $\alpha$ and $\beta$ denote the forward and backward maps in

$$\mathrm{Hom}(Fx, y) \leftrightarrow \mathrm{Hom}(x, Gy) \tag{6.1}$$

respectively. Let $f : x \to Gy$. Then $f = \alpha(\beta(f))$. But $\alpha(\beta(f)) = G\beta(f) \circ e_x$ so we get existence of a factorisation. For uniqueness, suppose $f = h \circ e_x$. Since $G$ is full there is a $l : Fx \to y$ such that $h = Gl$. So $\alpha(l) = \alpha(\beta(f))$. But $\alpha$ is a bijection so $l = \beta(f)$ and hence $h = G\beta(f)$ which gives uniqueness. ■

## 6.2 Properties of sheaves of rings

**Thm 6.2.1.** *Let $\mathscr{F}$ be a sheaf of rings on $X$, $U = \cup_i U_i$ and $s \in \mathscr{F}(U)$. Then $s$ is invertible iff $s|_{U_i}$ is invertible for all $i$.*

*Proof.* The forwards direction is trivial. Now suppose $s|_{U_i}$ is invertible for all $i$. Then there are $t_i \in \mathscr{F}(U_i)$ such that $t_i s|_{U_i} = 1$. But then, since inverses are unique we must have $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$ since they are both the inverse of $s|_{U_i \cap U_j}$. Thus there is a section $t \in \mathscr{F}(U)$ that restricts to the $t_i$. Checking locally it follows that $ts = 1$ and so $s$ is invertible. ■

## 6.3 Locally ringed spaces

**Lemma 6.3.1.** *Let $(f, f^{\#}), (g, g^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be morphisms of locally ringed spaces. Let $\mathcal{U} = \{U_i\}$ be an open covering of $X$. If the morphisms agree on the restrictions to the $U_i$ then they are equal.*

*Proof.* We certainly have $f = g$. The result then follows from sheaf condition (A). ■

**Proposition 6.3.2.** *Let $X, Y$ and $\{Z_i\}_i$ be locally ringed spaces together with open immersions $f_i : Z_i \to X, g_i : Z_i \to Y$. Let $\alpha : X \to Y$ be a morphism such that $\alpha \circ f_i = g_i$ for all $i$ and $\alpha : f_i(Z_i) \cap f_j(Z_j) \to g_i(Z_i) \cap g_j(Z_j)$ is an isomorphism for all $i, j$. Then $\alpha$ is an isomorphism.*

*Proof.* We have that $\alpha : f_i(Z_i) \to g_i(Z_i)$ is an isomorphism for all $i$. So we can define inverses $\beta_i : g_i(Z_i) \to f_i(Z_i)$. They agree on overlaps and so they glue to give a global inverse $\beta$. ■

**Proposition 6.3.3.** *Let $f : X \to Y$ be a morphism of schemes and let $\{U_i\}$ be an open cover of $Y$ such that the restriction of $f$ to a morphism $f^{-1}(U_i) \to U_i$ is an isomorphism for all $i$. Then $f$ is an isomorphism.*

## 6.4 Restriction

*Remark* 6.4.1. Recall from chapter 2 that given $f : X \to Y$ we obtain functors $f_*, \lim_f, f^{-1}$ between $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$. These constructions were themselves functorial and give rise to contra/co-variant functors $\mathsf{Top} \to \mathsf{Set}$. The same also holds for $f_*, f^*$ as functors between $\mathsf{Mod}(X)$ and $\mathsf{Mod}(Y)$.

**Thm 6.4.2.** *Let $f : X \to Y$ be a continuous map and $U \subseteq X$, $V \subseteq Y$ be open subsets such that $f(U) \subseteq V$. Moreover, let $f|_{U,V}$ denote the map $U \to V$ arising from $f|_U$. Then for $\mathscr{F} \in \mathsf{Sh}(X)$ and $\mathscr{G} \in \mathsf{Sh}(Y)$ we have*

1. $(f^{-1}\mathscr{G})|_U \cong f|_U^{-1}\mathscr{G} \cong f|_{U,V}^{-1}(\mathscr{G}|_V)$
2. $(f_*\mathscr{F})|_V \cong (f|_{U,V})_*(\mathscr{F}|_U)$ *when* $U = f^{-1}(V)$

*where are isomorphisms are natural.*

*Proof.*
1. $f|_U = f \circ i_U$ and so we obtain the first isomorphism. $f|_U = i_V \circ f|_{U,V}$ and so we obtain the second isomorphism.
2. Straightforward calculation.

■

## 6.5 Results on schemes

**Thm 6.5.1.** *Let $(X, \mathscr{O}_X)$ be a scheme and $A$ a ring. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Sch}}(X, \mathrm{Spec}(A)) \leftrightarrow \mathrm{Hom}_{\mathsf{Ring}}(A, \Gamma(X, \mathscr{O}_X)). \tag{6.2}$$

*In other words $\Gamma \dashv \mathrm{Spec}$ as functors between $\mathsf{Sch}$ and $\mathsf{Ring}^{op}$.*

*Proof.* Given a morphism $(f, f^\#) : (X, \mathscr{O}_X) \to (\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$ we obtain map $A \to \Gamma(X, \mathscr{O}_X)$ from $f^\#(\mathrm{Spec}(A))$.

Conversely, suppose we have $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For an affine $U \subseteq X$, we have the map $A \to \Gamma(X, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$, and we thus obtain a map

$U \to \operatorname{Spec}(A)$. Let $U, V \subseteq X$ be affine and $W \subseteq U \cap V$ also be affine. The following diagram commutes

$$\begin{tikzcd} & & \Gamma(U, \mathscr{O}_X) \arrow[dr] & \\ A \arrow[r] & \Gamma(X, \mathscr{O}_X) \arrow[ur] \arrow[rr] \arrow[dr] & & \Gamma(W, \mathscr{O}_X) \\ & & \Gamma(V, \mathscr{O}_X) \arrow[ur] & \end{tikzcd} \tag{6.3}$$

and so

$$\begin{tikzcd} & U \arrow[dl] & \\ \operatorname{Spec}(A) & & W \arrow[ll] \arrow[ul] \arrow[dl] \\ & V \arrow[ul] & \end{tikzcd} \tag{6.4}$$

also commutes. So the morphisms agree on overlaps and so can be glued to get a morphism $X \to \operatorname{Spec}(A)$.

It is straightforward to check that this defines a bijection. ■

**Corollary 6.5.2.** *Let $(X, \mathscr{O}_X)$ be a scheme. There is a canonical morphism $X \to \operatorname{Spec}(\Gamma(X, \mathscr{O}_X)$ such that every morphism from $X$ to an affine scheme factors through this map uniquely.*

*Proof.* This follows from proposition 6.1.1. ■

### 6.5.1 Presheaves on the category of schemes

**Definition 6.5.3.** Let $F : \mathsf{Sch}^{op} \to \mathsf{Set}$ be a functor. We call $F$ locally sheafy if for any scheme $X$, $F|_{\mathsf{Top}(X)}$ is a sheaf of sets.

**Thm 6.5.4.** *Let $F, G : \mathsf{Sch}^{op} \to \mathsf{Set}$ be locally sheafy functors and suppose there is a natural transformation $\eta : F|_{\mathsf{Aff}^{op}} \Rightarrow G|_{\mathsf{Aff}^{op}}$. Then there is a unique natural transformation $\zeta : F \Rightarrow G$ such that $\zeta|_{\mathsf{Aff}} = \eta$.*

*Proof.* Let $X$ be a scheme and $s \in F(X)$. We wish to define $\zeta_X(s) \in G(X)$. For each affine piece $U$ of $X$, define $t_U = \eta_U(s|_U) \in G(U)$. Given any two affine pieces $U$ and $V$ we have $t_U|_{U \cap V} = \eta_{U \cap V}(s|_{U \cap V}) = t_V|_{U \cap V}$. Since the union of all affine pieces of $X$ is $X$ we obtain an element $t \in G(X)$ such that $t|_U = t_U$ for all affine $U \subseteq X$. Define $\zeta_X(s) = t$. Note that if $X$ was already affine then $\zeta_X = \eta_X$. We claim that $\zeta$ is a natural transformation.

Let $X, Y$ be schemes and $f : X \to Y$ a morphism (in $\mathsf{Sch}$). Let $U \subseteq Y$ and $V \subseteq f^{-1}(U) \subseteq X$ be affine pieces and $f|_{V,U} : V \to U$ denote the map such that $f \circ i_V = i_U \circ f|_{V,U}$. Then we know that

$$\begin{tikzcd}
F(U) \arrow[rrr, "\eta_U"] \arrow[ddd, "Ff_{V,U}"] & & & G(U) \arrow[ddd, "Ff_{V,U}"] \\
& F(Y) \arrow[ul] \arrow[r, "\zeta_Y"] \arrow[d, "Ff"] & G(Y) \arrow[ur] \arrow[d, "Gf"] & \\
& F(X) \arrow[dl] \arrow[r, "\zeta_X"] & G(X) \arrow[dr] & \\
F(V) \arrow[rrr, "\eta_V"] & & & G(V)
\end{tikzcd} \tag{6.5}$$

commutes except for the middle square. Thus $G(i_V) \circ (Gf \circ \zeta_Y) = G(i_V) \circ (\zeta_X \circ Ff)$. But we can vary the $U$ and $V$ so that the $V$ cover $X$. It follows that $Gf \circ \zeta_Y = \zeta_X \circ Gf$. Thus $\zeta$ is a natural transformation.

To see that $\zeta$ is unique, suppose $\xi : F \Rightarrow G$ is another natural transformation extending $\eta$. Then let $s \in F(X)$ and $U \subseteq X$ be an affine piece. We must have $G(i_U) \circ \zeta_X(s) = \eta_U \circ F(i_U) = G(i_U) \circ \xi_X(s)$. But we can vary $U$ to cover $X$ and so we must have $\zeta_X(s) = \xi_X(s)$ for all $s \in F(X)$ and hence $\zeta_X = \xi_X$ for all $X$ and hence $\zeta = \xi$. ■

**Corollary 6.5.5.** *Let $F, G : \mathsf{Sch}^{op} \to \mathsf{Set}$ be locally sheafy functors such that $F|_{\mathsf{Aff}^{op}} \cong G|_{\mathsf{Aff}^{op}}$. Then $F \cong G$.*

**Conjecture 6.5.6.** *There is an equivalence of categories between locally sheafy presheafs on $\mathsf{Sch}$ and locally sheafy presheafs on $\mathsf{Aff}$.*

*Proof.* Given $F : \mathsf{Aff}^{op} \to \mathsf{Set}$ define $\widetilde{F} : \mathsf{Sch}^{op} \to \mathsf{Set}$ by $X \mapsto \varprojlim_{U \subseteq X} F(U)$ where $U$ ranges over affine subsets of $X$ and send morphisms to the obvious things. ■

### 6.5.2 Basic open sets

**Proposition 6.5.7.** *Let $X$ be a scheme and $f \in \Gamma(X, \mathscr{O}_X)$. Then for affine $U \subseteq X$, $X_f \cap U = U_{f|_U}$ is a basic open set.*

*Proof.* It suffice to prove that for $X = \operatorname{Spec}(R)$ and $r \in R$, $X_r = D_r(R)$. But

$$X_r = \{\mathfrak{p} \lhd R : r/1 \notin \mathfrak{p}_\mathfrak{p}\} = \{\mathfrak{p} \lhd R : r \notin \mathfrak{p}\} = D_r(R). \tag{6.6}$$

■

**Proposition 6.5.8.** *Let $(f, f^\#) : X \to Y$ be a morphism of schemes and $r \in \Gamma(Y, \mathscr{O}_Y)$. Then $f^{-1}(Y_r) = X_{f^\#(Y)(r)}$.*

*Proof.* Recall that $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a morphism of local rings. Thus $r_{f(x)} \in \mathfrak{m}_{f(x)}$ iff $f_x^\#(r_{f(x)}) \in \mathfrak{m}_x$. But $f_x^\#(r_{f(x)}) = f^\#(Y)(r)_x$ and so

$$\begin{aligned} f^{-1}(Y_r) &= \{x \in X : r_{f(x)} \notin \mathfrak{m}_{f(x)}\} \\ &= \{x \in X : f^\#(Y)(r)_x \notin \mathfrak{m}_x\} = X_{f^\#(Y)(r)}. \end{aligned} \tag{6.7}$$

■