Local fields

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February 20, 2019

Contents

CHAPTER 1

Discrete valuation rings and Dedekind domains

1.1 Preliminaries

thm:int_dep **Thm 1.1.1.** *Let $S \subseteq R$ be rings and $x \in R$. The following are equivalent:*

1. *x is integral over S*
2. *$S[x]$ is a finite S-algebra*
3. *$S[x]$ is contained ina subring $T \subseteq R$ that is a finite S-alg*
4. *There is an $S[x]$-module M such that $Ann_{S[x]}(M) = 0$ and M is finitely generated as a S-module.*

Lemma 1.1.2. *Let $S \subseteq R$ be an integral extension of rings. Let $Q \lhd R$ be a prime ideal, and set $P = Q \cap S$. Then Q is maximal in R iff P is maximal in S.*

Lemma 1.1.3. *(Incomparability). Let $S \subseteq R$ be an integral extension of rings. Let $Q, Q' \lhd R$ be prime ideals with $Q \subseteq Q'$ and $Q \cap S = Q' \cap S$. Then $Q = Q'$.*

Lemma 1.1.4. *(Lying over). Let $S \subseteq R$ be an integral extension of rings. Let $P \lhd S$ be prime. Then there exists a prime $Q \lhd R$ such that $Q \cap S = P$.*

thm:going_up **Thm 1.1.5.** *(Going up). Let $S \subseteq R$ be an integral extension of rings. Let $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$ be a chain of primes in S and $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_m$ with $0 \leq m < n$ be a chain of primes in R with $Q_i \cap S = P_i$. Then the chain can be extended to $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n$ with $Q_i \cap S = P_i$.*

thm:going_down **Thm 1.1.6.** *(Going down). Let $S \subseteq R$ be an integral extension of domains such that S is normal. Let $P_1 \supseteq P_2 \supseteq \cdots \supseteq P_n$ be a chain of primes in S and $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_m$ with $0 \leq m < n$ be a chain of primes in R with $Q_i \cap S = P_i$. Then the chain can be extended to $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n$ with $Q_i \cap S = P_i$.*

thm:noeth_loc_ring **Thm 1.1.7.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then $\cap_n \mathfrak{m}^n = 0$.*

1.2 Discrete valuation rings

Definition 1.2.1. Let K be a field. A *discrete valuation* on K is a surjective function $v : K \to \mathbb{Z} \cup \{\infty\}$ satisfying

1. $v(x) = \infty$ iff $x = 0$
2. $v(xy) = v(x) + v(y)$
3. $v(x + y) \geq \inf(v(x), v(y))$.

If (K, v) is a field with a discrete valuation then $R = \{x \in K : v(x) \geq 0\}$ is a subring of K called the valuation ring of v.

Definition 1.2.2. A domain R is called a *discrete valuation ring* if there is a valuation v on $K = \mathrm{Frac}(R)$ for which R is the valuation ring.

Lemma 1.2.3. *Any DVR is a Noetherian local domain of Krull dimension 1.*

Proof. Let $t \in R$ be such that $v(t) = 1$. Then all the ideals of R are in the chain

$$R \supsetneq (t) \supsetneq (t^2) \supsetneq (t^3) \supsetneq \cdots \supseteq 0. \tag{1.1}$$

■

Thm 1.2.4. *Let R be a Noetherian local domain with Krull dimenstion 1 and maximal ideal $\mathfrak{m}$. The following are equivalent*

1. *R is a DVR*
2. *$\mathfrak{m}$ is principal*
3. *R is normal.*

Proof. From the conditions on R we know the only prime ideals are 0 and $\mathfrak{m}$.

(1) $\Rightarrow$ (3) Let $x \in K = \mathrm{Frac}(R)$ and suppose it is integral over R. Then there are $a_0, \ldots, a_{n-1} \in R$ such that $x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0 = 0$. Suppose that $v(x) < 0$. Then

$$nv(x) = v(x^n) \geq \inf(v(a_i x^i)) \geq (n-1)v(x) \tag{1.2}$$

and so $v(x) \geq 0$ which is a contradiction and so $v(x) \geq 0$ and hence $x \in r$.

(3) $\Rightarrow$ (2) Let $0 \neq x \in \mathfrak{m}$. Then $\mathrm{rad}((x)) = \mathfrak{m}$ and so since R is Noetherian there is an $n \in \mathbb{N}$ such that $\mathfrak{m}^n \subseteq (x) \subseteq \mathfrak{m}$. Let n be minimal with this property.

Remark 1.2.5. Morally we now have $(x) = \mathfrak{m}^n$.

Let $y \in \mathfrak{m}^{n-1}$ be such that $y \notin (x)$. Set $z = x/y \in K$. Then $z^{-1} \notin R$ so z^{-1} is not integral over R. But this means that $z^{-1}\mathfrak{m} \not\subseteq \mathfrak{m}$ my theorem 1.1.1. But $y\mathfrak{m} \subset (x)$ and so $z^{-1}\mathfrak{m} \subseteq R$ and hence $z^{-1}\mathfrak{m} = R$ and so $\mathfrak{m} = (z)$.

(2) $\Rightarrow$ (1) Define obvious valuation. ■

Proposition 1.2.6. *R is a DVR iff it is a Noetherian local ring and its maximal ideal is generated by a non-nilpotent element π.*

Proof. By theorem 1.1.7 every element in R can be written as $\pi^n u$ for u a unit and $n \in \mathbb{N}_0$. It follows that R must be an integral domain, and so this expression is unique. Setting $v(\pi^n u) = n$ one can easily check that this defines valuation on $\mathrm{Frac}(R)$ with valuation ring R. ■

1.3 Dedekind domains

Definition 1.3.1. A *Dedekind domain* is a normal Noetherian domain of Krull dimension 1.

Lemma 1.3.2. *Let R be a Noetherian domain of Krull dimension 1. Then R is a Dedekind domain iff every local ring $R_{\mathfrak{p}}$ for $\mathfrak{p} \neq 0$ is a DVR.*

Definition 1.3.3. Let R be a Dedekind domain with $K = \mathrm{Frac}(R)$. We call an R-submodule of K a fractional ideal if it is finitely generated as an R-module.

Remark 1.3.4. One can define the product of fractional ideals in the obvious way. Then the products of fractional ideals are fractional ideals.

Proposition 1.3.5. *Let R be a Dedekind domain. Then all fractional ideals are invertible.*

1.3.1 Examples

PIDs

Thm 1.3.6. *Let R be a PID. Then R is a Dedekind domain.*

Proof. Certainly Noetherian. Also clearly has Krull dimension 1. Finally, $R_{\mathfrak{p}}$ is a local PID and hence a DVR for all $0 \neq \mathfrak{p} \lhd R$. ■

Rings of integers

Thm 1.3.7. *Let $K/\mathbb{Q}$ be a number field and $\mathcal{O}_K$ its ring of algebraic integers. Then $\mathcal{O}_K$ is a Dedekind domain.*

Proof. If $\alpha \in \mathrm{Frac}(\mathcal{O}_K) = K$ is integral over $\mathcal{O}_K$ then it is integral over $\mathbb{Z}$ and so lies in $\mathcal{O}_K$. Thus $\mathcal{O}_K$ is normal. To see that $\mathcal{O}_K$ is Noetherian note that it is finite over $\mathbb{Z}$ and hence Noetherian. Finally, $\mathcal{O}_K$ has Krull dimension 1 since it is integral over $\mathbb{Z}$ and $\mathbb{Z}$ has Krull dimension 1. ■

Coordinate rings

Thm 1.3.8. *Let V be an affine variety defined over an algebraically closed field k. Then $k[V]$ is a Dedekind domain iff V is non-singular, irreducible and of dimension* 1*.*

Proof. $k[V]$ is always Noetherian. V is irreducible iff $k[V]$ is a domain. V is of dimension 1 iff $k[V]$ has Krull dimension 1.

Now suppose $k[V]$ is a Dedekind domain. Then we know that all the local rings are normal Noetherian local domains i.e. DVRs and hence V must be non-singular. Conversely, suppose V is non-singular. Then all the local rings of $k[V]$ are DVRs and so $k[V]$ is a Dedekind domain. ∎