Smooth representations of locally profinite groups

Emile T. Okada

April 5, 2019

Contents

CHAPTER 1

Pro-$\mathcal{C}$ groups

1.1 Topological preliminaries

1.2 Pro-$\mathcal{C}$ groups

Thm 1.2.1. *Let $\mathcal{C}$ be a formation of finite groups. Then the following are equivalent.*

1. *G is a pro-$\mathcal{C}$ group;*
2. *G is compact Hausdorff totally disconnected, and for each open normal subgroup U of G, $G/U \in \mathcal{C}$;*
3. *G is compact and the identity element 1 of G admits a fundamental system $\mathcal{U}$ of open neighbourhoods U such that $\bigcap_{U\in\mathcal{U}} U = 1$ and each U is an open normal subgroup of G with $G/U \in \mathcal{C}$;*
4. *The identity element 1 of G admits a fundamental system $\mathcal{U}$ of open neighbourhoods U such that each U is a normal subgroup of G with $G/U \in \mathcal{C}$, and*

$$\varprojlim_{U\in\mathcal{U}} G/U. \tag{1.1}$$

CHAPTER 2

Smooth Representations of Locally Profinite Groups

2.1 Locally profinite groups

Definition 2.1.1. *A locally profinite group* is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G.

Proposition 2.1.2. *Let G be a locall profinite group.*

1. *Closed subgroups of G are locally profinite.*
2. *Quotients of G by closed normal subgroups are locally profinite.*

Proposition 2.1.3. *Let G be a compact locally profinite group then the map*

$$G \to \varinjlim G/K \tag{2.1}$$

is a topological isomorphism, where K ranges over all open normal subgroups of G.

Proposition 2.1.4. *A topological group G is locally profinite iff G is locally compact and totally disconnected.*

Proposition 2.1.5. *Let $\{K_n\}_n$ be a decreasing sequence of compact open subgroups of G such that $\cap_n K_n = \{e\}$. Then for any neighbourhood U of e there is an n such that $K_n \subseteq U$.*

2.2 Smooth representations

Definition 2.2.1. Let G be a locally profinite group and (π, V) a complex representation of G. (π, V) is *smooth* if for every $v \in V$ there is a compact open subgroup K of G such that $v \in V^K$.

(π, V) is *admissable* if the space V^K is finite dimensional for each compact open subgroup K of G.

Proposition 2.2.2. *Let (π, V) be a smooth representation. Then subrepresentations and quotients are also smooth.*

2.2.1 Characters

Proposition 2.2.3. *Let $\psi : G \to \mathbb{C}^\times$ be a group homomorphism. The following are equivalent*

1. *ψ is continuous,*
2. *$\ker\psi$ is open,*
3. *$\ker\psi$ contains an open set,*
4. *the corresponding representation on $\mathbb{C}$ is smooth.*

Proof. (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1) clear. (1) $\Rightarrow$ (2) Let U be an open subset of $\mathbb{C}^\times$. Then $\psi^{-1}(U)$ is open and so contains an open compact subgroup K. For U sufficiently small, it contains no non-trivial subgroups of $\mathbb{C}^\times$ and so $K \subseteq \ker\psi$. ∎

Definition 2.2.4. We call a homomorphism $\psi : G \to \mathbb{C}^\times$ that satisfies any of the above conditions a chcaracter of G.

Proposition 2.2.5. *If $\psi : G \to \mathbb{C}^\times$ is a character and G is a union of its open compact subgroups, then $\psi(G) \subseteq S^1$.*

2.2.2 Semisimplicity

Proposition 2.2.6. *If G is compact then any smooth representation is semisimple.*

Proof. Let $v \in V$ and $K \subseteq G$ be an open compact subgroup such that $v \in V^K$. G is compact and so $|G : K| < \infty$. Thus $W = \mathbb{C}Gv$ is finite dimensional. Moreover, if we let $K' = \cap_{g\in G/K} gKg^{-1}$ then this is a open normal subgroup of G and K' acts triviall on W. Thus W descends to a G/K' representation. But G/K' is finite and so W is a sum of its simple submodules. It follows that the same holds for V and so V is semisimple. ∎

Corollary 2.2.7. *If G is compact then any irreducible smooth representation is finite dimensional.*

Corollary 2.2.8. *Let G be a locally profinite group, and let K be a compact open subgroup of G. Let (V, π) be a smooth representation of G. Then $Res_K^G V$ is semisimple.*

Proposition 2.2.9. *Let G be a locally profinite group, K a compact open subgroups of G and (π, V) a smooth representation of G.*

1.
$$V = \bigoplus_{\phi \in \hat{K}} V^\rho. \tag{2.2}$$

2. *Let (σ, W) be a representation of G and $f : V \to W$ a G-homomorphism. Then for every $\rho \in \hat{K}$ we have $f(V^\rho) \subseteq W^\rho$ and $W^\rho \cap f(V) = f(V^\rho)$.*

Corollary 2.2.10. *Let $U \to V \to W$ be a sequence of smooth representations of G. The sequence is exact iff $U^K \to V^K \to W^K$ is exact (as vector spaces) for all compact open subgroups K of G.*

Definition 2.2.11. If H is a subgroup of G, we define

$$V(H) = \operatorname{span}\{v - \pi(h)v : v \in V, h \in H\}. \tag{2.3}$$

Corollary 2.2.12. *Let G be a locally profinite group, and let (π, V) be a smooth representation of G. Let K be a compact open subgroup of G. Then*

$$V(K) = \bigoplus_{\rho \in \hat{K} \setminus \{1\}} V^\rho \tag{2.4}$$

is the unique K-complement to V^K in V.

Proof. Consider the map $V \to V^K$ given by quotienting by $\bigoplus_{\rho \in \hat{K} \setminus \{1\}} V^\rho$. $V(K)$ must lie in the kernel so we have on inclusion. Conversly, if U is an irreducible K-subrepresentation of V not isormophic to the trivial representation the $U(K) = U$ and so we get the other inclusion. ∎

Proposition 2.2.13. *Let (π, V) be an aribitrary representation. Define $V^\infty = \bigcup_K V^K$, where K ranges over compact open subgroups of G. Then V^∞ is a smooth subrepresentation of G.*

Proposition 2.2.14. *Let (π, V) be a smooth representation of G, and (σ, W) be an arbitrary representation. Then every morphism $f : V \to W$ factors through W^∞.*

Corollary 2.2.15. $(-)^\infty : \mathsf{Rep}_G \to \mathsf{Smo}_G$ *is a functor.*

Thm 2.2.16. *Let $i : \mathsf{Smo}_G \to \mathsf{Rep}_G$ be the inclusion functor. Then $i \dashv (-)^\infty$.*

2.2.3 Induction

Definition 2.2.17. Let G be a locally profinite group, H a closed or open subgroup and (σ, W) be a smooth representation of H. Define $\text{s-Ind}_H^G(W) = (\operatorname{Ind}_H^G(W))^\infty$. Note that if (π, V) is a smooth G representation then so is $\operatorname{Res}_H^G(V)$. Write s-Res for the functor between $\mathsf{Smo}_G \to \mathsf{Smo}_H$.

Proposition 2.2.18. *s-Res$_H^G \dashv$ s-Ind$_H^G$.*

Proof. Let V be a G-representation and W be a H-representation. Let α_W : s-$\mathrm{Ind}_H^G(W) \to W$ be the H-homomorphism $f \mapsto f(e)$. Then we have the maps

$$\mathrm{Hom}(\text{s-}\mathrm{Res}_H^G(V), W) \leftrightarrow \mathrm{Hom}(V, \text{s-}\mathrm{Ind}_H^G(W)) \tag{2.5}$$
$$\phi \mapsto (v \mapsto (g \mapsto \phi(gv)) \tag{2.6}$$
$$\alpha_W \circ \psi \leftarrow\!\shortmid \psi. \tag{2.7}$$

It is straightforward to check that these maps are mutually inverse. ∎

Proposition 2.2.19. *s-Ind_H^G is exact.*

Definition 2.2.20. Let G be a locally profinite group, H a closed or open subgroup and (σ, W) be a smooth representation of H. Define c-$\mathrm{Ind}_H^G(W)$ to be the subset of s-$\mathrm{Ind}_H^G(W)$ consisting of functions with compact support modulo H i.e. the image of $\mathrm{supp}(f)$ in $H\backslash G$ is compact. It is an easy check to see that c-$\mathrm{Ind}_H^G(W)$ yields a subrepresentation of s-$\mathrm{Ind}_H^G(W)$.

Lemma 2.2.21. *Let G be a locally profinite group, H a subgroup, and K a open compact subgroup. Then*

1. *K-orbits in $H\backslash G$ are open and compact.*
2. *If a subset $C \subseteq H\backslash G$ is compact it lies in the union of finitely many K-orbits.*

Proposition 2.2.22. *Let $f \in$ s-$\mathrm{Ind}_H^G(W)$. Then f has compact support modulo H iff $\mathrm{supp}(f) \subseteq H \cdot C$ for some $C \subseteq G$ compact.*

Proof. $f \in$ s-$\mathrm{Ind}_H^G(W)$ so there is a compact open subgroup K such that K stabilises f. It follows that the support of f is a union of double (H, K) cosets. Let $q : G \to H\backslash G$ be the quotient map. Then $q(\mathrm{supp}(f))$ is a union of K-orbits.

($\Rightarrow$) Suppose $q(\mathrm{supp}(f))$ is compact. By the lemma it is a finite union of K-orbits. Thus $\mathrm{supp}(f)$ is a union of finitely many double (H, K)-cosets. Let $g_1, \ldots, g_n$ be double coset representatives. Then $\mathrm{supp}(f) = H \cdot (\cup_i g_i K)$ where $\cup_i g_i K$ is compact (and open).

($\Leftarrow$) Suppose $\mathrm{supp}(f) \subseteq H \cdot C$ with C compact. Then $q(H \cdot C) = q(C)$ is compact and so lies in a finite union of K orbits. But $q(\mathrm{supp}(f)) \subseteq q(C)$ and so $q(\mathrm{supp}(f))$ must be a finite union of K-orbits and hence must be compact. ∎

Remark 2.2.23. The proposition is also true if we insist that C is open.

Proposition 2.2.24. *c-Ind_H^G is exact.*

Proposition 2.2.25. *Let H be an open subgroup of G, and $\phi \in \mathrm{Ind}_H^G(W)$ be compactly supported modulo H. Then $\phi \in$ c-$\mathrm{Ind}_H^G(W)$.*

Definition 2.2.26. Let H be an open subgroup of G and W an H representation. Then there is a H-homomorphism $\alpha_W^c : W \to$ c-Ind_H^G given by $w \mapsto f_w$ where f_w is the function that sends h to $h.w$ and is 0 outside of H. By the previous proposition, this does indeed lie in c-$\mathrm{Ind}_H^G(W)$.

Lemma 2.2.27. *Let H be an open subgroup of G, and let W be a representation of H. Then*

1. *The map α_W^c is an H-isomorphism with the space of functions $f \in$ c-$\mathrm{Ind}_H^G(W)$ such that $\mathrm{supp}(f) \subseteq H$.*
2. *If $\mathcal{W}$ is a basis for W and $\mathcal{G}$ a choice of representatives for G/H, then $\{gf_w : w \in \mathcal{W}, g \in \mathcal{G}\}$ is a basis for c-$\mathrm{Ind}_H^G(W)$.*

Thm 2.2.28. *Let H be an open subgroup of G, W an H-representation and V a G-representation. Then there is a natural bijection*

$$\mathrm{Hom}_G(c\text{-}Ind_H^G(W), V) \leftrightarrow \mathrm{Hom}_H(W, Res_H^G(V)). \tag{2.8}$$

Proof. We have the maps

$$\mathrm{Hom}_G(\text{c-Ind}_H^G(W), V) \leftrightarrow \mathrm{Hom}_H(W, \mathrm{Res}_H^G(V)) \tag{2.9}$$
$$\phi \mapsto \phi \circ \alpha_W^c \tag{2.10}$$
$$(gf_w \mapsto g\psi(w)) \leftarrow\!\shortmid \psi. \tag{2.11}$$

It is starightforward to check that the second map is well-defined and that these maps are mutually inverse. ■

2.3 Irreducible representations and the contragredient

Remark 2.3.1. From now on assume that G/K is countable for any compact open subgroup K of G.

Lemma 2.3.2. *Let V be an irreducible smooth representation of G. Then* $\dim_{\mathbb{C}} V$ *is countable.*

Lemma 2.3.3. *(Schur's lemma). If V is an irreducible smooth representation of G, then* $\mathrm{End}_G(V) = \mathbb{C}$.

Corollary 2.3.4. *Let V be an irreducible smooth representation of G. Then the central character of V is smooth.*

Corollary 2.3.5. *If G is abelian then any irreducible smooth representation of G is 1-dimensional.*

Definition 2.3.6. Let V be a smooth G-representation. We define the contragredient, or smooth dual, of V to be $\check{V} = (V^*)^\infty$.

Remark 2.3.7. If $K \leq G$ is a compact open subgroup of G, then for any $f \in (\check{V})^K$, $f(V(K)) = 0$.

prop:dual **Proposition 2.3.8.** *Restriction to V^K induces an isomorphism*

$$(\check{V})^K \cong (V^K)^*. \tag{2.12}$$

Thm 2.3.9. *The canonical morphism $V \to \check{\check{V}}$ is an isomorphism iff V is admissable.*

Proposition 2.3.10. *The contravariant functor $\vee : \mathsf{Rep}(G) \to \mathsf{Rep}(G)$ is exact.*

Proof. Follows from proposition 2.3.8. ∎

Corollary 2.3.11. *V is irreducible iff $\check{V}$ is irreducible.*

Proposition 2.3.12. *Let V and W be smooth representations of G, and $\mathcal{P}(V, W)$ be the space of G-invariant bilinear pairings $V \times W \to \mathbb{C}$. Then there are isomorphisms*

$$\mathrm{Hom}_G(V, \check{W}) \cong \mathcal{P}(V, W) \cong \mathrm{Hom}_G(W, \check{V}). \tag{2.13}$$

2.4 Measures

Proposition 2.4.1. *Let $C_c^\infty(G)$ be the space of locally constant functions on G with compact support. Then $(C_c^\infty(G), \lambda)$ and (C_c^∞, ρ) are both smooth.*

Remark 2.4.2. Suppose a function $f : G \to \mathbb{C}$ is fixed by $\rho(K)$ (or $\lambda(K)$) for K a compact open subgroup K of G. Then f has compact support iff $\mathrm{supp}(f) \subseteq C$ for some compact set C.

Definition 2.4.3. A right Haar integral on G is a non-zero G-homomorphism $I : (C_c^\infty(G), \rho) \to \mathbb{C}$ such that $I(f) \geq 0$ for any $f \in C_c^\infty(G)$, $f \geq 0$.

Thm 2.4.4. *There exists a unique right Haar integral $I : C_c^\infty(G) \to \mathbb{C}$ up to scaling.*

Proof. Let K be a compact open subgroup of G and write ${}^K C_c^\infty$ for the subspace $(C_c^\infty(G))^{\lambda(K)}$. Then ${}^K C_c^\infty(G) = \text{c-Ind}_K^G 1_K$. It follows that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G({}^K C_c^\infty(G), \mathbb{C}) = 1. \tag{2.14}$$

If $f_{K,g}$ denotes the indicator function on the coset Kg, then the map

$$I_K :{}^K C_c^\infty(G) \to \mathbb{C}, f_{K,g} \mapsto 1 \tag{2.15}$$

is a G-homomorphism and so all G-homomorphisms ${}^K C_c^\infty \to \mathbb{C}$ are a multiple of this map.

Now let $\{K_n\}_n$ be a descending sequence of compact open subgroups of G such that $\cap_n K_n = \{e\}$. Then $\{{}^{K_n} C_c^\infty(G)\}_n$ is an ascending sequence of subspaces of $C_c^\infty(G)$ such that $C_c^\infty(G) = \cup_n {}^{K_n} C_c^\infty(G)$. Let $I_n = I_{K_n}/|K_1 : K_n|$. Then

$$I_{n+1}(f_{g,K_n}) = |K_n : K_{n+1}|/|K_1 : K_{n+1}| = 1/|K_1 : K_n| = I_n(f_{g,K_n}). \tag{2.16}$$

It follows that $I_{n+1}|_{K_n C_c^\infty(G)} = I_n$ and so we can define a G-homomorphism $I : C_c^\infty(G) \to \mathbb{C}$. It is clear that this map is a right Haar measure.

Now suppose I' is another Haar measure. Then there are $\alpha_n \in \mathbb{C}$ such that $I'|_{K_n C_c^\infty(G)} = \alpha_n \cdot I_n$ for all n. Evaluating at the f_{g,K_n} gives that $\alpha_n = \alpha_{n+1} =:$ α for all n and so $I' = \alpha I$. ■

Remark 2.4.5. If $f \geq 0$ and there exists a $g \in G$ such that $f(g) > 0$ then $I(f) > 0$.

Definition 2.4.6. Define $\vee : C_c^\infty(G) \to C_c^\infty(G)$ by $f \mapsto \check{f}$ where $\check{f}(g) = f(g^{-1})$. Then $\vee : (C_c^\infty(G), \lambda) \to (C_c^\infty(G), \rho)$ is a G-isomorphism.

Remark 2.4.7. $\vee$ induces a bijection between left and right Haar measures.

Definition 2.4.8. Let I be a left Haar measure on G. For a non-empty compact open subset S of G, let Γ_S denote its characteristic function. We define

$$\mu_G(S) = I(\Gamma_S). \tag{2.17}$$

Then $\mu_G(gS) = \mu_G(S)$ for all $g \in G$.

Remark 2.4.9. We have that $I(f) = \int_G f d\mu_G$ for $f \in C_c^\infty(G)$.

Definition 2.4.10. We can extend the domain of Haar integration as follows. Let μ_G be a left Haar measure on G, and f be a function on G invariant under left translation by a compact open subgroup K of G. If the series

$$\sum_{g \in K \backslash G} \int_{Kg} |f(x)| d\mu_G(x) \tag{2.18}$$

converges define

$$\int_G f(x) d\mu_G(x) = \sum_{g \in K \backslash G} \int_{Kg} f(x) d\mu_G(x). \tag{2.19}$$

Proposition 2.4.11. *This definition does not depend on K and is left translation invariant.*

Proof. Let K' be any other compact open subgroup of G. Then $K \cap K'$ has finite index in K and K'. It follows that

$$\begin{aligned}
\sum_{g \in K \backslash G} \int_{Kg} |f(x)| d\mu_G(x) &= \sum_{g \in K \backslash G} \sum_{h \in K \cap K' \backslash K} \int_{K \cap K' hg} |f(x)| d\mu_G(x) \\
&= \sum_{g \in K \cap K' \backslash G} \int_{K \cap K' g} |f(x)| d\mu_G(x) \\
&= \sum_{g \in K' \backslash G} \sum_{h \in K \cap K' \backslash K'} \int_{K \cap K' hg} |f(x)| d\mu_G(x) \\
&= \sum_{g \in K' \backslash G} \int_{K' g} |f(x)| d\mu_G(x)
\end{aligned} \tag{2.20}$$

and all series converge. It follows that the same series but without absolute values converge, and so we obtain the first part of the proposition.

For the second part let $y \in g$. Then $\{yg : g \in K\backslash G\}$ is a set of coset representatives for $yKy^{-1}\backslash G$ and

$$\begin{aligned}\sum_{g\in K\backslash G}\int_{yKy^{-1}\cdot yg}|\lambda_y f(x)|d\mu_G(x) &= \sum_{g\in K\backslash G}\int_G 1_{yKy^{-1}yg}(x)|\lambda_y f(x)|d\mu_G(x)\\ &= \sum_{g\in K\backslash G}\int_G \lambda_y(1_{Kg}(x)|f(x)|)d\mu_G(x)\\ &= \sum_{g\in K\backslash G}\int_{Kg}|f(x)|d\mu_G(x). \qquad (2.21)\end{aligned}$$

Thus $\int_G \lambda_y f d\mu_G$ is defined and the above calculation, but without absolute values, shows that it is equal to $\int_G f d\mu_G$. ■

Proposition 2.4.12. *Let G_1, G_2 be locally profinite groups. Then the natural map $C_c^\infty(G_1) \otimes_{\mathbb{C}} C_c^\infty(G_2) \to C_c^\infty(G_1 \times G_2)$ is an isomorphism that respects both left and right translation.*

Proposition 2.4.13. *If μ_1, μ_2 are left Haar measures then the map*

$$\mu : C_c^\infty(G_1 \times G_2) \to \mathbb{C} \qquad (2.22)$$

defined via the above isomorphism is also a left Haar measure.

Proposition 2.4.14. *Let $f \in G_1 \times G_2$. Then the function*

$$f_1(g_1) = \int_{G_2} f(g_1, g_2)d\mu_2(g_2) \qquad (2.23)$$

lies in $C_c^\infty(G_2)$ and

$$\int_{G_1\times G_2} f(g)d\mu_G(g) = \int_{G_1} f_1(g_1)d\mu_1(g_1). \qquad (2.24)$$

Definition 2.4.15. Let μ_G be a left Haar measure on G. For $g \in G$, $f \mapsto \int_G \rho_g f d\mu_G$ is another left Haar measure. It follows that there is a unique $\delta_G(g) \in \mathbb{R}_+^\times$ such that

$$\delta_G(g)\int_G \rho_g f d\mu_G = \int_G f d\mu_G. \qquad (2.25)$$

This map $\delta_G : G \to \mathbb{R}_+^\times$ is a homomorphism.

Proposition 2.4.16. *δ_G is trivial on open compact subgroups of G.*

Proposition 2.4.17. *A homomorphism $\psi : G \to R_+^\times$ is a character iff it is trivial on compact open subgroups.*

Corollary 2.4.18. *δ_G is a character.*

Proposition 2.4.19. *δ_G is trivial iff G is unimodular.*

Proposition 2.4.20. *The functional $f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x)$ is a right Haar integral.*

Proof.

$$\begin{aligned}\rho_y f \mapsto \int_G \delta_G(x)^{-1} \rho_y f(x) d\mu_G(x) &= \delta_G(y) \int_G \rho_y(\delta_G(x)^{-1} f(x)) d\mu_G(x) \\ &= \int_G \delta_G(x)^{-1} f(x) d\mu_G(x). \qquad (2.26)\end{aligned}$$

■

Definition 2.4.21. Let H be a closed subgroup of G, $\theta : H \to \mathbb{C}^\times$ a character and $C_c^\infty(H\backslash G, \theta) = \text{c-Ind}_H^G(\theta)$.

Definition 2.4.22. Let μ_H be a left Haar measure on H. Define the map $\sim: (C_c^\infty(G), \rho) \to C_c^\infty(H\backslash G, \theta)$ by

$$\widetilde{f}(g) = \int_H (\theta\delta_H)^{-1} \rho_g f d\mu_H. \qquad (2.27)$$

This map is a G-homomorphism and

$$\widetilde{(\lambda_h f)} = (\theta\delta_H)(h)^{-1} \widetilde{f} \qquad (2.28)$$

for $f \in C_c^\infty(G), h \in H$.

Lemma 2.4.23. *$\sim$ is surjective.*

Proof. Let K be an open compact subgroup of G. Then each double coset HgK supports at most a 1-dimensional subspace of $C_c^\infty(H\backslash G, \theta)^K$ and these spaces span $C_c^\infty(H\backslash G, \theta)^K$. But each $1_{gK} \in (C_c^\infty(G))^K$ maps to a non-zero element of $C_c^\infty(H\backslash G, \theta)^K$ with support HgK and so the map is surjective. ■

Corollary 2.4.24. *Let $\theta : H \to \mathbb{C}^\times$ be a character of H and I a right Haar intgral on G. Then $\mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C}) \neq 0$ iff I factors through $C_c^\infty(H\backslash G, \theta)$.*

Corollary 2.4.25. $\dim_{\mathbb{C}} \mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C}) = 0$ *or* 1.

Remark 2.4.26. Let K be a open compact subgroup of G, $g \in G$ and $f = 1_{gK}$. Suppose $\delta_G|_H = \theta\delta_H$. Let $x \in G$. Then

$$\widetilde{f}(x) = \int_H (\theta\delta_H)(h)^{-1} 1_{gKx^{-1}}(h) d\mu_H(h). \qquad (2.29)$$

But $h \in gKx^{-1}$ iff $x = h^{-1}gk$ for some $k \in K$. Thus $\widetilde{f}(x)$ is 0 if $x \notin HgK$. If $x \in HgK$ write $x = h_0 g k_0$ and $L = gKg^{-1} \cap H$. Then

$$\begin{aligned}\widetilde{f}(x) &= \int_H (\theta\delta_H)(h)^{-1} 1_{Lh_0^{-1}}(h) d\mu_H(h) \\ &= (\theta\delta_H)(h_0) \int_H \rho_{h_0}((\theta\delta_H)^{-1} 1_L)(h) d\mu_H(h) \\ &= \theta(h_0) \int_L (\theta\delta_H)(h)^{-1} d\mu_H(h). \end{aligned} \tag{2.30}$$

But δ_G is trivial on L and so $\widetilde{f}(x) = \theta(h_0)\mu_H(L)$. It follows that

$$\widetilde{1_{h_i gK}}(hgk) = \theta(h)\delta_G(h_i)^{-1}\mu_H(L). \tag{2.31}$$

lem:I_ker **Lemma 2.4.27.** *Suppose $\delta_G|_H = \theta\delta_H$ and let I denote the right Haar integral*

$$f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x). \tag{2.32}$$

If $\widetilde{f} = 0$ then $I(f) = 0$.

Proof. Suppose f is fixed by K. It suffices to check the case when f is of the form $\sum_i \alpha_i 1_{h_i gK}$ for $\alpha_i \in \mathbb{C}$ and the $h_i gK$ distinct cosets. Then by the remark $\widetilde{f} = 0$ implies that $\sum_i \alpha_i \delta_G(h_i)^{-1} = 0$. But

$$I(1_{h_i gK}) = \delta_G(h_i g)^{-1}\mu_G(K) \tag{2.33}$$

and so

$$I(f) = \mu_G(K)\delta_G(g)^{-1} \sum_i \alpha_i \delta_G(h_i)^{-1} = 0. \tag{2.34}$$

■

Thm 2.4.28. *Let $\theta : H \to \mathbb{C}^\times$ be a character of H. The following are equivalent:*

1. $\mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C}) \neq 0$
2. $\theta\delta_H = \delta_G|_H$.

Proof. (1) $\Rightarrow$ (2) Let $0 \neq I_\theta \in \mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C})$ be such that the right Haar integral $I : f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x)$ is equal to $I_\theta(\widetilde{f})$. Note that elements of the form $\lambda_h f - (\theta\delta_H)(h)^{-1} f$ map to zero under $\sim$ and so we get

$$\begin{aligned} 0 = I(\lambda_h f - (\theta\delta_H)(h)^{-1} f) &= \int_G \delta_G(x)^{-1}(\lambda_h f - (\theta\delta_H)(h)^{-1} f) d\mu_G(x) \\ &= \left(\delta_G(h)^{-1} - (\theta\delta_H)(h)^{-1}\right) I(f). \end{aligned} \tag{2.35}$$

Picking an f such that $I(f) \neq 0$ we get that $\delta_G|_H = \theta\delta_H$.

(2) $\Rightarrow$ (1) By lemma 2.4.27, if $\widetilde{f} = 0$ then $I(f) = 0$. The result follows. ■

Corollary 2.4.29. *Suppose $\theta\delta_H = \delta_G|_H$. Then there is a non-zero $I_\theta : (C_c^\infty(G), \rho) \to \mathbb{C}$ such that $I_\theta(f) \geq 0$, whenever $f \geq 0$.*

Definition 2.4.30. If H is a closed subgroup of G define $\delta_{H\backslash G} = \delta_H^{-1}\delta_G|_H : H \to \mathbb{R}_+^\times$. Write $\mu_{H\backslash G}$ for

$$I_{\delta_{H\backslash G}}(f) = \int_{H\backslash G} f(g) d\mu_{H\backslash G}(g) \tag{2.36}$$

where $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$.

2.4.1 Duality theorem

Fix measures μ_G, μ_H and write $\mu_{H\backslash G}$ for the corresponding semi-invariant measure on $H\backslash G$.

Proposition 2.4.31. *Let W be a H-representation. Given $\phi \in$ c-$\mathrm{Ind}_H^G W, \Phi \in$ s-$\mathrm{Ind}_H^G(\delta_{H\backslash G} \otimes \check{W})$ define $f_{\Phi,\phi} : G \to \mathbb{C}$ by*

$$f_{\Phi,\phi}(g) = \langle \Phi(g), \phi(g) \rangle. \tag{2.37}$$

Then $f_{\Phi,\phi}$ lies in $C_c^\infty(H\backslash G, \delta_{H\backslash G})$.

Proof. We clearly have

$$f_{\Phi,\phi}(hg) = \delta_{H\backslash G}(h) f_{\Phi,\phi}(g), h \in H, g \in G. \tag{2.38}$$

We also have $gf_{\Phi,\phi} = f_{g\phi,g\Phi}$. Thus, if K is a compact open subgroup that fixes both ϕ and Φ then K also fixes $f_{\Phi,\phi}$. Finally it remains to check that f has compact support module H. But $\mathrm{supp}(f_{\Phi,\phi}) \subseteq \mathrm{supp}(\phi) = H\mathrm{supp}(\phi)$. ∎

Remark 2.4.32. Let $F =$ s-$\mathrm{Ind}_H^G \circ (\delta_{H\backslash G} \otimes -) \circ \vee$ and $G =$ c-Ind_H^G. If $h : V \to W$ is a homomorphism between H-representations then for $\phi \in G(V), \Phi \in F(W)$ we have

$$f_{F(h)\Phi,\phi}(g) = \langle \Phi(g) \circ h, \phi(g) \rangle = \langle \Phi(g), h \circ \phi(g) \rangle = f_{\Phi, G(h)\phi}(g). \tag{2.39}$$

Definition 2.4.33. Define the pairing

$$(-,-)_W : \text{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W}) \times \text{c-Ind}_H^G W \to \mathbb{C} \tag{2.40}$$

by

$$(\Phi, \phi)_W \mapsto \int_{H\backslash G} f_{\Phi,\phi} d\mu_{H\backslash G}. \tag{2.41}$$

This pairing is clearly G-invariant. By the remark the induced map

$$\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W}) \to (\text{c-Ind}_H^G W)^\vee \tag{2.42}$$

is natural in W.

Lemma 2.4.34. *Let K be a compact open subset of G, $\mathcal{G}$ a set of representatives for $H\backslash G/K$, and for each $g \in \mathcal{G}$, let $\mathcal{W}_g$ be a basis for $W^{H\cap gKg^{-1}}$. Then for each $g \in \mathcal{G}, w \in \mathcal{W}_g$ there is a unique $f_{g,w}$ with support HgK and $f_{g,w}(g) = w$, and the collection of all of these form a basis for $(c\text{-}Ind_H^G W)^K$.*

Proof. It is clear that the $f_{g,w}$ exist and that they are linearly independent. To see that they span $(\text{c-Ind}_H^G W)^K$, note that if $f \in (\text{c-Ind}_H^G W)^K$ then $\mathrm{supp}(f)$ is the union of finitely many double cosets of $H\backslash G/K$. Noting that f multiplies by the indicators on the various double cosets are still in $(\text{c-Ind}_H^G W)^K$, we may thus reduce to the case when $\mathrm{supp}(f) = HgK$ for some $g \in \mathcal{G}$. But note that $f(g) \in W^{H\cap g^K g^{-1}}$. Taking the appropriate linear combination of $f_{g,w}$'s gives the result. ∎

Remark 2.4.35. We have that

$$(\delta_{H\backslash G} \otimes \check{W})^{H\cap gKg^{-1}} = \check{W}^{H\cap gKg^{-1}} = \left(W^{H\cap gKg^{-1}}\right)^* \tag{2.43}$$

since $\delta_{H\backslash G}$ is trivial on $H \cap gKg^{-1}$. It follows that the dual basis of $\mathcal{W}_g$ give a basis for $(\delta_{H\backslash G} \otimes \check{W})^{H\cap gKg^{-1}}$. Write $f_{g,\check{w}}, g \in \mathcal{G}, w \in \mathcal{W}_g^*$ for the elements of $\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes W)$ that arise in the same way as in the lemma. Then by a similar argument as above, $\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes W)$ consists of all functions f such that $f|_{HgK}$ is a finite linear combination of $f_{g,\check{w}}$'s.

Note moreover that for $g \in \mathcal{G}, w \in \mathcal{W}_g, \check{w} \in \mathcal{W}_g^*$,

$$(f_{g,\check{w}}, f_{g,w}) = \int_{H\backslash G} 1_{HgK}\langle \check{w}, w\rangle d\mu_{H\backslash G} = \mu_{H\backslash G}(HgK)\langle \check{w}, w\rangle \tag{2.44}$$

and

$$(f_{g,\check{w}}, f_{g',w}) = 0 \tag{2.45}$$

when $g' \in \mathcal{G}$ and $g \neq g'$.

Proposition 2.4.36. *The pairing $(-,-)$ is perfect.*

Proof. It suffices to show that the induced map

$$\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W})^K \to ((\text{c-Ind}_H^G W)^K)^* \tag{2.46}$$

is an isomorphism for any compact open subgroup K of G. But this just follows from the remark. ∎

Corollary 2.4.37. *There is a natural isomorphism*

$$(c\text{-}Ind_H^G W)^\vee \cong s\text{-}Ind_H^G(\delta_{H\backslash G} \otimes \check{W}). \tag{2.47}$$

2.5 The Hecke Algebra

Definition 2.5.1. Let $f_1, f_2 \in C_c^\infty(G)$ and define

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu_G(x). \tag{2.48}$$

Lemma 2.5.2. *Let $f_1, f_2 \in C_c^\infty(G)$. Then the map $(x, g) \mapsto f_1(x) f_2(x^{-1}g)$ is in $C_c^\infty(G \times G)$.*

Proof. Let K be a compact open subgroup such that $\rho(K)$ fixes f_1, f_2 and $\lambda(K)$ fixes f_2. Then for $k_1, k_2 \in K$, $(xk_1, gk_2) \mapsto f_1(x) f_2(x^{-1}g)$ and so it is fixed by $K \times K$. It remains to check that it has compact support. But its support is

$$\{(x, g) : x \in \operatorname{supp}(f_1), g \in x \cdot \operatorname{supp}(f_2)\}. \tag{2.49}$$

This is the image of $\operatorname{supp}(f_1) \times \operatorname{supp}(f_2)$ under the homeomorphism $G \times G \to G \times G : (x, y) \mapsto (x, xy)$ and so is compact. ■

Proposition 2.5.3. *If $f_1, f_2 \in C_c^\infty(G)$ then $f_1 * f_2 \in C_c^\infty(G)$.*

Proof. Note that for a fixed $g \in G$ the map $x \mapsto f_1(x) f_2(x^{-1}g)$ is in $C_c^\infty(G)$ and so $f_1 * f_2$ is defined everywhere. Let K be a compact open subgroup of G such that $\rho(K)$ fixes f_2. Then it is clear that $\rho(K)$ also fixes $f_1 * f_2$. To see that the support is compact note that $f_1 * f_2(g) \neq 0$ only if $\operatorname{supp}(f_1) \cap g^{-1}\operatorname{supp}(f_2) \neq \emptyset$. But $\operatorname{supp}(f_1)$ is compact and $\operatorname{supp}(f_2)$ is open and so only finitely many cosets of $\operatorname{supp}(f_2)$ can intersect $\operatorname{supp}(f_1)$. Thus $\operatorname{supp}(f_1 * f_2)$ is contained a finite union of cosets of $\operatorname{supp}(f_2)$ and so is compact. ■

Remark 2.5.4. It is easy to check that $*$ is associative.

Definition 2.5.5. The Hecke algebra of G is $\mathcal{H}(G) = (C_c^\infty(G), *)$. This is an associative algebra.

For a compact open subgroup K of G define $e_K := 1_K / \mu_G(K)$.

Remark 2.5.6. e_K is idempotent.

Remark 2.5.7. For any $f \in C_c^\infty(G), k \in K, g \in G$ we have $e_K * f(kg) = e_K * f(g)$. In other words, $e_K * f$ is fixed by $\lambda(K)$. Similarly $f * e_K$ is fixed by $\rho(K)$.

Proposition 2.5.8. *Let K be a compact open subgroup of G and $f \in C_c^\infty(G)$. Then f is fixed by $\lambda(K)$ iff $e_K * f = f$.*

Proof. It is clear that if f is fixed by $\lambda(K)$ then $e_K * f(g) = f(g)$ for all $g \in G$. Conversely, suppose $e_K * f = f$. Then the result follows from the remark. ■

Remark 2.5.9. Similarly f is fixed by $\rho(K)$ iff $f * e_K = f$.

Corollary 2.5.10. *The space $\mathcal{H}(G, K) := e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$, with unit e_K.*

Corollary 2.5.11.

$$\mathcal{H}(G,K) = \{f \in \mathcal{H}(G) : f(k_1 g k_2) = f(g), g \in G, k_1, k_2 \in K\}. \tag{2.50}$$

Definition 2.5.12. Let M be a left $\mathcal{H}(G)$-module. We say that M is smooth if $\mathcal{H}(G) * M = M$. Since $\mathcal{H}(G)$ is a union of the $\mathcal{H}(G,K)$ this is equivalent to saying for every $m \in M$ there is a compact open subgroup K such that $e_K * m = m$.

Write $\mathcal{H}(G) - \mathrm{Mod}$ for the category of smooth $\mathcal{H}(G)$-modules.

Definition 2.5.13. Let (π, V) be a smooth G representaion. We can turn V into a smooth $\mathcal{H}(G)$-module by defining for $f \in \mathcal{H}(G), v \in V$

$$\pi(f)v = \int_G f(g)\pi(g)v d\mu_G(g). \tag{2.51}$$

Remark 2.5.14. Let K be a compact open subgroup of G such that $\rho(K)$ fixes f and K fixes v. Then map $g \mapsto f(g)\pi(g)v$ is fixed by $\rho(K)$ and has compact support. Thus the integral is defined and is equal to the finite sum

$$\sum_{g \in G/K} f(g)\pi(g)v. \tag{2.52}$$

It is then clear that $\pi(e_K)v = v$ for $v \in V^K$.

Remark 2.5.15. If $V = (C_c^\infty(G), \lambda)$ then the $\mathcal{H}(G)$-module action is given by $\lambda(\phi)f = \phi * f$.

If $V = (C_c^\infty(G), \rho)$ then the $\mathcal{H}(G)$-module action is given by $\rho(\phi)f = f * \check{\phi}$.

Proposition 2.5.16. *The above procedure defines a functor* $\mathsf{Smo}_G \to \mathcal{H}(G) -$ *Mod which is the identity on morphisms.*

Proof. It is easy to check that if $f_1, f_2 \in C_c^\infty(G), v \in V$ then $\pi(f_1)(\pi(f_2)v) = \pi(f_1 * f_2)v$. Thus V is a $\mathcal{H}(G)$-module. By the remark, V is moreover a smooth $\mathcal{H}(G)$-module. It is clear that G-homomorphisms are also $\mathcal{H}(G)$-homomorphisms. ■

Lemma 2.5.17. *Let M be a smooth $\mathcal{H}(G)$-module. Then $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \cong M$.*

Proof. Let $\theta : \mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \to M$ be the canonical map. Suppose $\sum_i f_i \otimes m_i$ is in the kernel. Let K be a compact open subgroup that fixes each of the f_i by translation on both sides and is such that $m_i \in e_K * M$ for all i. Then $e_K * m_i = m_i$ for all i and so

$$\sum_i f_i \otimes m_i = e_K \otimes \sum_i f_i * m_i = 0. \tag{2.53}$$

Thus the map is injective. But it is surjective by definition of smoothness. Hence we have an isomorphism. ■

Corollary 2.5.18. *Let M be a smooth $\mathcal{H}(G)$-module. Then M is naturally a G-representation.*

Proof. G acts on $\mathcal{H}(G)$ by left translation, and hence on $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M$. ■

Remark 2.5.19. If $m \in M$ and K is a compact open subgroup of G such that $e_K * m = m$, then for $g \in G$, $gm = 1_{gK} * m/\mu_G(K)$. But then for the induced $\mathcal{H}(G)$-module strucutre we have

$$\pi(1_{gK})m = \int_G 1_{gK}(x)1_{xK} * md\mu_G(x)/\mu_G(K) = 1_{gK} * m. \tag{2.54}$$

This suffices to show that the induces module strucutre is just the original module structure.

Corollary 2.5.20. *The above procedure defines a functor $\mathcal{H}(G)-Mod \to \mathsf{Smo}_G$ which is the identity on morphisms.*

Remark 2.5.21. Conversely if we start with a smooth G-representation V, then for $v \in V^K$ we have $gv = 1_{gK} * v/\mu_G(K) = \int_G 1_{gK}(x)\pi(x)vd\mu_G(x)/\mu_G(K) = \pi(g)v$. We thus have the following result.

Thm 2.5.22. *The functors $\mathsf{Smo}_G \to \mathcal{H}(G) - Mod$ and $\mathcal{H}(G) - Mod \to \mathsf{Smo}_G$ are mutually inverse.*

Proposition 2.5.23. *Let V be a smooth G-representation. Then the operator e_K* is the projection onto V^K along $V(K)$. The space V^K is an $\mathcal{H}(G,K)$-module on which e_K acts as the identity.*

Proof. Let $k \in K$ and $v \in V$. Then

$$k(e_K * v) = e_K * (kv) = e_K * v \tag{2.55}$$

where the last equality follows from δ_G being trivial on K. Thus e_K is a K-homomorphism with image in V^K. It follows that it must send $V(K)$ to 0. Moreover it is idempotent and the identity on V^K. This gives the result. ■

Lemma 2.5.24. *Let V be an irreducible smooth G-representation. Then V^K is either 0 or a simple $\mathcal{H}(G,K)$-module.*

Proof. Suppose $V^K \neq 0$. Then let M be a non-zero $\mathcal{H}(G,K)$-submodule of V. Then $\mathcal{H}(G)M = V$ by irreducibility and so

$$V^K = e_K * V = e_K * \mathcal{H}(G)M = \mathcal{H}(G,K)M = M. \tag{2.56}$$

■

`prop:bij` **Proposition 2.5.25.** *The map $V \mapsto V^K$ induces a bijection between*

1. *equivalence classes of smooth representations of G such that $V^K \neq 0$*

2. equivalance classes of simple $\mathcal{H}(G,K)$-modules.

Proof. Let M be a simple $\mathcal{H}(G,K)$-module and let $U = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$. Then $U^K = e_K * \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M = e_K \otimes M \cong M$. Let X be a maximal G-subspace of U such that $X^K = 0$ (exists by Zorn). This subspace is unique since $(X+X')^K = X^K + X'^K$. Note that X is maximal such that $X \cap U^K = X \cap e_K \otimes M = 0$. If $X \subsetneq W$ is a G-subspace of U then W must meet $e_K \otimes M$ and so must contain $e_K \otimes M$ (as M is simple) and so must equal to U. It follows that $V = U/X$ is irreducible and $V^K = M$ as $\mathcal{H}(G,K)$-modules. Note that the isomorphism class of V depends only on that of M.

Thus we now have maps going in both directions and we know that one composition is the identity. To see that the other composition is the identity, let V be an irreducible G-representation and $M = V^K$. We have a map $U = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M \to V$, $f \otimes m \mapsto f * m$. The image is non-zero subrepresentation of V and so the map must be surjective. Moreover, the image of X is a submodule that does not intersect V^K and so must be zero. Thus X lies in the kernel of the map. Now suppose u lie in the both U^K and the kernel of the map. Then $u = e_K \otimes m$ some $m \in M$. But then $e_K * m = 0$ and $e_K * m = m$ and so $e_K \otimes m = 0$. Thus the kernel lies inside X. It follows that $V \cong U/X$ as required. ■

Corollary 2.5.26. *Let V be a smooth non-zero representation of G. Then V is irreducible iff for any open compact open subgroup K of G, the space V^K is either zero or $\mathcal{H}(G,K)$-simple.*

Proof. ($\Rightarrow$) Done. ($\Leftarrow$) Let V a G-representation with a non-zero sub-representation U. Let $W = V/U$ and K be a compact open subgroup of G such that $U^K, W^K \neq 0$. Then $0 \to U^K \to V^K \to W^K \to 0$ is exact and so V^K is not a simple $\mathcal{H}(G,K)$-module. ■

Definition 2.5.27. Let $(\rho, V) \in \hat{K}$ and define

$$e_V(x) = \frac{\dim V}{\mu_G(K)} \operatorname{tr}(\rho(x^{-1})) 1_K(x). \tag{2.57}$$

Recall that since K is compact, the kernel of ρ is also a compact open subgroup $K' \leq K$ such that K/K' is finite. It follows that ρ is constant on double cosets $K' \backslash G / K'$ and so $e_{K'} * e_\rho = e_\rho * e_{K'} = e_\rho$. Thus $e_\rho \in \mathcal{H}(K,K') \subseteq \mathcal{H}(G,K')$.

Proposition 2.5.28. *The map $\mathcal{H}(K,K') \to \mathbb{C}[K/K']$, $1_{gK'}/\mu_G(K') \mapsto gK'$ is an algebra isomorphism that respects their respective actions on V.*

Remark 2.5.29. Under this isomorphism e_V gets sent to the idempotent for V is $\mathbb{C}[K/K']$.

Corollary 2.5.30. *1. The function $e_V \in \mathcal{H}(G)$ is idempotent.*

2. If W is a smooth G-representation of G, then e_ρ is the K-projection $V \to V^\rho$.

Remark 2.5.31. Replacing V^K with V^ρ and $\mathcal{H}(G,K)$ with $e_\rho * \mathcal{H}(G) * e_\rho$ we get an exact analogue of proposition 2.5.25.