Smooth representations of locally profinite groups

Emile T. Okada

October 9, 2019

Contents

CHAPTER 1

Pro-$\mathcal{C}$ groups

1.1 Topological preliminaries

1.2 Pro-$\mathcal{C}$ groups

Thm 1.2.1. *Let $\mathcal{C}$ be a formation of finite groups. Then the following are equivalent.*

1. *G is a pro-$\mathcal{C}$ group;*
2. *G is compact Hausdorff totally disconnected, and for each open normal subgroup U of G, $G/U \in \mathcal{C}$;*
3. *G is compact and the identity element 1 of G admits a fundamental system $\mathcal{U}$ of open neighbourhoods U such that $\bigcap_{U\in\mathcal{U}} U = 1$ and each U is an open normal subgroup of G with $G/U \in \mathcal{C}$;*
4. *The identity element 1 of G admits a fundamental system $\mathcal{U}$ of open neighbourhoods U such that each U is a normal subgroup of G with $G/U \in \mathcal{C}$, and*
$$\varprojlim_{U\in\mathcal{U}} G/U. \tag{1.1}$$

CHAPTER 2

Smooth Representations of Locally Profinite Groups

2.1 Locally profinite groups

Definition 2.1.1. *A locally profinite group* is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G.

Proposition 2.1.2. *Let G be a locall profinite group.*

1. *Closed subgroups of G are locally profinite.*
2. *Quotients of G by closed normal subgroups are locally profinite.*

Proposition 2.1.3. *Let G be a compact locally profinite group then the map*

$$G \to \varinjlim G/K \tag{2.1}$$

is a topological isomorphism, where K ranges over all open normal subgroups of G.

Proposition 2.1.4. *A topological group G is locally profinite iff G is locally compact and totally disconnected.*

Proposition 2.1.5. *Let $\{K_n\}_n$ be a decreasing sequence of compact open subgroups of G such that $\cap_n K_n = \{e\}$. Then for any neighbourhood U of e there is an n such that $K_n \subseteq U$.*

2.2 Smooth representations

Definition 2.2.1. Let G be a locally profinite group and (π, V) a complex representation of G. (π, V) is *smooth* if for every $v \in V$ there is a compact open subgroup K of G such that $v \in V^K$.

(π, V) is *admissable* if the space V^K is finite dimensional for each compact open subgroup K of G.

Proposition 2.2.2. *Let (π, V) be a smooth representation. Then subrepresentations and quotients are also smooth.*

2.2.1 Characters

Proposition 2.2.3. *Let $\psi : G \to \mathbb{C}^\times$ be a group homomorphism. The following are equivalent*

1. *ψ is continuous,*
2. *$\ker \psi$ is open,*
3. *$\ker \psi$ contains an open set,*
4. *the corresponding representation on $\mathbb{C}$ is smooth.*

Proof. (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1) clear. (1) $\Rightarrow$ (2) Let U be an open subset of $\mathbb{C}^\times$. Then $\psi^{-1}(U)$ is open and so contains an open compact subgroup K. For U sufficiently small, it contains no non-trivial subgroups of $\mathbb{C}^\times$ and so $K \subseteq \ker \psi$. ∎

Definition 2.2.4. We call a homomorphism $\psi : G \to \mathbb{C}^\times$ that satisfies any of the above conditions a chcaracter of G.

Proposition 2.2.5. *If $\psi : G \to \mathbb{C}^\times$ is a character and G is a union of its open compact subgroups, then $\psi(G) \subseteq S^1$.*

Definition 2.2.6. Let $\psi \in \hat{F}$, $\psi \neq 1$. The level of ψ is least $d \in \mathbb{Z}$ such that $\mathfrak{p}^d \subseteq \ker \psi$.

Proposition 2.2.7. *Let $\psi \in \hat{F}$, $\psi \neq 1$, have level d. Let $a \in F$. The map $a\psi : x \mapsto \psi(ax)$ is a character of F. If $a \neq 0$, the character has level $d - \nu_F(a)$.*

Proposition 2.2.8. *Let $\psi \in \hat{F}$, $\psi \neq 1$, have level d. Then the map $a \mapsto a\psi$ is a group isomorphism $F \to \hat{F}$.*

Proof. Let $\theta \in \hat{F}, \theta \neq 1$ have level l. Let $\bar{w}$ be a prime element of F and $u \in U_F$. Then $u\bar{w}^{d-l}\psi$ has level d and so agrees with ψ on $\mathfrak{p}^l$. Note that $u\bar{w}^{d-l}\psi$ and $u'\bar{w}^{d-l}\psi$ for $u, u' \in U_F$ agree on $\mathfrak{p}^{l-1}$ iff $u \equiv u' \pmod{\mathfrak{p}}$. Moreover, $\mathfrak{p}^{l-1}$ has $q-1$ characters that are trivial on $\mathfrak{p}^l$. As u ranges over U_F/U_F^1, the $q-1$ characters $u\bar{w}^{d-l}\psi|_{\mathfrak{p}^{l-1}}$ are distinct, non-trivial, but trivial on $\mathfrak{p}^l$. Thus, one of them must equal to $\theta|_{\mathfrak{p}^{l-1}}$. Call the corresponding u, u_1. Iterating we obtain a sequence of $u_n \in U_F$ such that $u_n\bar{w}^{d-l}\psi$ agrees with θ on $\mathfrak{p}^{l-n}$ and $u_{n+1} \equiv u_n \pmod{\mathfrak{p}^n}$. Thus the sequence $\{u_n\}$ must converge to some $u \in U_F$ and we have $\theta = u\bar{w}^{d-l}\psi$. ∎

2.2.2 Semisimplicity

Proposition 2.2.9. *If G is compact then any smooth representation is semi-simple.*

Proof. Let $v \in V$ and $K \subseteq G$ be an open compact subgroup such that $v \in V^K$. G is compact and so $|G : K| < \infty$. Thus $W = \mathbb{C}Gv$ is finite dimensional. Moreover, if we let $K' = \cap_{g \in G/K} gKg^{-1}$ then this is a open normal subgroup of G and K' acts triviall on W. Thus W descends to a G/K' representation. But G/K' is finite and so W is a sum of its simple submodules. It follows that the same holds for V and so V is semisimple. ■

Corollary 2.2.10. *If G is compact then any irreducible smooth representation is finite dimensional.*

Corollary 2.2.11. *Let G be a locally profinite group, and let K be a compact open subgroup of G. Let (V, π) be a smooth representation of G. Then $Res_K^G V$ is semisimple.*

Proposition 2.2.12. *Let G be a locally profinite group, K a compact open subgroups of G and (π, V) a smooth representation of G.*

1. $$V = \bigoplus_{\phi \in \hat{K}} V^\rho. \tag{2.2}$$

2. *Let (σ, W) be a representation of G and $f : V \to W$ a G-homomorphism. Then for every $\rho \in \hat{K}$ we have $f(V^\rho) \subseteq W^\rho$ and $W^\rho \cap f(V) = f(V^\rho)$.*

Corollary 2.2.13. *Let $U \to V \to W$ be a sequence of smooth representations of G. The sequence is exact iff $U^K \to V^K \to W^K$ is exact (as vector spaces) for all compact open subgroups K of G.*

Definition 2.2.14. If H is a subgroup of G, we define

$$V(H) = \operatorname{span}\{v - \pi(h)v : v \in V, h \in H\}. \tag{2.3}$$

Corollary 2.2.15. *Let G be a locally profinite group, and let (π, V) be a smooth representation of G. Let K be a compact open subgroup of G. Then*

$$V(K) = \bigoplus_{\rho \in \hat{K} \setminus \{1\}} V^\rho \tag{2.4}$$

is the unique K-complement to V^K in V.

Proof. Consider the map $V \to V^K$ given by quotienting by $\bigoplus_{\rho \in \hat{K} \setminus \{1\}} V^\rho$. $V(K)$ must lie in the kernel so we have on inclusion. Conversly, if U is an irreducible K-subrepresentation of V not isormophic to the trivial representation the $U(K) = U$ and so we get the other inclusion. ■

Proposition 2.2.16. *Let (π, V) be an aribitrary representation. Define $V^\infty = \bigcup_K V^K$, where K ranges over compact open subgroups of G. Then V^∞ is a smooth subrepresentation of G.*

Proposition 2.2.17. *Let (π, V) be a smooth representation of G, and (σ, W) be an arbitrary representation. Then every morphism $f : V \to W$ factors through W^∞.*

Corollary 2.2.18. $(-)^\infty : \mathsf{Rep}_G \to \mathsf{Smo}_G$ *is a functor.*

Thm 2.2.19. *Let $i : \mathsf{Smo}_G \to \mathsf{Rep}_G$ be the inclusion functor. Then $i \dashv (-)^\infty$.*

2.2.3 Induction

Definition 2.2.20. Let G be a locally profinite group, H a closed or open subgroup and (σ, W) be a smooth representation of H. Define $\text{s-Ind}_H^G(W) = (\text{Ind}_H^G(W))^\infty$. Note that if (π, V) is a smooth G representation then so is $\text{Res}_H^G(V)$. Write s-Res for the functor between $\mathsf{Smo}_G \to \mathsf{Smo}_H$.

Proposition 2.2.21. *s-Res$_H^G \dashv$ s-Ind$_H^G$.*

Proof. Let V be a G-representation and W be a H-representation. Let $\alpha_W : \text{s-Ind}_H^G(W) \to W$ be the H-homomorphism $f \mapsto f(e)$. Then we have the maps

$$\text{Hom}(\text{s-Res}_H^G(V), W) \leftrightarrow \text{Hom}(V, \text{s-Ind}_H^G(W)) \tag{2.5}$$
$$\phi \mapsto (v \mapsto (g \mapsto \phi(gv)) \tag{2.6}$$
$$\alpha_W \circ \psi \leftarrow\!\shortmid \psi. \tag{2.7}$$

It is straightforward to check that these maps are mutually inverse. ∎

Proposition 2.2.22. *s-Ind$_H^G$ is exact.*

Definition 2.2.23. Let G be a locally profinite group, H a closed or open subgroup and (σ, W) be a smooth representation of H. Define $\text{c-Ind}_H^G(W)$ to be the subset of $\text{s-Ind}_H^G(W)$ consisting of functions with compact support modulo H i.e. the image of $\text{supp}(f)$ in $H\backslash G$ is compact. It is an easy check to see that $\text{c-Ind}_H^G(W)$ yields a subrepresentation of $\text{s-Ind}_H^G(W)$.

Lemma 2.2.24. *Let G be a locally profinite group, H a subgroup, and K a open compact subgroup. Then*

1. *K-orbits in $H\backslash G$ are open and compact.*
2. *If a subset $C \subseteq H\backslash G$ is compact it lies in the union of finitely many K-orbits.*

Proposition 2.2.25. *Let $f \in$ s-Ind$_H^G(W)$. Then f has compact support modulo H iff supp$(f) \subseteq H \cdot C$ for some $C \subseteq G$ compact.*

Proof. $f \in \text{s-Ind}_H^G(W)$ so there is a compact open subgroup K such that K stabilises f. It follows that the support of f is a union of double (H, K) cosets. Let $q : G \to H\backslash G$ be the quotient map. Then $q(\text{supp}(f))$ is a union of K-orbits.

($\Rightarrow$) Suppose $q(\text{supp}(f))$ is compact. By the lemma it is a finite union of K-orbits. Thus $\text{supp}(f)$ is a union of finitely many double (H, K)-cosets. Let $g_1, \ldots, g_n$ be double coset representatives. Then $\text{supp}(f) = H \cdot (\cup_i g_i K)$ where $\cup_i g_i K$ is compact (and open).

($\Leftarrow$) Suppose $\text{supp}(f) \subseteq H \cdot C$ with C compact. Then $q(H \cdot C) = q(C)$ is compact and so lies in a finite union of K orbits. But $q(\text{supp}(f)) \subseteq q(C)$ and so $q(\text{supp}(f))$ must be a finite union of K-orbits and hence must be compact. ■

Remark 2.2.26. The proposition is also true if we insist that C is open.

Proposition 2.2.27. *c-Ind_H^G is exact.*

Proposition 2.2.28. *Let H be an open subgroup of G, and $\phi \in \mathit{Ind}_H^G(W)$ be compactly supported modulo H. Then $\phi \in$ c-$\mathit{Ind}_H^G(W)$.*

Proof. Since H is open and ϕ is compactly supported modulo H, $supp(\phi) = \cup_{i=1}^n H g_i$ for some $g_i \in G$. Let $L = \cap_{i=1}^n H^{g_i}$ and let K be a compact open subgroup of G such that $f(g_1), \ldots, f(g_n) \in W^K$. Then $L \cap K$ is a compact open subgroup of G (as L is open and hence closed) and for $x \in K \cap K$ we have

$$\begin{aligned}(x\phi)(hg_i) = \phi(hg_i x) &= h\phi(g_i x g_i^{-1} g_i) \\ &= h(g_i x g_i^{-1})\phi(g_i) = h\phi(g_i) = \phi(hg_i).\end{aligned} \tag{2.8}$$

This $L \cap K$ fixes ϕ and so $\phi \in \text{c-Ind}_H^G(W)$. ■

Definition 2.2.29. Let H be an open subgroup of G and W an H representation. Then there is a H-homomorphism $\alpha_W^c : W \to \text{c-Ind}_H^G$ given by $w \mapsto f_w$ where f_w is the function that sends h to $h.w$ and is 0 outside of H. By the previous proposition, this does indeed lie in $\text{c-Ind}_H^G(W)$.

Lemma 2.2.30. *Let H be an open subgroup of G, and let W be a representation of H. Then*

1. *The map α_W^c is an H-isomorphism with the space of functions $f \in$ c-$\mathit{Ind}_H^G(W)$ such that $supp(f) \subseteq H$.*

2. *If $\mathcal{W}$ is a basis for W and $\mathcal{G}$ a choice of representatives for G/H, then $\{gf_w : w \in \mathcal{W}, g \in \mathcal{G}\}$ is a basis for c-$\mathit{Ind}_H^G(W)$.*

Thm 2.2.31. *Let H be an open subgroup of G, W an H-representation and V a G-representation. Then there is a natural bijection*

$$\text{Hom}_G(\textit{c-Ind}_H^G(W), V) \leftrightarrow \text{Hom}_H(W, \mathit{Res}_H^G(V)). \tag{2.9}$$

Proof. We have the maps

$$\mathrm{Hom}_G(\text{c-Ind}_H^G(W), V) \leftrightarrow \mathrm{Hom}_H(W, \mathrm{Res}_H^G(V)) \tag{2.10}$$
$$\phi \mapsto \phi \circ \alpha_W^c \tag{2.11}$$
$$(gf_w \mapsto g\psi(w)) \leftarrow\!\shortmid \psi. \tag{2.12}$$

It is starightforward to check that the second map is well-defined and that these maps are mutually inverse. ■

Proposition 2.2.32. *Let V be a smooth representation of G, admitting χ as a central character. Let K be an open subgroup of G such that KZ/Z is compact.*

1. *Let $v \in V$. The KZ-space spanned by v is finite dimensional, and is a sum of irreducible KZ-spaces.*

2. *As representation of KZ, the space V is semisimple.*

2.3 Irreducible representations and the contragredient

Remark 2.3.1. From now on assume that G/K is countable for any compact open subgroup K of G.

Lemma 2.3.2. *Let V be an irreducible smooth representation of G. Then $\dim_{\mathbb{C}} V$ is countable.*

Lemma 2.3.3. *(Schur's lemma). If V is an irreducible smooth representation of G, then $\mathrm{End}_G(V) = \mathbb{C}$.*

Corollary 2.3.4. *Let V be an irreducible smooth representation of G. Then the central character of V is smooth.*

Corollary 2.3.5. *If G is abelian then any irreducible smooth representation of G is 1-dimensional.*

Definition 2.3.6. Let V be a smooth G-representation. We define the contragredient, or smooth dual, of V to be $\check{V} = (V^*)^\infty$.

Remark 2.3.7. If $K \leq G$ is a compact open subgroup of G, then for any $f \in (\check{V})^K$, $f(V(K)) = 0$.

prop:dual **Proposition 2.3.8.** *Restriction to V^K induces an isomorphism*

$$(\check{V})^K \cong (V^K)^*. \tag{2.13}$$

Thm 2.3.9. *The canonical morphism $V \to \check{\check{V}}$ is an isomorphism iff V is admissable.*

Proposition 2.3.10. *The contravariant functor $\vee : \mathsf{Rep}(G) \to \mathsf{Rep}(G)$ is exact.*

Proof. Follows from proposition 2.3.8. ∎

Corollary 2.3.11. *Let V be an admissable representation. Then V is irreducible iff $\check{V}$ is irreducible.*

Proposition 2.3.12. *Let V and W be smooth representations of G, and $\mathcal{P}(V,W)$ be the space of G-invariant bilinear pairings $V \times W \to \mathbb{C}$. Then there are isomorphisms*

$$\mathrm{Hom}_G(V,\check{W}) \cong \mathcal{P}(V,W) \cong \mathrm{Hom}_G(W,\check{V}). \tag{2.14}$$

2.4 Measures

Proposition 2.4.1. *Let $C_c^\infty(G)$ be the space of locally constant functions on G with compact support. Then $(C_c^\infty(G),\lambda)$ and (C_c^∞,ρ) are both smooth.*

Remark 2.4.2. Suppose a function $f : G \to \mathbb{C}$ is fixed by $\rho(K)$ (or $\lambda(K)$) for K a compact open subgroup K of G. Then f has compact support iff $\mathrm{supp}(f) \subseteq C$ for some compact set C.

Definition 2.4.3. A right Haar integral on G is a non-zero G-homomorphism $I : (C_c^\infty(G),\rho) \to \mathbb{C}$ such that $I(f) \geq 0$ for any $f \in C_c^\infty(G)$, $f \geq 0$.

Thm 2.4.4. *There exists a unique right Haar integral $I : C_c^\infty(G) \to \mathbb{C}$ up to scaling.*

Proof. Let K be a compact open subgroup of G and write ${}^K C_c^\infty$ for the subspace $(C_c^\infty(G))^{\lambda(K)}$. Then ${}^K C_c^\infty(G) = \text{c-Ind}_K^G 1_K$. It follows that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G({}^K C_c^\infty(G),\mathbb{C}) = 1. \tag{2.15}$$

If $f_{K,g}$ denotes the indicator function on the coset Kg, then the map

$$I_K :{}^K C_c^\infty(G) \to \mathbb{C}, f_{K,g} \mapsto 1 \tag{2.16}$$

is a G-homomorphism and so all G-homomorphisms ${}^K C_c^\infty \to \mathbb{C}$ are a multiple of this map.

Now let $\{K_n\}_n$ be a descending sequence of compact open subgroups of G such that $\cap_n K_n = \{e\}$. Then $\{{}^{K_n} C_c^\infty(G)\}_n$ is an ascending sequence of subspaces of $C_c^\infty(G)$ such that $C_c^\infty(G) = \cup_n {}^{K_n} C_c^\infty(G)$. Let $I_n = I_{K_n}/|K_1 : K_n|$. Then

$$I_{n+1}(f_{g,K_n}) = |K_n : K_{n+1}|/|K_1 : K_{n+1}| = 1/|K_1 : K_n| = I_n(f_{g,K_n}). \tag{2.17}$$

It follows that $I_{n+1}|_{{}^{K_n} C_c^\infty(G)} = I_n$ and so we can define a G-homomorphism $I : C_c^\infty(G) \to \mathbb{C}$. It is clear that this map is a right Haar measure.

Now suppose I' is another Haar measure. Then there are $\alpha_n \in \mathbb{C}$ such that $I'|_{{}^{K_n} C_c^\infty(G)} = \alpha_n \cdot I_n$ for all n. Evaluating at the f_{g,K_n} gives that $\alpha_n = \alpha_{n+1} =: \alpha$ for all n and so $I' = \alpha I$. ∎

Remark 2.4.5. If $f \geq 0$ and there exists a $g \in G$ such that $f(g) > 0$ then $I(f) > 0$.

Definition 2.4.6. Define $\vee : C_c^\infty(G) \to C_c^\infty(G)$ by $f \mapsto \check{f}$ where $\check{f}(g) = f(g^{-1})$. Then $\vee : (C_c^\infty(G), \lambda) \to (C_c^\infty(G), \rho)$ is a G-isomorphism.

Remark 2.4.7. $\vee$ induces a bijection between left and right Haar measures.

Definition 2.4.8. Let I be a left Haar measure on G. For a non-empty compact open subset S of G, let Γ_S denote its characteristic function. We define

$$\mu_G(S) = I(\Gamma_S). \tag{2.18}$$

Then $\mu_G(gS) = \mu_G(S)$ for all $g \in G$.

Remark 2.4.9. We have that $I(f) = \int_G f d\mu_G$ for $f \in C_c^\infty(G)$.

Definition 2.4.10. We can extend the domain of Haar integration as follows. Let μ_G be a left Haar measure on G, and f be a function on G invariant under left translation by a compact open subgroup K of G. If the series

$$\sum_{g \in K\backslash G} \int_{Kg} |f(x)| d\mu_G(x) \tag{2.19}$$

converges define

$$\int_G f(x) d\mu_G(x) = \sum_{g \in K\backslash G} \int_{Kg} f(x) d\mu_G(x). \tag{2.20}$$

Proposition 2.4.11. *This definition does not depend on K and is left translation invariant.*

Proof. Let K' be any other compact open subgroup of G. Then $K \cap K'$ has finite index in K and K'. It follows that

$$\begin{aligned} \sum_{g \in K\backslash G} \int_{Kg} |f(x)| d\mu_G(x) &= \sum_{g \in K\backslash G} \sum_{h \in K \cap K'\backslash K} \int_{K\cap K'hg} |f(x)| d\mu_G(x) \\ &= \sum_{g \in K\cap K'\backslash G} \int_{K\cap K'g} |f(x)| d\mu_G(x) \\ &= \sum_{g \in K'\backslash G} \sum_{h \in K \cap K'\backslash K'} \int_{K\cap K'hg} |f(x)| d\mu_G(x) \\ &= \sum_{g \in K'\backslash G} \int_{K'g} |f(x)| d\mu_G(x) \end{aligned} \tag{2.21}$$

and all series converge. It follows that the same series but without absolute values converge, and so we obtain the first part of the proposition.

For the second part let $y \in g$. Then $\{yg : g \in K\backslash G\}$ is a set of coset representatives for $yKy^{-1}\backslash G$ and

$$\begin{aligned}\sum_{g\in K\backslash G}\int_{yKy^{-1}\cdot yg}|\lambda_y f(x)|d\mu_G(x) &= \sum_{g\in K\backslash G}\int_G 1_{yKy^{-1}yg}(x)|\lambda_y f(x)|d\mu_G(x)\\ &= \sum_{g\in K\backslash G}\int_G \lambda_y(1_{Kg}(x)|f(x)|)d\mu_G(x)\\ &= \sum_{g\in K\backslash G}\int_{Kg}|f(x)|d\mu_G(x). \qquad (2.22)\end{aligned}$$

Thus $\int_G \lambda_y f d\mu_G$ is defined and the above calculation, but without absolute values, shows that it is equal to $\int_G f d\mu_G$. ■

Proposition 2.4.12. *Let G_1, G_2 be locally profinite groups. Then the natural map $C_c^\infty(G_1)\otimes_{\mathbb{C}} C_c^\infty(G_2) \to C_c^\infty(G_1\times G_2)$ is an isomorphism that respects both left and right translation.*

Proposition 2.4.13. *If μ_1, μ_2 are left Haar measures then the map*

$$\mu : C_c^\infty(G_1\times G_2) \to \mathbb{C} \qquad (2.23)$$

defined via the above isomorphism is also a left Haar measure.

Proposition 2.4.14. *Let $f \in G_1\times G_2$. Then the function*

$$f_1(g_1) = \int_{G_2} f(g_1, g_2)d\mu_2(g_2) \qquad (2.24)$$

lies in $C_c^\infty(G_2)$ and

$$\int_{G_1\times G_2} f(g)d\mu_G(g) = \int_{G_1} f_1(g_1)d\mu_1(g_1). \qquad (2.25)$$

Definition 2.4.15. Let μ_G be a left Haar measure on G. For $g \in G$, $f \mapsto \int_G \rho_g f d\mu_G$ is another left Haar measure. It follows that there is a unique $\delta_G(g) \in \mathbb{R}_+^\times$ such that

$$\delta_G(g)\int_G \rho_g f d\mu_G = \int_G f d\mu_G. \qquad (2.26)$$

This map $\delta_G : G \to \mathbb{R}_+^\times$ is a homomorphism.

Proposition 2.4.16. *δ_G is trivial on open compact subgroups of G.*

Proposition 2.4.17. *A homomorphism $\psi : G \to \mathbb{R}_+^\times$ is a character iff it is trivial on compact open subgroups.*

Corollary 2.4.18. *δ_G is a character.*

Proposition 2.4.19. *δ_G is trivial iff G is unimodular.*

Proposition 2.4.20. *The functional $f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x)$ is a right Haar integral.*

Proof.

$$\rho_y f \mapsto \int_G \delta_G(x)^{-1} \rho_y f(x) d\mu_G(x) = \delta_G(y) \int_G \rho_y(\delta_G(x)^{-1} f(x)) d\mu_G(x)$$
$$= \int_G \delta_G(x)^{-1} f(x) d\mu_G(x). \tag{2.27}$$

■

Proposition 2.4.21. *Let K be a compact open subgroup of G and $g \in G$. Then $\mu_G(gKg^{-1}) = \delta_G(g)^{-1}\mu_G(K)$.*

Definition 2.4.22. Let H be a closed subgroup of G, $\theta : H \to \mathbb{C}^\times$ a character and $C_c^\infty(H\backslash G, \theta) = \text{c-Ind}_H^G(\theta)$.

Definition 2.4.23. Let μ_H be a left Haar measure on H. Define the map $\sim$: $(C_c^\infty(G), \rho) \to C_c^\infty(H\backslash G, \theta)$ by

$$\widetilde{f}(g) = \int_H (\theta\delta_H)^{-1} \rho_g f d\mu_H = \int_H \theta(h^{-1}) f(hg) d\mu_H^R(h). \tag{2.28}$$

This map is a G-homomorphism and

$$\widetilde{(\lambda_h f)} = (\theta\delta_H)(h)^{-1} \widetilde{f} \tag{2.29}$$

for $f \in C_c^\infty(G), h \in H$.

lem:t_surj **Lemma 2.4.24.** *$\sim$ is surjective.*

Proof. Let K be an open compact subgroup of G. Then each double coset HgK supports at most a 1-dimensional subspace of $C_c^\infty(H\backslash G, \theta)^K$ and these spaces span $C_c^\infty(H\backslash G, \theta)^K$. But each $1_{gK} \in (C_c^\infty(G))^K$ maps to a non-zero element of $C_c^\infty(H\backslash G, \theta)^K$ with support HgK and so the map is surjective. ■

Corollary 2.4.25. *Let $\theta : H \to \mathbb{C}^\times$ be a character of H and I a right Haar intgral on G. Then $\mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C}) \neq 0$ iff I factors through $C_c^\infty(H\backslash G, \theta)$.*

Corollary 2.4.26. $\dim_{\mathbb{C}} \mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C}) = 0$ *or* 1.

Remark 2.4.27. Let K be a open compact subgroup of G, $g \in G$ and $f = 1_{gK}$. Suppose $\delta_G|_H = \theta\delta_H$. Let $x \in G$. Then

$$\widetilde{f}(x) = \int_H (\theta\delta_H)(h)^{-1} 1_{gKx^{-1}}(h) d\mu_H(h). \tag{2.30}$$

But $h \in gKx^{-1}$ iff $x = h^{-1}gk$ for some $k \in K$. Thus $\widetilde{f}(x)$ is 0 if $x \notin HgK$. If $x \in HgK$ write $x = h_0gk_0$ and $L = gKg^{-1} \cap H$. Then

$$\begin{aligned}\widetilde{f}(x) &= \int_H (\theta\delta_H)(h)^{-1} 1_{Lh_0^{-1}}(h) d\mu_H(h) \\ &= (\theta\delta_H)(h_0) \int_H \rho_{h_0}((\theta\delta_H)^{-1} 1_L)(h) d\mu_H(h) \\ &= \theta(h_0) \int_L (\theta\delta_H)(h)^{-1} d\mu_H(h). \qquad (2.31)\end{aligned}$$

But δ_G is trivial on L and so $\widetilde{f}(x) = \theta(h_0)\mu_H(L)$. It follows that

$$\widetilde{1_{h_igK}}(hgk) = \theta(h)\delta_G(h_i)^{-1}\mu_H(L). \qquad (2.32)$$

lem:I_ker **Lemma 2.4.28.** *Suppose $\delta_G|_H = \theta\delta_H$ and let I denote the right Haar integral*

$$f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x). \qquad (2.33)$$

If $\widetilde{f} = 0$ then $I(f) = 0$.

Proof. Suppose f is fixed by K. It suffices to check the case when f is of the form $\sum_i \alpha_i 1_{h_igK}$ for $\alpha_i \in \mathbb{C}$ and the h_igK distinct cosets. Then by the remark $\widetilde{f} = 0$ implies that $\sum_i \alpha_i \delta_G(h_i)^{-1} = 0$. But

$$I(1_{h_igK}) = \delta_G(h_ig)^{-1}\mu_G(K) \qquad (2.34)$$

and so

$$I(f) = \mu_G(K)\delta_G(g)^{-1} \sum_i \alpha_i \delta_G(h_i)^{-1} = 0. \qquad (2.35)$$

■

Thm 2.4.29. *Let $\theta : H \to \mathbb{C}^\times$ be a character of H. The following are equivalent:*

1. $\mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C}) \neq 0$
2. $\delta_G|_H = \theta\delta_H$.

Proof. (1) $\Rightarrow$ (2) Let $0 \neq I_\theta \in \mathrm{Hom}_G((C_c^\infty(H\backslash G, \theta), \rho), \mathbb{C})$ be such that the right Haar integral $I : f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x)$ is equal to $I_\theta(\widetilde{f})$. Note that elements of the form $\lambda_h f - (\theta\delta_H)(h)^{-1} f$ map to zero under $\sim$ and so we get

$$\begin{aligned}0 = I(\lambda_h f - (\theta\delta_H)(h)^{-1} f) &= \int_G \delta_G(x)^{-1}(\lambda_h f - (\theta\delta_H)(h)^{-1} f) d\mu_G(x) \\ &= \left(\delta_G(h)^{-1} - (\theta\delta_H)(h)^{-1}\right) I(f). \qquad (2.36)\end{aligned}$$

Picking an f such that $I(f) \neq 0$ we get that $\delta_G|_H = \theta\delta_H$.

(2) $\Rightarrow$ (1) By lemma 2.4.28, if $\widetilde{f} = 0$ then $I(f) = 0$. The result follows. ■

Corollary 2.4.30. *Suppose $\theta\delta_H = \delta_G|_H$. Then there is a non-zero $I_\theta : (C_c^\infty(G), \rho) \to \mathbb{C}$ such that $I_\theta(f) \geq 0$, whenever $f \geq 0$.*

Definition 2.4.31. If H is a closed subgroup of G define $\delta_{H\backslash G} = \delta_H^{-1}\delta_G|_H : H \to \mathbb{R}_+^\times$. Write $\mu_{H\backslash G}$ for

$$I_{\delta_{H\backslash G}}(f) = \int_{H\backslash G} f(g) d\mu_{H\backslash G}(g) \tag{2.37}$$

where $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$.

prop:semi_inv_eq **Proposition 2.4.32.** *Let $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$. If f is fixed by the compact open subgroup K of G then*

$$\int_{H\backslash G} f(g) d\mu_{H\backslash G}(g) = \sum_{g\in H\backslash G/K} \frac{\mu_G({}^gK)}{\mu_H(H\cap {}^gK)} f(g) \tag{2.38}$$

where ${}^gK = gKg^{-1}$.

Proof. Suppose $\operatorname{supp}(f) \subseteq HgK$. Write ${}^gH = H \cap {}^gK$. Then $f = \widetilde{h}$ where $h = c1_{gK}$ for some constant c. Thus $f(g) = \widetilde{h}(g) = c\mu_H({}^gH)$ and so $c = f(g)/\mu_H({}^gH)$. But

$$I_{\delta_{H\backslash G}}(f) = I(h) = \int_G \delta_G(x)^{-1} c 1_{gK}(x) d\mu_G(x) = \frac{\mu_G(K)f(g)}{\mu_H({}^gH)\delta_G(g)}. \tag{2.39}$$

The result follows from $\mu_G(K)/\delta_G(g) = \mu_G({}^gK)$. ■

2.4.1 Left invariant measures

Remark 2.4.33. For $f \in C_c^\infty(G)$ and $g \in G$ we have $\delta_G(g)I(\lambda_g f) = I(f)$.

Definition 2.4.34. Write $\int d\mu_H^R$ for I. Let $\theta : H \to \mathbb{C}^\times$ be a character and define $C_c^\infty(G/H, \theta)$ in the obvious way. Define $\delta_G^R : G \to \mathbb{C}^\times$ by $\delta_G^R = \delta_G^{-1}$.

Definition 2.4.35. Define the map $\sim: (C_c^\infty(G), \lambda) \to C_c^\infty(G/H, \theta)$ by

$$\widetilde{f}(g) = \int_H (\theta\delta_H^R)^{-1} \lambda_{g^{-1}} f d\mu_H^R = \int_H \theta^{-1}\lambda_{g^{-1}} f d\mu_H. \tag{2.40}$$

This map is a G-homomorphism and

$$\widetilde{\rho_h f} = (\theta\delta_H^R)(h)\widetilde{f} \tag{2.41}$$

for $f \in C_c^\infty(G)$ and $h \in H$.

Remark 2.4.36. Let $\theta : H \to \mathbb{C}^\times$ be a character of H, K a compact open subgroup of G and $g \in G$. Suppose $\delta_G^R|_H = \theta\delta_H^R$. Then

$$\widetilde{1_{Kg}}(kgh) = \theta(h)\mu_H(H \cap g^{-1}Kg) \tag{2.42}$$

and $\widetilde{1_{Kg}}$ is 0 outside of KgH.

Thm 2.4.37. *Let $\theta : H \to \mathbb{C}^\times$ be a character of H. The following are equivalent:*

1. $\mathrm{Hom}_G((C_c^\infty(G/H,\theta),\lambda),\mathbb{C}) \neq 0$
2. $\delta_G^R|_H = \theta\delta_H^R$.

prop:left_semi_inv **Proposition 2.4.38.** *Let $f \in C_c^\infty(G/H, \delta_{G/H})$. If f is fixed by the compact open subgroup K of G then*

$$\int_{G/H} f(g)d\mu_{G/H}(g) = \sum_{g\in K\backslash G/H} \frac{\mu_G(K^g)}{\mu_H(H\cap K^g)} f(g) \tag{2.43}$$

where $K^g = g^{-1}Kg$.

Proof. Identical to the proof of proposition 2.4.32. ■

2.4.2 Relatively invariant measures

Definition 2.4.39. A measure η on G is said to be relatively invariant with left multiplier χ_l and right multiplier χ_r if for $f \in C_c^\infty(G)$

$$\eta(\lambda_g f) = \chi_l(g)\eta(f), \quad \eta(\rho_g f) = \chi_r(g)\eta(f). \tag{2.44}$$

We say that η is relatively left (resp. right) invariant if it satisfies the respective condition above.

Proposition 2.4.40. *The multipliers χ_l, χ_r are characters $G \to \mathbb{R}_+^\times$.*

Proof. They are both clearly homomorphisms from G to $\mathbb{R}_+^\times$. It thus suffices to show that they are trivial on compact open subgroups of G. But this is obvious. ■

Proposition 2.4.41. *Let $\chi : G \to \mathbb{R}_+^\times$ be a character. Then the measure $\eta : C_c^\infty(G) \to \mathbb{R}_{\geq 0}$ defined by*

$$\eta(f) = \int_G f(g)\chi(g)d\mu(g) \tag{2.45}$$ eq:rel_meas

satisfies $\eta(\lambda_g f) = \chi(g)\eta(f)$. Conversely if η is a measure that satisfies $\eta(\lambda_g f) = \chi(g)\eta(f)$ then there is a unique left Haar measure μ so that η may be expressed as in equation 2.45.

Proof. Obvious. ■

Corollary 2.4.42. *If η is relatively left invariant, then it is relatively invariant with $\chi_r = (\chi_l\delta_G)^{-1}$.*

Corollary 2.4.43. *If η is realtively invariant, then $\chi_l\chi_r\delta_G = 1$.*

Corollary 2.4.44. *The space of relatively left invariant measures with mulilier χ is one dimensional.*

Relatively invariant measures on quotient spaces

Let X be a locally compact toplogical space and let H be a locally compact group acting properly on X on the right. If $\chi : H \to \mathbb{R}_+^\times$ is a character of H and μ_X a measure on X we say that μ_X is relatively invariant with multiplier χ if $\mu_X(\rho_{h^{-1}} f) = \chi(h)\mu_X(f)$ for all $f \in C_c^\infty(X)$.

Definition 2.4.45. For $f \in C_c(X)$ define

$$f^1(x) = \int_H f(xh) d\mu_H(h). \tag{2.46}$$

It is clear that this function factors through the projection map $\pi : X \to X/H$. Let $f^b : X/H \to \mathbb{C}$ denote the map so that $f^1 = f^b \circ \pi$. The map $f \mapsto f^b$ is surjective and its kernel is contained in the kernel of μ_X. Thus we can define a measure μ_X^b on X/H by $\mu_X^b(g) = \mu_X(f)$ where f is such that $f^b = g$.

Thm 2.4.46. *Let the locally compact group H act propoerly on the right of a locally compact space X and let μ_H be a given measure on H. Then given a relatively invariant measure on X with multiplier χ there exists a unique measure $\mu_{X/H}$ that has the property*

$$\mu_X(f) = \mu_{X/H}(f^b) \tag{2.47}$$

iff $\chi = \delta_H$. If this condition is satisfied, then $\mu_{X/H}$ is given by μ_X^b.

2.4.3 Duality theorem

Fix measures μ_G, μ_H and write $\mu_{H\backslash G}$ for the corresponding semi-invariant measure on $H\backslash G$.

Proposition 2.4.47. *Let W be a H-representation. Given $\phi \in$ c-Ind$_H^G W, \Phi \in$ s-Ind$_H^G(\delta_{H\backslash G} \otimes \check{W})$ define $f_{\Phi,\phi} : G \to \mathbb{C}$ by*

$$f_{\Phi,\phi}(g) = \langle \Phi(g), \phi(g) \rangle. \tag{2.48}$$

Then $f_{\Phi,\phi}$ lies in $C_c^\infty(H\backslash G, \delta_{H\backslash G})$.

Proof. We clearly have

$$f_{\Phi,\phi}(hg) = \delta_{H\backslash G}(h) f_{\Phi,\phi}(g), h \in H, g \in G. \tag{2.49}$$

We also have $gf_{\Phi,\phi} = f_{g\phi,g\Phi}$. Thus, if K is a compact open subgroup that fixes both ϕ and Φ then K also fixes $f_{\Phi,\phi}$. Finally it remains to check that f has compact support module H. But $\mathrm{supp}(f_{\Phi,\phi}) \subseteq \mathrm{supp}(\phi) = H\mathrm{supp}(\phi)$. ■

Remark 2.4.48. Let $F = \text{s-Ind}_H^G \circ (\delta_{H\backslash G} \otimes -) \circ \vee$ and $G = \text{c-Ind}_H^G$. If $h : V \to W$ is a homomorphism between H-representations then for $\phi \in G(V), \Phi \in F(W)$ we have

$$f_{F(h)\Phi,\phi}(g) = \langle \Phi(g) \circ h, \phi(g) \rangle = \langle \Phi(g), h \circ \phi(g) \rangle = f_{\Phi,G(h)\phi}(g). \tag{2.50}$$

Definition 2.4.49. Define the pairing

$$(-,-)_W : \text{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W}) \times \text{c-Ind}_H^G W \to \mathbb{C} \tag{2.51}$$

by

$$(\Phi, \phi)_W \mapsto \int_{H\backslash G} f_{\Phi,\phi} d\mu_{H\backslash G}. \tag{2.52}$$

This pairing is clearly G-invariant. By the remark the induced map

$$\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W}) \to (\text{c-Ind}_H^G W)^\vee \tag{2.53}$$

is natural in W.

Lemma 2.4.50. *Let K be a compact open subset of G, $\mathcal{G}$ a set of representatives for $H\backslash G/K$, and for each $g \in \mathcal{G}$, let $\mathcal{W}_g$ be a basis for $W^{H\cap gKg^{-1}}$. Then for each $g \in \mathcal{G}, w \in \mathcal{W}_g$ there is a unique $f_{g,w}$ with support HgK and $f_{g,w}(g) = w$, and the collection of all of these form a basis for $(\text{c-Ind}_H^G W)^K$.*

Proof. It is clear that the $f_{g,w}$ exist and that they are linearly independent. To see that they span $(\text{c-Ind}_H^G W)^K$, note that if $f \in (\text{c-Ind}_H^G W)^K$ then $\text{supp}(f)$ is the union of finitely many double cosets of $H\backslash G/K$. Noting that f multiplies by the indicators on the various double cosets are still in $(\text{c-Ind}_H^G W)^K$, we may thus reduce to the case when $\text{supp}(f) = HgK$ for some $g \in \mathcal{G}$. But note that $f(g) \in W^{H\cap gKg^{-1}}$. Taking the appropriate linear combination of $f_{g,w}$'s gives the result. ∎

Remark 2.4.51. We have that

$$(\delta_{H\backslash G} \otimes \check{W})^{H\cap gKg^{-1}} = \check{W}^{H\cap gKg^{-1}} = \left(W^{H\cap gKg^{-1}}\right)^* \tag{2.54}$$

since $\delta_{H\backslash G}$ is trivial on $H \cap gKg^{-1}$. It follows that the dual basis of $\mathcal{W}_g$ give a basis for $(\delta_{H\backslash G} \otimes \check{W})^{H\cap gKg^{-1}}$. Write $f_{g,\check{w}}, g \in \mathcal{G}, w \in \mathcal{W}_g^*$ for the elements of $\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes W)$ that arise in the same way as in the lemma. Then by a similar argument as above, $\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes W)$ consists of all functions f such that $f|_{HgK}$ is a finite linear combination of $f_{g,\check{w}}$'s.

Note moreover that for $g \in \mathcal{G}, w \in \mathcal{W}_g, \check{w} \in \mathcal{W}_g^*$,

$$(f_{g,\check{w}}, f_{g,w}) = \int_{H\backslash G} 1_{HgK}\langle \check{w}, w\rangle d\mu_{H\backslash G} = \mu_{H\backslash G}(HgK)\langle \check{w}, w\rangle \tag{2.55}$$

and

$$(f_{g,\check{w}}, f_{g',w}) = 0 \tag{2.56}$$

when $g' \in \mathcal{G}$ and $g \neq g'$.

Proposition 2.4.52. *The pairing $(-,-)$ is perfect.*

Proof. It suffices to show that the induced map

$$\text{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W})^K \to ((\text{c-Ind}_H^G W)^K)^* \tag{2.57}$$

is an isomorphism for any compact open subgroup K of G. But this just follows from the remark. ■

Corollary 2.4.53. *There is a natural isomorphism*

$$(\textit{c-Ind}_H^G W)^\vee \cong \textit{s-Ind}_H^G(\delta_{H\backslash G} \otimes \check{W}). \tag{2.58}$$

2.5 The Hecke Algebra

Definition 2.5.1. Let $f_1, f_2 \in C_c^\infty(G)$ and define

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu_G(x). \tag{2.59}$$

Lemma 2.5.2. *Let $f_1, f_2 \in C_c^\infty(G)$. Then the map $(x, g) \mapsto f_1(x) f_2(x^{-1}g)$ is in $C_c^\infty(G \times G)$.*

Proof. Let K be a compact open subgroup such that $\rho(K)$ fixes f_1, f_2 and $\lambda(K)$ fixes f_2. Then for $k_1, k_2 \in K$, $(xk_1, gk_2) \mapsto f_1(x) f_2(x^{-1}g)$ and so it is fixed by $K \times K$. It remains to check that it has compact support. But its support is

$$\{(x, g) : x \in \text{supp}(f_1), g \in x \cdot \text{supp}(f_2)\}. \tag{2.60}$$

This is the image of $\text{supp}(f_1) \times \text{supp}(f_2)$ under the homeomorphism $G \times G \to G \times G : (x, y) \mapsto (x, xy)$ and so is compact. ■

Proposition 2.5.3. *If $f_1, f_2 \in C_c^\infty(G)$ then $f_1 * f_2 \in C_c^\infty(G)$.*

Proof. Note that for a fixed $g \in G$ the map $x \mapsto f_1(x) f_2(x^{-1}g)$ is in $C_c^\infty(G)$ and so $f_1 * f_2$ is defined everywhere. Let K be a compact open subgroup of G such that $\rho(K)$ fixes f_2. Then it is clear that $\rho(K)$ also fixes $f_1 * f_2$. To see that the support is compact note that $f_1 * f_2(g) \neq 0$ only if $\text{supp}(f_1) \cap g^{-1}\text{supp}(f_2) \neq \emptyset$. But $\text{supp}(f_1)$ is compact and $\text{supp}(f_2)$ is open and so only finitely many cosets of $\text{supp}(f_2)$ can intersect $\text{supp}(f_1)$. Thus $\text{supp}(f_1 * f_2)$ is contained a finite union of cosets of $\text{supp}(f_2)$ and so is compact. ■

Remark 2.5.4. It is easy to check that $*$ is associative.

Definition 2.5.5. The Hecke algebra of G is $\mathcal{H}(G) = (C_c^\infty(G), *)$. This is an associative algebra.

For a compact open subgroup K of G define $e_K := 1_K/\mu_G(K)$.

Remark 2.5.6. e_K is idempotent.

Remark 2.5.7. For any $f \in C_c^\infty(G), k \in K, g \in G$ we have $e_K * f(kg) = e_K * f(g)$. In other words, $e_K * f$ is fixed by $\lambda(K)$. Similarly $f * e_K$ is fixed by $\rho(K)$.

Proposition 2.5.8. *Let K be a compact open subgroup of G and $f \in C_c^\infty(G)$. Then f is fixed by $\lambda(K)$ iff $e_K * f = f$.*

Proof. It is clear that if f is fixed by $\lambda(K)$ then $e_K * f(g) = f(g)$ for all $g \in G$. Conversely, suppose $e_K * f = f$. Then the result follows from the remark. ■

Remark 2.5.9. Similarly f is fixed by $\rho(K)$ iff $f * e_K = f$.

Corollary 2.5.10. *The space $\mathcal{H}(G, K) := e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$, with unit e_K.*

Corollary 2.5.11.

$$\mathcal{H}(G, K) = \{f \in \mathcal{H}(G) : f(k_1 g k_2) = f(g), g \in G, k_1, k_2 \in K\}. \tag{2.61}$$

Definition 2.5.12. Let M be a left $\mathcal{H}(G)$-module. We say that M is smooth if $\mathcal{H}(G) * M = M$. Since $\mathcal{H}(G)$ is a union of the $\mathcal{H}(G, K)$ this is equivalent to saying for every $m \in M$ there is a compact open subgroup K such that $e_K * m = m$.

Write $\mathcal{H}(G) - \mathrm{Mod}$ for the category of smooth $\mathcal{H}(G)$-modules.

Definition 2.5.13. Let (π, V) be a smooth G representaion. We can turn V into a smooth $\mathcal{H}(G)$-module by defining for $f \in \mathcal{H}(G), v \in V$

$$\pi(f)v = \int_G f(g)\pi(g)v d\mu_G(g). \tag{2.62}$$

Remark 2.5.14. Let K be a compact open subgroup of G such that $\rho(K)$ fixes f and K fixes v. Then map $g \mapsto f(g)\pi(g)v$ is fixed by $\rho(K)$ and has compact support. Thus the integral is defined and is equal to the finite sum

$$\sum_{g \in G/K} f(g)\pi(g)v. \tag{2.63}$$

It is then clear that $\pi(e_K)v = v$ for $v \in V^K$.

Remark 2.5.15. If $V = (C_c^\infty(G), \lambda)$ then the $\mathcal{H}(G)$-module action is given by $\lambda(\phi)f = \phi * f$.

If $V = (C_c^\infty(G), \rho)$ then the $\mathcal{H}(G)$-module action is given by $\rho(\phi)f = f * \check{\phi}$.

Proposition 2.5.16. *The above procedure defines a functor $\mathsf{Smo}_G \to \mathcal{H}(G) -$ Mod which is the identity on morphisms.*

Proof. It is easy to check that if $f_1, f_2 \in C_c^\infty(G), v \in V$ then $\pi(f_1)(\pi(f_2)v) = \pi(f_1 * f_2)v$. Thus V is a $\mathcal{H}(G)$-module. By the remark, V is moreover a smooth $\mathcal{H}(G)$-module. It is clear that G-homomorphisms are also $\mathcal{H}(G)$-homomorphisms. ■

Lemma 2.5.17. *Let M be a smooth $\mathcal{H}(G)$-module. Then $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \cong M$.*

Proof. Let $\theta : \mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \to M$ be the canonical map. Suppose $\sum_i f_i \otimes m_i$ is in the kernel. Let K be a compact open subgroup that fixes each of the f_i by translation on both sides and is such that $m_i \in e_K * M$ for all i. Then $e_K * m_i = m_i$ for all i and so

$$\sum_i f_i \otimes m_i = e_K \otimes \sum_i f_i * m_i = 0. \tag{2.64}$$

Thus the map is injective. But it is surjective by definition of smoothness. Hence we have an isomorphism. ■

Corollary 2.5.18. *Let M be a smooth $\mathcal{H}(G)$-module. Then M is naturally a G-representation.*

Proof. G acts on $\mathcal{H}(G)$ by left translation, and hence on $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M$. ■

Remark 2.5.19. If $m \in M$ and K is a compact open subgroup of G such that $e_K * m = m$, then for $g \in G$, $gm = 1_{gK} * m/\mu_G(K)$. But then for the induced $\mathcal{H}(G)$-module strucutre we have

$$\pi(1_{gK})m = \int_G 1_{gK}(x)1_{xK} * m d\mu_G(x)/\mu_G(K) = 1_{gK} * m. \tag{2.65}$$

This suffices to show that the induces module strucutre is just the original module structure.

Corollary 2.5.20. *The above procedure defines a functor $\mathcal{H}(G)-Mod \to \mathsf{Smo}_G$ which is the identity on morphisms.*

Remark 2.5.21. Conversely if we start with a smooth G-representation V, then for $v \in V^K$ we have $gv = 1_{gK} * v/\mu_G(K) = \int_G 1_{gK}(x)\pi(x)v d\mu_G(x)/\mu_G(K) = \pi(g)v$. We thus have the following result.

Thm 2.5.22. *The functors $\mathsf{Smo}_G \to \mathcal{H}(G) - Mod$ and $\mathcal{H}(G) - Mod \to \mathsf{Smo}_G$ are mutually inverse.*

Proposition 2.5.23. *Let V be a smooth G-representation. Then the operator e_K* is the projection onto V^K along $V(K)$. The space V^K is an $\mathcal{H}(G,K)$-module on which e_K acts as the identity.*

Proof. Let $k \in K$ and $v \in V$. Then

$$k(e_K * v) = e_K * (kv) = e_K * v \tag{2.66}$$

where the last equality follows from δ_G being trivial on K. Thus e_K is a K-homomorphism with image in V^K. It follows that it must send $V(K)$ to 0. Moreover it is idempotent and the identity on V^K. This gives the result. ■

Lemma 2.5.24. *Let V be an irreducible smooth G-representation. Then V^K is either 0 or a simple $\mathcal{H}(G,K)$-module.*

Proof. Suppose $V^K \neq 0$. Then let M be a non-zero $\mathcal{H}(G,K)$-submodule of V. Then $\mathcal{H}(G)M = V$ by irreducibility and so

$$V^K = e_K * V = e_K * \mathcal{H}(G)M = \mathcal{H}(G,K)M = M. \tag{2.67}$$

■

prop:bij **Proposition 2.5.25.** *The map $V \mapsto V^K$ induces a bijection between*

1. *equivalence classes of smooth representations of G such that $V^K \neq 0$*
2. *equivalance classes of simple $\mathcal{H}(G,K)$-modules.*

Proof. Let M be a simple $\mathcal{H}(G,K)$-module and let $U = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$. Then $U^K = e_K * \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M = e_K \otimes M \cong M$. Let X be a maximal G-subspace of U such that $X^K = 0$ (exists by Zorn). This subspace is unique since $(X + X')^K = X^K + X'^K$. Note that X is maximal such that $X \cap U^K = X \cap e_K \otimes M = 0$. If $X \subsetneq W$ is a G-subspace of U then W must meet $e_K \otimes M$ and so must contain $e_K \otimes M$ (as M is simple) and so must equal to U. It follows that $V = U/X$ is irreducible and $V^K = M$ as $\mathcal{H}(G,K)$-modules. Note that the isomorphism class of V depends only on that of M.

Thus we now have maps going in both directions and we know that one composition is the identity. To see that the other composition is the identity, let V be an irreducible G-representation and $M = V^K$. We have a map $U = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M \to V$, $f \otimes m \mapsto f * m$. The image is non-zero subrepresentation of V and so the map must be surjective. Moreover, the image of X is a submodule that does not intersect V^K and so must be zero. Thus X lies in the kernel of the map. Now suppose u lie in the both U^K and the kernel of the map. Then $u = e_K \otimes m$ some $m \in M$. But then $e_K * m = 0$ and $e_K * m = m$ and so $e_K \otimes m = 0$. Thus the kernel lies inside X. It follows that $V \cong U/X$ as required. ■

Corollary 2.5.26. *Let V be a smooth non-zero representation of G. Then V is irreducible iff for any open compact open subgroup K of G, the space V^K is either zero or $\mathcal{H}(G,K)$-simple.*

Proof. ($\Rightarrow$) Done. ($\Leftarrow$) Let V a G-representation with a non-zero sub-representation U. Let $W = V/U$ and K be a compact open subgroup of G such that $U^K, W^K \neq 0$. Then $0 \to U^K \to V^K \to W^K \to 0$ is exact and so V^K is not a simple $\mathcal{H}(G,K)$-module. ■

Definition 2.5.27. Let $(\rho, V) \in \hat{K}$ and define

$$e_V(x) = \frac{\dim V}{\mu_G(K)} \operatorname{tr}(\rho(x^{-1})) 1_K(x). \tag{2.68}$$

Recall that since K is compact, the kernel of ρ is also a compact open subgroup $K' \leq K$ such that K/K' is finite. It follows that ρ is constant on double cosets $K' \backslash G / K'$ and so $e_{K'} * e_\rho = e_\rho * e_{K'} = e_\rho$. Thus $e_\rho \in \mathcal{H}(K,K') \subseteq \mathcal{H}(G,K')$.

Proposition 2.5.28. *The map $\mathcal{H}(K, K') \to \mathbb{C}[K/K']$, $1_{gK'}/\mu_G(K') \mapsto gK'$ is an algebra isomorphism that respects their respective actions on V.*

Remark 2.5.29. Under this isomorphism e_V gets sent to the idempotent for V is $\mathbb{C}[K/K']$.

Corollary 2.5.30. 1. *The function $e_V \in \mathcal{H}(G)$ is idempotent.*

2. *If W is a smooth G-representation of G, then e_ρ is the K-projection $V \to V^\rho$.*

Remark 2.5.31. Replacing V^K with V^ρ and $\mathcal{H}(G, K)$ with $e_\rho * \mathcal{H}(G) * e_\rho$ we get an exact analogue of proposition 2.5.25.

Proposition 2.5.32. *For $i = 1, 2$, let G_i be a locally profinite group, K_i a compact open subgroup of G_i and $\rho_i \in \hat{K}_i$. Let V be a vector space which is also both a G_1 and G_2 representation (denoted π_1 and π_2 respectively) such that their actions commute. Then V is natrually a $G_1 \times G_2$ representation, denoted $\pi_1 \times \pi_2$, and*

$$\pi_1(e_{\rho_1}) * (\pi_2(e_{\rho_2}) * V) = \pi_1 \times \pi_2(e_{\rho_1 \otimes \rho_2}) * V = V^{\rho_1 \otimes \rho_2}. \tag{2.69}$$

2.6 Intertwining

Definition 2.6.1. Let K_1, K_2 be compact open (or just closed) subgroups of G and let $(\rho_i, U_i) \in \hat{K}_i$ for $i = 1, 2$. The element $g \in G$ intertwines U_1 with U_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(U_1^g, U_2) \neq 0, \tag{2.70}$$

where U_i^g denotes the representation $x \mapsto \rho_1(gxg^{-1})$ of the group $K_1^g = g^{-1}K_1 g$. As a property of g, this depends only on the double coset $K_1 g K_2$.

prop:intertwine **Proposition 2.6.2.** *For $i = 1, 2$, let K_i be a compact open subgroup of G and let $(\rho_i, U_i) \in \hat{K}_i$. Let (π, V) be an irreducible smooth representation of G which contains both U_1 and U_2. Then there exists a $g \in G$ that intertwines U_1 with U_2.*

Proof. Let e_2 denote the K_2 projection $V \to V^{\rho_2}$. Since $g^{-1}V^{\rho_1} = V^{\rho_1^g}$ and $\sum_{g \in G} g^{-1}V^{\rho_1} = V$ there is a $g \in G$ such that $g^{-1}V^{\rho_1} \to V^{\rho_2}$ is non-zero. It is then easy to see that this g intertwines U_1 with U_2. ∎

Remark 2.6.3. Let K_1, K_2, U_1, U_2 be as in the proposition. Since V is semisimple as a $K_1^g \cap K_2$-representation

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_{K_1^g \cap K_2}(U_1^g, U_2) &= \dim_{\mathbb{C}} \mathrm{Hom}_{K_1^g \cap K_2}(U_2, U_1^g) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{K_1 \cap K_2^{g^{-1}}}(U_2^{g^{-1}}, U_1). \end{aligned} \tag{2.71}$$

It follows that g intertwines U_1 with U_2 iff g^{-1} intertwines U_2 with U_1.

Definition 2.6.4. We say that U_1 and U_2 intertwine in G is there is a $g \in G$ that intertwines U_1 with U_2.

If we have a since pair (K, ρ) we say g intertwines ρ if it intertwines ρ with itself.

Proposition 2.6.5. *Let K be a compact open subgroup of G, $g \in G$, and $\rho \in \hat{K}$. The following are equivalent*

1. *there exists a $f \in e_\rho * \mathcal{H}(G) * e_\rho$ such that $f|_{KgK} \neq 0$,*
2. *g intertwines ρ.*

Proof. Let $C^\infty(KgK)$ be the space of G-smooth functions on the coset KgK. $C^\infty(KgK)$ is naturally a smooth $K \times K$-representation, which we call π, via $(k_1, k_2)f : x \mapsto f(k_1^{-1}xk_2)$. Let $H = \{(k, g^{-1}kg) : k \in K \cap gKg^{-1}\} \subseteq K \times K$. Then the map $C^\infty(KgK) \to \mathbb{C}$, $f \mapsto f(g)$ is a H-homomorphism (H acts trivially on $\mathbb{C}$). We thus obatin a map $C^\infty(KgK) \to \text{s-Ind}_H^{K\times K}\mathbb{C} =: V$. Now, given an $\phi \in V$ define $f_\phi \in C^\infty(KgK)$ by $f_\phi(k_1gk_2) = \phi(k_1^{-1}, k_2)$ for $k_1, k_2 \in K$. This is well-defined since $\phi \in V$ and is also G-smooth. It is clear that these maps are inverse $K \times K$-homomorphisms and so $V \cong C^\infty(KgK)$ as $K \times K$-representations. Now,

$$\begin{aligned} e_\rho * C^\infty(KgK) * e_\rho &= \lambda(e_\rho) * (\rho(e_{\check{\rho}}) * C^\infty(KgK)) \\ &= \pi(e_{\rho\otimes\check{\rho}}) * C^\infty(KgK) \cong V^{\rho\otimes\check{\rho}}, \end{aligned} \tag{2.72}$$

as $e_{\rho\otimes\check{\rho}}$ naturally lives in $\mathcal{H}(K\times K)$. Thus (1) holds iff $V^{\rho\otimes\check{\rho}} \neq 0$ iff $\mathrm{Hom}_{K\times K}(\rho\otimes\check{\rho}, V) \neq 0$ iff $\mathrm{Hom}_H(\rho\otimes\check{\rho}, \mathbb{C}) \neq 0$. Since H is compact, this is equivalent to $\rho\otimes\check{\rho}$ having a fixed vector which in turn is equivalent to the representation $k \mapsto \rho(k) \otimes \check{\rho}(g^{-1}kg)$ of $K \cap gKg^{-1}$ having a fixed vector. This is equivalent to $\mathrm{Hom}_{K\cap gKg^{-1}}(\rho^g, \rho) \neq 0$ as required. ■

Definition 2.6.6. Let K be an open subgroup of G, containing and compact module Z. Let (ρ, W) be be an irreducible smooth representation of K. Write $\mathcal{H}(G, \rho)$ for the space of functions $f : G \to \mathrm{End}_{\mathbb{C}}(W)$ which are are compactly supported modulo Z and satisfy

$$f(k_1gk_2) = k_1f(g)k_2, \quad k_i \in K, g \in G. \tag{2.73}$$

Note that any $f \in \mathcal{H}(G, \rho)$ has support a finite union of double cosets $K\backslash G/K$.

Let $\dot{\mu}$ be a Haar measure on G/Z. For $\phi_1, \phi_2 \in \mathcal{H}(G, \rho)$, we set

$$\phi_1 * \phi_2(g) = \int_{G\backslash Z} \phi_1(x)\phi_2(x^{-1}g)d\dot{\mu}(x), g \in G. \tag{2.74}$$

The function $\phi_1 * \phi_2$ lies in $\mathcal{H}(G, \rho)$, an under this operation $\mathcal{H}(G, \rho)$ is an associative $\mathbb{C}$-algebra with unit (given by $x \mapsto 1_K(x)x/\dot{\mu}(K/Z)$).

Remark 2.6.7. We have $e_\rho * \mathcal{H}(G) * e_\rho \otimes \mathrm{End}_{\mathbb{C}}(W) \cong \mathcal{H}(G, \rho)$.

Proposition 2.6.8. *Let $g \in G$. There exists $\phi \in \mathcal{H}(G,\rho)$ with support KgK iff g intertwines ρ.*

Proof. Let $f \in \mathrm{End}_{\mathbb{C}}(W)$. For $g \in G$, the assignment $k_1 g k_2 \mapsto k_1 f k_2$, is well-defined and given an element of $\mathcal{H}(G,\rho)$ iff for $k \in K^g \cap K$ we have $f \circ \rho(k) = \rho^g(k) \circ f$. This is the case iff $f \in \mathrm{Hom}_{K \cap K^g}(\rho, \rho^g)$. But ρ and ρ^g are semisimple as $K \cap K^g$-representations and so

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K \cap K^g}(\rho, \rho^g) = \dim_{\mathbb{C}} \mathrm{Hom}_{K \cap K^g}(\rho^g, \rho) \tag{2.75}$$

and so the lemma follows. ∎

Corollary 2.6.9. *The space of functions $f \in \mathcal{H}(G,\rho)$ supported on KgK is isomorphic to* $\mathrm{Hom}_{K \cap K^g}(\rho^g, \rho)$.

Remark 2.6.10. $\text{c-Ind}_K^G W$ consists of functions $G \to W$ compactly supported modulo Z and satisfying $f(kg) = kf(g)$ for $k \in K, g \in G$.

Definition 2.6.11. Let $\phi \in \mathcal{H}(G,\rho)$ and $f \in \text{c-Ind}_K^G W$. Define

$$\phi * f(g) = \int_{G/Z} \phi(x) f(x^{-1}g) d\dot{\mu}(x), \quad g \in G. \tag{2.76}$$

This is clearly in $\text{c-Ind}_K^G W$. It easy to check that this moreover defines an action on $\text{c-Ind}_K^G W$ and so we obtain a group homomorphism $\mathcal{H}(G,\rho) \to \mathrm{End}_G(\text{c-Ind}_K^G W)$.

Proposition 2.6.12. *This homomorphism is an isomorphism.*

Proof. By Frobenius reciprocity we get a map

$$\mathcal{H}(G,\rho) \to \mathrm{Hom}_K(W, \text{c-Ind}_K^G W). \tag{2.77}$$

Now suppose we are given $\phi : W \to \text{c-Ind}_K^G W$ a K-homomorphism. Define $\Phi : G \to \mathrm{End}_{\mathbb{C}}(W)$ by

$$\Phi(g)(w) = \phi(w)(g). \tag{2.78}$$

Then $\Phi(k_1 g k_2)(w) = \phi(w)(k_1 g k_2) = k_1 \phi(w)(g k_2) = k_1 \phi(k_2 w)(g)$. Φ is compactly supported since W is finite dimensional, and so $\Phi \in \mathcal{H}(G,\rho)$. It is then easy to check that $\phi \mapsto \Phi/\dot{\mu}(K/Z)$ is the required inverse map. ∎

lem:cind_irr

Lemma 2.6.13. *Let K be a open subgroup of G containing and compact modulo Z, and let W be an irreducible smooth representation of K. Suppose $g \in G$ intertwines W iff $g \in K$. Then c-$\mathrm{Ind}_K^G W$ is irreducible.*

Proof. Let $X = \text{c-Ind}_K^G W$. The center Z acts via the cental character w_ρ and so X is a direct sum of K-isotypic components. Any K-homomorphism $W \to X$ has image in X^ρ. Thus

$$\mathrm{Hom}_K(W, X^\rho) = \mathrm{Hom}_K(W, X) \cong \mathrm{End}_G(X) \cong \mathcal{H}(G,\rho). \tag{2.79}$$

But $\mathcal{H}(G, \rho)$ must be 1-dimensional and so $W = X^\rho$. Now let Y be a non-zero G-subspace of X. Then

$$0 \neq \mathrm{Hom}_G(Y, X) \subseteq \mathrm{Hom}_G(Y, \text{s-Ind}_H^G W) \cong \mathrm{Hom}_K(Y, W). \tag{2.80}$$

Since Y is semisimple over K (since X is), we have $Y^\rho \neq 0$ and so $W = X^\rho = Y^\rho \subseteq Y$ since W is an irreducible K-representation. As W generates X over G we have that $Y = X$ and so X is irreducible. ■

CHAPTER 3

Irreducible representations of GL(2,F)

3.1 GL(n,F)

Remark 3.1.1. We have the following locally profinite groups.

1. $(F,+)$ is a locally profinite group with $\mathfrak{p}^n, n \in \mathbb{Z}$ a fundamental system of compact open neighbourhoods of 0.

2. $(F,\cdot)$ is a locally profinite group with $1+\mathfrak{p}^n, n \geq 1$ a fundamental system of compact open neighbourhoods of 1.

3. $(M_n(F),+)$ is a locally profinite group with multiplication of matrices continuous.

4. $(\mathbf{GL}_n(F),\cdot)$ is a locally profinite group with $K = \mathbf{GL}_n(\mathfrak{o})$ and $K_j = 1 + \mathfrak{p}^j M_n(\mathfrak{o})$ compact open and give a fundamental system of neighbourhoods of 1 in G.

Proposition 3.1.2. *Let $V = F^{\oplus n}$. Then V is a smooth G-representation.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors. Then e_1 is fixed by the compact open subgroup

$$\begin{pmatrix} 1 & \mathfrak{o} & \ldots & \mathfrak{o} \\ 0 & 1 & \ldots & \mathfrak{o} \\ 0 & \vdots & \ddots & \mathfrak{o} \\ 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{3.1}$$

Similarly for $e_2, \ldots, e_n$. The result follows. ∎

Definition 3.1.3. Let V be a n-dimensional vector space. An $\mathfrak{o}$-lattice in V is a finitely generated $\mathfrak{o}$-submodule L of V such that $V = FL$.

Proposition 3.1.4. *Let L be an $\mathfrak{o}$-lattice of V. Then there is an F-basis $\{x_1, \ldots, x_n\}$ of V such that $L = \sum_{i=1}^n \mathfrak{o}x_i$.*

Corollary 3.1.5. *$\mathfrak{o}$-lattices are compact open subgroups of V.*

cor:lat_syt **Corollary 3.1.6.** *$\mathfrak{o}$-lattices form a fundamental system of compact open subgroups of the origin.*

Proof. Open sets are locally a product of open sets of F. ■

Definition 3.1.7. A lattice of V is a compact open subgroup of V.

Proposition 3.1.8. *Let L be a subgroup of V. Then L is a lattice iff there are $\mathfrak{o}$-lattices L_1, L_2 in V such that $L_1 \subseteq L \subseteq L_2$.*

Proof. ($\Leftarrow$) $L_1 \subseteq L$ implies that L is open and hence closed. $L \subseteq L_2$ then implies that L is compact.

($\Rightarrow$) The existence of L_1 follows from corollary 3.1.6. It follows that $FL = V$. Now choose a basis $x_1, \ldots, x_n$ of V and consider the projection maps $\pi_i : V \to Fx_i$. Then $\pi_i(L)$ is a compact open subgroup of F and so is contained in some $\mathfrak{p}^{n_i}x_i$. Thus $L \subseteq \oplus_i \mathfrak{p}^{n_i}x_i$. ■

Corollary 3.1.9. *If V is both a lattice and an $\mathfrak{o}$-module then V is an $\mathfrak{o}$-lattice.*

Proof. Consider the ses $0 \to L_1 \to L \to L/L_1 \to 0$ and note that L/L_1 is finite and hence a f.g. $\mathfrak{o}$-module. ■

Definition 3.1.10. Suppose that V has a nondegenerate symmetric bilinear form $\langle -, - \rangle : V \times V \to \mathbb{C}$. Then given a lattice L we define its dual to be

$$L^* = \{x \in V : \langle x, y \rangle \in \mathfrak{o}, \forall y \in L\}. \tag{3.2}$$

Remark 3.1.11. If $L_1 \subseteq L_2$ then $L_2^* \subseteq L_1^*$.

Proposition 3.1.12. *If L is an $\mathfrak{o}$-lattice with basis $x_1, \ldots, x_n$ then $L^* = \bigoplus_i \mathfrak{o}y_i$ where $y_i \in V$ is such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all i, j.*

Corollary 3.1.13. *If L is an $\mathfrak{o}$-lattice then $L^{**} = L$.*

Corollary 3.1.14. *If L is a lattice then L^* is a lattice.*

Corollary 3.1.15. *If $L_1 \subseteq L_2$ are $\mathfrak{o}$-lattices then $|L_2 : L_1| = |L_1^* : L_2^*|$.*

Remark 3.1.16. It follows that $\mu(L)\mu(L^*)$ is a constant independent of L.

Corollary 3.1.17. *If L is an $\mathfrak{o}$-lattice then $\widehat{1_L} = \mu(L)1_{L^*}$.*

Proposition 3.1.18. *Let L be an $\mathfrak{o}$-lattice of V and $W \subseteq V$ a vector subspace. Then $L \cap W$ is an $\mathfrak{o}$-lattice of W.*

Proof. $L \cap W$ is clearly an open $\mathfrak{o}$-submodule of W and $F(L \cap W) = W$. Now let $\pi : V \to W$ be a projection map onto W. Then $L \cap W \subseteq \pi(L)$ and so is a closed subset of a compact subset and is hence itself compact. It follows that $L \cap W$ is a lattice and $\mathfrak{o}$-submodule of W and so is an $\mathfrak{o}$-lattice. ■

3.2 Structure theory

Proposition 3.2.1. *Let $K = \mathbf{GL}_2(\mathfrak{o})$. Then $G = BK$.*

Corollary 3.2.2. *$B\backslash G$ is compact.*

Proposition 3.2.3. *Let $\bar{\omega}$ be a prime element of F. Then*

$$\left\{ \begin{pmatrix} \bar{\omega}^a & 0 \\ 0 & \omega^b \end{pmatrix} : a \le b \in \mathbb{Z} \right\} \tag{3.3}$$

is a set of representatives for $K\backslash G/K$.

Corollary 3.2.4. *G/K is countable.*

Proposition 3.2.5. *If K' is a compact open subgroup of G, then there is a $g \in G$ such that $gK'g^{-1} \subseteq K$.*

Proof. Let e_1, e_2 be a basis vectors for V. Since V is a smooth, $K_i = \mathrm{stab}_K(e_i)$ is a compact open subgroup of K. It follows that Ke_i is a finite set. Let $S = Ke_1 \cup Ke_2$ and $L = \mathrm{span}_{\mathfrak{o}} S$. Then L is a K-stable $\mathfrak{o}$-lattice. The result follows. ■

Definition 3.2.6. The standard Iwahori sbugroup of G is the compact open subgroup

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in U_F, b \in \mathfrak{o}, c \in \mathfrak{p} \right\}. \tag{3.4}$$

Proposition 3.2.7. *(Iwahori decomposition). The product map*

$$I \cap N' \times I \cap T \times I \cap N \to I \tag{3.5}$$

is a homeomorphism for any ordering of the left hand side.

3.3 Haar measures

Lemma 3.3.1. *$C_c^\infty(F)$ is spanned by the characteristic functions of sets $a+\mathfrak{p}^m$ for $a \in F, m \in \mathbb{Z}$.*

Corollary 3.3.2. *Let $\Phi \in C_c^\infty(F), y \in F^\times$. Then*

$$\int_F \Phi(xy)d\mu(x) = |y|^{-1} \int_F \Phi(x)d\mu(x). \tag{3.6}$$

Corollary 3.3.3. *The functional $\Phi \mapsto \int_{F^\times} \Phi(x)|x|^{-1}d\mu(x)$ for $\Phi \in C_c^\infty(F^\times)$ is a Haar-measure on $F^\times$.*

Lemma 3.3.4. *The map $x \mapsto |\det x|$ for $x \in G$ is in $C_c^\infty(G)$.*

Proposition 3.3.5. *The functional $\Phi \mapsto \int_G \Phi(x)|\det x|^{-2}d\mu(x)$ for $\Phi \in C_c^\infty(G)$ is a left and right Haar integral on G. In particular, G is unimodular.*

Proposition 3.3.6. *Let V be a finite dimensional F-vector space and $\alpha \in \mathrm{End}(V)$. Then for $f \in C_c^\infty(V)$*

$$\int_V f(\alpha(v))dv = |\det(\alpha)|^{-1} \int_V f(v)dv. \tag{3.7}$$

Proof. It is easy to see that $\int_V \circ \alpha^*$ is a Haar measure on V and so there is a constant c_α such that $\int_V \circ \alpha^* = c_\alpha \cdot \int_V$. Etc. ∎

Definition 3.3.7. We have that $B = T \ltimes N$ and so $C_c^\infty(B) \cong C_c^\infty(T) \otimes_{\mathbb{C}} C_c^\infty(N)$. Define the linear functional

$$\Phi \mapsto \int_T \int_N \Phi(tn) d\mu_T(t) d\mu_N(n), \Phi \in C_c^\infty(B). \tag{3.8}$$

It is a straighforward check that this is a left Haar measure on B.

Lemma 3.3.8. *Let $\Phi \in C_c^\infty(N)$ and*

$$t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T. \tag{3.9}$$

Then

$$\int_N \Phi(tnt^{-1}) d\mu_N(n) = |a/b|^{-1} \int_N \Phi(n) d\mu_N(n). \tag{3.10}$$

Corollary 3.3.9. *The module δ_B of the group B is given by*

$$\delta_B : tn \mapsto |t_2/t_1|, \quad n \in N, t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T. \tag{3.11}$$

Lemma 3.3.10. *Let μ_K be a Haar measure on K. Then μ_K restricts to a semi-invariant measure on $B \cap K \backslash K$.*

Proof. Since K is compact, K is unimodular and so μ_K is also right invariant. The result follows. ∎

Corollary 3.3.11. *The space $C_c^\infty(B\backslash G, \delta_B^{-1})$ admits a positive semi-invariant measure $\dot\mu$. Moreover, K admits a Haar measure such that*

$$\int_{B\backslash G} f(g) d\dot\mu(g) = \int_K f(k) d\mu_K(k). \tag{3.12}$$

Proof. The first part follows from G being unimodular. For the second part note that δ_G is trivial on K and so restriction to K induces a map

$$C_c^\infty(B\backslash G, \delta_B^{-1}) \to C_c^\infty(K \cap B\backslash K, 1). \tag{3.13}$$

Since $G = BK$, this map is an injection. It is surjective by looking at $(-)^{K'}$ of the map for $K' \subseteq K$ a compact open subgroup of K. Thus it is a bijection. Note moreover that if $k \in K$ then ρ_k commutes with the map and so it is an isomorphism of K-representations. Write ϕ for the inverse map. Then the functional

$$f \mapsto \int_{B\backslash G} \phi(f)d\dot{\mu} \tag{3.14}$$

is a semi-invariant measure on $B \cap K\backslash K$. By the previous lemma the result follows. ■

3.4 Representations of Mirabolic Group

Definition 3.4.1. Let ϑ a character of N. Write $V(\vartheta)$ for the space spanned by elements of the form $nv - \theta(n)v$ for $n \in N, v \in V$. We set $V_\vartheta = V/V(\vartheta)$.

Remark 3.4.2. Let N_0 be a compact open subgroup of N. Then the map $V \to V^{N_0}$ given by

$$v \mapsto \frac{1}{\mu_N(N_0)} \int_{N_0} nvd\mu_N(n) \tag{3.15}$$

is a projection map onto V^{N_0}.

Lemma 3.4.3. *Let μ_N be a Haar measure on N and ϑ a character of N.*

1. *Let V be a smooth N-representation and $v \in V$. Then $v \in V(\vartheta)$ iff there is a compact open subgroup N_0 of N such that*

$$\int_{N_0} \vartheta(n)^{-1} nvd\mu_N(n) = 0. \tag{3.16}$$

2. *The process $V \mapsto V_\vartheta$ is an exact functor from Smo_N to the category of complex vector spaces.*

Proof. Consider first that case when ϑ is the trivial character on N. $N \cong F$ is a union of an ascending sequence of compact open subgoups. Thus if $v = \sum_i v_i - n_i v_i \in V(N)$ there is a compact N_0 containing all the v_i. The required integral is then 0 in this case.

Conversely suppose $v \in V$ and the integral is zero for some N_0 a compact open subgroup of N. Let N_1 be a compact normal subgroup of N_0 such that $v \in V^{N_1}$. V^{N_1} is naturally a N_0/N_1-representation. Thus $V^{N_1} = V^{N_1}(N_0/N_1) \oplus V^{N_0}$ and the map

$$w \mapsto \mu_N(N_0)^{-1} \int_{N_0} nwd\mu_N(n), w \in V^{N_1} \tag{3.17}$$

is the N_0-projection $V^{N_1} \to V^{N_0}$. The kernel is $V^{N_1}(N_0/N_1) \subseteq V(N)$ and so $v \in V(N)$.

Now suppose ϑ is arbitrary. Then $(\vartheta^{-1} \otimes V)(N) = V(\vartheta)$ and so the result follows from the previous working.

Part 2. then follows immediately. ■

Corollary 3.4.4. *Let V be a smooth N-representation. Then $V(N)_N = 0$ and $V(N)(N) = V(N)$.*

Corollary 3.4.5. *If $\vartheta \neq 1$ then $V(N)_\vartheta \cong V_\vartheta$.*

Proposition 3.4.6. *Ler V be a smooth N-representation, and $0 \neq v \in V$. Then there exists a character ϑ of N such that $v \notin V(\vartheta)$.*

Corollary 3.4.7. *Let V be a smooth N-representation. If $V_\vartheta = 0$ for all characters ϑ of N, then $V = 0$.*

Remark 3.4.8. Let V be a smooth M-representation. Then $V(N)$ is a M-subrepresentation and V_N is naturally a $S = M/N$-representation. But for $s \in S$, $sV(\vartheta) = V(\vartheta')$ where $\vartheta'(n) = \vartheta(s^{-1}ns)$. We thus get the following result.

Corollary 3.4.9. *Let V be a smooth N-representation. If $V_\vartheta = 0$ for some character ϑ of N and $V_N = 0$, then $V = 0$.*

Lemma 3.4.10. *Let ϑ, ϑ' be non-trivial characters of N. Then s-$\mathrm{Ind}_N^M \vartheta \cong$ s-$\mathrm{Ind}_N^M \vartheta'$ and similarly for c-Ind_N^M.*

Proposition 3.4.11. *Let ϑ be a non-trivial character on N and set $\mathcal{W} =$ s-$\mathrm{Ind}_N^M \vartheta, \mathcal{W}^c =$ c-$\mathrm{Ind}_N^M \vartheta$. Let $\alpha : \mathcal{W} \to \mathbb{C}$ denote the canonical map $f \mapsto f(1)$.*

1. *We have $\mathcal{W}(N) = \mathcal{W}^c(N) = \mathcal{W}^c$ and $(\mathcal{W}/\mathcal{W}^c)(N) = 0$.*
2. *The map α induces isomorphisms $\mathcal{W}_\vartheta \cong \mathbb{C}, \mathcal{W}^c_\vartheta \cong \mathbb{C}$.*

Corollary 3.4.12. *c-$\mathrm{Ind}_N^M \vartheta$ is irreducible over M.*

Proof. Let V be a M-subrepresentation of $\mathcal{W}^c$. As $\mathcal{W}^c_N = 0$, we have $V_N = 0 = (\mathcal{W}^c/V)_N$. Moreover, the sequence

$$0 \to V_\vartheta \to \mathcal{W}^c_\vartheta \to (\mathcal{W}^c/N)_\vartheta \to 0 \tag{3.18}$$

is exact and $\dim_\mathbb{C} \mathcal{W}^c_\vartheta = 1$. Thus $\dim_\mathbb{C} V_\vartheta$ is 0 or 1. In the first case $V = 0$. In the second case, $(\mathcal{W}^c/V)_\vartheta = 0$ and so $\mathcal{W}^c = V$. ∎

Definition 3.4.13. By Frobenius reciprocity we have

$$\mathrm{Hom}_N(V, V_\vartheta) \cong \mathrm{Hom}_M(V, \text{s-}\mathrm{Ind}_N^M V_\vartheta). \tag{3.19}$$

Let $q_\star$ be the map $V \to \text{s-}\mathrm{Ind}_N^M V_\vartheta$ corresponding to the projection map $q : V \to V_\vartheta$.

thm:c_ind **Thm 3.4.14.** *Let V be a smooth representation of M. The the map $q_\star : V \to$ s-$\mathrm{Ind}_N^M V_\vartheta$ induces an isomorphism $V(N) \cong$ c-$\mathrm{Ind}_N^M V_\vartheta$.*

Corollary 3.4.15. *Let V be an irreducible smooth representation of M. Either*

1. $\dim_\mathbb{C} V = 1$ *and V is the inflation of a character of $M/N \cong F^\times$, or*

2. $\dim_{\mathbb{C}} V$ *is infinite and* $V \cong$ *c-*$\mathrm{Ind}_N^M \vartheta$*, for any character* $\vartheta \neq 1$ *of* N.

In case (1), $\dim_{\mathbb{C}} V_N = 1$ *and* $V_\vartheta = 0$ *for* $\vartheta \neq 1$. *In case (2),* $V_N = 0$ *and* $\dim_{\mathbb{C}} V_\vartheta = 1$ *for all* $\vartheta \neq 1$.

3.5 Jacquet functor

Definition 3.5.1. Let V be a smooth G-representation. Then $V/V(N)$ is a smooth B-representation on which N acts trivially and so $V/V(N)$ is naturally a $T = B/N$-representation. We call V_N the Jacquet module of V at N. This process induces an exact additive functor $\mathsf{Smo}_G \to \mathsf{Smo}_T$.

Proposition 3.5.2. *The Jacquet functor is left adjoint to parabolic induction.*

Proposition 3.5.3. *Let* V *be a smooth irreducible representation of* G. *The following are equivalent:*

1. $V_N \neq 0$
2. V *is isomorphic to a subrepresentation of s-*$\mathrm{Ind}_B^G \chi$ *where* χ *is a character of* T.

Proof. (1) $\Leftarrow$ (2) Obvious. (1) $\Rightarrow$ (2) We just need to show that V_N is a finitely generated T-representation. ■

Definition 3.5.4. If $V_N = 0$ we call V cuspidal (or supercuspidal). Otherwise we say V is in the principal series.

Proposition 3.5.5. *Any character of* G *is of the form* $\phi \circ \det$*, for some character* ϕ *of* $F^\times$.

Proof. If χ is a character then its kernel contains $\mathbf{SL}_2(F)$ (the commutator subgroup of G). The result follows from the fact that det is surjective and open. ■

Definition 3.5.6. Let U be a smooth T-representation and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Write U^w for the twisted representation where t acts like wtw^{-1}.

Definition 3.5.7. Let U be a smooth T-representation and α_U : s-$\mathrm{Ind}_B^G U \to U$ be the canonical map. It is a map of B-representation, and give rise to a map $(\text{s-}\mathrm{Ind}_B^G U)_N \to U$ which we also call α_U.

Remark 3.5.8. $G = B \cup BwN = B \cup TwN$.

Lemma 3.5.9. *Let* U *be a smooth* T*-repre-sentation and* $f \in$ *s-*$\mathrm{Ind}_B^G U$. *Then* $f \in \ker \alpha_U$ *iff there is a compact open subgroup* N_0 *of* N *such that* $\mathrm{supp} f \subseteq BwN_0$.

Proof. There exists a compact open subgroup N_0' of N' such that N_0' fixes f. Thus f vanishes on BN_0'. But supp(f) is a union of right B cosets and

$$B\begin{pmatrix}1 & 0\\ x & 0\end{pmatrix} = Bw\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}. \tag{3.20}$$

It is easy to see that if U_0 is a compact open subgroup of F then $F\backslash(U_0\backslash\{0\})^{-1}$ is contained in a compact open subgroup of F and so there exists a compact open subgroup N_0 such that $\mathrm{supp}(f) \subseteq BwN_0$. ∎

Remark 3.5.10. $\{1\} \cup \{wn : n \in N\}$ is a complete set of right B coset representatives.

Lemma 3.5.11. *For $f \in C_c^\infty(N), t \in T$ we have*

$$\int_N f(txt^{-1})d\mu_N(x) = \delta_B(t)\int_N f(x)d\mu_N(x). \tag{3.21}$$

Definition 3.5.12. Given $f \in \ker \alpha_U$ define $f_N : T \to U$ by

$$f_N(x) = \int_N f(xwn)d\mu_N(n) = xf_N(1). \tag{3.22}$$

Lemma 3.5.13. *There exists a compact open subgroup N_0 of N such that*

$$\int_{N_0} f(xn)d\mu_N(n) = 0, \quad \forall x \in G. \tag{3.23}$$

iff $f_N(t) = 0$ for all $t \in T$.

Proof. ($\Rightarrow$) Let N_1 be a compact open subgroup such that $\mathrm{supp}(f) \subseteq BwN_1$. Let $\{a_1, \ldots, a_k\}$ be elements such that $N_1 \subseteq \cup_i a_i N_0$. Then

$$\begin{aligned} 0 &= \sum_i \int_{N_0} f(wa_i n)d\mu_N(n) = \int_{\cup_i a_i N_0} f(wn)d\mu_N(n) \\ &= \int_N f(wn)d\mu_N(n) = f_N(1). \end{aligned} \tag{3.24}$$

It follows that $f_N(t) = 0$ for all $t \in T$. ($\Leftarrow$) Let N_1 be as before and set $N_0 = N_1$. Then since $f_N(1) = 0$ we have

$$\int_{N_0} f(wn)d\mu_N(n) = 0. \tag{3.25}$$

Note that $G = B \cup BwN$. If $x \in B$ then trivially

$$\int_{N_0} f(xn)d\mu_N(n) = 0. \tag{3.26}$$

If $x \in BwN$, wlog $x = wn$ for some n. Then

$$\int_{N_0} f(wnn_0)d\mu_N(n_0) = \int_{nN_0} f(wn_0)d\mu_N(n_0) = 0 \tag{3.27}$$

since if $n \in N_0$, ok, and otherwise, f is zero on wnN_0. ∎

Lemma 3.5.14. *(Restriction-Induction Lemma). Let U be a smooth T-representation. Then there is an exact sequence of T-representations*

$$0 \to U^w \otimes \delta_B^{-1} \to (\textit{s-Ind}_B^G U)_N \xrightarrow{\alpha_U} U \to 0. \tag{3.28}$$

Proof. Let V be the kernel of the canonical B-homomorphism s-$\mathrm{Ind}_B^G U \to U$. It suffices to determine what V_N is. By the previous lemmma, the kernel of the map $f \mapsto f_N(1)$ is $V(N)$. Thus we obtain a bijection $V_N \to U$. Moreover for $f \in V, t \in T$

$$\begin{aligned}(tf)_N(x) &= \int_N f(xwnt)d\mu_N(n)\\ &= \delta_B(t^{-1})\int_N f(xt^w wn)d\mu_N(n)\\ &= \delta_B(t^{-1})(t^w f_N)(x).\end{aligned} \tag{3.29}$$

Thus the map $V_N \to \delta_B^{-1} \otimes U^w$ is an isomorphism of T-representations. ∎

Proposition 3.5.15. *Let V be a smooth irreducible representation of G which is not cuspidal. Then V is admissable.*

Proof. $V_N \neq 0$ and so V is isomorphic to a subrepresentation of s-$\mathrm{Ind}_B^G\chi$ for some character χ of T. It thus suffices to show that s-$\mathrm{Ind}_B^G\chi$ is admissable. Let $K_0 = \mathbf{GL}_2(\mathfrak{o})$ and $K \subseteq K_0$ a compact open subgroup. Then since $B\backslash G/K$ is finite, (s-$\mathrm{Ind}_B^G\chi)^K$ is also finite dimensional. ∎

Definition 3.5.16. Let ϕ be a character of $F^\times$ and (π, V) be a smooth representation of G. Define $(\phi\pi, V)$ to be the representation $\phi \circ \det \otimes V$.

Proposition 3.5.17. *Let χ be a character of T. Then s-$\mathit{Ind}_B^G(\phi{\cdot}\chi) \cong \phi$s-$\mathit{Ind}_B^G\chi$.*

Proof. The map ϕs-$\mathrm{Ind}_B^G\chi \to$ s-$\mathrm{Ind}_B^G(\phi \cdot \chi)$ given by $f \mapsto (x \mapsto \phi \circ \det(x) f(x))$ is an isomorphism of G-representations. ∎

3.6 Irreducibility criterion

thm:irr_crit **Thm 3.6.1.** *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, and set $X =$ s-$\mathit{Ind}_B^G\chi$.*

1. *X is reducible iff $\chi_1\chi_2^{-1}$ is either the trivial character or $x \mapsto |x|^2$,*
2. *If X is reducible then:*
   a) *the composition length of X is 2,*
   b) *one composition factor has dimension 1, the other of infinite dimension,*
   c) *X has a 1-dimensional G-subspace iff $\chi_1\chi_2^{-1} = 1$*
   d) *X has a 1-dimensional G-quotient iff $\chi_1\chi_2^{-1}(x) = |x|^2$, $x \in F^\times$.*

Definition 3.6.2. Let $V = \{f \in X : f(1) = 0\}$ and $W = V(N)$. For $f \in V$ define $f_N \in C_c^\infty(N)$ by $f_N(n) = f(wn), n \in N$.

Proposition 3.6.3. *The map $V \to C_c^\infty(N), f \mapsto f_N$ is a N-isomorphism.*

Remark 3.6.4. We can give $C_c^\infty(N)$ the structure of a M-representation by letting $s = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ act by

$$s.\phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi_2(a)\phi\left(\begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}\right) \tag{3.30}$$

where $\phi \in C_c^\infty(N)$. It is then an easy check to see that the map $V \to C_c^\infty(N)$ is an isomorphism of M-representations.

Remark 3.6.5. Let ϑ be a character of N. Then the map $C_c^\infty(N) \to C_c^\infty(N), f \mapsto \vartheta f$ is a linear isomorphism that sends $V(N)$ to $V(\vartheta)$.

Proposition 3.6.6. *Then W is an irreducible B-representation.*

Proof. By the restriction-induction lemma, V_N has dimension 1. It follows by the previous remark that V_ϑ also has dimension 1. Since $W \neq 0$, the inclusion $W \to V$ induces an isomorphism $W_\vartheta \to V_\vartheta$. It follows that $W_\vartheta \cong \vartheta$ and so by theorem 3.4.14 that

$$W = W(N) \cong \text{c-Ind}_N^M \vartheta \tag{3.31}$$

which we know is irreducible. ∎

cor:comp_series **Corollary 3.6.7.** *X has composition length 3 as a B or M-representation. Two of the composition factors have dimension one, and the third is of infinite dimension. It has composition length at most 3 as a G-representation.*

Proposition 3.6.8. *The following are equivalent:*

1. *$\chi_1 = \chi_2$,*
2. *X has a one-dimensional N-subspace.*

When these conditions hold,

1. *X has a unique one dimensional N-subspace X_0,*
2. *X_0 is a G-subspace, and it is not contained in V.*

Proof. (1) ⇒ (2) Wlog $\chi_1 = \chi_2 = 1$. The contant functions then form a one-dimensional G-subspace of X not contained in V. (2) ⇒ (1) Suppose $f \in X$ spans an N-stable subspace of dimension 1. Then N acts on f as a character and so $\mathrm{supp}(f)$ is a union of $B\backslash G/N$ double cosets. If $f(1) = 0$ then $\mathrm{supp}(f) = BwN$, but this is impossible since $f \in V$ implies that $\mathrm{supp}(f) = BwN_0$ for some compact open $N_0 \subseteq N$. Thus $f \notin V$ and so the canonical

N-map $X \to \mathbb{C} = X/V$ identifies $\mathbb{C}f$ with the trivial representation and so N fixes f. Moreover

$$\begin{aligned} f(w) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f(w) &= f\left(\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \right) \\ &= \chi_1(-1)\chi_1^{-1}\chi_2(x) f\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \right). \end{aligned} \tag{3.32}$$

But for sufficiently large $|x|$, f is fixed by $\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}$ and so

$$f(w) = \chi_1(-1)\chi_1^{-1}\chi_2(x)f(1) \tag{3.33}$$

for sufficiently large $|x|$. It follows that $\chi_1 = \chi_2 =: \phi$. Moreover $f(g) = \phi(\det g)f(1)$ and so $\mathbb{C}f$ is also a G-subspace.

This shows that (1) $\Leftrightarrow$ (2). Moreover, we have showed that the one-dimensional subspace is uniquely determined, is a G-subspace and is not contained in V. ∎

Proof. (of theorem 3.6.1). ($\Rightarrow$) Suppose X is reducible. Then it has composition length 2 or 3. By corollary 3.6.7, X has either a finite dimensional G-subspace or G-quotient. If X has a finite dimensional subspace then it has a one dimensional N-subspace, call it L, and $\chi_1 = \chi_2 =: \phi$. Moreover, G acts on L as the character $\phi \circ \det$, and $L \cap V = 0$. Thus $Y := X/L \cong V$ as B-representations. Now suppose X has G-length 3. Then Y has G-length 2. But V has B-length 2 and a unique B-quotient which is of dimension 1. This gives a G-quotient of Y on which G must act as a character $\phi' \circ \det$. This would force $\phi' \otimes \phi'$ to be a factor of $Y_N \cong (\phi \otimes \phi) \cdot \delta_B^{-1}$, but this is clearly impossible. Thus Y has G-length 2.

Now suppose X has a finite dimensional G-quotient. Then $\check{X}$ has a finite dimensional G-subspace and by the duality theorem $\check{X} \cong \text{s-Ind}_B^G(\delta_B^{-1}\check{\chi})$. We are thus in the previous case and the result follows. Thus we have shown that if X is reducible it has the stated form.

($\Leftarrow$) This follows from the previous proposition and its dual. ∎

3.7 Classification of irreducible representations

Proposition 3.7.1. *Let χ, ξ be characters of T. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\textit{s-Ind}_B^G\chi, \textit{s-Ind}_B^G\xi) = \begin{cases} 1 & \text{if } \xi = \chi \text{ or } \xi = \chi^w\delta_B^{-1} \\ 0 & \text{otherwise} \end{cases}. \tag{3.34}$$

Proof. We have that $(\text{s-Ind}_B^G\chi)_N$ fits into the short exact sequence

$$0 \to \chi^w\delta_B^{-1} \to (\text{s-Ind}_B^G\chi)_N \to \chi \to 0. \tag{3.35}$$

If $\chi = \chi^w \delta_B^{-1}$ then $\chi_1(x) = |x|\chi_2(x)$ and so $\mathrm{Ind}_B^G\chi$ is irreducible. The result is then obvious.

If $\chi \neq \chi^w \delta_B^{-1}$ then the sequence splits (WHY?). The result is then obvious. ■

Definition 3.7.2. The irreducible quotient of $\text{s-Ind}_B^G 1_T$ is called St_G, the Steinberg representation of G.

Remark 3.7.3. Let ϕ be a character of $F^\times$. Then we have an exact sequence

$$0 \to \phi_G \to \text{s-Ind}_B^G \phi_T \to \phi \cdot \mathrm{St}_G \to 0. \tag{3.36}$$

Taking the dual of this sequence when $\phi = 1$ we get

$$0 \to \check{\mathrm{St}}_G \to \text{s-Ind}_B^G \delta_B^{-1} \to 1_G \to 0. \tag{3.37}$$

The proposition then implies that $\mathrm{St}_G \cong \check{\mathrm{St}}_G$.

Definition 3.7.4. Let U be a smooth representation of T. Define

$$\boldsymbol{\iota}_B^G U = \text{s-Ind}_B^G(\delta_B^{-1/2} \otimes U). \tag{3.38}$$

This has the connvenient property that $(\boldsymbol{\iota}_B^G U)^\vee \cong \boldsymbol{\iota}_B^G \check{U}$. Using this functor instead of parabolic induction we get the following result.

Proposition 3.7.5.

1. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Then $\boldsymbol{\iota}_B^G\chi$ is reducible iff $\chi_1\chi_2^{-1}$ is equal to one of $x \mapsto |x|^{\pm 1}$. Equivalently iff $\chi = \phi \cdot \delta_B^{\pm 1/2}$.*

2. *Let χ, ξ be characters of T. Then $\mathrm{Hom}_G(\boldsymbol{\iota}_B^G\chi, \boldsymbol{\iota}_B^G\xi)$ is not zero iff $\xi = \chi$ or $\xi = \chi^w$.*

Thm 3.7.6. *The following is a complete list of isomorphism classes of irreducible non-cuspidal representations of G*

1. *$\boldsymbol{\iota}_B^G\chi$, where $\chi \neq \phi \cdot \delta_B^{\pm 1/2}$ and χ is a character of $F^\times$,*

2. *$\phi \circ \det$ where ϕ ranges over the character of $F^\times$,*

3. *the special representations, $\phi \cdot St_G$ where ϕ ranges over the characters of $F^\times$.*

The classes in this list are all distinct except we have $\boldsymbol{\iota}_B^G\chi \cong \boldsymbol{\iota}_B^G\chi^w$.

Proof. Follows from above work. ■

3.8 Cuspidal representations

Definition 3.8.1. Let (π, V) be a smooth G-representation. Let $v \in V, \check{v} \in \check{V}$ and define $\gamma_{\check{v}\otimes v} : g \mapsto \langle \check{v}, gv \rangle$. This is a smooth function and is clearly bilinear in $v, \check{v}$ and so we have a map $\check{V} \otimes V \to C_c^\infty(G)$. If we endow $\check{V} \otimes V$ with the obvious strucutre of a $G \times G$-representation, and let $G \times G$ act on $C_c^\infty(G)$ by left translation in the first factor and right translation in the second factor, then this map is a $G \times G$-homomorphism. Write $C(\pi)$ for the image of this map. We call $C(\pi)$ the matrix coefficients of π.

Remark 3.8.2. If V is irreducible and $z \in Z$ then for $\gamma \in C(\pi)$ we have

$$\gamma(zg) = w_\pi(z)\gamma(g) \tag{3.39}$$

where w_π is the central character of V.

Definition 3.8.3. We say that V is γ-cuspidal if every $\gamma \in C(\pi)$ is compactly supported modulo Z.

Proposition 3.8.4. *Let V be an irreducible γ-cuspidal representation of G. Then V is admissable.*

Proof. Suppose V is not admissable. Then let K be a compact open subgroup such that V^K is not finite dimensional. Note that V^K has countably infinite dimension and so $\check{V}^K = (V^K)^*$ has uncountable dimension. But for a fixed non-zero $v \in V^K$ we have an injective map $\Gamma_v : \check{V}^K \to C(\pi), \check{v} \mapsto \gamma_{\check{v}\otimes v}$. If f lies in the image of this map then is satisfies

$$f(zkgk') = w_\pi(z)f(g), \quad g \in G, z \in Z, k, k' \in K \tag{3.40}$$

and is supported by a finite union of double cosets $ZKgK$. Thus the dimension of $\Gamma_v(\check{V}^K)$ is at most countable. This is a contradiction. ∎

Proposition 3.8.5. *Let V be an irreducible admissable representation of G, and suppose that some non-zero coefficient of π is compactly supported modulo Z. Then π is γ-cuspidal.*

Lemma 3.8.6. *Let V be a cuspidal representation. Let $t = \begin{pmatrix} \bar{w} & 0 \\ 0 & 1 \end{pmatrix}$, $v \in V$, and $\check{v} \in \check{V}$. Then there exists an $m \geq 0$ such that $\gamma_{\check{v}\otimes v}(t^n) = 0$ for all $n \geq m$.*

Thm 3.8.7. *Let V be an irreducible smooth representation of G. Then V is cuspidal iff it is γ-cuspidal.*

Proof. $(\Rightarrow)$ Let $T^+ = \{t^n : n \in \mathbb{N}_0\}$. Then T^+ is a set of representatives for $ZK\backslash G/K$ where $K = \mathbf{GL}_2(\mathfrak{o})$. Let $f = \gamma_{\check{v}\otimes v}$ be a non-zero coefficient of π, and K' a compact open normal subgroup of K fixing both $\check{v}$ and v. Let $k_1, \ldots, k_n$ be coset representatives of K/K' If $g \in G$, then there is a $n \geq 0$ such that

$$ZKgK = ZKt^nK = \bigcup_{i,j} ZK'k_i^{-1}t^nk_jK' \tag{3.41}$$

and so $\operatorname{supp}(f) \subseteq \bigcup_{i,j} ZK'(\operatorname{supp}(f_{ij}) \cap T^+)K'$ where f_{ij} is the function $x \mapsto k_i x k_j^{-1}$. By the previous lemma, this set is compactly supported modulo Z. It follows that π is γ-cuspidal.

($\Leftarrow$) Boring and technical. ∎

Corollary 3.8.8. *Every irreducible smooth representation of G is admissable.*

3.9 Compact induction and cuspidal representations

thm:cind_irr

Thm 3.9.1. *Let K be an open subgroups of $G = \mathbf{GL}_2(F)$, containing and compact modulo Z. Let W be a smooth representation of K and suppose that $g \in G$ intertwines W iff $g \in K$. Then c-$\mathrm{Ind}_K^G W$ is cuspidal and irreducible.*

Proof. Write $X = \text{c-Ind}_K^G W$. By lemma 2.6.13 we know that X is irreducible and so it suffices to show that X is γ-cuspidal.

Let $\psi : W \to X$ be the canonical K-homomorphism which identifies W with the K-subrepresentation of functions supported in K. Both G and K are unimodular and so $\check{X} \cong \text{s-Ind}_K^G \check{W}$. The maps $\check{W} \to \text{c-Ind}_K^G \check{W} \to \text{s-Ind}_K^G \check{W}$ identifies $\check{W}$ with the functions in $\check{X}$ with support contained in K. Let $w \in W \subseteq X, \check{w} \in \check{W} \subseteq \check{X}$. Then $\gamma_{\check{w} \otimes w}$ has suuport contained in K and is non-zero. Thus X is γ-cuspidal as required. ∎

3.9.1 Example

Let $G = \mathbf{GL}_2(F)$, $K = \mathbf{GL}_2(\mathfrak{o})$, $K_1 = 1 + \mathfrak{p}M_2(\mathfrak{o})$ and $I_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$.

Lemma 3.9.2. *For $i = 1, 2$, let $\widetilde{U}_i$ be an irreducible representation of $\mathbf{GL}_2(k)$, and let U_i denote inflation of $\widetilde{U}_i$ to K. Suppose that $\widetilde{U}_1$ is cuspidal.*

1. *The representations U_i intertwine in G iff $\widetilde{U}_1 \cong \widetilde{U}_2$.*

2. *An element $g \in G$ intertwines U_1 iff $g \in ZK$.*

Thm 3.9.3. *Let V be a smooth irreducible representation of G, and suppose that V contains the trivial character of K_1. Then exactly one of the following holds:*

1. *V contains a representation U of K, inflated from an irreducible cuspidal representation $\widetilde{U}$ of $\mathbf{GL}_2(k)$,*

2. *V contains the trivial character of I_1.*

In the first case, V is cuspidal, and there exists a representation W of ZK such that $\mathrm{Res}_K^{ZK} W \cong U$ and $V \cong$ c-$\mathrm{Ind}_{ZK}^G W$.

Proof. V^{K_1} is a K-representation on which K_1 acts trivially and so is naturally a direct sum of $\mathbf{GL}_2(k)$-representations. Let U be one of the irreducible K-summands in flated from $\widetilde{U}$. Then either $\widetilde{U}$ is cuspidal or it is not. If it is not cuspidal then it contains the trivial character of $N(k)$, and so U contains the trivial character on I_1. It follows from the previous lemma and proposition 2.6.2 that both of these cases cannot happen.

Finally, assume $\widetilde{U}$ is cuspidal. U is clearly also a KZ-representation and so we have a non-trivial KZ-homomorphism $U \to V$. By Frobenius reciprocity this gives a G-homomorphism $\text{c-Ind}_{KZ}^{G} U \to V$. But by the previous lemma and theorem 3.9.1, $\text{c-Ind}_{KZ}^{G} U$ is irreducible and so $V \cong \text{c-Ind}_{KZ}^{G} U$. ■

CHAPTER 4

Orbital Integrals and Nilpotent Orbits

4.1 Structure theory

Definition 4.1.1. Given $\psi = \alpha + n$ where $\alpha \in \Phi$, $n \in \mathbb{Z}$, define

$$\mathfrak{g}_\psi = \{x_\alpha(t) : t \in \mathfrak{p}^n\}. \tag{4.1}$$

Define $T_0 = \{(t_1, \ldots, t_n) \in T : t_i \in R^\times\}$ and for $r \in \mathbb{R}$

$$\begin{aligned} T_r &= \{t \in T_0 : \nu(1 - \chi(t)) \geq r, \forall \chi \in X^*(T)\} \\ &= \{t \in T_0 : t_j \in 1 + \bar{w}^{\lceil r \rceil}.R, 1 \leq j \leq n\}. \end{aligned} \tag{4.2}$$

$$\mathfrak{t}_r = \{t : t_j \in \bar{w}^{\lceil r \rceil}.R, 1 \leq j \leq n\}. \tag{4.3}$$

Given $x \in \mathcal{A}(T)$ and $r \in \mathbb{R}$ define

$$\mathfrak{g}_{x,r} = \mathfrak{t}_r \oplus \sum_{\psi \in \Psi : \psi(x) \geq r} \mathfrak{g}_\psi \tag{4.4}$$

$$\mathfrak{g}_{x,r^+} = \mathfrak{t}_r \oplus \sum_{\psi \in \Psi : \psi(x) > r} \mathfrak{g}_\psi \tag{4.5}$$

$$G_{x,r} = \langle T_r, U_\psi \rangle_{\phi \in \Psi : \psi(x) \geq r} \tag{4.6}$$

$$G_{x,r^+} = \langle T_r, U_\psi \rangle_{\phi \in \Psi : \psi(x) > r}. \tag{4.7}$$

Remark 4.1.2. 1. If $r \leq s$ then $\mathfrak{g}_{x,r} \supseteq \mathfrak{g}_{x,s}$, $G_{x,r} \supseteq G_{x,s}$.

2. $\mathfrak{g}_{x,r}.\mathfrak{g}_{y,s} \subseteq \mathfrak{g}_{x,r+s}$.

3. $G_{x,r}$ stabilises $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,r^+}$.

Lemma 4.1.3. *Let $x, y \in \mathcal{B}$, and let $r \in \mathbb{R}$. Then $\mathfrak{g}_{x,r} \subseteq \mathfrak{g}_{y,r} + \mathcal{N}$.*

Definition 4.1.4. Let $r \in \mathbb{R}$. Define $\mathfrak{g}_r = \cup_{x \in \mathcal{B}} \mathfrak{g}_{x,r}$. We have

$$\mathfrak{g}_r = \{X \in \mathfrak{g} : \nu(e_X) \geq r \text{ for all eigenvalues } e_X \text{ of } X\} \tag{4.8}$$

and $\mathcal{N} = \bigcap_r \mathfrak{g}_r$.

Definition 4.1.5. A G-domain of $\mathfrak{g}$ is an invariant clopen subset of $\mathfrak{g}$.

Thm 4.1.6. *Let $r \in \mathbb{R}$. Then*

$$\mathfrak{g}_r = \cap_{x \in \mathcal{B}} (\mathfrak{g}_{x,r} + \mathcal{N}). \tag{4.9}$$

Corollary 4.1.7. *$\mathfrak{g}_r$ is a G-domain.*

4.2 Nilpotent orbits

Definition 4.2.1. Let $Z \in \mathcal{N}$ and $s \in \mathbb{R}$. Define

$$\mathcal{B}(Z,s) = \{z \in \mathcal{B}(G) : Z \in \mathfrak{g}_{z,s}\}. \tag{4.10}$$

This is a nonempty and convex subset of $\mathcal{B}(G)$. In fact it is a union of generalised s-facets (what are these?).

Lemma 4.2.2. *$\mathcal{B}(Z,S)$ is closed.*

Definition 4.2.3.

$$\mathcal{B}(Y,H,X) = \mathcal{B}(X,r) \cap \mathcal{B}(Y,-r). \tag{4.11}$$

This set is a nonempty, closed and convex subset of $\mathcal{B}(G)$ and is a union of generalised r-facets.

Thm 4.2.4. *Fix $X \in \mathcal{N} \backslash \{0\}$ and suppose (Y,H,X) is an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$. Let $\lambda \in X_*^k(G)$ be adapted to (Y,H,X) and fix $x \in \mathcal{B}(Y,H,X)$. Under appropriate hypotheses we have that*

$$^{G_{x,0^+}}(X + C_{\mathfrak{g}_{x,r^+}}) = X + \mathfrak{g}_{x,r^+}. \tag{4.12}$$

4.3 Nilpotent elements and Jacobson-Morozov

In this section we outline some results and notation associated to nilpotent elements in semisimple Lie algebras.

Thm 4.3.1. *(Jacobson-Morozov) Let e be a nilpotent element in a semisimple Lie algebra $\mathfrak{g}$. Then e can be included in an $\mathfrak{sl}_2$-triple $\{e,h,f\}$ and all such $\mathfrak{sl}_2$-triples are conjugate by an element of $Z_G(e)$.*

Now, given a nilpotent element e and an $\mathfrak{sl}_2$-triple $\{e,h,f\}$ containing e we define the following objects:

1. Let $\mathfrak{g}_\lambda$ denote the λ weight space of the adjoint action of the $\mathfrak{sl}_2$ subalgebra $\langle e, h, f\rangle$ on $\mathfrak{g}$,
2. $\mathfrak{p}_0 = \bigoplus_{\lambda \geq 0} \mathfrak{g}_\lambda$ and P_0 the parabolic subgroup of G with Lie algebra $\mathfrak{p}_0$,
3. $\mathfrak{n}_\lambda = \bigoplus_{\mu > \lambda} \mathfrak{g}_\mu$,
4. N_0 the unipotent radical of P_0 (with Lie algebra $\mathfrak{n}_0$),
5. M_0 the centraliser of h in G,
6. $V_0 = \mathrm{Ad}(M_0) \cdot X_0$.

Then $P_0 = M_0 \cdot N_0$ is the Levi decomposition of P_0, and V_0 is open in $\mathfrak{g}_2$.
When $\mathfrak{g} = \mathfrak{gl}_n$ we additionally the following objects:

1. Let l be so that $e^l = 0 \neq e^{l-1}$ and set $V_i = \ker e^i$,
2. $\mathscr{P} = \{p \in \mathfrak{g} : p(V_i) \subseteq V_i\}$, P the subgroup of G with lie algebra $\mathscr{P}$,
3. $\mathscr{U} = \{u \in \mathfrak{g} : u(V_i) \subseteq V_{i-1}\}$, U the subgroup of G with lie algebra $\mathscr{U}$,
4. $\mathscr{U}^{(2)} = \{u \in \mathfrak{g} : u(V_i) \subseteq V_{i-2}\}$.

The following results hold.

Proposition 4.3.2. *1.* $\ker ad(e) \subseteq \mathscr{P}$*,*

2. $ad(e)$ is surjective from $\mathscr{P} \to \mathscr{U}$ and from $\mathscr{U} \to \mathscr{U}^{(2)}$,

3. If $ad(e)(x) \in \mathscr{U}$ then $x \in \mathscr{P}$,

4. If $Ad(g)(e) \in \mathscr{U}$ then $g \in P$,

5. $Ad(P)(e)$ is dense in $\mathscr{U}$. The complement of $Ad(P)(e)$ in $\mathscr{U}$ is a proper closed subvariety of $\mathscr{U}$.

4.4 Orbital integrals

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined ovr a (non-dicrete) locally compact field k of characterstic zero, and let G be its group of k-rational points. Let $\mathcal{O}(x)$ denote the conjugacy class of $x \in G$ and dy^* denote the invariant measure on G/G_x. We wish to show that for $f \in C_c^\infty(\mathfrak{g})$ and x nilpotent

$$\int_{G/G_x} f(yxy^{-1}) dy^* \tag{4.13}$$

converges and hence that $f \mapsto \int_{G/G_x} f(yxy^{-1}) dy^*$ defines a distribution on $\mathfrak{g}$.

Lemma 4.4.1. *The N_0-orbit of X_0 is $X_0 + \mathfrak{n}_2$ and $Ad(P_0) \cdot (X_0)$, the P_0-orbit of X_0, is $V_0 + \mathfrak{n}_2$.*

Proof. ($\subseteq$) Since $[X_0, \mathfrak{n}_0] \subseteq \mathfrak{n}_2$ it follows that $Ad(N_0) \cdot X_0 \subseteq X_0 + \mathfrak{n}_2$. ($\supseteq$) First note that $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ and that $\cdots \supseteq \mathfrak{n}_{-1} \supseteq \mathfrak{n}_0 \supseteq \mathfrak{n}_1 \supseteq \cdots$. Now let $Y \in \mathfrak{n}_2$. We wish to show that $X_0 + Y \in \mathrm{Ad}(N_0) \cdot X_0$. To do this we construct a sequence $Z_s \in \mathfrak{n}_2, s \geq 2$ such that $Y - Z_s \in \mathfrak{n}_s$ and $X_0 + Z_s \in \mathrm{Ad}(N_0) \cdot X_0$. The existence of such a sequence is sufficient since $\mathfrak{g}$ is finite dimensional and so there must exist an n sufficiently large so that $\mathfrak{g}_n = 0$ and hence that $Z_n = Y$. By construction of the sequence we then get that $X_0 + Y = X_0 + Z_n \in \mathrm{Ad}(N_0) \cdot X_0$ as required.

We now construct such a sequence. Let $Z_2 = 0$ and suppose Z_s has been defined. Let $Y' \in \mathfrak{g}_{s+1}$ be such that $Y - Z_s \in Y' + \mathfrak{n}_{s+1}$. Since $s \geq 2$, it follows from basic $\mathfrak{sl}_2$ representation theory that $\mathrm{ad}(X_0)$ surjects $\mathfrak{g}_{s-1}$ onto $\mathfrak{g}_{s+1}$. Thus there exists a $Z \in \mathfrak{g}_{s-1}$ such that $\mathrm{ad}(X_0)(Z) = -Y'$. It then follows that

$$\mathrm{Ad}(\exp(Z)) \cdot (X_0 + Z_s) = X_0 + Z_s + \mathrm{ad}(Z)(X_0) + W = X_0 + Z_s + Y' + W \quad (4.14)$$

where $W \in \mathfrak{n}_{s+1}$. Let $Z_{s+1} = Z_s + Y' + W$. Then

$$Z_{s+1} \equiv Z_s + Y' \equiv Y \pmod{\mathfrak{n}_{s+1}}. \quad (4.15)$$

It follows that Z_{s+1} has the required properties and so we have the required sequence.

The last statement follows immediately. ■

lem:phi

Lemma 4.4.2. *Let $B(,)$ be a non-degenerate G-invariant bilinear form. Let $Z_1, Z_2, \ldots, Z_r$ and $Z'_1, Z'_2, \ldots, Z'_r$ be bases for $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ respectively such that $B(Z_i, Z'_j) = \delta_{ij}$. For $X \in \mathfrak{g}_2$, let $[X, Z'_i] = \sum_j c_{ji}(X) Z_j$ and $\phi(X) = |\det(c_{ij}(X))|^{1/2}$. then $\phi(X_0) > 0$ and*

$$\phi(Ad(m) \cdot (X)) = |\det(Ad(m)|_{\mathfrak{g}_1})| \phi(X) \quad (4.16)$$

for all $X \in \mathfrak{g}_2$ and $m \in M_0$.

Proof. By elementary $\mathfrak{sl}_2$ representation theory the map $\mathrm{ad}(X_0) : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is surjective. Since they have equal dimension it follows that $\phi(X_0) > 0$.

For $m \in M_0$ let

$$\mathrm{Ad}(m) \cdot Z_i = \sum_j a_{ji}(m) Z_j \quad (4.17)$$

$$\mathrm{Ad}(m) \cdot Z'_i = \sum_j b_{ji}(m) Z'_j, \quad (4.18)$$

and $A(m) = (a_{ij}(m)), B(m) = (b_{ij}(m)), C(X) = (c_{ij}(X))$. Since B is G invariant we get that

$$a_{ji}(m) = B(\mathrm{Ad}(m) \cdot Z_i, Z'_j) = B(Z_i, \mathrm{Ad}(m^{-1}) \cdot Z'_j) = b_{ij}(m^{-1}) \quad (4.19)$$

and so $A(m) = B(m^{-1})^t$. Moreover $\mathrm{ad}(\mathrm{Ad}(m) \cdot X) = \mathrm{Ad}(m) \cdot \mathrm{ad}(x) \cdot \mathrm{Ad}(m^{-1})$ and so

$$C(\mathrm{Ad}(m) \cdot X) = A(m) \cdot C(X) \cdot A(m)^t. \quad (4.20)$$

Since $A(m)$ represents the map $\mathrm{Ad}(m)|_{\mathfrak{g}_1}$ with respect to the basis $Z_1, \ldots, Z_r$, the result follows. ■

Lemma 4.4.3. *Let*

$$\Lambda(f) = \int_{V_0+\mathfrak{n}_2} \phi(X) f(X+Z) dX dZ. \tag{4.21}$$

Then there is a constant c such that for all $f \in C_c^\infty(V_0 + \mathfrak{n}_2)$,

$$\Lambda(f) = c \int_{P_0/G_0} f(Ad(p) \cdot X_0) dp^*. \tag{4.22}$$

Proof. Let $q \in P_0$, $q = mn$ with $m \in M_0, n \in N_0$ and define $f^q(Y) = f(\mathrm{Ad}(q)^{-1} \cdot Y)$,

$$\delta_1(q) = |\det(\mathrm{Ad}(m)|_{\mathfrak{n}_2})| = |\det(\mathrm{Ad}(q)|_{\mathfrak{n}_2})|, \tag{4.23}$$
$$\delta_2(q) = |\det(\mathrm{Ad}(m)|_{\mathfrak{g}_2})|, \tag{4.24}$$
$$\delta_3(q) = |\det(\mathrm{Ad}(m)|_{\mathfrak{g}_1})|. \tag{4.25}$$

Then

$$\begin{aligned}\Lambda(f^q) &= \int_{V_0} \phi(X) dX \int_{\mathfrak{n}_2} f^q(X+Z) dZ \\ &= \delta_1(q) \int_{V_0} \phi(X) dX \int_{\mathfrak{n}_2} f(\mathrm{Ad}(q)^{-1} \cdot X + Z) dZ. \end{aligned} \tag{4.26}$$

But $\mathrm{Ad}(q)^{-1} \cdot X = \mathrm{Ad}(m)^{-1} \cdot X + Z'$ for some $Z' \in \mathfrak{n}_2$ and so

$$\begin{aligned}\Lambda(f^q) &= \delta_1(q) \int_{V_0} \phi(X) dX \int_{\mathfrak{n}_2} f(\mathrm{Ad}(m)^{-1} \cdot X + Z) dZ \\ &= \delta_1(q) \int_{\mathfrak{n}_2} dZ \int_{V_0} f(\mathrm{Ad}(m)^{-1} \cdot X + Z) \phi(X) dX \\ &= \delta_1(q)\delta_2(m) \int_{\mathfrak{n}_2} dZ \int_{V_0} f(X+Z) \phi(\mathrm{Ad}(m) \cdot X) dX \\ &= \delta_1(q)\delta_2(m)\delta_3(m)\Lambda(f) \end{aligned} \tag{4.27}$$

where we use the fact that $\phi(\mathrm{Ad}(m) \cdot X) = \delta_3(m)\phi(X)$ for the last equality. But $\delta_1(q)\delta_2(q)\delta_3(q) = \Delta_{P_0}(q)^{-1}$ and so Λ is a relatively invariant measure with the same multiplier as

$$f \mapsto \int_{P_0/G_0} f(\mathrm{Ad}(p) \cdot X_0) dp^*. \tag{4.28}$$

It follows that they are proportional. ■

lem:prod **Lemma 4.4.4.** *There exists a constant c such that for all $f \in C_c^\infty(G/G_0)$,*

$$\int_{G/G_0} f(x^*) dx^* = c \int_{K \times P_0/G_0} f(up^*) du dp^*. \tag{4.29}$$

Thm 4.4.5. *There exists a constant c such that for all $f \in C_c^\infty(\mathfrak{g})$*

$$\int_{G/G_0} f(Ad(x)\cdot X_0)dx^* = c\int_{V_0+\mathfrak{n}_2} \phi(X)\bar{f}(X+Z)dXdZ \tag{4.30}$$

where ϕ is as defined in lemma 4.4.2, and dX and dZ are Haar measures on the vector spaces $\mathfrak{g}_2$ and $\mathfrak{n}_2$ respectively and

$$\bar{f}(Y) = \int_K f(Ad(u)\cdot Y)du. \tag{4.31}$$

In particular the nilpotent orbital integral converges and hence defines a distribution on $\mathfrak{g}$.

Proof. By combining the previous two lemmas we get that for all $f \in C_c^\infty(\mathfrak{g})$

$$\begin{aligned}\int_{G/G_0} f(\mathrm{Ad}(x)\cdot X_0)dx^* &= \int_{K\times P_0/G_0} f(\mathrm{Ad}(up^*)\cdot X_0)dudp^* \\ &= \int_{P_0/G_0} \bar{f}(p^*)dp^* \\ &= c'\int_{V_0+\mathfrak{n}_2} \phi(X)\bar{f}(X+Z)dXdZ. \end{aligned}\tag{4.32}$$

■

4.4.1 GL(n)

Lemma 4.4.6. *The map $\beta : C_c^\infty(G/P,\delta_P) \to \mathbb{C}$ given by $f \mapsto \int_K fdk$ is a G-homomorphism.*

Proof. Define the map $\alpha : C_c^\infty(G) \to C_c^\infty(G/P,\delta_P)$ by

$$\alpha(f)(g) = \int_P f(gp)d\mu_P^R(p). \tag{4.33}$$

Then $\alpha(f)(gp_0) = \delta_P(p_0)\alpha(f)(g)$ and so $\alpha(f) \in C_c^\infty(G/P,\delta_P)$ and α is clearly a G-homomorphism. By a similar argument as in lemma 2.4.24, α is surjective. But $\beta(\alpha(f)) = \int_G fd\mu_G$ is a G-homomorphism. It follows that β must also be a G-homomorphism. ■

Remark 4.4.7. If P is a parabolic subgroup of G then there is a compact open subgroup K of G such that $G = K.P$ and so $C_c^\infty(G/P,\delta_P) = C^\infty(G/P,\delta_P)$ since $G/P \cong K/K\cap P$ is compact.

Lemma 4.4.8. *Let P_0 be the stabiliser of X_0 under the adjoint action of P on $\mathscr{U}$ and $U = Ad(P)X_0$. Then the orbit map $P/P_0 \to U$ which sends pP_0 to $Ad(p)X_0$ is a P-equivariant homeomorphism.*

Proof. The map is certainly a P-equivariant continuous bijection. Moreover, since $F^{\times}I \subseteq P_0$ it follows that $P = (P \cap K).P_0$ and thus that P/P_0 is compact. Since U is Hausdorff it follows that the map is a homeomorphism. ■

cor:dim1 **Corollary 4.4.9.** *Let χ be a character of P and U be as in the lemma. Then*

$$\dim \mathrm{Hom}_H((C_c^\infty(U), \lambda), (\mathbb{C}, \chi)) \leq 1 \tag{4.34}$$

with equality iff $\chi = \Delta_P^{-1}$.

Proof. By the lemma U is a homogeneous space. The result follows immediately. ■

lem:int **Lemma 4.4.10.** *There exists a constant c such that for all $f \in C_c^\infty(\mathscr{U})$,*

$$\int_{P/P_0} f(Ad(p)X_0)d\mu_{P/P_0}(p) = c\int_{\mathscr{U}} f(u)du. \tag{4.35}$$

Proof. Let $U = \mathrm{Ad}(P)X_0$. Then since $\Delta_P^{-1}(p) = |\det(\mathrm{Ad}(p)|_{\mathscr{U}})|$, the functionals

$$f \mapsto \int_{P/P_0} f(\mathrm{Ad}(p)X_0)d\mu_{P/P_0}(p) \tag{4.36}$$

$$f \mapsto \int_U f(u)d\mu_{\mathscr{U}}(u) \tag{4.37}$$

are both relatively invariant measures on $C_c^\infty(U)$ with multiplier Δ_P^{-1}. By corollary 4.4.9, they must be proportional. But U is an open subvariety of $\mathscr{U}$ of codim ≥ 1 and so for all $f \in C_c^\infty(\mathscr{U})$

$$\int_U f(u)d\mu_{\mathscr{U}}(u) = \int_{\mathscr{U}} f(u)d\mu_{\mathscr{U}}(u). \tag{4.38}$$

■

lem:g_inv **Lemma 4.4.11.** *Let (σ, V) be a smooth representation of G, P a parabolic subgroup and K a compact open subgroup of G so that $G = P.K$. If $\theta \in \check{V}$ satisfies $p.\theta = \delta_P(p)\theta$ then $\check{\sigma}(\chi_K) * \theta$ is G-invariant.*

Proof. Let $\phi = \sigma(\chi_K) * \theta$. We wish to show that $\phi(g.v) = \phi(v)$ for all $v \in V, g \in G$. But the map $f_v : g \mapsto (v, g.\theta) \in C_c^\infty(G/P, \delta_P)$ by assumption and so $\phi(v) = \beta(f_v))$. Moreover, the map $V \to C_c^\infty(G/P, \delta_P) : v \mapsto f_v$ is clearly a G-homomorphism. It follows that the composition $v \mapsto f_v \mapsto \beta(f_v) = \phi(v)$ is a G-homomorphism as required. ■

Thm 4.4.12. *Let $\boldsymbol{G} = \mathbf{GL}_n$. Then there exists a constant c such that for all $f \in C_c^\infty(\mathfrak{g})$ we have*

$$\int_{G/G_0} f(yX_0y^{-1})dy^* = c\int_{\mathscr{U}} \left(\int_K Ad^*k(f)(u)dk\right) du. \tag{4.39}$$

Proof. The result follows by lemma 4.4.4 and lemma 4.4.10. ■

4.5 Fourier transforms

Definition 4.5.1. G acts on $C_c^\infty(\mathfrak{g})$ in the obvious way. Write $J(\mathfrak{g})$ for the space of invariant distributions on $\mathfrak{g}$. For $T \in J(\mathfrak{g})$, define its fourier transform $\hat{T}$ by

$$\hat{T}(f) = T(\hat{f}) \tag{4.40}$$

for $f \in C_c^\infty(\mathfrak{g})$.

Thm 4.5.2. *Let $T \in J(\mathfrak{g})$. Then there exists a $\hat{T} \in L^1_{loc}(\mathfrak{g})$ such that for all $f \in C_c^\infty(\mathfrak{g})$*

$$\hat{T}(f) = \int_{\mathfrak{g}} \hat{T}(X) f(X) dX. \tag{4.41}$$

Definition 4.5.3. Let K be a compact open subgroup of G. For $X \in \mathfrak{g}^{rss}$ define the function

$$\eta_X : \mathfrak{g} \to \mathbb{C}, \quad Y \mapsto \int_K \Lambda(\langle Y, {}^k X\rangle) dk. \tag{4.42}$$

Thm 4.5.4. *Let K be any compact open subgroup of G and let dk be the normalised Haar measure on K. For all $X \in \mathfrak{g}^{rss}$ we have*

$$\hat{\mathcal{O}}_\mu(X) = \mathcal{O}_\mu(\eta_X). \tag{4.43}$$

Thm 4.5.5. *Let $r \in \mathbb{R}$. If $T \in J(\mathfrak{g}_r)$, then $\hat{T}$ is represented on $\mathfrak{g}^{rss}$ by $X \mapsto T(\eta_X)$.*

Proof. The hard part of the proof is showing that the map from $\mathfrak{g}^{rss}$ to $C^\infty(\mathfrak{g})$, $X \mapsto \eta_{X,r} := \eta_X 1_{\mathfrak{g}_r}$ is locally constant. Assuming this result we get that there exists a finite collection of open compact disjoint subsets $\{w_i\}_i$ of $\mathfrak{g}^{rss}$ such that $X \mapsto \eta_X$ and f are constant on w_i

$$\begin{aligned}
\int_{\mathfrak{g}} f(X) T(\eta_X) dX &= \sum_i |w_i| f(X_i) T(\eta_{X_i}) \\
&= T\left(\sum_i |w_i| f(X_i) \eta_{X_i}\right) \\
&= T\left(Y \mapsto \int_{\mathfrak{g}} f(X) \eta_X(Y) dX\right) \\
&= T\left(Y \mapsto \int_{\mathfrak{g}} f(X) \int_K \Lambda(\langle Y, {}^k X\rangle)) dk dX\right) \\
&= T\left(Y \mapsto \int_K \hat{f}({}^k Y) dk\right) \\
&= T(\hat{f}).
\end{aligned} \tag{4.44}$$

∎

Definition 4.5.6. Define the subspace D_r of $C_c^\infty(\mathfrak{g})$ by

$$D_r := \sum_{x \in \mathcal{B}} C_c(\mathfrak{g}/\mathfrak{g}_{x,r}) \tag{4.45}$$

and define D_{r^+} similarly.

Proposition 4.5.7. *The fourier transform gives bijective maps*

$$\begin{aligned} D_{r^+} &\leftrightarrow C_c^\infty(\mathfrak{g}_{-r}) \\ D_r &\leftrightarrow C_c^\infty(\mathfrak{g}_{(-r)^+}). \end{aligned}$$

4.6 Character distribution

Definition 4.6.1. Let $f \in C_c^\infty(G)$ and (π, V) be an admissable representation of G. Note that the map $\pi(f) : V \to V$ can be thought of as a map $V^k \to V^K$ where K is a compact open subgroup of G that fixes f by both left and right translation. Define $\mathrm{tr}(\pi(f))$ to be the trace of this map. The character distribution of π is then the map $\Theta_\pi : C_c^\infty(G) \to \mathbb{C}$ given by $f \mapsto \mathrm{tr}(\pi(f))$.

Lemma 4.6.2. *Let π_1, π_2 be two unitary admissable representations of G with $\Theta_{\pi_1} = \Theta_{\pi_2}$. Then they are equivalent.*

Thm 4.6.3. *Let P be a parabolic subgroup of G, $P = MN$ its Levi decomposition, K a maximal compact open subgroup of G such that $G = PK$, ρ an admissable representation of M and θ_ρ a locally integrable function on M representing Θ_ρ. Then $\pi :=$ s-$\mathrm{Ind}_P^G \rho$ is admissable and*

$$\Theta_\pi(f) = \int_K \int_M \int_N f(kmnk^{-1}) \delta_P(m)^{-1/2} \theta_\rho(m) dm dn dk. \tag{4.46}$$

Lemma 4.6.4. *Let $K' \subseteq K$ be compact open subgroups of G and let (σ, V) be an irreducible smooth representation of K. Then*

$$\mathrm{tr} \int_{K'} \sigma(k) dk \neq 0 \tag{4.47}$$

iff σ is trivial on K'.

Proof. ($\Leftarrow$) Trivial. ($\Rightarrow$) Let $L = \ker \sigma$. This is a compact open subgroup of K since V must be finite dimensional. Moreover, $K'/L \cap K'$ embedds naturally as a subgroup of K/L and

$$\mathrm{tr} \int_{K'} \sigma(k) dk = \sum_{x \in K'/L \cap K'} \mathrm{tr}\, \sigma(x) \mu_G(L \cap K'). \tag{4.48}$$

Since this is nonzero, σ is a component of $\mathrm{Ind}_{K'/L \cap K'}^{K/L} 1$. ∎

Definition 4.6.5. Let (π, V) be an admissable representation of G. We define the depth, $\rho(\pi)$ of (π, V) to be

$$\rho(\pi) = \min(r \in \mathbb{Q}_{\geq} : \exists x \in \mathcal{O}, V^{G_{x,r^+}} \neq 0). \tag{4.49}$$

Proposition 4.6.6. *If $(x, r) \in \mathcal{A}(T) \times \mathbb{R}_{\geq 0}$ with $V^{G_{x,r^+}} \neq 0$ then $r \geq \rho(\pi)$.*

4.6.1 Elementrary Kirillov theory

Proposition 4.6.7. *Let $r \in \mathbb{R}_{\geq}$. Then*

$$\widehat{G_{x,r^+}/G_{x,(2r)^+}} \cong \mathfrak{g}_{x,-2r}/\mathfrak{g}_{x,-r}. \tag{4.50}$$

Proof. First note that $G_{x,r^+}/G_{x,(2r)^+} \cong \mathfrak{g}_{x,r^+}/\mathfrak{g}_{x,(2r)^+}$. Now let $\bar{X} \in \mathfrak{g}_{x,-2r}/\mathfrak{g}_{x,-r}$. Define $\psi_{\bar{X}} : \mathfrak{g}_{x,r^+}/\mathfrak{g}_{x,(2r)^+} \to \mathbb{C}$ by $\bar{Y} \mapsto \Lambda(\mathrm{tr}(X.Y))$. It is easy to see that this is well-defined.

This map is a bijection (why?). ∎

Proposition 4.6.8. *Let $r, s \in \mathbb{R}_{\geq}$ and $x, y \in \mathcal{B}$. Fix a character $\bar{\sigma}$ of $G_{x,r^+}/G_{x,(2r)^+}$ and a character $\bar{\tau}$ of $G_{x,s^+}/G_{x,(2s)^+}$. Let X_σ and X_τ be representatives of $\bar{\sigma}$ and $\bar{\tau}$ respectively. Then there exists an $g \in G$ such that*

$${}^{g}(X_\tau + \mathfrak{g}_{y,-s}) \cap (X_\sigma + \mathfrak{g}_{x,-r}) \neq \emptyset. \tag{4.51}$$

Proof. Let $V_\sigma \subseteq V$ be a one-dimensional subspace of V on which G_{x,r^+} acts by σ, and similarly define V_τ. Since V is irreducible there exists a $g \in G$ such that the image of gV_σ under the projection $V \to V_\tau$ is nonzero. Rest is blah. ∎

Proposition 4.6.9. *Let $s > \rho(\pi)$. If $\hat{\Theta}_\pi(f) \neq 0$ then supp$(f) \cap \mathfrak{g}_{(-s)^+} \neq 0$.*

Proof. Note there is a $x \in \mathcal{A}$ such that the trivial representation of $G_{x,\rho(\pi)}$ occurs in π. Now let $x \in mathcalB$, and $\bar{\tau} \in \widehat{G_{y,s}/G_{y,s^+}}$ such that τ occurs in π. Let $X_\tau + \mathfrak{g}_{y,(-s)^+}$ be the coset corresponding to $\bar{\tau}$. By the previous proposition there exists an $g \in G$ such that ${}^{g}\mathfrak{g}_{x-\rho(\pi)} \cap (X_\tau + \mathfrak{g}_{y,(-s)^+}) \neq 0$. But

$$\mathfrak{g}_{x,-\rho(\pi)} \subseteq \mathfrak{g}_{y,-\rho(\pi)} + \mathcal{N} \subseteq \mathfrak{g}_{y,(-s)^+} + \mathcal{N}. \tag{4.52}$$

Thus wlog X_τ is nilpotent.

Now for $h \in C_c^\infty(\mathfrak{g}_{\rho(\pi)^+})$ define $\widetilde{h} \in C_c^\infty(G_{\rho(\pi)^+})$ by $\widetilde{h}(g) = h(g-1)$. Then define the distribution $\Theta_{\pi,\mathfrak{g}}$ on $\mathfrak{g}$ by $\Theta_{\pi,\mathfrak{g}}(f) = \Theta_\pi(\widetilde{f}_{\rho(\pi)})$ where $f_{\rho(\pi)} = f \cdot 1_{\mathfrak{g}_{\rho(\pi)^+}}$. ∎

4.7 Homogeneity results

Thm 4.7.1. *Let $r = k/n$ for $0 \leq k < n$. Then*

$$\mathrm{res}_{D_{r^+}} J(\mathfrak{g}_{r^+}) = \mathrm{res}_{D_{r^+}} J(\mathcal{N}). \tag{4.53}$$

Proof. ($\mathbf{GL}_1(k)$) If $r = 0$, then $D_{r^+} = C_c(k/\mathfrak{p})$. Thus we need to show

$$\operatorname{res}_{C_c(k/\mathfrak{p})} J(\mathfrak{p}) = \operatorname{res}_{C_c(k/\mathfrak{p})} J(\mathcal{N}). \tag{4.54}$$

The right hand side is 1-dimensional and spanned by the distribution $f \mapsto f(0)$. Moreover, the rhs is a subspace of the lhs. Since $\mathbf{GL}_1(k)$ is abelian all distributions are invariant. Now, if $f \in C_c(k/\mathfrak{p})$ and $T \in J(\mathfrak{P})$ then writing

$$f = \sum_{\bar{X} \in k/\mathfrak{p}} c_{\bar{X}} 1_{X+\mathfrak{p}} \tag{4.55}$$

we obtain that $T(f) = T(1_{\mathfrak{p}})f(0)$. Thus the lhs is also one dimensional. This concludes the proof. ∎

Proof. ($\mathbf{GL}_2(k)$) Let $r = 0$ and fix $T \in J(\mathfrak{g}_{0^+})$. We will show that T is determined by its restriction to $C(\mathfrak{k}_0/\mathfrak{k}_1) + C(\mathfrak{b}_0/\mathfrak{b}_{1/2})$.

Fix $f \in D_{0^+}$. WLOG $f = 1_{X+\mathfrak{g}_{x,0^+}}$ where $x \in \mathcal{B}$ and $X \in \mathfrak{g}$. If $(X + \mathfrak{g}_{x,0^+}) \cap \mathfrak{g}_{0^+} = \emptyset$ then $T(f) = 0$. Thus suppose the intersection is $\neq \emptyset$. Since $\mathfrak{g}_{0^+} \subseteq \mathfrak{g}_{x,0^+} + \mathcal{N}$ we have $(X + \mathfrak{g}_{x,0^+}) \cap \mathcal{N} \neq \emptyset$ and so wlog $X \in \mathcal{N}$.

Up to conjugacy $\mathfrak{g}_{x,0^+} = \mathfrak{k}_1$ or $\mathfrak{b}_{1/2}$.

1. If $\mathfrak{g}_{x,0^+} = \mathfrak{k}_1$ then we must have $x = x_0$. We have that $X \in \mathcal{N} \cap (\mathfrak{g}_{x_0,-m} \backslash \mathfrak{g}_{x_0,(-m)^+}) = \mathcal{N} \cap (\mathfrak{k}_{-m} \backslash \mathfrak{k}_{1-m})$ for some $m > 0$ (if $m = 0$ then we are done). Since we are free to conjugate by K_0, we may assume that

$$X = \begin{pmatrix} 0 & \varpi^{-m}u \\ 0 & 0 \end{pmatrix} \tag{4.56}$$

with $u \in R^{\times}$. But

$$X \in \mathfrak{g}_{y_0,(1/2-m)} = \varpi^{-m} \begin{pmatrix} \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} \tag{4.57}$$

where y_0 is as in the notes and

$$\begin{aligned} T(1_{X+\mathfrak{k}_1}) &= \frac{1}{q^2} \sum_{\bar{t} \in T_m/T_{m^+}} T(1_{{}^tX+\mathfrak{k}_1}) \\ &= \frac{1}{q^2} \sum_{\bar{\alpha},\bar{\beta} \in R/\mathfrak{p}} T\left(\left[X + \begin{pmatrix} 0 & u(\alpha-\beta) \\ 0 & 0 \end{pmatrix} + \mathfrak{k}_1\right]\right) \\ &= \frac{1}{q} T(1_{X+\mathfrak{b}_{1/2}}). \end{aligned} \tag{4.58}$$

Thus we have succeeded in writing $T(f)$ in terms of $T(f')$ where $f' \in D_{0^+}$ which is supported closer to the origin with respect to some other point in the building.

2. Now suppose $\mathfrak{g}_{x,0^+} = \mathfrak{b}_{1/2}$. Then $x = y$ for any y in the interior of C_0. Free to conjugate by $G_{y,0} = B_0$ we may assume that X is one of

$$\begin{pmatrix} 0 & \bar{w}^{-m}u \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ \bar{w}^{1-m}u & 0 \end{pmatrix} \tag{4.59}$$

where $u \in R^\times$ and $m > 0$ (if $m = 0$ we would be done).

■

4.8 Generalised Gelfand-Garaev representations

Definition 4.8.1. Let N be a nilpotent element in $\mathfrak{g}^F$ with associated $\mathfrak{sl}_2$ triple $N = e, h, f$. Let $\mathcal{O}_U$ be the coadjoint orbit of the element $-f$. By Kirillov theory $\mathcal{O}_U$ determines a $|\mathcal{O}_U|^{1/2}$-dimensional representation η_N of U^F. The induced representation $\Gamma_N = \mathrm{Ind}_{U^F}^{G^F}\eta_N$ is called the generalised Gelfand-Graev representation of G^F associated to N. Write γ_N for the character of Γ_N.

Proposition 4.8.2. *1. γ_N only depends on the $Ad(G^F)$-orbit of N.*

2. The support of γ_N is contained in the closure of $Ad(G)(\exp(N))$.

Proposition 4.8.3. *For $N' \in \mathcal{N}^F$, let $\mathcal{O}(N')$ be the $Ad(G)$-orbit of N' in $\mathfrak{g}$.*

1. If $\hat{\gamma}_N(N') \neq 0$, then N must lie in the closure of $\mathcal{O}(N')$,

2. if $N \in \mathcal{O}(N')$ and $\hat{\gamma}_N(N') \neq 0$, then N' is in the G^F-orbit of N,

3. $\hat{\gamma}_N(N) = q^{r(N)}\#C_G(N)$.

Thm 4.8.4. *Let $e, h, f \in \mathfrak{g}_{x,0^+}$ be an $\mathfrak{sl}_2$ triple. Let $f_{x,\mathcal{O}}$ be the generalised Gelfand-Graev representation of $M_x = G_{x,0}/G_{x,0^+}$ attached to $\bar{e} \in \mathfrak{m}_x = \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}$. Then*

1. $f_{x,\mathcal{O}}$ is supported on the topologically unipotent set.

2. $\hat{\mu}_{\mathcal{O}'}(f_{x,\mathcal{O}}) = 0$ unless $\mathcal{O}$ lies in the closure $\overline{\mathcal{O}'}$.

3. Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two nilpotent orbits in $\mathfrak{g}$ which belong to the same nilpotent orbit in $\mathfrak{g}$

a) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct in $\mathfrak{g}$, then $\hat{\mu}_{\mathcal{O}_1}(f_{x,\mathcal{O}_2}) = 0$.

b) If $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ in $\mathfrak{g}$, then $\hat{\mu}_{\mathcal{O}}(f_{x,\mathcal{O}}) \neq 0$.

4. For any irreducible smooth admissible representation π

$$\Theta_\pi(f_{x,\mathcal{O}}) = \sum_{\sigma \in \hat{M}_x} m_\pi(\sigma)\langle f_{x,\mathcal{O}}, \sigma\rangle. \tag{4.60}$$

4.9 Fourier transforms of nilpotent orbital integrals on generalised Gelfand Graev characters

Lemma 4.9.1. *Let $Y = y + \mathfrak{g}_{x_0,0^+}$ and $Z = z + \mathfrak{g}_{x_0,0^+}$ be G^F conjugate in $\mathfrak{m}_{x_0} = \mathfrak{g}_{x_0,0}/\mathfrak{g}_{x_0,0^+}$. If $\mathcal{O}$ is a conjugacy class in G then*

$$\mu_{\mathcal{O}}([Y]) = \mu_{\mathcal{O}}([Z]). \tag{4.61}$$

Proof. Since Y and Z are G^F conjugate there exists a $g_0 \in G_{x_0,0}$ such that ${}^{g_0}(y + \mathfrak{g}_{x_0,0^+}) = z + \mathfrak{g}_{x_0,0^+}$. Let x be a representative for $\mathcal{O}$. Then

$$\begin{aligned}
\mu_{\mathcal{O}}([Y]) &= \int_{G/C_G(x)} 1_{{}^{g_0}(y+\mathfrak{g}_{x_0,0^+})}({}^g x)dg^* \\
&= \int_{G/C_G(x)} 1_{y+\mathfrak{g}_{x_0,0^+}}({}^{g_0^{-1}g}x)dg^* \\
&= \int_{G/C_G(x)} 1_{y+\mathfrak{g}_{x_0,0^+}}({}^g x)dg^* = \mu_{\mathcal{O}}([Z])
\end{aligned} \tag{4.62}$$

using the left invariance of the measure. ■

cor:f_nil **Corollary 4.9.2.** *Let $x \in \mathcal{B}$, and $e, h, f \in \mathfrak{g}_{x,0}$ be an $\mathfrak{sl}_2$ triple. Let $f_{x,\mathcal{O}}$ be the character of the generalised Gelfand-Graev representation of M_x^F attached to $\bar{e} \in \mathfrak{m}_x^F$. Then for any conjugacy class $\mathcal{O}'$ of G we have*

$$\hat{\mu}_{\mathcal{O}'}(f_{x,\mathcal{O}}) = vol(\mathfrak{g}_{x,0}) \cdot |G^F| \cdot \sum_N \frac{\hat{\gamma}_{\bar{e}}(N)}{C_{G^F}(N)} \mu_{\mathcal{O}'}(\mathcal{O}' \cap N) \tag{4.63}$$

where the sum is over a set of representatives for the nilpotent conjugacy classes of G^F.

Corollary 4.9.3. *The matrix of test functions is upper triangular.*

Proof. Suppose $\mathcal{O} \nsubseteq \bar{\mathcal{O}}'$. Let $e \in \mathcal{O} \cap \mathfrak{g}_{x,0}$ where $x \in \mathcal{B}$ and let $f_{x,\mathcal{O}}$ be the corresponding generalised Gelfand-Graev character. Let $N = X + \mathfrak{g}_{x,0^+} \in \mathfrak{m}_x^F$ be a nilpotent element. Then $\mu_{\mathcal{O}'}(\mathcal{O}' \cap N) \neq 0$ only if $X \in \bar{\mathcal{O}}'$. But $\hat{\gamma}_{\bar{e}}(N) \neq 0$ only if $\bar{e}$ lies in the closure of the orbit of N. In other words, it is 0 unless $e + \mathfrak{g}_{x,0^+} \cap \mathcal{O}(N) \neq \emptyset$.

$$\hat{\gamma}_{\bar{e}}(N)\mu_{\mathcal{O}'}(\mathcal{O}' \cap N). \tag{4.64}$$

We need $\mathcal{O} \subseteq \overline{\mathcal{O}(X)}$ and $\mathcal{O}(X) \subseteq \overline{\mathcal{O}'}$ for this to be nonzero. ■

Remark 4.9.4. $\hat{\mu}_{\mathcal{O}}(f_{x,\mathcal{O}})$ is easy to compute and is equal to

$$\mathrm{vol}(\mathfrak{g}_{x_0,0^+}) \cdot q^{r(\bar{e})} \cdot |G^F| \cdot \mu_{\mathcal{O}}(\mathcal{O} \cap (e + \mathfrak{g}_{x_0,0^+})). \tag{4.65}$$

GL(2) example calculation

Let $G = \mathbf{GL}_2$. Then nilpotent conjugacy classes have representatives

$$X_{\lambda_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_{\lambda_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.66}$$

The centralisers are

$$C_{\mathfrak{g}}(X_{\lambda_1}) = \mathfrak{g}, \quad C_{\mathfrak{g}}(X_{\lambda_2}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\} \tag{4.67}$$

and so

$$r(X_{\lambda_1}) = 0, \quad r(X_{\lambda_2}) = 1. \tag{4.68}$$

Let $Y_{\lambda_2} = X_{\lambda_2}^T$ and $\Sigma_{\lambda_2} = -Y_{\lambda_2} + C_{\mathfrak{g}}(X_{\lambda_2})$. Then the number of elements in $\Sigma_{\lambda_2}\, G^F$ conjugate to X_{λ_2} is 1. Thus we get

	X_{λ_1}	X_{λ_2}
$\hat{\gamma}_{X_{\lambda_1}}$	$\lvert G^F\rvert$	$\lvert G^F\rvert$
$\hat{\gamma}_{X_{\lambda_2}}$	0	$q \cdot \lvert C_{G^F}(X_{\lambda_2})\rvert$.

We now calculate the nilpotent orbital integrals. Note that $\mathcal{O}_{\lambda_i} \cap (X_{\lambda_i} + \mathfrak{g}_{x_0,0^+}) = {}^{G_{x_0,0^+}} X_{\lambda_i}$. Therefore

$$\begin{aligned} \mu_{\mathcal{O}_{\lambda_i}}(\mathcal{O}_{\lambda_i} \cap (X_{\lambda_i} + \mathfrak{g}_{x_0,0^+})) &= \int_{G/C_G(X_{\lambda_i})} 1_{{}^{G_{x_0,0^+}} X_{\lambda_i}}({}^g X_{\lambda_i}) dg^* \\ &= \int_{G/C_G(X_{\lambda_i})} 1_{G_{x_0,0^+} C_G(X_{\lambda_i})}(g) dg^* \\ &= \frac{\mu_G(G_{x_0,0^+})}{\mu_H(H \cap G_{x_0,0^+})} \end{aligned} \tag{4.69}$$

where $H = C_G(X_{\lambda_i})$ and we used proposition 2.4.38. It remains to determine $\mu_{\mathcal{O}_{\lambda_2}}(\mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0^+})$. To do this we first determine $\mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0^+}$. Note that $G_{x_0,0}$ acts on both $\mathcal{O}_{\lambda_2}$ and $\mathfrak{g}_{x_0,0^+}$ by conjugation and so $\mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0^+}$ splits into a disjoint union of $G_{x_0,0}$ orbits.

Lemma 4.9.5. *Let* $t_n = \begin{pmatrix} \bar{\omega}^n & 0 \\ 0 & 1 \end{pmatrix}$. *Then*

$$\mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0} = \coprod_{n \geq 0} {}^{G_{x_0,0} t_n} X_{\lambda_2}. \tag{4.70}$$

Proof. We certainly have ($\supseteq$). For ($\subseteq$) let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $X \in \mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0}$ iff $a, b, c, d \in R$, $X \neq 0$ and $\operatorname{tr} X = \det X = 0$. Thus $d = -a$ and $a^2 + bc = 0$.

1. If $a = 0$ then $b = 0$ or $c = 0$ and the result is obvious.

2. If $b=0$ or $c=0$ then $a=0$ and the result is obvious.

Thus it remains to consider the case when $a,b,c\neq 0$. In this case $b/a=-a/c$ and so either $b/a\in R$ or $c/a\in R$. If $b/a\in R$ then

$$\begin{pmatrix}1 & b/a\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\begin{pmatrix}1 & -b/a\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ c & 0\end{pmatrix}. \tag{4.71}$$

It is clear that this lies in the $G_{x_0,0}$ orbit of ${}^{t_{\nu(c)}}X_{\lambda_2}$. If $c/a\in R$ then

$$\begin{pmatrix}1 & 0\\ -c/a & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\begin{pmatrix}1 & 0\\ c/a & 1\end{pmatrix}=\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}. \tag{4.72}$$

which lies in the $G_{x_0,0}$ orbit of ${}^{t_{\nu(b)}}X_{\lambda_2}$.

It is straightforward to see that the orbits are distinct. This proves the lemma. ■

Corollary 4.9.6.

$$\mathcal{O}_{\lambda_2}\cap\mathfrak{g}_{x_0,0^+}=\coprod_{n\geq 1}{}^{G_{x_0,0}t_n}X_{\lambda_2}. \tag{4.73}$$

Proof. Let $A\in\mathfrak{g}_{x_0,0}=M_2(R)$. Then $A\in\mathcal{O}_{\lambda_2}$ iff $\bar{\omega}A\in\mathcal{O}_{\lambda_2}$. Thus $\mathcal{O}_{\lambda_2}\cap\mathfrak{g}_{x_0,0^+}=\bar{\omega}(\mathcal{O}_{\lambda_2}\cap\mathfrak{g}_{x_0,0})$. But $\bar{\omega}\cdot\ {}^{t_n}X_{\lambda_2}=\ {}^{t_{n+1}}X_{\lambda_2}$ and so the result follows. ■

We are now able to calculate $\mu_{\mathcal{O}_{\lambda_2}}(\mathcal{O}_{\lambda_2}\cap\mathfrak{g}_{x_0,0^+})$.

$$\begin{aligned}\mu_{\mathcal{O}_{\lambda_2}}([\mathcal{O}_{\lambda_2}\cap\mathfrak{g}_{x_0,0^+}]) &= \int_{G/C_G(X_{\lambda_2})} 1_{\mathcal{O}_{\lambda_2}\cap\mathfrak{g}_{x_0,0^+}}({}^{g}X_{\lambda_2})dg^*\\ &= \int_{G/C_G(X_{\lambda_2})}\sum_{n\geq 1} 1_{{}^{G_{x_0,0}t_n}X_{\lambda_2}}({}^{g}X_{\lambda_2})dg^*\\ &= \sum_{n\geq 1}\int_{G/C_G(X_{\lambda_2})} 1_{G_{x_0,0}t_nC_G(X_{\lambda_2})}(g)dg^*\\ &= \sum_{n\geq 1}\frac{\mu_G({}^{t_n^{-1}}G_{x_0,0})}{\mu_H(H\cap{}^{t_n^{-1}}G_{x_0,0})}\end{aligned} \tag{4.74}$$

where $H=C_G(X_{\lambda_2})$. Now, let

$$H_n=H\cap{}^{t_n^{-1}}G_{x_0,0}=\left\{\begin{pmatrix}a & \bar{\omega}^{-n}b\\ 0 & a\end{pmatrix}: a\in R^\times, b\in R\right\} \tag{4.75}$$

$$U_n=\left\{\begin{pmatrix}1 & \bar{\omega}^{-n}R\\ 0 & 1\end{pmatrix}\right\},\quad Z=R^\times I_2. \tag{4.76}$$

Note that these are all abelian groups and that $H_n=ZU_n=H_0U_n$. Thus

$$|H_n:H_0|=|H_0U_n:H_0|=|U_n:U_n\cap H_0|=|\bar{\omega}^{-n}R:R|=q^n \tag{4.77}$$

and so $\mu_H(H_n) = q^n \mu_H(H_0) = q^n \cdot \mu_H(H \cap G_{x_0,0})$. Now, $H \cap G_{x_0,0^+}$ is the kernel of the map reduction mod $\mathfrak{p}$ map $H \cap G_{x_0,0} \to \mathbf{GL}_2(\mathbb{F}_q)$. The image of this map is

$$C_{G^F}(X_{\lambda_2}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{F}_q, a \neq 0 \right\} \tag{4.78}$$

which has size $q(q-1)$. It follows that

$$\mu_H(H \cap G_{x_0,0}) = q(q-1) \cdot \mu_H(H \cap G_{x_0,0^+}). \tag{4.79}$$

We also have that

$$\mu_G({}^{t_n^{-1}}G_{x_0,0}) = \mu_G(G_{x_0,0}) = |G^F| \cdot \mu_G(G_{x_0,0^+}) \tag{4.80}$$

where we use the fact that G is unimodular. Thus

$$\begin{aligned} \mu_{\mathcal{O}_{\lambda_2}}([\mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0^+}]) &= \frac{|G^F|}{|C_G(X_{\lambda_2})|} \cdot \frac{\mu_G(G_{x_0,0^+})}{\mu_H(H \cap G_{x_0,0^+})} \sum_{n \geq 1} \frac{1}{q^n} \\ &= (q+1) \cdot \mu_{\mathcal{O}_{\lambda_2}}([\mathcal{O}_{\lambda_2} \cap (X_{\lambda_2} + \mathfrak{g}_{x_0,0^+})]). \end{aligned} \tag{4.81}$$

We can now finally compute the matrix $(\hat{\mu}_{\mathcal{O}_j}(f_{x,\mathcal{O}_i}))_{i,j}$. If we normalise the nilpotent orbital integrals so that the diagonal of this matrix consists only of 1's then we have that

$$\mu_{\mathcal{O}_{\lambda_1}}([\mathcal{O}_{\lambda_1} \cap \mathfrak{g}_{x_0,0^+}]) = \frac{1}{\mathrm{vol}(\mathfrak{g}_{x_0,0^+}) \cdot |G^F|} \tag{4.82}$$

$$\mu_{\mathcal{O}_{\lambda_2}}([\mathcal{O}_{\lambda_2} \cap (X_{\lambda_2} + \mathfrak{g}_{x_0,0^+}]) = \frac{1}{\mathrm{vol}(\mathfrak{g}_{x_0,0^+}) \cdot q \cdot |G^F|} \tag{4.83}$$

$$\mu_{\mathcal{O}_{\lambda_2}}([\mathcal{O}_{\lambda_2} \cap \mathfrak{g}_{x_0,0^+}]) = \frac{q+1}{\mathrm{vol}(\mathfrak{g}_{x_0,0^+}) \cdot q \cdot |G^F|}. \tag{4.84}$$

Plugging all of this into corollary 4.9.2 we get that

$$\hat{\mu}_{\mathcal{O}_{\lambda_2}}(f_{x_0,\mathcal{O}_{\lambda_1}}) = q+1. \tag{4.85}$$

Thus $(\hat{\mu}_{\mathcal{O}_j}(f_{x,\mathcal{O}_i}))_{i,j}$ is given by

	$\mu_{\mathcal{O}_{\lambda_1}}$	$\mu_{\mathcal{O}_{\lambda_2}}$
$f_{x_0,\mathcal{O}_{\lambda_1}}$	1	q+1
$f_{x_0,\mathcal{O}_{\lambda_2}}$	0	1.

GL(n) - Approach 1

Let λ be a partition of n and let $\mathcal{O}_\lambda + \mathfrak{g}_{x_0,0^+}$ denote the set

$$\bigcup_{N \in \mathcal{O}_\lambda(G^F)} N. \tag{4.86}$$

Note that this set is closed under conjugation by $G_{x_0,0}$. Moreover, we have that for any $N \in \mathcal{O}_\lambda(G^F)$, $\mu_{\mathcal{O}_\eta}(\mathcal{O}_\lambda + \mathfrak{g}_{x_0,0^+}) = |\mathrm{ccl}_{G^F}(N)|\mu_{\mathcal{O}_\eta}(N)$ and so

$$\hat{\mu}_{\mathcal{O}_\eta}(f_{x,\mathcal{O}_\lambda}) = \mathrm{vol}(\mathfrak{g}_{x,0}) \cdot \sum_\nu \hat{\gamma}_{\bar{e}}(X_\nu)\mu_{\mathcal{O}_\eta}(\mathcal{O}_\nu + \mathfrak{g}_{x_0,0^+}). \tag{4.87}$$

Write $\mathcal{N}_{x_0}$ for $\mathfrak{g}_{x_0,0} \cap \mathcal{N}$ and $\mathcal{N}_{x_0,\lambda}$ for $\mathcal{N}_{x_0} \cap \mathcal{O}_\lambda$. Both of these sets are closed under conjugation by $G_{x_0,0}$ and $\mathcal{N}_{x_0} = \bigcup_{\lambda \vdash n} \mathcal{N}_{x_0,\lambda}$. We can thus write each $\mathcal{N}_{x_0,\lambda}$ as a union of $G_{x_0,0}$ conjugacy classes:

$$\mathcal{N}_{x_0,\lambda} = \bigcup_\alpha \mathcal{C}_{x_0,\lambda,\alpha}. \tag{4.88}$$

Let $g_{\lambda,\alpha} \in \mathbf{GL}_n(k)$ be such that ${}^{g_{\lambda,\alpha}}X_\lambda \in \mathcal{C}_{x_0,\lambda,\alpha}$. Since $\mathcal{N}_{x_0} \subseteq \bigcup_{\lambda \vdash n} \mathcal{O}_\lambda + \mathfrak{g}_{x_0,0^+}$ each $\mathcal{C}_{x_0,\lambda,\alpha}$ must belong to a unique $\mathcal{O}_\eta + \mathfrak{g}_{x_0,0^+}$ for some $\eta \vdash n$. Define $\mathcal{C}(\lambda,\eta)$ to be the set $\{\alpha : \mathcal{C}_{x_0,\lambda,\alpha} \subseteq \mathcal{O}_\eta + \mathfrak{g}_{x_0,0^+}\}$. Then

$$\mathcal{O}_\lambda \cap (\mathcal{O}_\eta + \mathfrak{g}_{x_0,0^+}) = \bigcup_{\alpha \in \mathcal{C}(\lambda,\eta)} {}^{G_{x_0,0}g_{\lambda,\alpha}}X_\lambda \tag{4.89}$$

and so

$$\begin{aligned}\mu_{\mathcal{O}_\lambda}(\mathcal{O}_\eta + \mathfrak{g}_{x_0,0^+}) &= \int_{G/C_G(X_\lambda)} \sum_{\alpha \in \mathcal{C}(\lambda,\eta)} 1_{G_{x_0,0}g_{\lambda,\alpha}C_G(X_\lambda)}(g)dg^* \\ &= \sum_{\alpha \in \mathcal{C}(\lambda,\eta)} \frac{\mu_G(G_{x_0,0})}{\mu_H(H \cap G_{x_0,0}^{g_{\lambda,\alpha}})},\end{aligned} \tag{4.90}$$

where $H = C_G(X_\lambda)$. Note that

$$\mu_{\mathcal{O}_\lambda}(\mathcal{O}_\lambda + \mathfrak{g}_{x_0,0^+}) = \frac{\mu_G(G_{x_0,0})}{\mu_H(H \cap G_{x_0,0})} \tag{4.91}$$

so if we normalise the invariant measure so that $\hat{\mu}_{\mathcal{O}_\lambda}(f_{x_0,\mathcal{O}_\lambda}) = 1$ then

$$\frac{\mu_G(G_{x_0,0})}{\mu_H(H \cap G_{x_0,0})} = \Big(\mathrm{vol}(\mathfrak{g}_{x_0,0})q^{r(X_\lambda)}|C_{G^F}(X_\lambda)|\Big)^{-1}. \tag{4.92}$$

Thus

$$\mu_{\mathcal{O}_\lambda}(\mathcal{O}_\eta + \mathfrak{g}_{x_0,0^+}) = \frac{1}{\mathrm{vol}(\mathfrak{g}_{x_0,0})q^{r(X_\lambda)}|C_{G^F}(X_\lambda)|} \sum_{\alpha \in \mathcal{C}(\lambda,\eta)} \frac{\mu_H(H \cap G_{x_0,0})}{\mu_H(H \cap G_{x_0,0}^{g_{\lambda,\alpha}})}. \tag{4.93}$$

Remark 4.9.7. $G_{x_0,0}^{g_{\lambda,\alpha}} = G_{g_{\lambda,\alpha}^{-1}x_0,0}$ and so if we let $\Omega(\lambda,\alpha) = \{x_0, g_{\lambda,\alpha}^{-1}x_0\}$ then

$$\frac{\mu_H(H \cap G_{x_0,0})}{\mu_H(H \cap G_{x_0,0}^{g_{\lambda,\alpha}})} = \frac{|C_{G_{x_0,0}}(X_\lambda) : C_{G_{\Omega(\lambda,\alpha)}}(X_\lambda)|}{|C_{G_{g_{\lambda,\alpha}^{-1}x_0,0}}(X_\lambda) : C_{G_{\Omega(\lambda,\alpha)}}(X_\lambda)|}. \tag{4.94}$$

Results on double cosets

prop:double_cosets **Proposition 4.9.8.** *Let G be a group, $H, K \leq G$, $L \leq H$ subgroups and g an element of G. Then*

1. *$\{\sigma g : \sigma \in K\backslash G/H\}$ is a set of representatives for $K\backslash G/H^g$,*
2. *$\{\sigma\tau : \sigma \in K\backslash G/H, \tau \in K^\sigma \cap H\backslash H/L\}$ is a set of representatives for $K\backslash G/L$.*

prop:par_double_cosets **Proposition 4.9.9.** *Let $n \in \mathbb{N}$ and $A = (a_1, \ldots, a_k), B = (b_1, \ldots, b_l)$ be such that $n = \sum_i a_i = \sum_i b_i$ and $a_i, b_i \geq 0$ for all i. Then*

$$P_A\backslash G/P_B \longleftrightarrow S_A\backslash S_n/S_B \longleftrightarrow T(A, B) \tag{4.95}$$

where

$$T(A, B) = \{(c_{ij})_{1\leq i\leq k, 1\leq j\leq l} : \sum_j c_{ij} = a_i, \sum_i c_{ij} = b_j, c_{ij} \geq 0\}. \tag{4.96}$$

Proof. Let $A_i = \sum_{m=1}^i a_i$ and $B_i = \sum_{m=1}^i b_i$. For $\sigma \in S_n$ define $\Phi(\sigma) = (c_{ij})_{ij}$ where

$$c_{ij} = |\sigma([A_{i-1} + 1, A_i]) \cap [B_{i-1} + 1, B_i]|. \tag{4.97}$$

It is clear that this map descends to a well-defined map on double cosets $S_A\backslash S_n/S_B \to T(A, B)$. It is a straightforward check to see moreover that this map is a bijection. ∎

cor:pu_d_cosets **Corollary 4.9.10.** *Let n, A, B be as in the previous proposition. Let U_B denote the unipotent radical of P_B. Then*

$$P_A\backslash G/U_B \longleftrightarrow \{((c_{ij})_{ij}, f) : (c_{ij})_{ij} \in T(A, B), f \in \prod_j Gr(c_{\bullet j})\} \tag{4.98}$$

where $c_{\bullet j}$ denotes $(c_{1j}, \ldots, c_{lj})$ and $Gr(c_{\bullet j})$ is the grassmannian associated to this sequence.

Proof. By proposition 4.9.8, $\{(\sigma, f) : \sigma \in P_A\backslash G/P_B, f \in P_A^\sigma \cap P_B\backslash P_B/U_B\}$ is a set of representatives for $P_A\backslash G/U_B$. Moreover for a given $\sigma \in P_A\backslash G/P_B$ corresponding to $(c_{ij})_{ij} \in T(A, B)$, $P_A^\sigma \cap P_B$ consists of the block matrices in P_B where the (i, j)th block looks like

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,k} \\ 0 & A_{2,2} & \ldots & A_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{k,k} \end{pmatrix} \tag{4.99}$$

when $j \geq i$, and is 0 otherwise. Here $A_{a,b}$ is a matrix of size $c_{ai} \times c_{bj}$. But we have the semidirect product decomposition $P_B = M_B.U_B$ where $M_B = \prod_i \mathbf{GL}_{b_i}$ and $U_B \lhd P_B$ and so the action of $P_A^\sigma \cap P_B$ on P_B/U_B is isomorphic to the action of $P_A^\sigma \cap P_B$ on M_B where $g = mu \in P_A^\sigma \cap P_B, m \in M_B, u \in U_B$ acts by multiplication by m. It is clear that the orbits of this action are exactly the orbits of the action by $\prod_j P_{c_{\bullet j}}$ on M_B by left multiplaction and hence can be parametrised by an element of $\prod_j Gr(c_{\bullet j})$. ■

Results on parabolic subgroups

Definition 4.9.11. Let $n \in \mathbb{N}$, $\lambda \dashv n$ be a partition and X_λ a nilpotent element with Jordan decomposition corresponding to λ. Let (e, h, f) be an $\mathfrak{sl}_2$-triple with $e = X_\lambda$ and $\mathfrak{g}_k$ be the weight-space of weight k. To this $\mathfrak{sl}_2$ triple we may associate a parabolic subgroup P_0 of G with Lie algebra equal to $\mathfrak{p}_0 = \bigoplus_{K \geq 0} \mathfrak{g}_k$. Let $A = (a_1, \dots, a_k)$ be the type of the standard parabolic which is conjugate to P_0 and define $\Theta(\lambda) = A$.

Proposition 4.9.12. *The function Θ is well defined.*

Thm 4.9.13. *The function Θ can be computed via the following operations on the Young diagram of λ:*

1. *Center all the rows,*

2. *Count the number of boxes along each column.*

Proof. Blah blah blah. ■

Definition 4.9.14. Let P be a parabolic subgroup which stablises the flag $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = V$. Recall that P has unipotent radical $U = \{g \in P : (g - I)V_i \subseteq V_{i-1}\}$. Define $U^{(2)} = \{g \in P : (g - I)V_i \subseteq V_{i-2}\}$.

Proposition 4.9.15. $U^{(2)} \lhd P$.

GL(n) - Approach 2

We have that $\hat{\mu}_{\mathcal{O}_\lambda} = \Theta_{\text{s-Ind}_{P(\lambda)}^G 1}$. Thus

$$\hat{\mu}_{\mathcal{O}_\lambda}(f_{x,\mathcal{O}_\eta}) = \langle \Gamma_{x,\mathcal{O}_\eta}, (\text{s-Ind}_{P(\lambda)}^G 1)^{G_{x,0}} \rangle. \tag{4.100}$$

But $(\text{s-Ind}_{P(\lambda)}^{G} 1)^{G_{x,0+}}$ is isomorphic to $\text{s-Ind}_{P(\lambda)^F}^{G^F} 1$ as $G_{x,0}$-representations. We also have that $\Gamma_{x,\mathcal{O}_\lambda} = \text{Ind}_{K_\eta}^{G^F} \xi_\eta$. Thus

$$\begin{aligned}
\hat{\mu}_{\mathcal{O}_\lambda}(f_{x,\mathcal{O}_\eta}) &= \langle 1, \text{Res}_{P(\lambda)^F}^{G^F} \text{Ind}_{K_\eta}^{G^F} \xi_\eta \rangle \\
&= \langle 1, \bigoplus_{g \in P(\lambda)^F \backslash G^F / K_\eta} \text{Ind}_{P(\lambda) \cap \ {}^g K_\eta}^{P(\lambda)} g \circ \text{Res}_{K_\eta \cap P(\lambda)^g}^{K_\eta} \xi_\eta \rangle \\
&= \sum_{g \in P(\lambda)^F \backslash G^F / K_\eta} \langle 1, g \circ \text{Res}_{K_\eta \cap P(\lambda)^g}^{K_\eta} \xi_\eta \rangle \\
&= \sum_{g \in P(\lambda)^F \backslash G^F / K_\eta} \langle 1, \text{Res}_{K_\eta \cap P(\lambda)^g}^{K_\eta} \xi_\eta \rangle \\
&= \sum_{g \in P(\lambda)^F \backslash G^F / K_\eta} [K_\eta \cap P(\lambda)^g \subseteq \ker \xi_\eta]. \qquad (4.101)
\end{aligned}$$

Remark 4.9.16. This is a 'basis invariant' invariant sum. By this we mean the following: let b be a change of basis matrix and set $K'_\eta = \ {}^b K_\eta$. Then

$$\sum_{g \in P(\lambda)^F \backslash G^F / K_\eta} [K_\eta \cap P(\lambda)^g \subseteq \ker \xi_\eta] = \sum_{g \in P(\lambda)^F \backslash G^F / K'_\eta} [K'_\eta \cap P(\lambda)^g \subseteq \ {}^b \ker \xi_\eta]. \qquad (4.102)$$

It follows that we can work with K_η in any basis we like as long as we view the kernel of ξ_η in the same basis. Since $K_\eta \subseteq U_\eta \subseteq P_\eta$ where P_η is a parabolic subgroup of G^F, we will work in a basis with respect to which P_η is a standard parabolic.

Proposition 4.9.17. *There is a bijection*

$$P_A \backslash G / K_\eta \longleftrightarrow \{(\sigma, f, k) : \sigma \in S_A \backslash S_n / S_B, f \in \prod_j Gr(\Phi(\sigma)_{\bullet j}), k \in P_A^{\sigma f} \cap U_B \backslash\} \qquad (4.103)$$

By proposition 4.9.8 and corollary 4.9.10

$$\{\sigma f k : \sigma \in\} \qquad (4.104)$$

$$\sum_{\substack{\sigma \in S_A \backslash S_n / S_B \\ f \in \prod_j Gr(\Phi(\sigma)_{\bullet j})}} \sum_{k \in P(\lambda)^{\sigma f} \cap U_B \backslash U_B / K} . \qquad (4.105)$$

4.9.1 Wave front sets

lem:nil_wavefront **Lemma 4.9.18.** *Let (π, V) be a smooth representation of G and $x \in \mathcal{B}$. If $\mathcal{O}$ is a nilpotent wavefront then $\Theta_\pi(f_{x,\mathcal{O}}) \neq 0$.*

lem:fin_red **Lemma 4.9.19.** *Let (π, V) be a smooth representation of G, $x \in \mathcal{B}$ and $\mathcal{O}$ a nilpotent orbit of $\mathfrak{g}$. Then $\Theta_\pi(f_{x,\mathcal{O}}) = 0$ iff $\Gamma_{\bar{e}}$ and $V^{G_{x,0+}}$ have no irreducible constituents in common.*

Proposition 4.9.20. *The wave front set of a smooth admissible representation (π, V) of depth zero can be determined from the wave front sets of the finite field representations. In particular they are the maximal nilpotent orbits such that $V^{G_{x,0+}}$ has an irreducible component in common with the corresponding generalised Gelfand-Graev representation.*

Proof. We have that $\Theta_\pi = \sum_{\mathcal{O}'} c_{\mathcal{O}'}\hat{\mu}_{\mathcal{O}'}$. But $\hat{\mu}_{\mathcal{O}'}(f_{x,\mathcal{O}}) \neq 0$ only if $\mathcal{O} \subseteq \overline{\mathcal{O}'}$. Thus $\Theta_\pi(f_{x,\mathcal{O}}) = 0$ if $\mathcal{O}$ is strictly larger than a wavefront nilpotent. Finally, if $\Theta_\pi(f_{x,\mathcal{O}}) \neq 0$ then $\mathcal{O} \subseteq WF(\pi)$ and so $\mathcal{O}$ is less than some wavefront nilpotent. Together with lemma 4.9.18 it follows that the wavefront nilpotents are the maximal nilpotent orbits such that $\Theta_\pi(f_{x,\mathcal{O}}) \neq 0$. But by lemma 4.9.19 it follows that the wavefront nilpotents are the maximal nilpotent orbits such that $V^{G_{x,0+}}$ has an irreducible constituent in common with the corresponding generalised Gelfand-Graev representation. ■