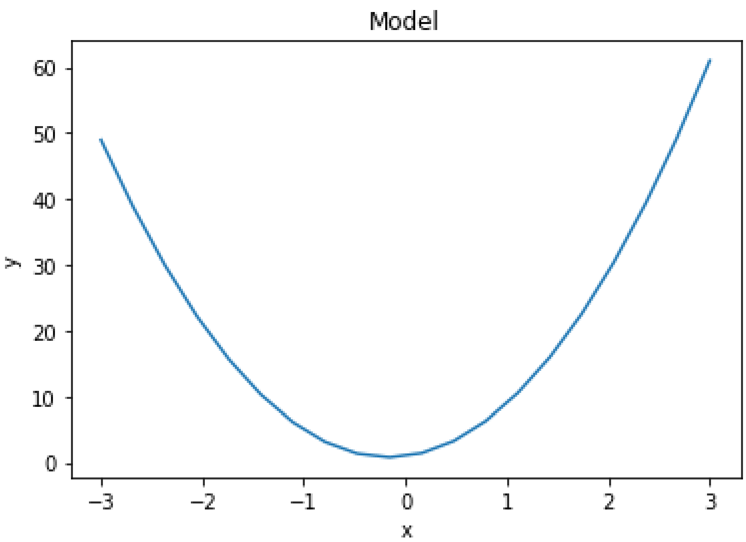
In [1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm, laplace
import math
```

# Assignment 2

## Task 2.1

First I do the same as in assignment 1: defines a arbitrary polynomial model.

In [18]:
```
def arbitrary_poly(params):
    poly_model = lambda x: sum([p*(x**i) for i, p in enumerate(params)])
    return poly_model

# params: [theta_0, theta_1, ... , theta_n], where n = model order and theta_
true_params = [1,2,6]
y_model = arbitrary_poly(true_params)

# Plot true model
x = np.linspace(start=-3, stop=3, num=20)
plt.figure()
plt.plot(x, y_model(x))
plt.xlabel("x")
plt.ylabel("y")
plt.title("Model");
```

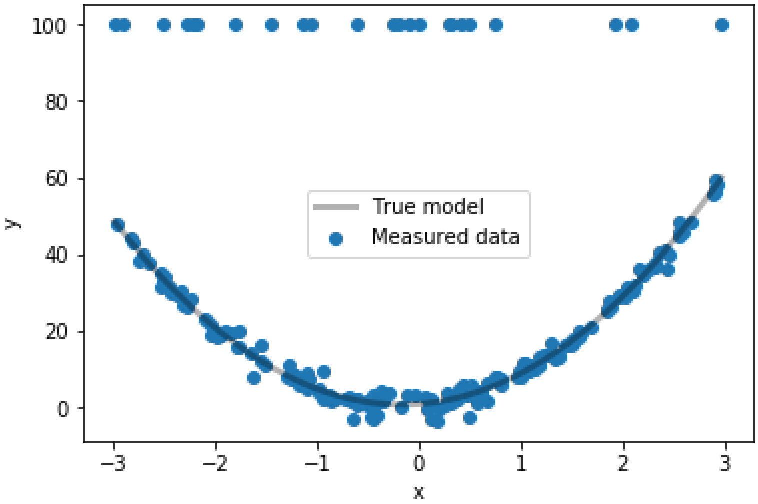



## Then I generate the noise measurements with gaussian distribution:

In [19]:
```
# Hyperparameters for the type of noise-generating distribution.
loc = 0           # location (mean) parameter
scale = 1         # scaling (std dev) parameter
magnitude = 1.2   # noise magnitude
N = 201          # number of samples

np.random.seed(123)  # Non-random generation between code executions. Comment
```

```
# Generate data points
range_low = -3
range_high = 3
u = np.sort(np.random.uniform(range_low,range_high,N))
y_true = y_model(u)


# Generate noisez

pdf = laplace.pdf

normVariance = 1 # Input as the scale parameter in the normal distribution

laplaceVariance = 1

alfa = 0

gamma = 0.1

noiseNorm = magnitude * np.random.normal(loc, normVariance, int(alfa * N))

noiseLaplace = magnitude * np.random.laplace(loc, laplaceVariance, int((1-alf



#NoiseFault = np.concatenate((noiseLaplace, faultyMeasurement))

#np.random.shuffle(NoiseFault)
#print(NoiseFault[:10])

# Add noise to the generated data points - thus simulating measurement
print(len(noiseLaplace))
y = y_true

for i in range(0, N):
    faulty = np.random.binomial(1, gamma)
    if faulty:
        y[i] = 100
    else:
        y[i] = y_true[i] + noiseLaplace[i]

# Plot measured data
plt.scatter(u, y, label=r"Measured data")
u0 = np.linspace(min(u), max(u), N)
plt.plot(u0, y_model(u0), "k", alpha=0.3, lw=3, label="True model")
plt.legend()
plt.xlabel("x")
plt.ylabel("y");
```

```
201
```



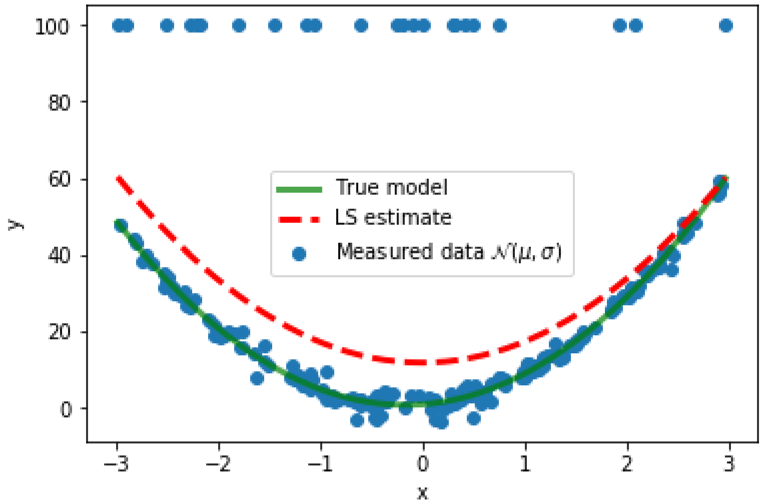

# Task 2.3

## LS:

In [20]:

```
y=y_true
# Matrix form
u_tensor_0 = np.reshape(u,(N,1))
#print(u_tensor_0)
ones_vec = np.ones((N,1))

u_tensor = ones_vec
#print(ones_vec)

#print(u_tensor)

for i in range(1,len(true_params)):
    u_tensor = np.append(u_tensor, np.power(u_tensor_0, i) ,axis=1)
#print(u_tensor1)

#-------------
u_transpose_dot_u = np.dot(u_tensor.T,u_tensor)  # calculating dot product
u_transpose_dot_u_inv = np.linalg.inv(u_transpose_dot_u) #calculating inverse

u_transpose_dot_y = np.dot(u_tensor.T,y)  # calculating dot product


LS_params = np.dot(u_transpose_dot_u_inv,u_transpose_dot_y)
LS_params_rounded = ["{:.2f}".format(round(i, 2)) for i in LS_params.tolist()
#print(f"LS parameters:         {LS_params_rounded}")
#print(f"True model parameters: {true_params}")

diffParams = []
for i in range(0, len(true_params)):
    diffParams.append(float(true_params[i] - float(LS_params_rounded[i])))

#print("The differnence between the estimated theata and the real Theta is: "
#print(diffParams)

# Recreate model based on LS estimate:
LS_params = LS_params.tolist()
LS_estimate = arbitrary_poly(LS_params)

# Plot true vs. estimated model
plt.scatter(u, y, label=r"Measured data $\mathcal{N}(\mu, \sigma)$")
u0 = np.linspace(min(u), max(u), N)
```

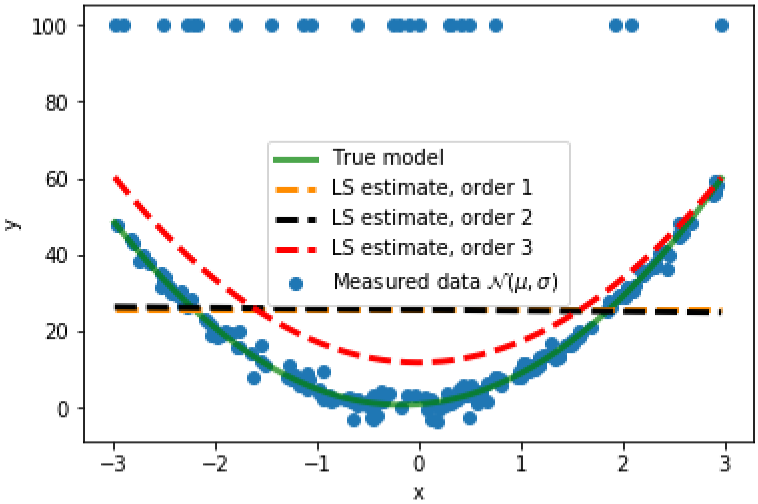
```
plt.plot(u0, y_model(u0), "g", alpha=0.7, lw=3, label="True model")
plt.plot(u0, LS_estimate(u0), "r--",  lw=3, label="LS estimate")
#plt.xlim(0, 10)
plt.legend()
plt.xlabel("x")
plt.ylabel("y");
```




In [21]:

```
y=y_true
# Matrix form
def LSorderFunc(order, u, y, N):
    u_tensor_0 = np.reshape(u,(N,1))
    #print(u_tensor_0)
    ones_vec = np.ones((N,1))

    u_tensor = ones_vec
    #print(ones_vec)

    #print(u_tensor)

    for i in range(1,order):
        u_tensor = np.append(u_tensor, np.power(u_tensor_0, i) ,axis=1)
    #print(u_tensor1)

    #-------------
    u_transpose_dot_u = np.dot(u_tensor.T,u_tensor)  # calculating dot produc
    u_transpose_dot_u_inv = np.linalg.inv(u_transpose_dot_u) #calculating inv

    u_transpose_dot_y = np.dot(u_tensor.T,y)  # calculating dot product


    LS_params = np.dot(u_transpose_dot_u_inv,u_transpose_dot_y)
    LS_params_rounded = ["{:.2f}".format(round(i, 2)) for i in LS_params.toli
    #print(f"LS parameters:         {LS_params_rounded}")
   # print(f"True model parameters: {true_params}")

    diffParams = []
    for i in range(0, order):
        diffParams.append(float(true_params[i] - float(LS_params_rounded[i]))

    #print("The diffrence between the estimated theata and the real Theta i
    #print(diffParams)

    # Recreate model based on LS estimate:
    LS_params = LS_params.tolist()
    LS_estimate = arbitrary_poly(LS_params)
    return LS_params, LS_estimate
```

```
LS_params_1, LS_estimate_1 = LSorderFunc(1, u, y, N)
LS_params_2, LS_estimate_2 = LSorderFunc(2, u, y, N)
LS_params_3, LS_estimate_3 = LSorderFunc(3, u, y, N)
#print(LS_params_1, LS_params_2, LS_params_3)
# Plot true vs. estimated model

plt.scatter(u, y,  label=r"Measured data $\mathcal{N}(\mu, \sigma)$")
u0 = np.linspace(min(u), max(u), N)
plt.plot(u0, y_model(u0), "g", alpha=0.7, lw=3, label="True model")
plt.plot(u0, LS_estimate_1(u0), color="darkorange" ,linestyle="--",  lw=3, la
plt.plot(u0, LS_estimate_2(u0), color="black" ,linestyle="--",  lw=3, label="

plt.plot(u0, LS_estimate_3(u0), "r--",  lw=3, label="LS estimate, order 3")
#plt.xlim(0, 10)
plt.legend()
plt.xlabel("x")
plt.ylabel("y");
```




# ML:

## Function for calculating the log likelihood function:

In [22]:

```
y=y_true
def log_lik(par_vec, y, x):
        # Use the distribution class chosen earlier
    # If the standard deviation parameter is negative, return a large value:
    if par_vec[-1] < 0:
        return(1e8)
    # The likelihood function values:
    lik = pdf(y,
              loc = sum([p*(x**i) for i, p in enumerate(par_vec[:-1])]),
              scale = par_vec[-1])

    #This is similar to calculating the likelihood for Y - XB
    # res = y - par_vec[0] - par_vec[1] * x
    # lik = norm.pdf(res, loc = 0, sd = par_vec[2])

    # If all logarithms are zero, return a large value
    if all(v == 0 for v in lik):
        return(1e8)
    # Logarithm of zero = -Inf
    return(-sum(np.log(lik[np.nonzero(lik)])))
```

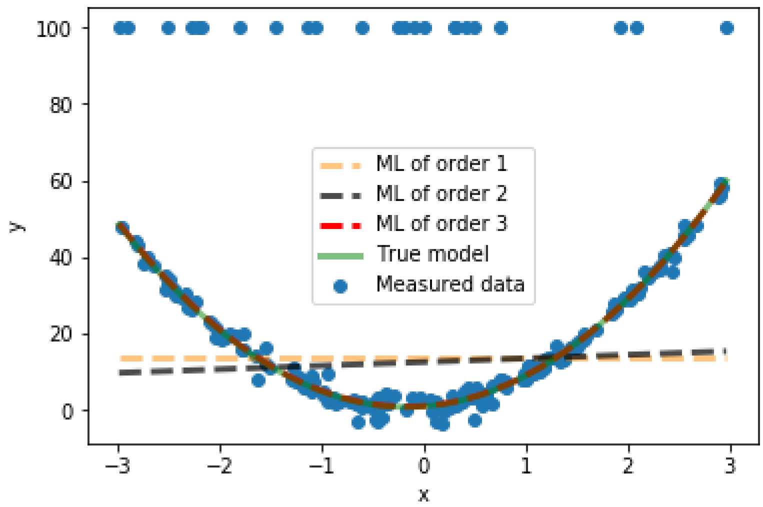
## Function for calculating the MLE:

In [23]:
```
y=y_true
pdf = laplace.pdf
def MLEfunction(order,u, y, N):

    import scipy.optimize as optimize

    # The likelihood function includes the scale (std dev) parameter which is
    # therefore the initial guess verctor has length n+2 [theta_0_hat, theta_
    init_guess = np.zeros(order+1)
    init_guess[-1] = N

    # Do Maximum Likelihood Estimation:
    opt_res = optimize.minimize(fun = log_lik,
                                x0 = init_guess,
                                options={'disp': False},
                                args = (y, u))

    MLE_params = opt_res.x[:-1]
    MLE_estimate = arbitrary_poly(MLE_params)

    MLE_params_rounded = ["{:.2f}".format(round(i, 2)) for i in MLE_params.to
    #print(f"\nMLE parameters of order :      {MLE_params_rounded}")

    return MLE_params, MLE_estimate
```

## Calculating ML for differnt orders:

In [24]:
```
y = y_true
params1,estimate1 = MLEfunction(1, u, y, N)
params2, estimate2 = MLEfunction(2, u, y, N)
params3, estimate3 = MLEfunction(3, u, y, N)

print(f"\nTrue model parameters: {true_params}")

# Plot measured data
plt.scatter(u, y, label=r"Measured data")
u0 = np.linspace(min(u), max(u), N)
plt.plot(u0, estimate1(u0), linestyle='--', color='darkorange', alpha=0.5, lw
plt.plot(u0, estimate2(u0), linestyle='--', color='black', alpha=0.7, lw=3, l
plt.plot(u0, estimate3(u0), linestyle='--', color='red', alpha=1, lw=3, label
plt.plot(u0, y_model(u0), 'g', alpha = 0.5, lw = 3, label="True model")
plt.legend()
plt.xlabel("x")

plt.ylabel("y");
```

```
True model parameters: [1, 2, 6]
```




In [25]:

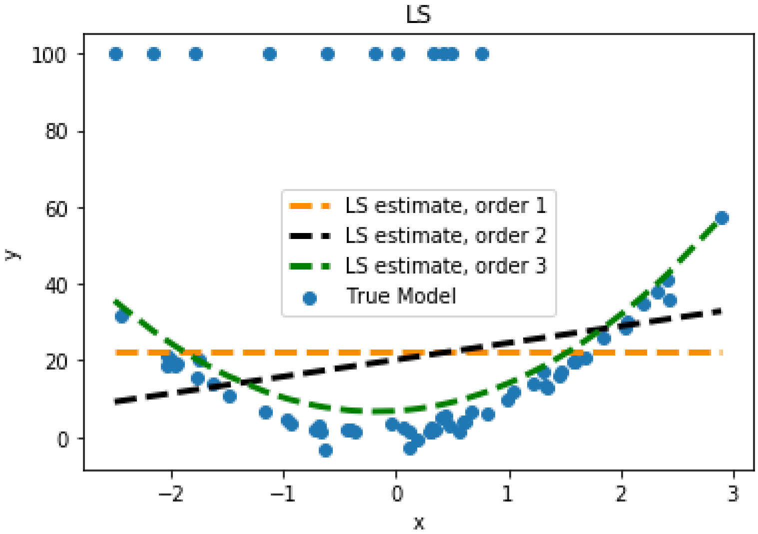
```
y=y_true
def getModels(order, u, y):
    LS_params = []
    LS_estimates =[]
    MLE_params = []
    MLE_estimates = []
    N = len(u)
    for i in range(order):
        tempPar, tempEst = LSorderFunc(i+1, u, y, N)
        LS_params.append(tempPar)
        LS_estimates.append(tempEst)
        tempPar, tempEst = MLEfunction(i+1,u, y, N)
        MLE_params.append(tempPar)
        MLE_estimates.append(tempEst)
    return LS_params, LS_estimates, MLE_params, MLE_estimates


def createNewSets(u,y):
    NoiseXY = np.array([u,y]).T
    np.random.shuffle(NoiseXY)


    training_set = NoiseXY[::3]
    training_set = training_set[training_set[:,0].argsort()].T

    test_set =NoiseXY[1::3]

    test_set = test_set[test_set[:,0].argsort()].T

    validation_set = NoiseXY[2::3]
    validation_set = validation_set[validation_set[:,0].argsort()].T

    true_training = y_model(training_set[0])
    true_test = y_model(test_set[0])
    true_validation = y_model(validation_set[0])
    return training_set, test_set, validation_set
training_set, test_set, validation_set = createNewSets(u,y)

true_training = y_model(training_set[0])
true_test = y_model(test_set[0])
true_validation = y_model(validation_set[0])
def singlePerf(y_t, y_hat):
    return  sum(abs(y_t - y_hat))

def modelSelect(performance):
    return performance.index(min(performance))
```

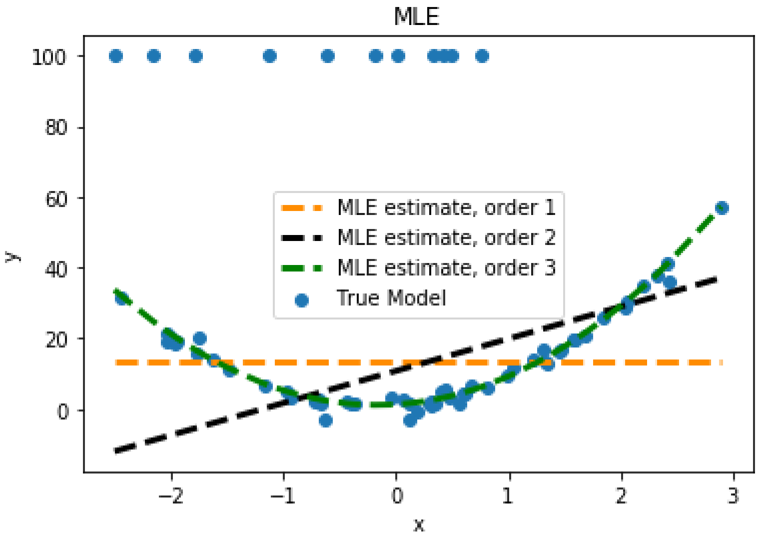
```
def modelsPerf(order, LS_estimates, MLE_estimates, validation_set, test_set):
    LS_valid_dev = []
    MLE_valid_dev = []

    for i in range(order):
        tmpLS = singlePerf(validation_set[1], LS_estimates[i](validation_set[
        LS_valid_dev.append(tmpLS)
        tmpMLE = singlePerf(validation_set[1], MLE_estimates[i](validation_se
        MLE_valid_dev.append(tmpMLE)

    LS_opt_ind = modelSelect(LS_valid_dev)
    LS_opt_model = LS_estimates[LS_opt_ind]

    MLE_opt_ind = modelSelect(MLE_valid_dev)
    MLE_opt_model = MLE_estimates[MLE_opt_ind]
    #test
    print(f"Best LS model: {LS_opt_ind+1}")
    print(f"Best ML model: {MLE_opt_ind+1}")
    LS_perf_test = singlePerf(test_set[1], LS_opt_model(test_set[0]))
    MLE_perf_test = singlePerf(test_set[1], MLE_opt_model(test_set[0]))
    print(f"\nLS_model_{LS_opt_ind + 1}_dev_test: {LS_perf_test}")
    print(f"ML_model_{MLE_opt_ind + 1}_dev_test: {MLE_perf_test}")

    return LS_opt_model, MLE_opt_model, LS_perf_test, MLE_perf_test


LS_params, LS_estimates, MLE_params, MLE_estimates= getModels(
    3, training_set[0], training_set[1])
LS_opt_model, MLE_opt_model, LS_perf_test, MLE_per_test = modelsPerf(
    3, LS_estimates, MLE_estimates, validation_set, test_set)
plt.figure(1)
plt.scatter(validation_set[0],validation_set[1] ,  label="True Model")


plt.plot(validation_set[0], LS_estimates[0](validation_set[0]), color="darkor
plt.plot(validation_set[0], LS_estimates[1](validation_set[0]), color="black"
#plt.scatter(validation_set[0], LS_estimates[2](validation_set[0]))
plt.plot(validation_set[0], LS_estimates[2](validation_set[0]), color="green"
#plt.xlim(0, 10)
plt.legend()
plt.title("LS")
plt.xlabel("x")
plt.ylabel("y")

plt.figure(2)
plt.scatter(validation_set[0],validation_set[1] ,  label="True Model")
plt.title("MLE")

plt.plot(validation_set[0], MLE_estimates[0](validation_set[0]), color="darko
plt.plot(validation_set[0], MLE_estimates[1](validation_set[0]), color="black
#plt.scatter(validation_set[0], LS_estimates[2](validation_set[0]))
plt.plot(validation_set[0], MLE_estimates[2](validation_set[0]), color="green
#plt.xlim(0, 10)
plt.legend()
plt.xlabel("x")
plt.ylabel("y")
```

```
Best LS model: 3
Best ML model: 3

LS_model_3_dev_test: 860.3564044742878
ML_model_3_dev_test: 701.2416926058229
```

Out[25]:

```
Text(0, 0.5, 'y')
```



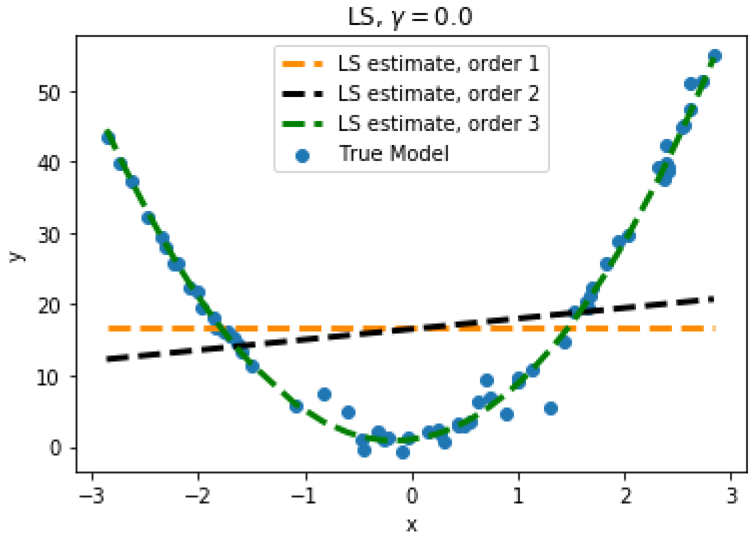



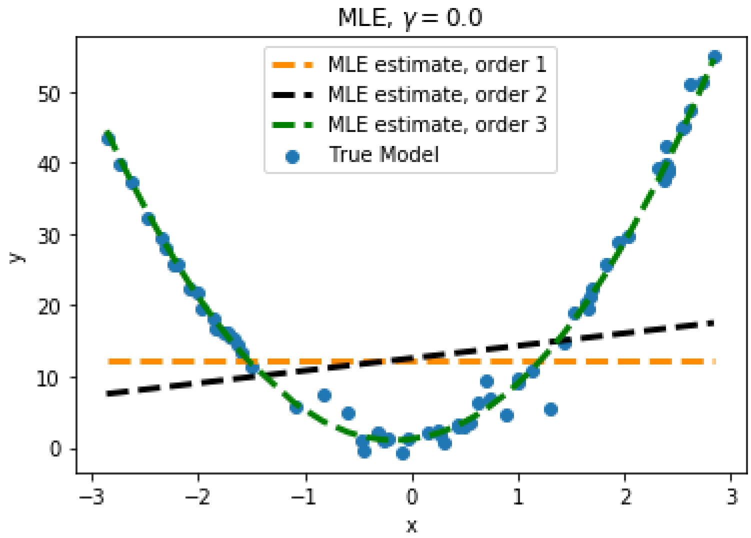

In [28]:

```
def plots4Gammas(magnitude, loc, N, gammaRange):
    range_low = -3
    range_high = 3
    u = np.sort(np.random.uniform(range_low,range_high,N))
    y_true = y_model(u)
    pdf = laplace.pdf
    alfa = 0
    normVariance = 1 # Input as the scale parameter in the normal distributio

    laplaceVariance = 1
    noiseNorm = magnitude * np.random.normal(loc, normVariance, int(alfa * N)

    noiseLaplace = magnitude * np.random.laplace(loc, laplaceVariance, int((1
    gammaArray = []
    bestLS = []
    bestMLE = []
    for i in range(gammaRange+1):
        gamma = i/gammaRange
        gammaArray.append(gamma)



        # Add noise to the generated data points - thus simulating measuremen

        y = [0]*N
        for j in range(0, N):
            faulty = np.random.binomial(1, gamma)
            if faulty:
```

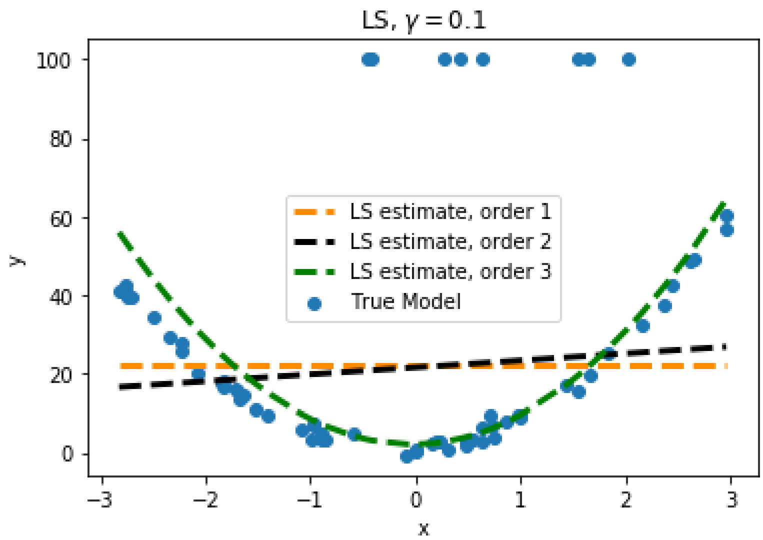
```
                y[j] = 100
            else:
                y[j] = y_true[j] + noiseLaplace[j]
        if gamma==1.0:
            print(y)
        training_set, test_set, validation_set = createNewSets(u,y)

        LS_params, LS_estimates, MLE_params, MLE_estimates= getModels(
            3, training_set[0], training_set[1])
        LS_opt_model, MLE_opt_model, LS_perf_test, MLE_perf_test = modelsPerf
            3, LS_estimates, MLE_estimates, validation_set, test_set)
        bestLS.append(LS_perf_test)
        bestMLE.append(MLE_perf_test)

        plt.figure(num=2*i+1)
        plt.scatter(validation_set[0],validation_set[1] ,  label="True Model"
        plt.title("LS, $\gamma = $" f"{gamma}")

        plt.plot(validation_set[0], LS_estimates[0](validation_set[0]), color
        plt.plot(validation_set[0], LS_estimates[1](validation_set[0]), color
        #plt.scatter(validation_set[0], LS_estimates[2](validation_set[0]))
        plt.plot(validation_set[0], LS_estimates[2](validation_set[0]), color
        #plt.xlim(0, 10)
        plt.legend()
        plt.xlabel("x")
        plt.ylabel("y")

        plt.figure(num=2*i+2)
        plt.scatter(validation_set[0],validation_set[1] ,  label="True Model"
        plt.title(r"MLE, $\gamma = $" f"{gamma}")

        plt.plot(validation_set[0], MLE_estimates[0](validation_set[0]), colo
        plt.plot(validation_set[0], MLE_estimates[1](validation_set[0]), colo
        #plt.scatter(validation_set[0], LS_estimates[2](validation_set[0]))
        plt.plot(validation_set[0], MLE_estimates[2](validation_set[0]), colo
        #plt.xlim(0, 10)
        plt.legend()
        plt.xlabel("x")
        plt.ylabel("y")
        plt.show()
    return bestLS, bestMLE, gammaArray
bestLS, bestMLE, gammaArray = plots4Gammas(magnitude=1, loc=0, N=201, gammaRa
```

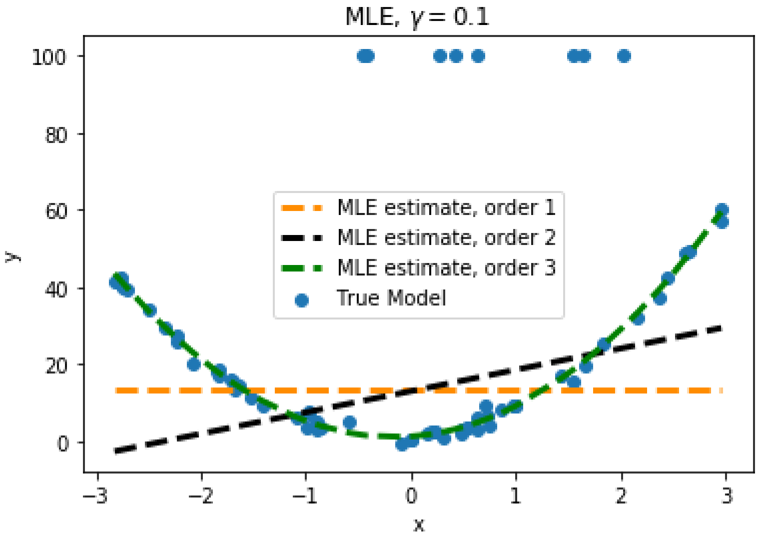
```
Best LS model: 3
Best ML model: 3

LS_model_3_dev_test: 52.858503824812466
ML_model_3_dev_test: 54.32082451497303
```



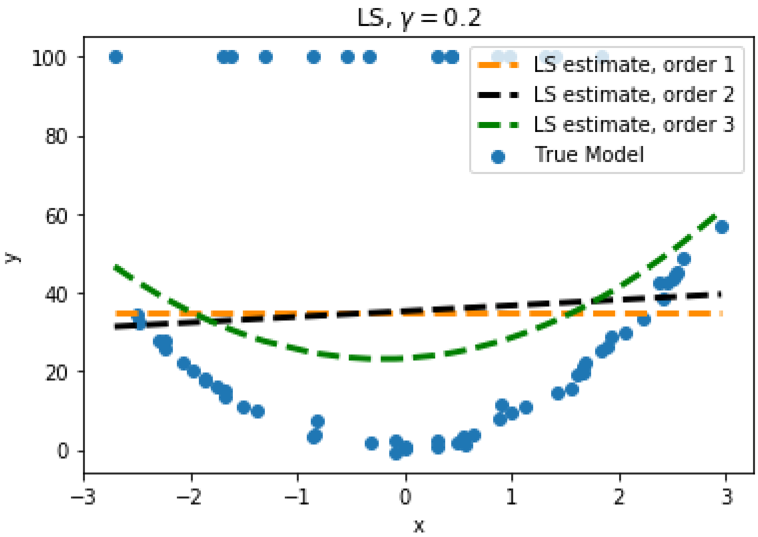



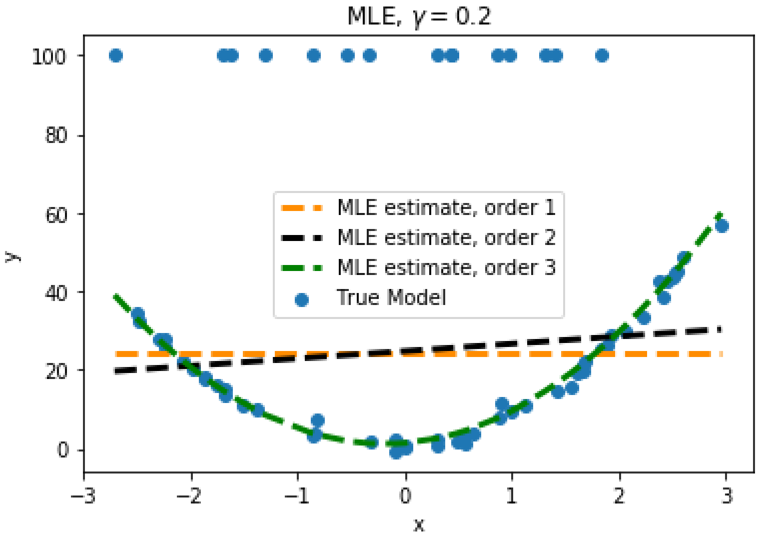

```
Best LS model: 3
Best ML model: 3

LS_model_3_dev_test: 919.5934668760548
ML_model_3_dev_test: 790.7144965384142
```



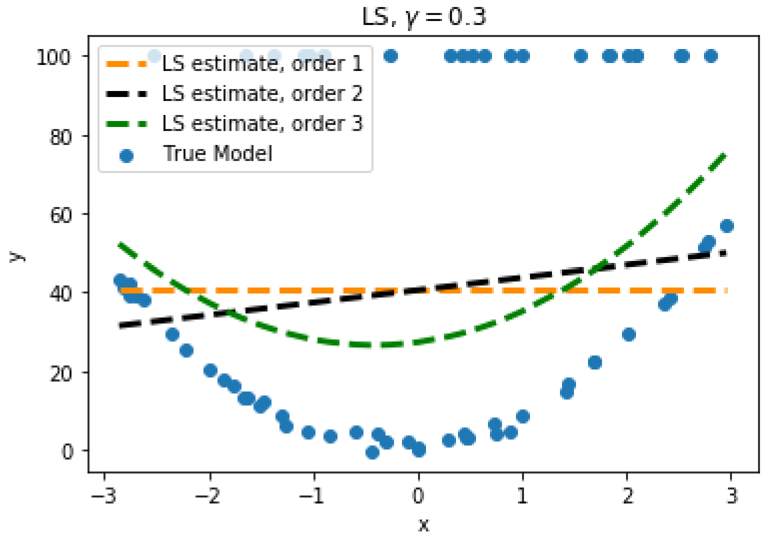


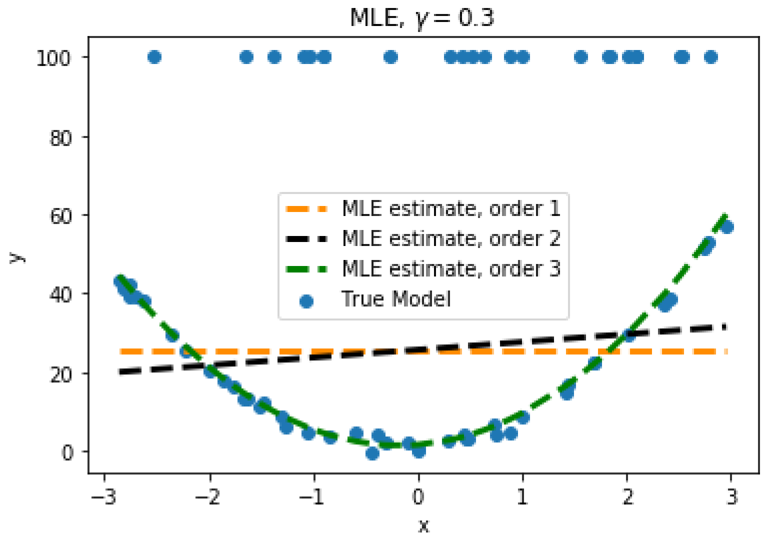

```
Best LS model: 3
Best ML model: 3

LS_model_3_dev_test: 2002.9794477995722
ML_model_3_dev_test: 1576.2762271162285
```



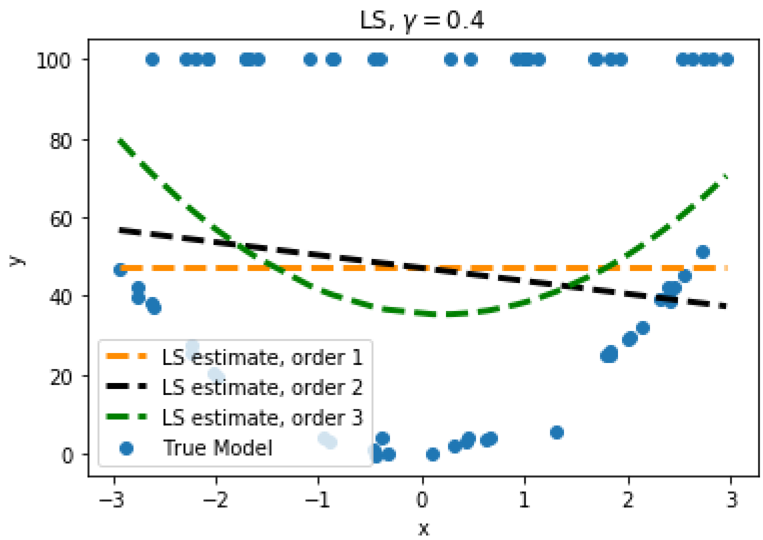



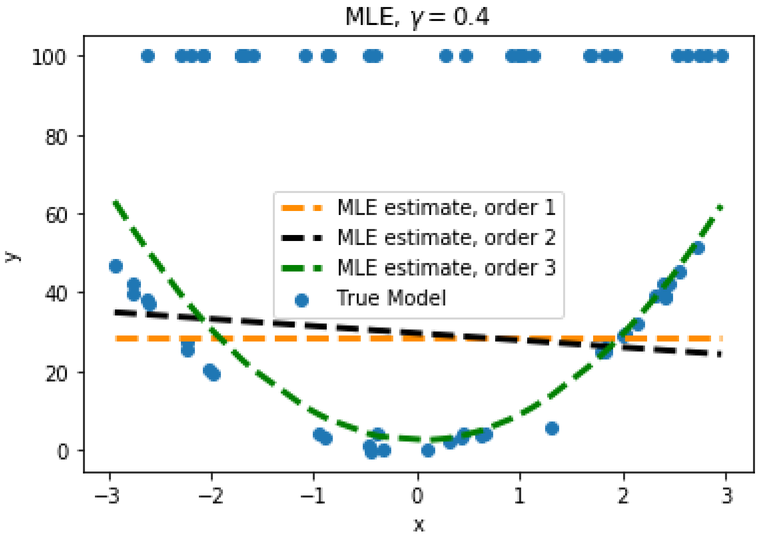

```
Best LS model: 3
Best ML model: 3

LS_model_3_dev_test: 2099.4683884741885
ML_model_3_dev_test: 1404.1356129940887
```



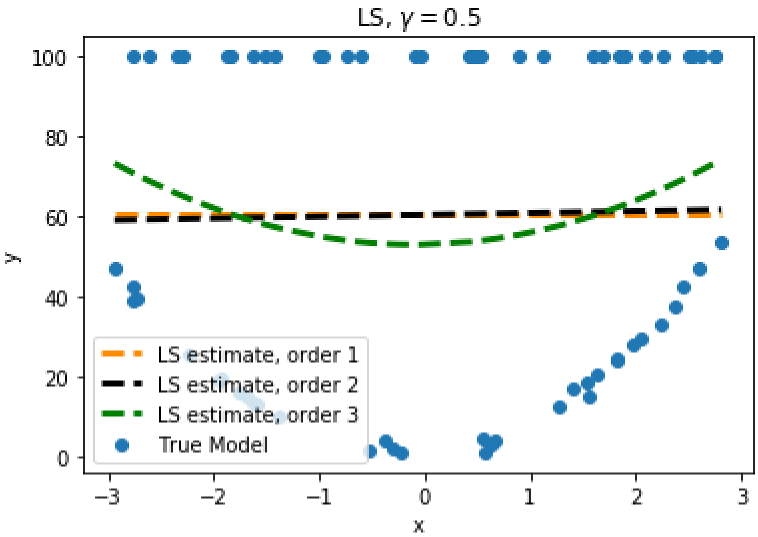



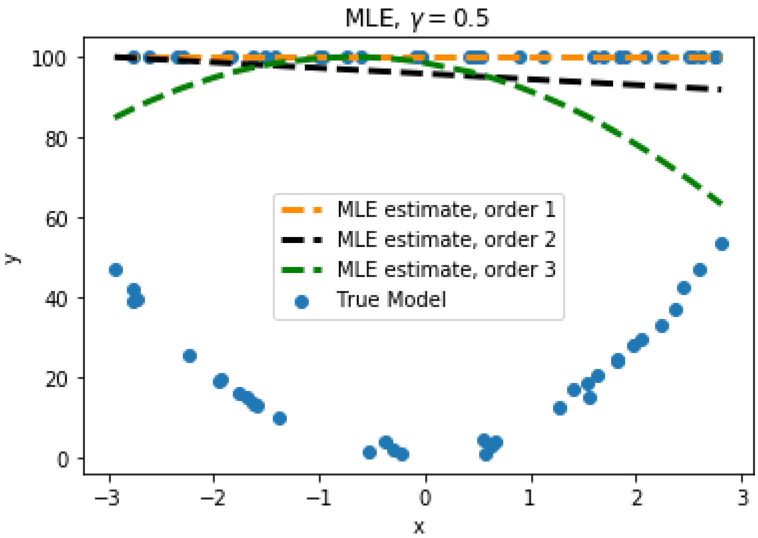

```
Best LS model: 1
Best ML model: 3

LS_model_1_dev_test: 2566.339247140539
ML_model_3_dev_test: 2133.4935713205946
```



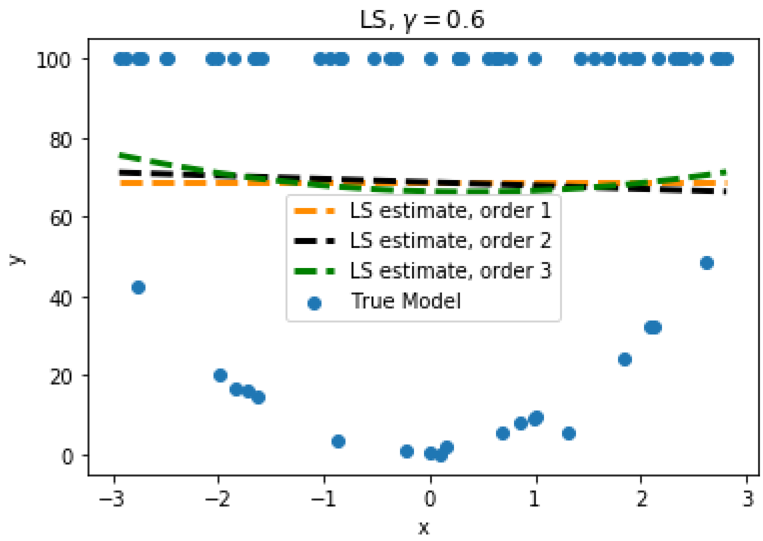


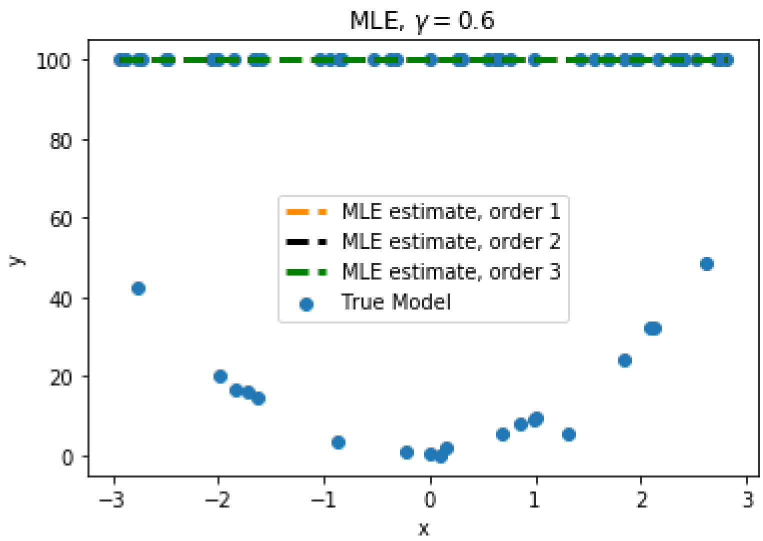

```
Best LS model: 2
Best ML model: 3

LS_model_2_dev_test: 2871.8213372236582
ML_model_3_dev_test: 3129.754679857139
```



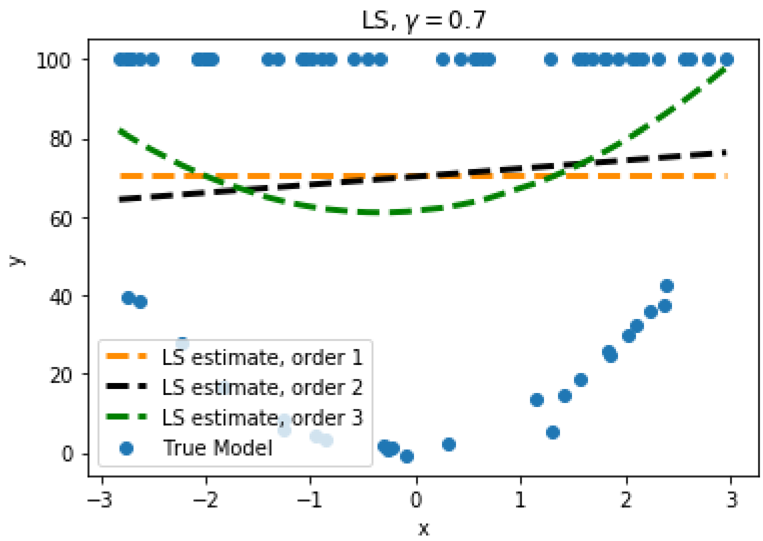



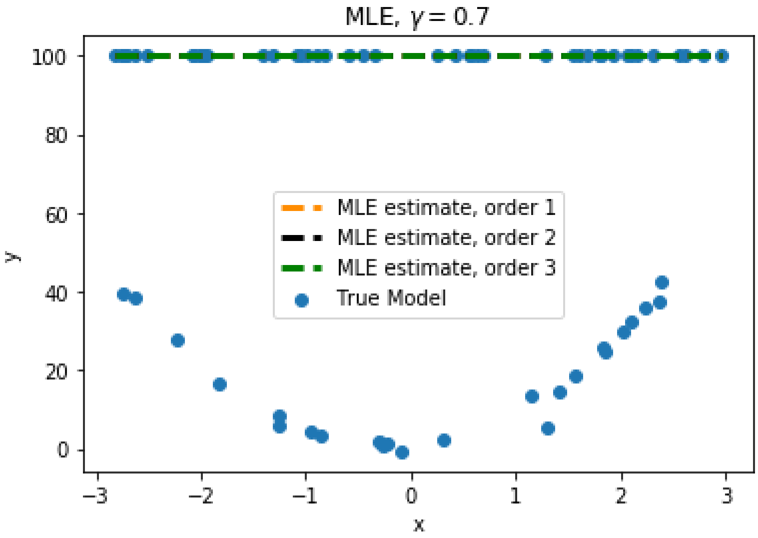

```
Best LS model: 3
Best ML model: 2

LS_model_3_dev_test: 2680.798784385445
ML_model_2_dev_test: 2406.1154523196824
```



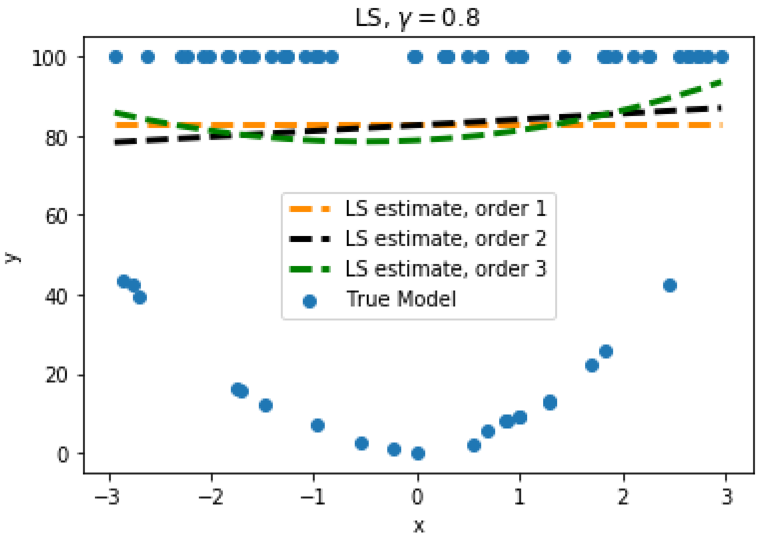



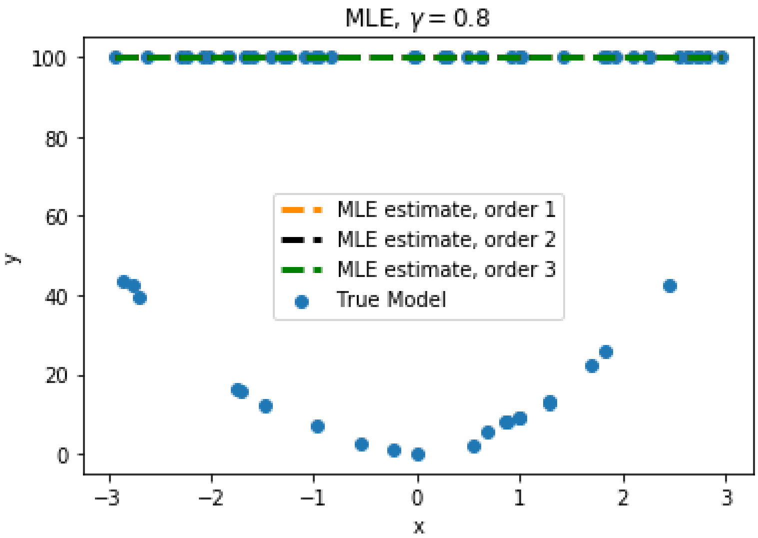

```
Best LS model: 3
Best ML model: 2

LS_model_3_dev_test: 2485.6882641933785
ML_model_2_dev_test: 1714.9808835307326
```



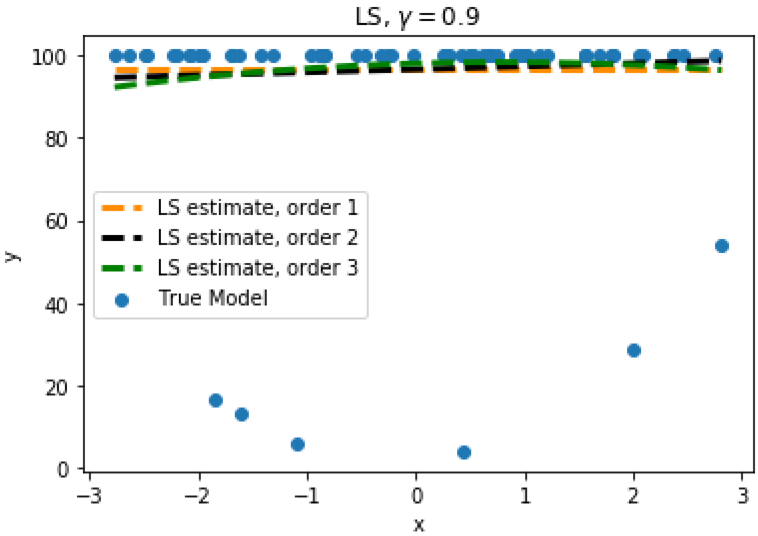


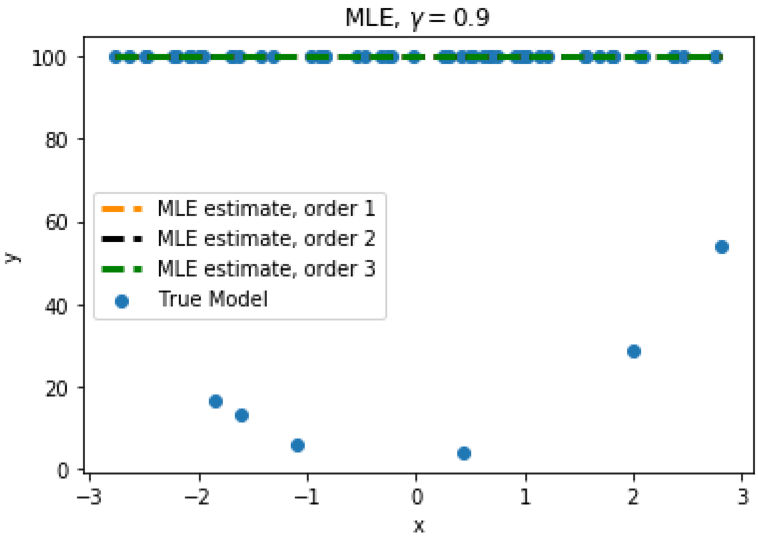

```
Best LS model: 3
Best ML model: 2

LS_model_3_dev_test: 1547.9531758511675
ML_model_2_dev_test: 599.7980101630301
```




```
Best LS model: 3
Best ML model: 2

LS_model_3_dev_test: 732.2309709522534
ML_model_2_dev_test: 554.1104142638515
```




```
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100, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100]
Best LS model: 1
Best ML model: 3

LS_model_1_dev_test: 0.0
ML_model_3_dev_test: 1.3491539903043304e-06
```
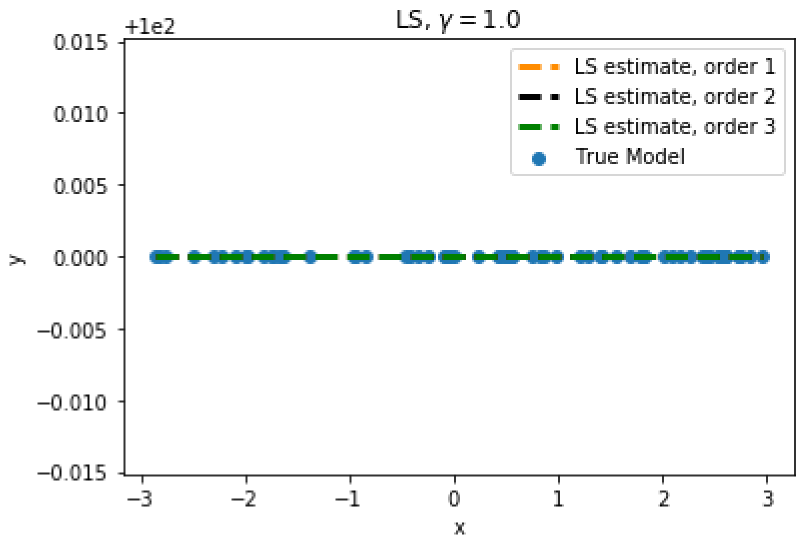
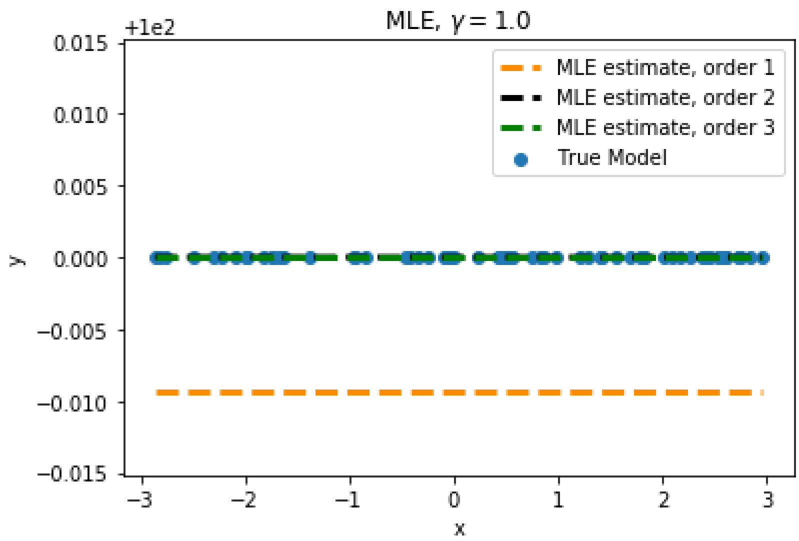
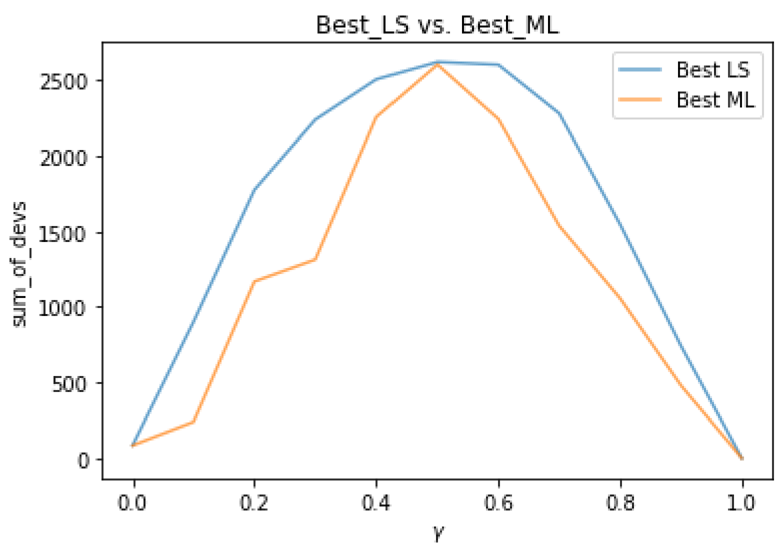




In [27]:

```
plt.plot(gammaArray[:], bestLS[:], alpha=0.7, label="Best LS")
plt.plot(gammaArray[:], bestMLE[:], alpha=0.7, label="Best ML")
plt.legend()
plt.xlabel(r"$\gamma$")
plt.ylabel("sum_of_devs")
plt.title("Best_LS vs. Best_ML")
print(gammaArray)
```

```
[0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]
```



In [ ]:

```

```