Danmarks
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Assignment 2

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2.1: Stability

Let the process X_t be given by

$$X_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3} \tag{1}$$

where ε_t is a white noise process with $\sigma = 0.1$.
Figure 1 shows the simulation of the process with 200 samples.

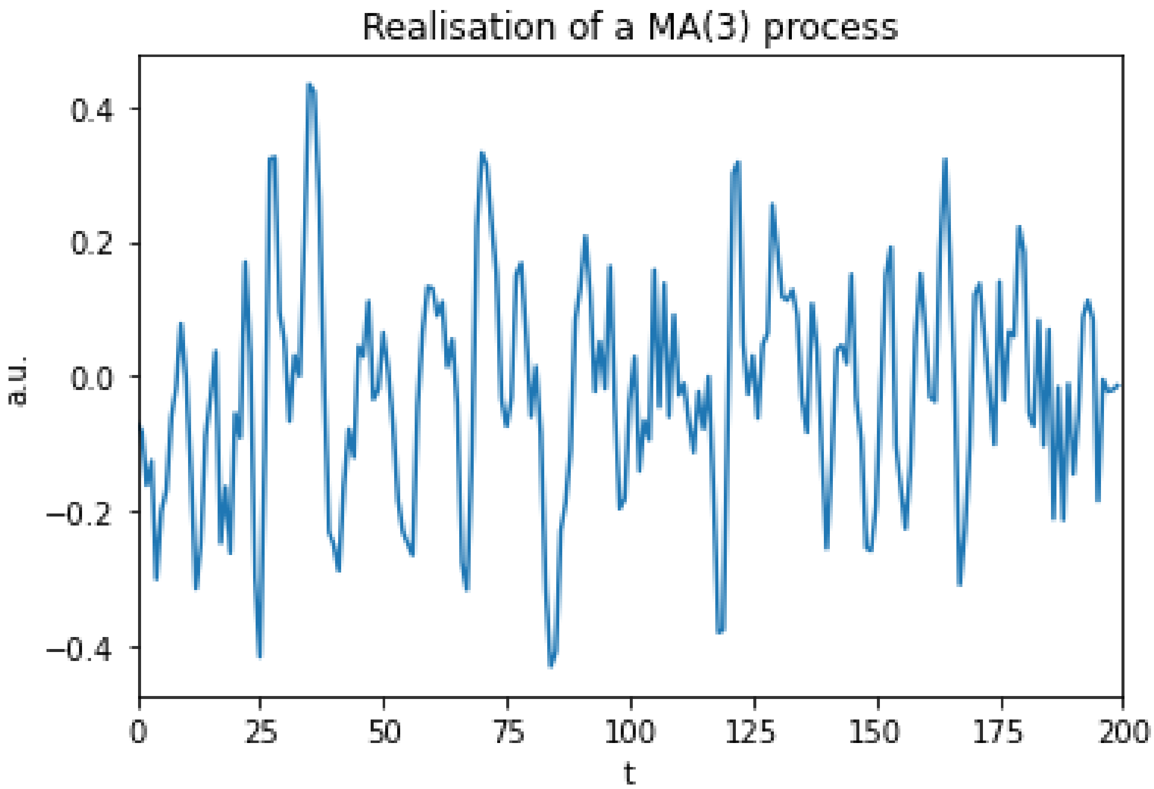

Figure 1: One simulation consisting of 200 samples of the MA(3) process given in Equation 1 with $\sigma = 0.1$.

2.1.1. Second order moment representation

The second order moment representation are given as the mean and autocovariance of the process. The mean is calculated analytically by taking the expected value of Y_t:

$$E[Y_t] = E[\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}] \tag{2a}$$
$$E[Y_t] = E[\varepsilon_t] + E[\varepsilon_{t-1}] + E[\varepsilon_{t-2}] + E[\varepsilon_{t-3}] \tag{2b}$$
$$E[Y_t] = 0 \tag{2c}$$

The autocovariance can be calculated by looking at how the variance of different terms in the process affects each other. ε_{t-3} will affect Y_t as a function of the variance propagated through the other terms. For example the autocovariance for Y_t is given as:

$$\gamma(t) = (1 + \theta_1 + \theta_2 + \theta_3)\sigma^2 = (1 + 1 + 1 + 1)\sigma^2 = 4\sigma^2 \tag{3}$$

By using the same approach as above for the prior terms as well the following autocovariance can be found:

$$\gamma(k) = \begin{cases} 4\sigma^2 = 0.04 & |k| = 0 \\ 3\sigma^2 = 0.03 & |k| = 1 \\ 2\sigma^2 = 0.02 & |k| = 2 \\ \sigma^2 = 0.01 & |k| = 3 \\ 0 & |k| = 4, 5, ... \end{cases} \tag{4}$$

2.2.2. Stationarity and invertibility

This is a MA process and all MA processes are stationary (Madsen, 2007, p. 118). This fact can be further investigated by looking at how the mean and variance are in different parts of the process. By separating the process into four equally sized parts and calculating the mean and variance for each part I get the following table:

Part	Mean:	Variance:
1	-0.006	0.018
2	-0.005	0.031
3	0.048	0.024
4	0.033	0.024

Table 1: Table showing how the mean and variance of the time series evolves through time. The time series were split into four different parts and the mean and variance were calculated for each part. Both the mean and variance values are all in the same scale, thus this process looks to be stationary.

As you can see from Table 1 all the values, both for mean and variance, are somewhat constant through the whole time series. This indicates that the process is indeed stationary, just as expected.

For a process to be invertible the roots of the polynomial describing the MA term of the process needs to be inside the unit circle (Madsen, 2007, p. 118). Using the lag operator the MA term of this process are described as:

$$X_t = (1 + B^{-1} + B^{-2} + B^{-3})\varepsilon_t$$

Further transformation of this expression into the z-domain gives the following equation:

$$X_t = (1 + z^{-1} + z^{-2} + z^{-3})\varepsilon_t$$

The roots of this polynomial is: $-1, -j, j$, where j is the imaginary unit. And since length of these roots are 1, they will not lie within the unit circle and the thus the process is not invertible.

2.1.3. 10 simulations

In Figure 2 10 simulations of 200 observations of the process is plotted.

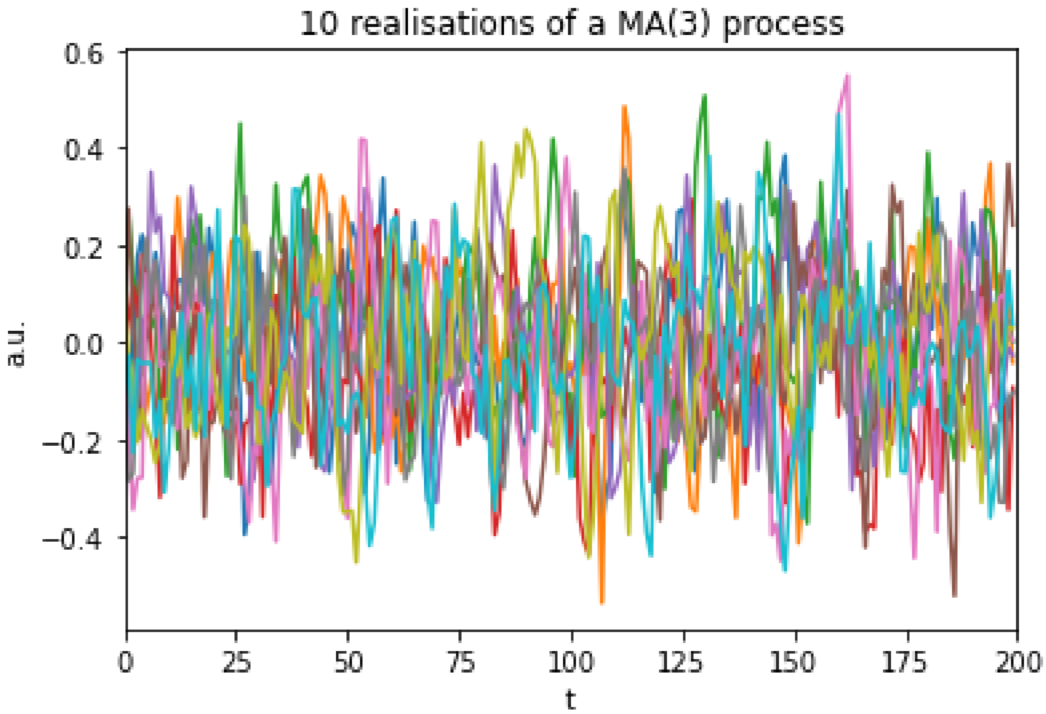

Figure 2: 10 realisations of MA(3) process all of 200 samples. Note how the different realisations differs very much.

2.1.4. ACF for each realisation

In Figure 3 the autocorrelation function for all the 10 realisations of the MA(3) process is plotted. Note how all the different processes has very similar autocorrelations.

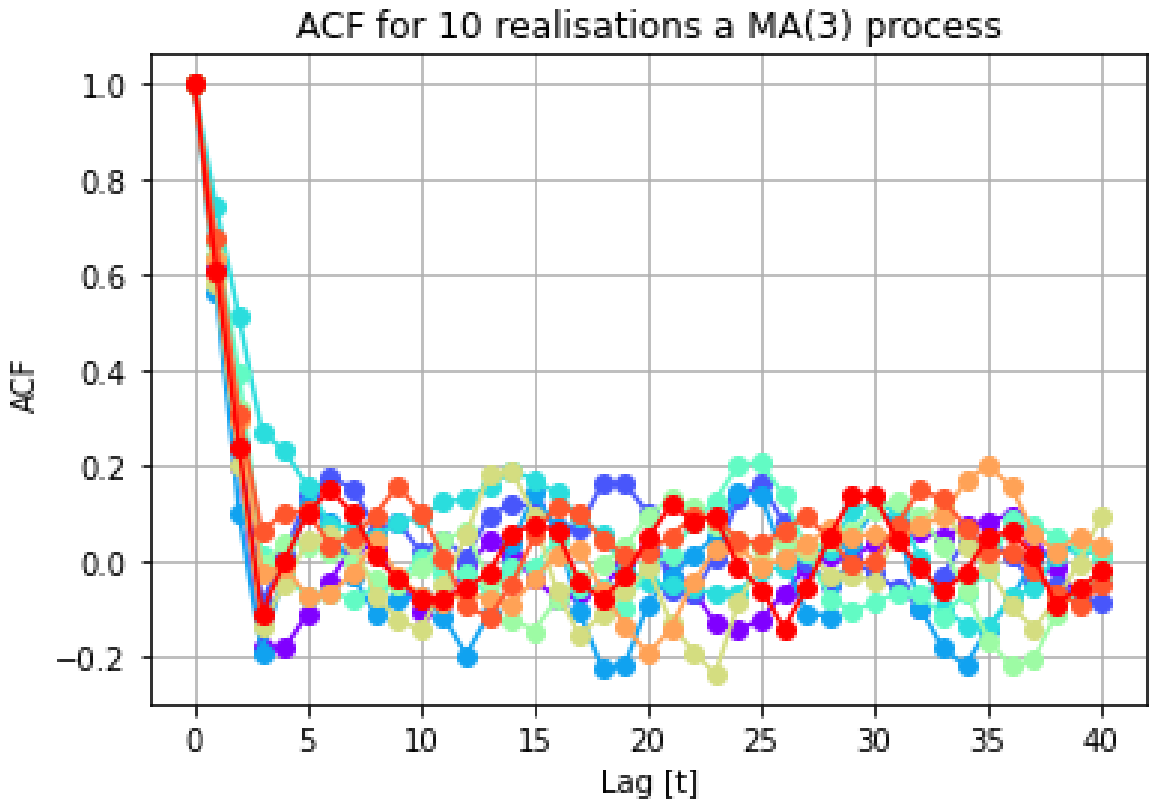

Figure 3: Autocorrelation function for 10 realisations of a MA(3) process. Note how all the different realisations of the process has almost the same autocorrelation function.

2.1.5. PACF for each realisation

Figure 4 shows the partial autocorrelation plots for the 10 different realisations of the MA(3) process.

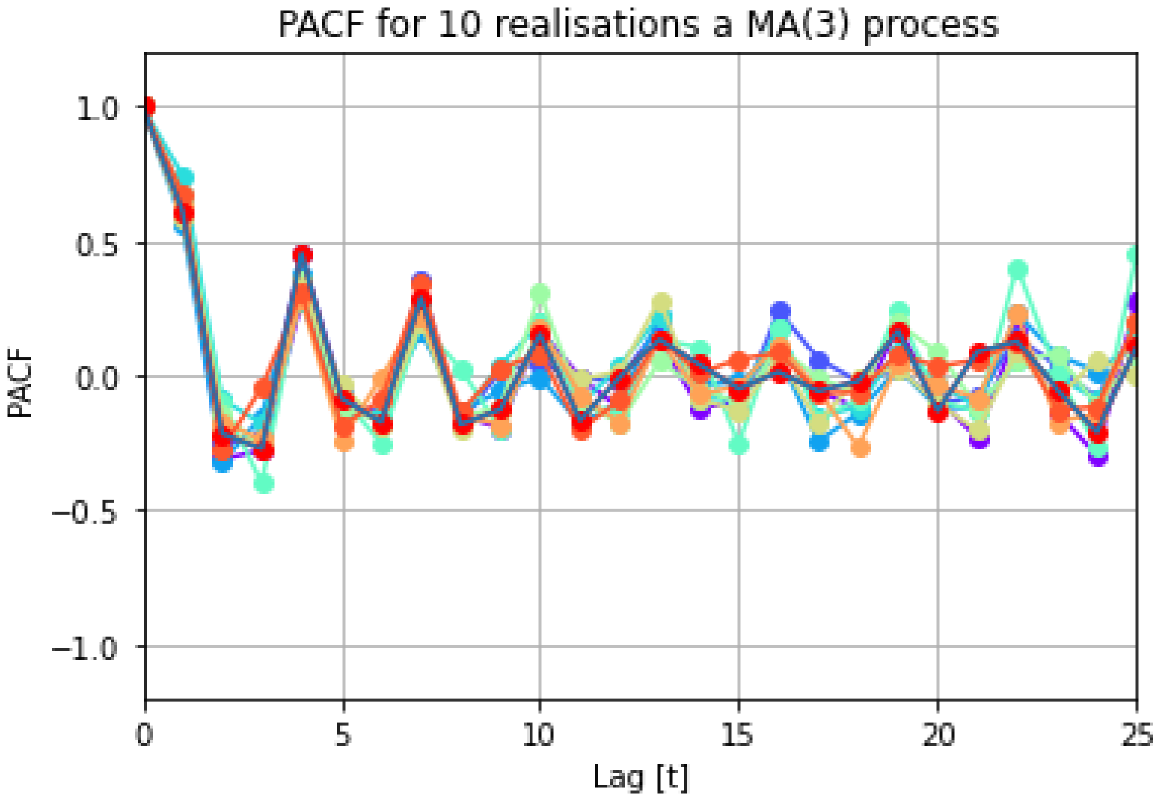

Figure 4: Partial autocorrelation function for 10 realisations of a MA(3) process. Note how all the different realisations of the process has almost the same autocorrelation function.

2.1.6. Variance for each realisation

In Table 2 the variance and mean of all the 10 different realisations of the MA(3) process are shown. They all seem to have approximately the same value both for the variance and mean.

Realisation:	Variance:	Mean:
1	0.03	-0.012
2	0.037	-0.002
3	0.043	0.019
4	0.039	0.012
5	0.038	-0.001
6	0.039	-0.042
7	0.038	0.028
8	0.047	0.013
9	0.033	-0.008
10	0.036	-0.02

Table 2: Table showing the variance and mean for all the 10 different realisations of the process. They are all a measure on how the variance of each error term propagates through the model, and reflects the same autocovariance as earlier calculated. This is also the case with the mean.

2.1.7. Comparison and discussion

The mean was analytically calculated to be 0. This is due to all the different error terms in the process also having a mean of 0. As you can see in Table 2 the mean of the actual processes also lies just around 0. Due to the randomness of the simulation it differs from the actual mean of 0, but by simulation even longer the mean would approach 0.

The variance of the different processes are also shown in Table 2. The variance of each term in the process is a function of the variance of the 3 prior term, just as the autocovariance calculated in Equation 4. The autocovariance was analytically calculated to be 0.04 for each lag, something that seems to be the case in the realisations of the process as well. Again, the reasons for the slight deviations from the analytical result is the randomness of the process, and simulation for a longer time would make the variance get even closer to 0.04.

2.2: Predicting temperatures in a district heating network

A model for the forward temperature to a house in the heating network is given as:

$$(1 - 0.5B + 0.3B^2)(1 - 0.9B^{12})(Y_t - \mu) = \varepsilon_t \tag{5}$$

Where ε_t is a white noise process with $\sigma = 0.5$. The μ in the model is estimated to be 55.

To forecast the temperature in t = 2017M12 and 2018M1 I made and an ARIMA model representing the model given in Equation 5. The model given consists of one AR(2) term, B and B^2 and one seasonal AR(1) term, B^{12}.

The forecasts with a confidence interval is calculated to be:

t:	Forecast:	Prediction interval:
2017M12	58.71	± 0.98
2018M1	58.26	± 1.1

Table 3: Table with the forecasts from the model together with a upper and lower 95% prediction interval interval.

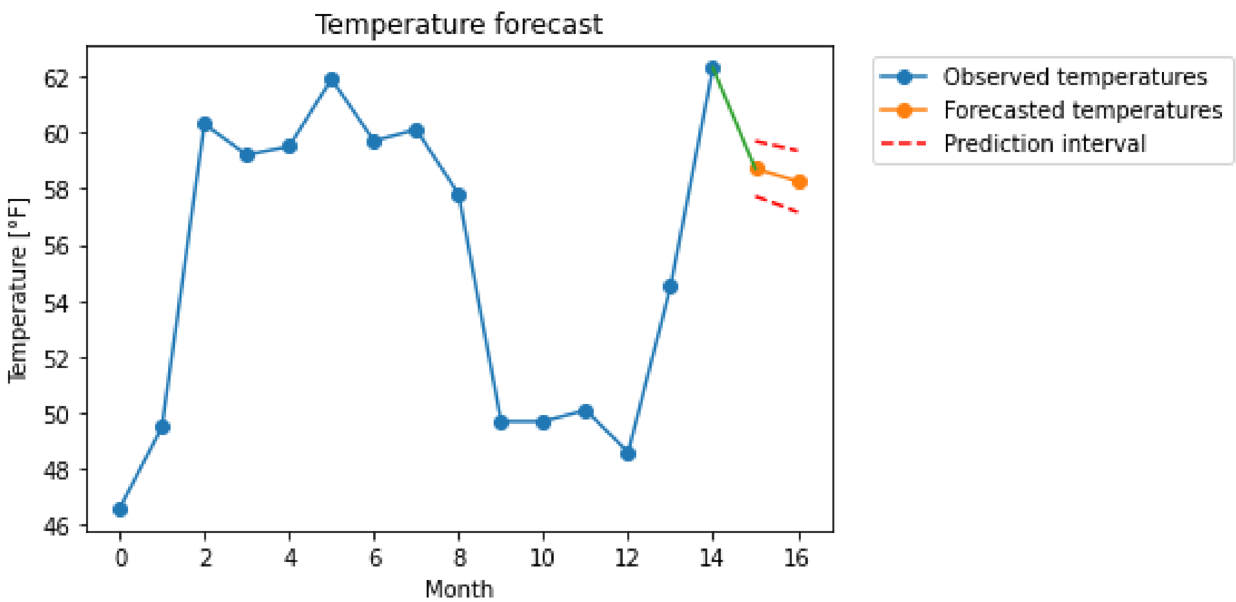

Figure 5: Plot of the original temperature data and the two forecasted values. The two red, dotted lines are the prediction interval of the forecasted data. Because of the $0.9B^{12}$ term in this model, the forecasted values has a close connection to what happened 12 months ago. The fact that the value 12 months ago were lower than now may be the reason for the dip from the last known value to the predictions.

2.3: Simulating seasonal processes

In this question we are going to simulate some seasonal ARIMA models and analyse behaviour of the different processes. The simulated models are all seasonal ARIMA models with seasonality of 12. They differ in the values of p, P, q and Q. Where p and q are the order of the AR and MA terms, respectively, and P and Q are the order of the seasonal AR and MA terms, respectively. The different coefficients for the terms are also given.

Plots of the autocorrelation function and partial autocorrelation function are useful tools when analysing seasonality and stationarity. The autocorrelation function are showing whether the different elements in a time series are positively correlated, negatively correlated or independent of each other. In other words the autocorrelation function describes the relationship between he current value of the time series and values in prior lags of the same time series. If the process is non stationary, the autocorrelation for small lags tends to be large and positive because the two observations are both close in size and in time. When a time series has seasonality this is often seen in the autocorrelation plot as large autocorrelations in the lags corresponding to the seasonality.

The partial autocorrelation function works in a slightly different way. Instead of explaining the correlation between the current lag and all the past lags, each spike in the plot explains only what the prior lags did not explain. So the partial autocorrelation for lag 3 is only the correlation that 1 and 2 did not explain. This is useful for detecting the order of the auto regressive term of the process, as it removes the correlation of all the terms with shorter lags and thus it shows how many lags the current lag are a function of.

Another useful tool for examining stationary is to inspect the roots of the AR polynomial of the process. A process is stationary if the roots of $\phi(z^{-1})$ with respect to z all lies within the unit circle. In mathematical terms the process is stationary if $|z| < 1$.

The sign convention for different ARIMA models may vary. A seasonal ARIMA process are given as:

$$\phi(B)\Phi(B^s)\nabla^d\nabla_s^D Y_t = \theta(B)\Theta(B^s)\varepsilon_t \tag{6}$$

I use the sign convention from the book. I define the following signs of $\phi(B)$, $\Phi(B^s)$, $\theta(B)$ and $\Theta(B^s)$:

$$\phi(B) = 1 + \phi_1 B + \phi_2 B^2 + ... \tag{7a}$$
$$\phi(B^s) = 1 + B^s \tag{7b}$$
$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + ... \tag{7c}$$
$$\Theta(B^s) = 1 + \Theta(B^s) \tag{7d}$$

In practice this will result in the ϕ, Φ, θ and Θ coefficients having the sign given in the task, not the opposite sign. This would have been the case if $\phi(B)$, $\Phi(B^s)$, $\theta(B)$ and $\Theta(B^s)$ were defined with $+$ instead of $-$.

2.3.1. $A(1,0,0) \times (0,0,0)_{12}$ with $\phi_1 = -0.85$

This model is a AR model of order 1. in Figure 6 a simulation of the process can be seen together with the autocorrelation and partial autocorrelation plots.

Firstly, for a model to be seasonal it needs to have a seasonal part. So just by inspecting the first model, it can be concluded that the it is not seasonal. This is also supported by the plots in Figure 6 as none of these shows any sign significant correlation corresponding to a seasonal trend.

When it comes to stability, the autocorrelation plot shows that close lags has high positive correlation, something that is usually a sign of non stationarity. But this is just an indicator and not always the case. So by finding the root of the AR polynomial I find the root to be 0.85. This root lies inside the unit circle and thus the process is in fact stationary.

$$1 - 0.85z^{-1} = 0 \implies z = 0.85.$$

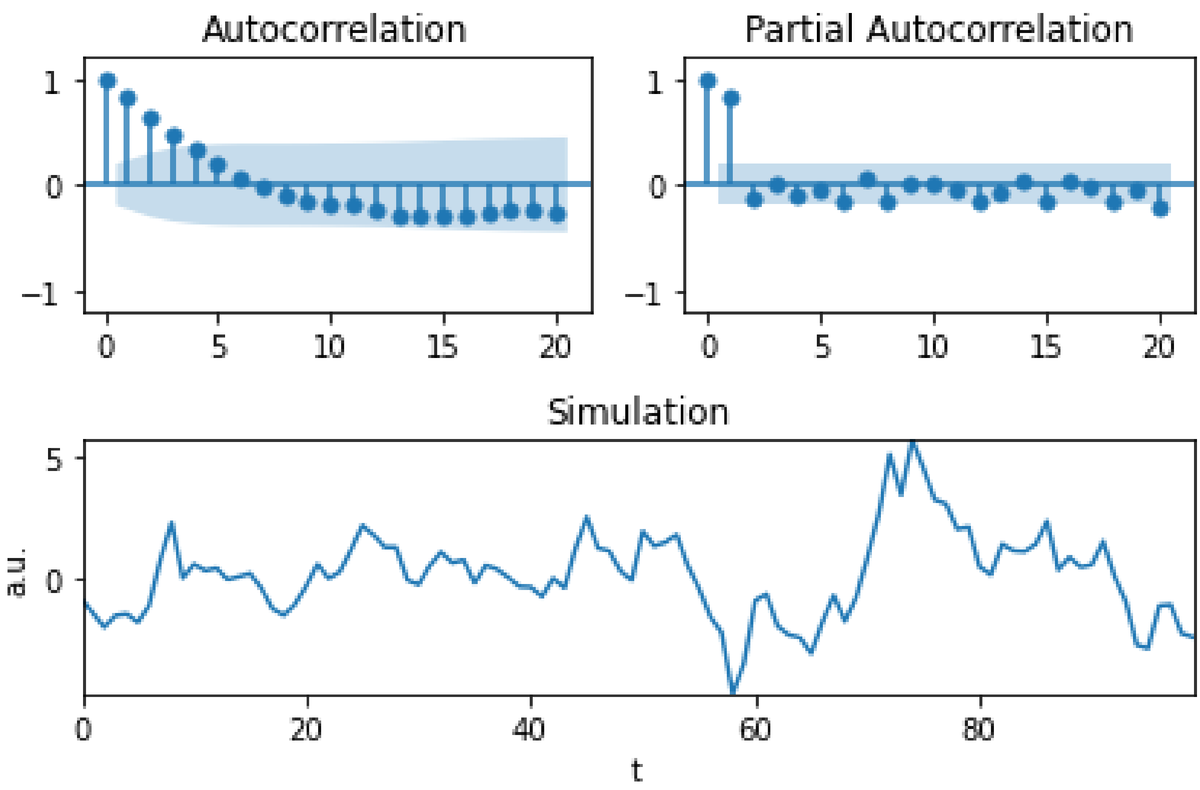

Figure 6: The autocorrelation and partial autocorrelation plots for a simulation of a $A(1,0,0) \times (0,0,0)_{12}$ process. The autocorrelation plot looks to have high positive correlations between close lags, something that is usually a sign of non stationarity. But inspection of the roots of the AR terms shows that the process is in fact stationary.

2.3.2. $A(0,0,0) \times (1,0,0)_{12}$ with $\Phi_1 = 0.85$

This model consists of only one seasonal AR term. A simulation together with the corresponding ACF and PACF plots can be seen in Figure 7. Just by looking at the simulation it is unclear whether the process is stationary or not. By inspecting the autocorrelation and partial autocorrelation plots in Figure 7 significant correlations at lag 12 can be seen in both plots. This is a sign of this model having a seasonality of 12. This is just as expected as process is simulated from a seasonal AR(1) model with seasonality of 12.

Stationarity can be analysed by taking a look at the autocorrelation plot. In Figure 7 it is clear that the autocorrelation are only significant for the first lag and decays to insignificant values for the next lags. This is a sign that there are no significant correlation between close lags, something that makes the process stationary.

The stationary assumed from the autocorrelation plot is also supported by the root of the AR term in the process. The root is found

$$1 + 0.85z^{-1} = 0 \implies z = -0.85 \tag{8}$$

This root is inside the unit circle, making the process stationary.

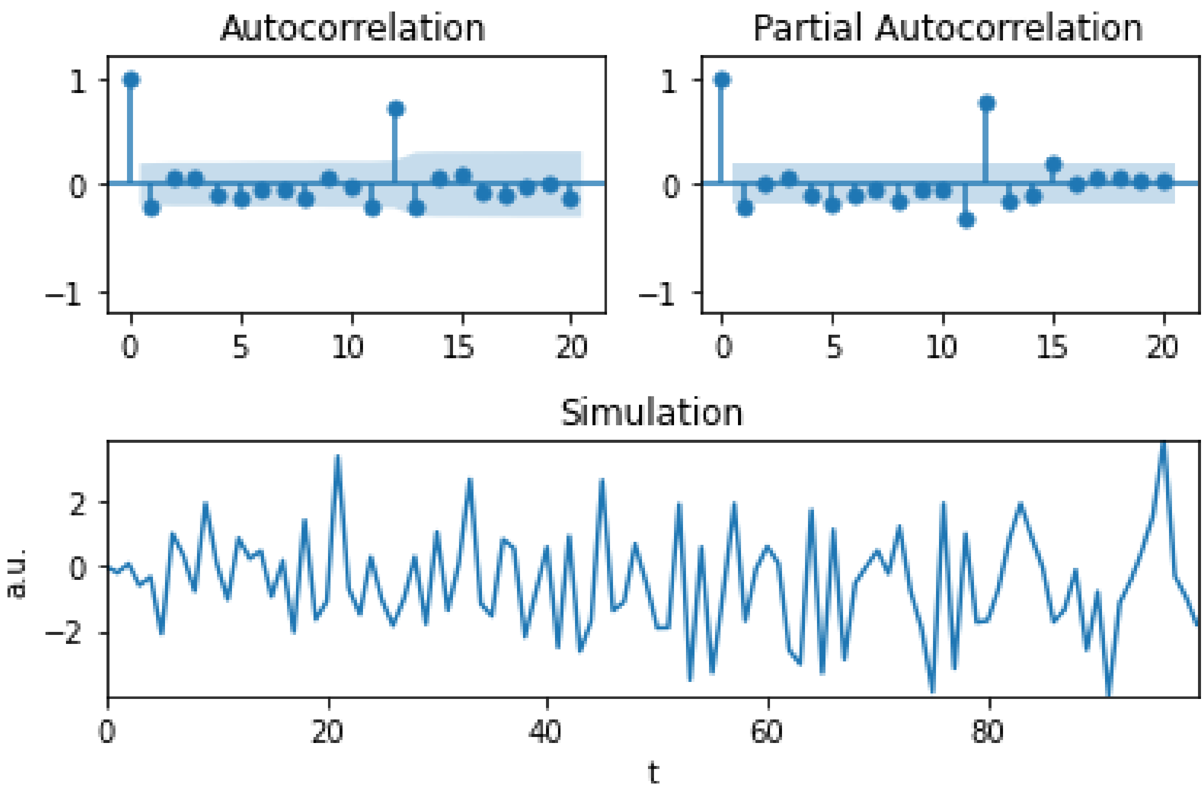

Figure 7: Simulation of a $A(0,0,0) \times (1,0,0)_{12}$ process with its autocorrelation and partial autocorrelation functions. The simulation shown no sign of seasonality, but by looking at the autocorrelation plots a seasonality of 12 can be found.

2.3.3. $A(1,0,0) \times (0,0,1)_{12}$ with $\phi_1 = -0.8$ and $\Theta_1 = 0.9$

This process consists of a AR term and a seasonal MA term. Although the simulation does not show any clear signs of seasonality, the seasonal MA term can be seen in the both the correlation plots as the significant correlation at around lag 12. Thus the model is actually seasonal with a correlation between the error at a lag and the error 12 lags ago.

Just by looking at the autocorrelation plot the process looks to be non stationary due to the slowly decaying high correlation between the close lags.

the positive AR term can bee seen in the autocorrelation plot in Figure 8 as the slowly decaying positive correlation between close lags. But by calculating the root of the AR polynomial to be 0.8, the process is actually stationary.

$$1 - 0.8z^{-1} = 0 \implies z = 0.8$$

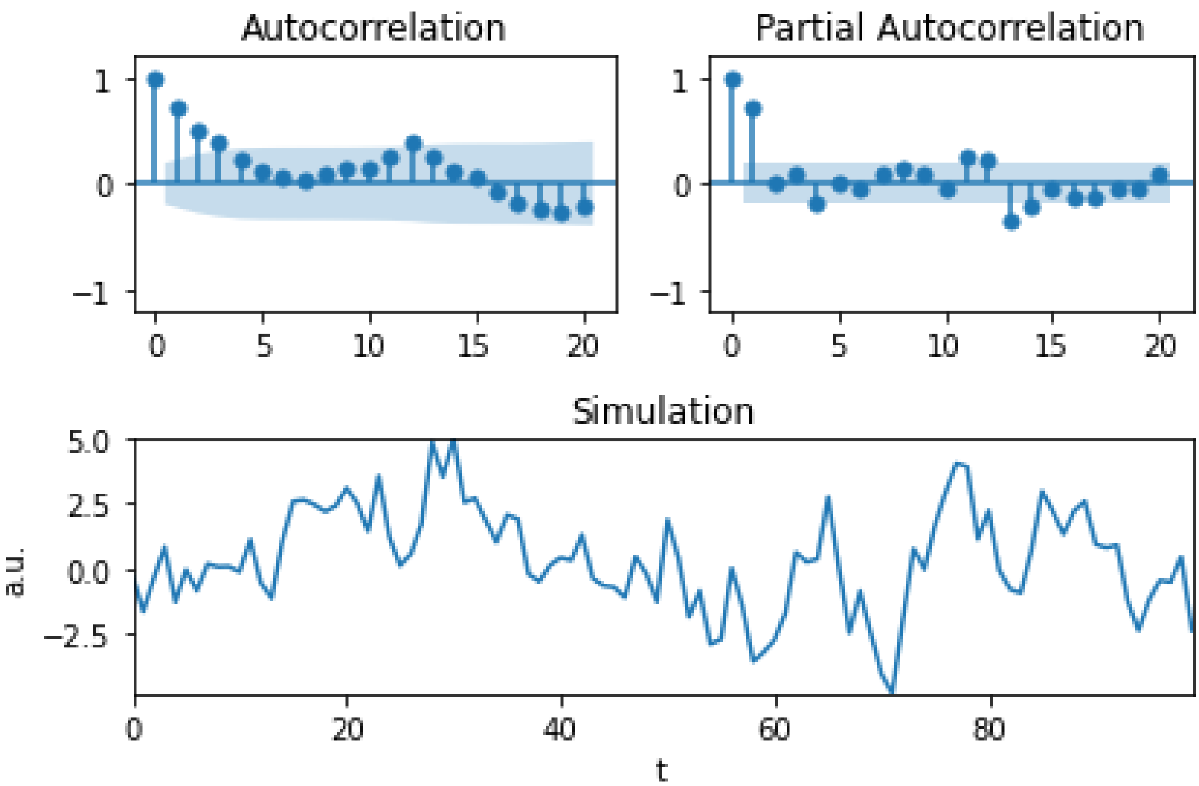

Figure 8: Simulation of a $A(1,0,0) \times (0,0,1)_{12}$ process with $\phi_1 = -0.8$ and $\Theta_1 = 0.9$. Plots of the autocorrelation function and the partial autocorrelation function is also included. This process has a seasonal MA term that can be seen as the spike at around 12 in the partial autocorrelation plot.

2.3.4. $A(1,0,0) \times (1,0,0)$ with $\phi_1 = 0.7$ and $\Phi_1 = 0.8$

This process consists of both a AR term and a seasonal AR term with a seasonality of 12. The package used to simulate this process is assuming normal ARMA processes. Thus this process needs to be transformed from a seasonal ARMA model to a normal ARMA model. Using the sign conventions stated in Equation 7 the equations describing the process becomes:

$$(1 + 0.7B)(1 + 0.8B^{12})Y_t = \varepsilon_t \tag{9}$$

Multiplying the two AR terms on the left hand side the following process description is obtained:

$$(0.7B + 0.8B^{12} + 0.56B^{13})Y_t = \varepsilon_t \tag{10}$$

The simulation of this model, together with autocorrelation and partial autocorrelation plots, is seen in Figure 9. Just by looking at the simulation the process does not seem to have any seasonality. But by inspecting the autocorrelation plots they show a significant correlation at lag 12, and thus also a seasonality of 12.

From the simulation this model also looks to be stationary as the variance and mean looks to be the same throughout the whole simulation. This is also supported by the autocorrelation plot where close correlations have opposite signs and drops down to insignificant values after just two lags.

The AR polynomial for this process is given in Equation 10, to find whether the roots of the polynomial are within the unit circle or not I found the a boundary for the radius of the roots:

$$(0.7z^{-1} + 0.8z^{-12} + 0.56z^{-13}) = 0 \implies |z| \leq 0.98$$

Thus all roots are inside the unit circle and the process is stationary.

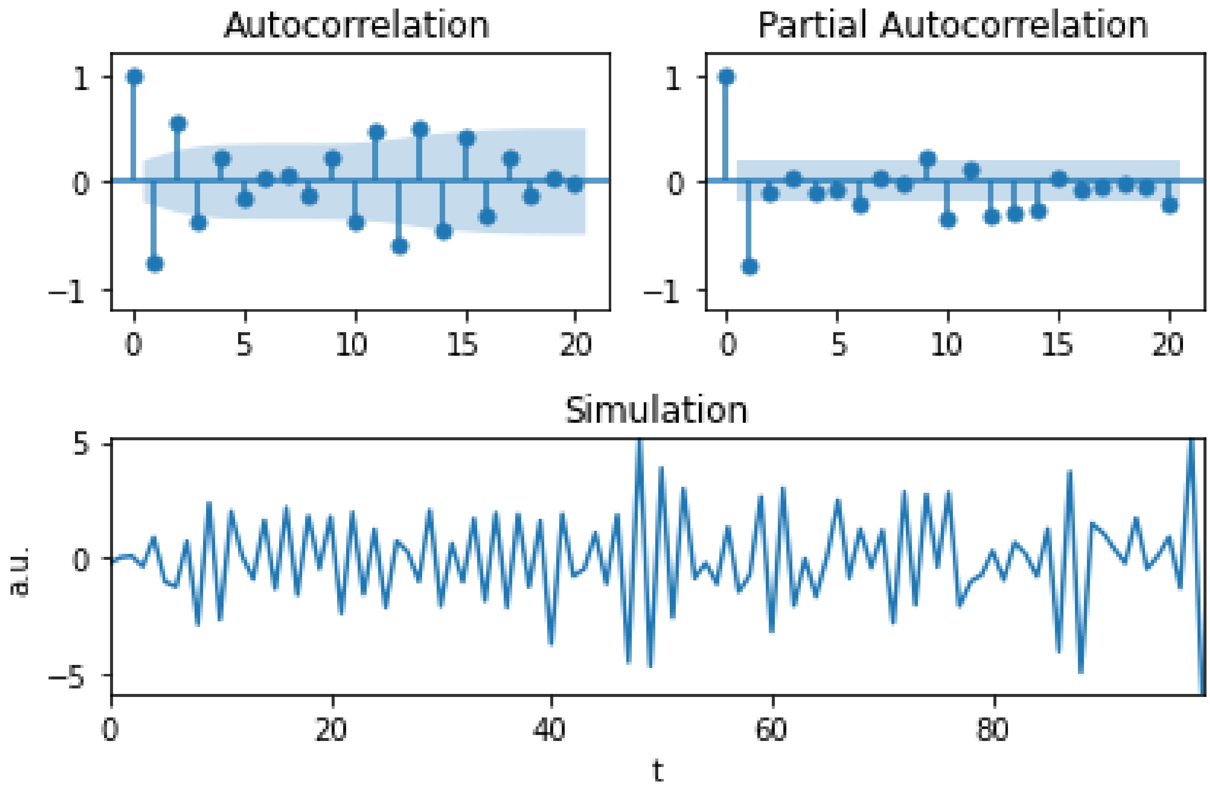

Figure 9: Simulation, autocorrelation and partial autocorrelation plots of $A(1, 0, 0) \times (1, 0, 0)$ with $\phi_1 = 0.7$ and $\Phi_1 = 0.8$. Although the autocorrelation plots shows sign of seasonality it is hard to spot in the simulation. And the process looks to be a stationary process, both when looking at the simulation and at the correlating plots.

2.3.5. $A(2, 0, 0) \times (1, 0, 0)$ with $\phi_1 = 0.5$, $\phi_2 = -0.3$ and $\Phi_1 = 0.8$

This process is also a seasonal ARMA process. Thus, to be able to model this process, it has to be transformed into a normal ARMA process. Again, following the sign convention in Equation 7, the model is transformed in the following way:

$$(1 + 0.5B - 0.3B^2)(1 + 0.8B^{12})Y_t = \varepsilon_t \tag{11a}$$
$$(1 + 0.5B - 0.3B^2 + 0.8B^{12} + 0.4B^{13} - 0.24B^{14})Y_t = \varepsilon_t \tag{11b}$$

The simulation of the above process, together with the autocorrelation and partial autocorrelation plots can be seen in Figure 10

The simulation itself shows no sign of seasonality. But also here the autocorrelation and partial autocorrelation plots comes in handy. Both correlation plots show significant correlations around lag 12, clearly indicating a seasonality of 12.

When it comes to stationarity, the simulation looks indeed to be stationary as both the mean and variance looks the same throughout the whole simulation. This is further supported by the autocorrelation plots decaying quickly to insignificant after just one lag.

Looking at the roots of the AR polynomial also implies that this is a stationary process. The solution of the polynomial gives this boundary on the radius of the roots:

$$1+0.5z^{-1}-0.3z^{-2}+0.8z^{-12}+0.4z^{-13}-0.24z^{-14}=0 \implies |z|\leq 0.98 \tag{12}$$

Thus all roots are within the unit circle and the process is stationary.

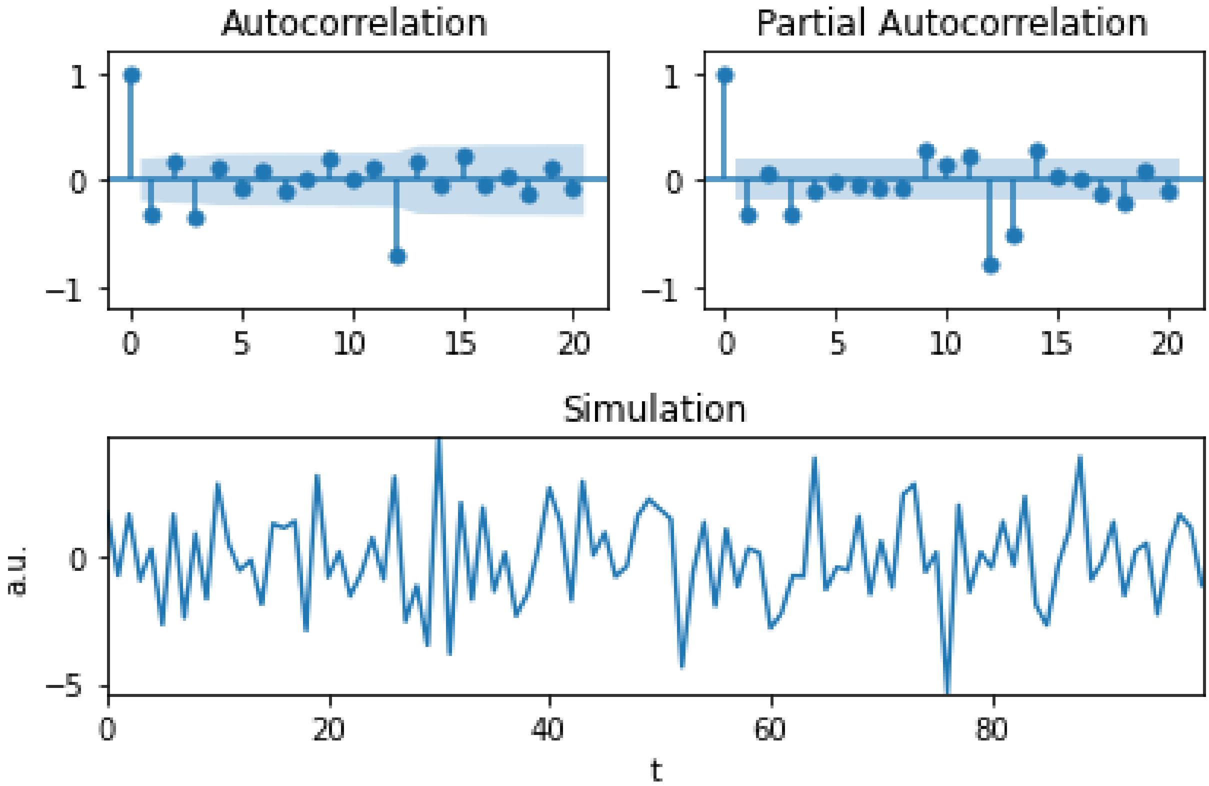

Figure 10: Simulation, autocorrelation and partial autocorrelation plots of $A(2,0,0)\times(1,0,0)$ with $\phi_1=0.5$, $\phi_2=-0.3$ and $\Phi_1=0.8$. The process looks to be a stationary process, both when looking at the simulation and at the correlating plots. The simulation seems to have the same mean and variance throughout the simulation and the correlation plots are quickly decaying towards insignificant values.

2.3.6. $A(0,0,1)\times(0,0,1)_{12}$ with $\theta_1=-0.4$ and $\Theta_1=0.8$

This process consists of both a normal moving average term and a seasonal moving average term. In order to model this process using as a normal ARMA process it needs to be transformed. By using the conventions in Equation 7 the model is transformed in the following way:

$$Y_t = (1-0.4B)(1+0.8B^{12})\varepsilon_t \tag{13a}$$
$$Y_t = (1-0.4B+0.8B^{12}-0.32B^{13})\varepsilon_t \tag{13b}$$

The simulation, together with the autocorrelation and partial autocorrelation plot can be seen in Figure 11.

For this process the seasonality are also hard to discover just by looking at the simulation. But when looking at both autocorrelation plots one can clearly see a significant spike at lag 12. This means that there are a correlation between the error at time step t and $t-12$. This error is randomly distributed, thus it is hard to spot some systematic seasonality just by looking at the simulation, although there exists one.

When it comes to stationarity the simulation looks stationary having the same variance and mean throughout the simulation. This is supported by the autocorrelation plots, both showing a quick decay to insignificant correlation after just a few lags. This is also supported by theorem 5.8 in Time Series Analysis by Henrik Madsen which says that pure MA processes always are stationary.

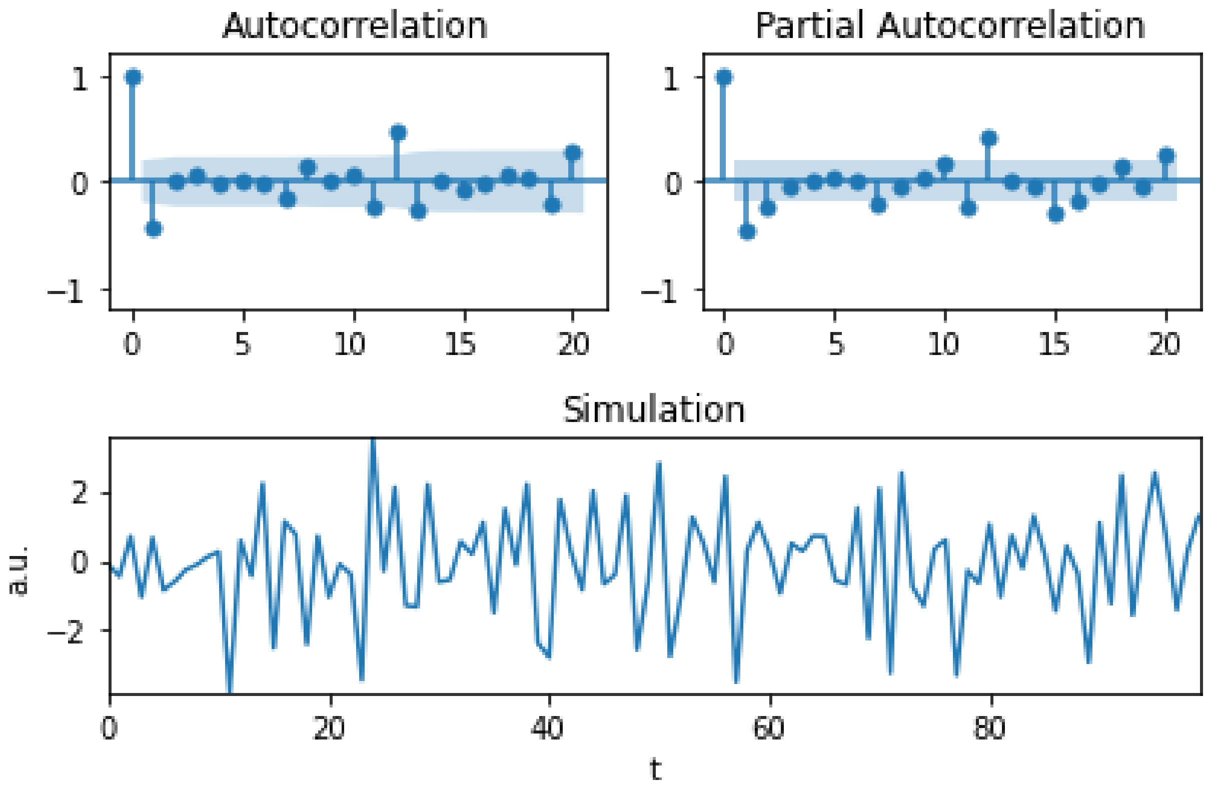

Figure 11: Simulation, autocorrelation and partial autocorrelation plot of $A(0,0,1)\times(0,0,1)_{12}$ with $\theta_1=-0.4$ and $\Theta_1=0.8$. Note how the simulation looks to have no seasonality, but both correlation plots shows a correlation at lag 12. This is due to the seasonality being a function of the random error, ε_t.

In conclusion, all processes with a seasonal term shows seasonality both in the autocorrelation and partial autocorrelation plot. This is just as expected. All processes are also stationary. In all cases this is supported both by the simulations having approximately the same mean and variance throughout the simulations, the autocorrelation and partial autocorrelation plots are decaying quickly to insignificant autocorrelations or decaying with opposite signs on each lags, and the polynomials for the AR term of the process having its roots inside the unit circle.

From this I conclude that the autocorrelation function for seasonal processes are a good tool for spotting seasonality. In most of the processes it is hard to spot the seasonal trend just by looking at the data, and even harder to determine the periodicity of the seasonality. And in all of these cases the autocorrelation plots shows clearly a seasonality of the right number of lags.

When it comes to stationary it is harder to determine from the autocorrelation plots whether a process is stationary or not. Some of the autocorrelation plots shows a more quick decrease in the correlation than others. Process 1 and 3 are examples were the autocorrelation function has high positive correlations between close lags while still being stationary. Thus it is useful to both look at the autocorrelation function and the actual process to determine stationarity. But if the coefficients are known they can be used to easily determine stationarity.

List of Figures

References

[1] Madsen H.(2007) *Time Series Analysis*, Chapman & Hall/CRC.