Lista 4

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Zadanie 1.

Teoria:

baza lub jej rozszerzenie — obraz

$F: V \to W$, jeśli $LIN(v_1, \dots, v_n) = V$ to $LIN(F(v_1), \dots, F(v_n)) = \operatorname{Im} F$

1) $F(x,y,z) = (2x+y,\ 3x-z,\ 5x+y-z,\ -2x+2y-2z)$

$v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$ } baza V

baza jądra

$F(v_1) = (2,3,5,-2)$
$F(v_2) = (1,0,1,2)$
$F(v_3) = (0,-1,-1,-2)$

$\begin{vmatrix} 2 & 3 & 5 & -2 \\ 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 3 & -6 \\ 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 2 & -4 \end{vmatrix}$ niezależny => baza

2) $G(x,y,z) = (x+y,\ y-2z,\ 3z,\ x-y)$

baza jak wcześniej

$G(1,0,0) = (1,0,0,1)$
$G(0,1,0) = (1,1,0,-1)$
$G(0,0,1) = (0,-2,3,0)$

widać, że niezależne (schodki)

3) $H(x,y,z) = (x+y,\ y+z)$

$H(0,0,1) = (0,1)$
$H(1,0,0) = (1,0)$
$H(0,1,0) = (1,1)$

baza, bo tworzą ten

Zadanie 2.

1) $H(x,y,z) = (x+y,\ y+z)$

$\begin{cases} x+y=0 \\ y+z=0 \end{cases} \Leftrightarrow \begin{cases} x=-y \\ y+z=0 \end{cases} \Leftrightarrow \begin{cases} x=-y \\ z=-y \end{cases}$

Wektory takie, że $H(\vec{v})=0$ mają postać $(-y, y, -y)$, zatem są wielokrotnościami $(-1,1,-1)$ — baza

2) $I(x,y,z) = (x+y,\ 2y+z,\ y-z)$

$\begin{cases} x+y=0 \\ 2y+z=0 \\ y-z=0 \end{cases} \Leftrightarrow \begin{cases} x=-y \\ 2y+z=0 \\ y=z \end{cases} \Leftrightarrow \begin{cases} x=-y \\ 3y=0 \\ y=z \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$

baza $= (0,0,0)$

3) $J(x,y,z) = (x+y,\ 2x+2y,\ 3x+3y)$

$\begin{cases} x+y=0 \\ 2x+2y=0 \\ 3x+3y=0 \end{cases} \Leftrightarrow \{ x=-y$

wektory są postaci $(-y, y, z)$, więc bazą będzie $\{(-1,1,0), (0,0,1)\}$

sprawdzić

~~Zadanie 3.~~

Zadanie 5.

$Id \cdot A = A$

$Id \cdot A = (c_{ij})$

$c_{ij} = \sum_{l=1}^{k} id_{il} \cdot a_{lj}$ — skoro tylko $id_{ii} \neq 0$: $= id_{ii} \cdot a_{ij} = 1 \cdot a_{ij} = a_{ij}$

$B \cdot Id = B$ identycznie

$\boxed{A \cdot (B+C)} = \boxed{A \cdot B + A \cdot C}$

$c_{ij} = \sum_{l=1}^{k} a_{il}(b_{lj} + c_{lj})$

widać, że identyczne

$d_{ij} = \sum_{l=1}^{k} (a_{il} b_{lj}) + (a_{il} c_{lj})$

$(A+B)C = A \cdot C + B \cdot C$

$\sum_{l=1}^{k} (a_{il} + b_{il}) \cdot c_{lj}$

widać

$\underbrace{\sum_{l=1}^{k} (a_{il} b_{lj})}_{(A \cdot C)_{ij}} + (a_{il} c_{lj})$

$\alpha(A \cdot B) = (\alpha A)B = A(\alpha B)$

$\alpha \sum_{l=1}^{k} a_{il} b_{lj} = \sum (\alpha a_{il}) b_{lj} = \sum a_{il} (\alpha b_{lj})$

$\boxed{A \cdot [B \mid C] = [AB \mid AC]}$???

to samo w drugą stronę też nie zbyt jak

ZADANIE 6.

$\boxed{(A \cdot B) \cdot C = A \cdot (B \cdot C)}$

$L = \sum_l (ab_{il} c_{lj})$

$ab_{il} = \sum_{x=1}^{k} a_{ix} b_{xl}$

$L = \sum_l \left(c_{lj} \sum_x a_{ix} b_{xl} \right) = \sum_l \left(\sum_x a_{ix} b_{xl} c_{lj} \right)$

$P = \sum_l a_{il} (bc_{lj})$

$P = \sum_l a_{il} \left(\sum_x b_{lx} c_{xj} \right) = \sum_l \left(\sum_x a_{il} b_{lx} c_{xj} \right)$

for (l) for (x) suma += $a_{il} b_{lx} c_{xj}$ $\overset{?}{=}$ for (x) for (l) suma += $a_{il} b_{lx} c_{xj}$

tak, bo i tak dodaje liczby rzeczywiste

!

równe

Zadanie 4.

$F: V \to W$

1) różnowartościowe => tylko $F(\vec{0}) = 0$, więc tylko $\vec{0} \in \ker(F)$ => 1 => 2

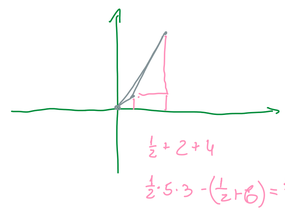
2) $\dim(\ker(F)) = 0 \Leftrightarrow$ zawiera tylko $\vec{0}$

3) $\ker(F)$ ma jeden wektor (zawiera ma 0, więc 2 => 3)

4) $\dim(\operatorname{Im}(F)) = \dim(V)$

$1 \to 2 \to 3 \Leftrightarrow 1,2,3$

4 4



$\frac{1}{2} + 2 + 4$

$\frac{1}{2} \cdot 5 \cdot 3 - \left(\frac{1}{2} + 6\right) = \frac{15}{2} - \frac{13}{2} = \frac{2}{2} = 1$

$3 \to 1 \Leftrightarrow (\neg 1 \Rightarrow \neg 3)$

$\exists v_1 \neq v_2 \quad F(v_1) = F(v_2)$

$v_1 - v_2 \neq \vec{0}$

$F(v_1 - v_2) = F(v_1) - F(v_2) = \vec{0}$, zatem

w $\ker(F)$ jest $v_1 - v_2$, więc jest tam coś poza $\vec{0}$ ↯

$\boxed{\vec{0} \in \ker(F)}$

$4 \to 1$

! Wzorek (tw. z wykładu:)

$\dim(V) = \dim(\operatorname{Im}(F)) + \dim(\ker(F))$

⇓

widać, że 4 ⇔ 2 □

Zadanie 9.

musi być tyle samo

$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = Id \cdot 9$

$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot 9 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot 9 = \begin{bmatrix} 9 & 18 & 18 \\ 18 & 9 & -18 \\ 18 & -18 & 9 \end{bmatrix}$

$\begin{bmatrix} A \\ A \end{bmatrix} \cdot \begin{bmatrix} A & A \end{bmatrix} = \begin{bmatrix} A^2 & A^2 \\ A^2 & A^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 & 9 & 0 & 0 \\ 0 & 9 & 0 & 0 & 9 & 0 \\ 0 & 0 & 9 & 0 & 0 & 9 \\ 9 & 0 & 0 & 9 & 0 & 0 \\ 0 & 9 & 0 & 0 & 9 & 0 \\ 0 & 0 & 9 & 0 & 0 & 9 \end{bmatrix}$

$\boxed{\begin{bmatrix} A \\ B \end{bmatrix} \cdot \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} A \cdot C & A \cdot D \\ B \cdot C & B \cdot D \end{bmatrix}}$

chyba tak wychodzi

Zadanie 8.

$\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}^4$

$\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}^2 = \begin{bmatrix} \alpha^2+1 & 2\alpha \\ 2\alpha & \alpha^2+1 \end{bmatrix}$ $\quad (\alpha+1)^2 - 2\alpha \quad 2\alpha$; $\quad 2\alpha \quad (\alpha^2+1)^2 - 2\alpha$

$\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \alpha^2+1 & 2\alpha \\ 2\alpha & \alpha^2+1 \end{bmatrix} = \begin{bmatrix} 2\alpha^3+2\alpha & 2\alpha^2+2\alpha+1 \\ 2\alpha^2+2\alpha+1 & 2\alpha^3+2\alpha \end{bmatrix} = \begin{bmatrix} \alpha^3+3\alpha & 3\alpha^2+1 \\ 3\alpha^2+1 & \alpha^3+3\alpha \end{bmatrix}$

$(\alpha+1)^3 - (3\alpha^2+1)$

$3\alpha^2+1$

$\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix} \cdot$

? ???

$(\alpha+1)^4$

$\alpha^4 + (\alpha-1)^4$

$\alpha^4 + (\alpha)\alpha^{4-1} + \dots \alpha + 1$

$\alpha^4 - (\alpha)\alpha^{4-1} + \dots \pm (\alpha) + 1$

$2\alpha^4 + 2\alpha^{4-2}$

$\frac{1}{2}\left[(\alpha+1)^4 + (\alpha-1)^4\right]$ = co drugiej potęgi a dodajemy wyrażenia liniowe

Zadanie 10.

$M = (m_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

$M^T = (m_{ji})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

$(A^T)^T = \left[(a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \right]^T = \left[(a_{ji})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \right]^T = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = A$

tak wystarczy?

$(A \cdot B)^T = B^T \cdot A^T$

$(A \cdot B) = \left(\sum_{l=1}^{k} a_{il} b_{lj} \right)_{\substack{i=1,\dots,m \\ j=1,\dots,m}}$

$(A \cdot B)^T = (c_{ji})_{\substack{i=1,\dots,m \\ j=1,\dots,m}}$ $\quad c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$

$A^T \cdot B^T =$ $\quad c_{ji} = \sum_{l=1}^{k} a_{jl} b_{li}$

$= (a_{ji}) \cdot (b_{ji}) =$

$= (c_{ji})$ $\quad c_{ji} = \sum_{l=1}^{k} a_{jl} b_{li}$

to samo

$(A+B)^T = A^T + B^T$

$(A+B)^T = (c_{ji})$

$\boxed{c_{ij} = a_{ij} + b_{ij}} \Rightarrow c_{ji} = a_{ji} + b_{ji}$

$A^T + B^T = (a_{ji}) + (b_{ji})$ — to samo

Zadanie 3.

$\vec{v} = a_7 x^7 + \dots + a_1 x + a_0$

$a, x \in \mathbb{Z}_5$

$F(x^i) = i x^{i-1} + i(i-1)x^{i-2} = i(x^{i-1} + (i-1)x^{i-2})$

$F(a_7 x^7 + \dots + a_0) = (a_7 x^7)' + (a_7 x^7)'' + \dots + (a_1 x)' + (a_1 x)'' + a_0 \cdot \overset{0}{F(1)}$

$F(x^7) = 7x^6 + 42x^5 = 2x^6 + 2x^5 \Rightarrow (0, \dots, 0, 1, 0)$

$F(x^6) = 6x^5 + 6 \cdot 5x^4 = x^5$

$\boxed{F(x^5) = 5x^4 + 5 \cdot 4x^3 = 0}$ => αx^5 nie zmienia

$F(x^4) = 4x^3 + 12x^2 = 4x^3 + 2x^2$

$F(x^3) = 3x^2 + 6x = 3x^2 + x$

$F(x^2) = 2x + 2$

$F(x) = \boxed{1} \to (0, 1, 0 \dots 0)$

$\boxed{F(x^0) = 0}$

baza jądra: $\begin{cases} (0,0,1,0..0) \\ (0, \dots, 0, 1) \end{cases}$

$F: A \to B$

baza dobrym A

baza jądra: $1, x^5$

baza obrazu: $1, x, x^2, x^3, x^5, x^6$

dobrym B

Zadanie 7. I przestrzeń liniowa

zrobić $\begin{cases} BA = AB \ i \ CA = AC \Rightarrow \\ \Rightarrow (B+C)A = A(B+C) \end{cases}$?

trzy $\begin{cases} BA = AB \Rightarrow (\alpha B)A = A(\alpha B) \\ (\alpha B)A = \alpha(BA) = \alpha(AB) = A(\alpha B) \end{cases}$ z zad. 5

=> przestrzeń liniowa

$(B+C)A = BA + CA = AB + AC = A(B+C)$ z zad. 5

II jeśli $AB = BA$ to $(A+B)^m = \sum_{i=0}^{m} \binom{m}{i} A^i B^{m-i}$

$(A+B)^1 = \binom{1}{1} A^0 B^1 + \binom{1}{1} A^1 B^0$ działa

$(A+B)^m = (A+B)(A+B)^{m-1} = (A+B) \sum_{j=0}^{m-1} \binom{m-1}{i} A^i B^{m-i-1} =$

$= \sum_{j=0}^{m-1} \binom{m-1}{i} A^{i+1} B^{m-i-1} + \sum_{j=0}^{m-1} \binom{m-1}{i} A^i B^{m-i} =$

$= \sum_{j=1}^{m} \binom{m-1}{j-1} A^i B^{m-i} + \sum_{j=0}^{m-1} \binom{m-1}{i} A^i B^{m-i} =$

$= \binom{m-1}{0} A^0 B^m + \sum_{j=1}^{m-1} \left[\binom{m-1}{j-1} + \binom{m-1}{i} \right] A^i B^{m+i} + \binom{m-1}{m-1} A^m B^0 =$

tyle samo

$= \binom{m}{0} A^0 B^m + \sum_{j=1}^{m-1} \binom{m}{i} A^i B^{m-j} + \binom{m}{m} A^m = \sum_{j=0}^{m} \binom{m}{i} A^i B^{m-j}$ □

indukcja

$\binom{m-1}{j-1} + \binom{m-1}{i} = \binom{m}{i}$ — sprawdzone

1 2 1
1 3 3 1
1 4 6 4 1

III $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $\quad B \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \cdot B$

$B \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} a+2b & a+b \\ c+2d & c+d \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \cdot B = \begin{bmatrix} a+c & b+d \\ 2a+c & 2b+d \end{bmatrix}$

$2b = c$

$\begin{cases} a = d \\ a = d \\ c = 2b \end{cases}$

$\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$

odwzorowanie liniowe

F

$1 \Leftrightarrow 4$

$\neg 1 \Leftrightarrow \neg 4$

$\neg 1 \Leftrightarrow \neg 2$

↓

nie różnowart.

↓

$F(v_1) = F(v_2)$

$F(v_1 - v_2) = 0$

↓

$\neg 2$

$\neg 2 \to F(v_2) = \vec{0}$, $F(\vec{0}) = \vec{0} \to \neg 1$

$\varphi \Rightarrow \varphi$

$\begin{cases} v_1 \neq v_2 \Rightarrow f(v_1) \neq f(v_2) \\ f(v_1) = f(v_2) \Rightarrow v_1 = v_2 \end{cases}$

$\neg\varphi \Rightarrow \neg\varphi$