An Introduction to Finite&Classical Model Theory Exercise List (Last updated 26th February 2023)

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2 Week 2 (06.03.2023)

2.1 Warm-Up

Call a formula *finitely satisfiable* if it has a *finite* model.

▶ **Exercise 2.1.** *Is it true that every finitely satisfiable formula is also satisfiable? Is it true that every satisfiable formula is also finitely satisfiable?*

▶ **Exercise 2.2.** *For a finite signature τ show that elementary equivalence of finite structures collapses to isomorphism. More precisely, show that if two finite τ-structures $\mathfrak{A}, \mathfrak{B}$ satisfy precisely the same* $\mathsf{FO}[\tau]$*-formulae then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

Hint: Prove first that one can completely describe a finite structure in FO.

2.2 Around Compactness Theorem

▶ **Exercise 2.3.** *During the lecture we proved that "even" is not definable in* $\mathsf{FO}[\emptyset]$*, i.e. that even is not definable over sets. Explain why such a proof cannot be adapted to the case of finite graphs or finite linear orders.*

The monadic fragment MFO of FO is composed of all first-order sentences that employ only unary predicates (and no constant symbols).

▶ **Exercise 2.4.** *Show that the is no* MFO *formula φ employing a unary predicate P for which for all finite $\mathfrak{A}$ we have $\mathfrak{A} \models \varphi$ if and only if $|P^{\mathfrak{A}}| = |A \setminus P^{\mathfrak{A}}|$, i.e. the interpretation of P and its complement are of equal sizes.*

Hint: Adapt the proof of inexpressivity of "even" in FO *given during the lecture.*

▶ **Exercise 2.5.** *Let $\mathcal{T}$ having models of arbitrary large finite sizes (e.g. for each $n \in \mathbb{N}$ there is a model $\mathfrak{A} \models \mathcal{T}$ with $|A| \geq n$). Show that $\mathcal{T}$ has an infinite model.*

▶ **Exercise 2.6.** *Disprove the following finitary version of Compactness Theorem.*

> *"Let $\mathcal{T}$ be an* FO *theory. If every subset $\mathcal{T}_0 \subseteq \mathcal{T}$ is finitely satisfiable, then $\mathcal{T}$ also is."*

▶ **Exercise 2.7.** *Employ Compactness of* FO *to show that there is a countable structure that is elementarily equivalent to $(\mathbb{Z}, \leq)$ but non-isomorphic with it.*

Hint: extend the signature with two constant symbols c, d, and consider the theory $\mathcal{T}$ of $(\mathbb{Z}, \leq)$ together with sentences φ_n for $n \in \mathbb{N}$ saying $\varphi_n =: \exists x_1 \cdots \exists x_n \; c < x_1 < x_2 < \cdots < x_n < d$.

▶ **Exercise 2.8.** *Show that "even" is not definable in finite linear orders, i.e. there is no* $\mathsf{FO}[\leq]$ *formula φ, such that for all finite structures $\mathfrak{A}$ that interpret $\leq$ as a linear order we have that $\mathfrak{A} \models \varphi$ if and only if A has even number of elements.*

Hint: IMO quite tricky. See Example 0.4.4 in Otto's notes for a sketch of the solution.

2.3 Around the Finite Model Property

We say that L has *the Finite Model Property* (FMP) if every satisfiable L-formula is also finitely satisfiable. We will give an example of logic with FMP and two logics without it. To do so, let FO^k denote the class of all first-order formulae that employ at most k variables (we stress that re-quantification of variables is possible)! Now show that:

▶ **Exercise 2.9.** *Consider a satisfiable FO^1 formula φ of the following form*

$$\bigwedge_{i=1}^{n} \forall x\ \theta_i(x) \quad \wedge \quad \bigwedge_{j=1}^{m} \exists x\ \chi_j(x),$$

where the formulae θ_i, χ_j are quantifier-free. Show that φ has a model of size linear in φ.
Remark: The following lemma can be shown: given $\varphi \in \mathsf{FO}^1[\tau]$ one can compute an equisatisfiable formula $\varphi' \in \mathsf{FO}^1[\tau']$ with $\tau \subseteq \tau'$ with a property that any model of φ' is model of φ and any model of φ can be extended (by interpreting predicates from $\tau' \setminus \tau$) to a model of φ'. Hence, we actually proved that FO^1 has FMP.

▶ **Exercise 2.10.** *Show that FO^k for any $k > 2$ as well as C^2 do not have the Finite Model Property. Here C^2 stands for an extension of FO^2 where in addition to standard quantifiers, we allow for using counting quantifiers $\exists^{\leq k}$ "at most k", $\exists^{\geq k}$ "at least k", and $\exists^{=k}$ "precisely k", for any $k \in \mathbb{N}$ (with their naïve semantics).*

Hint: Define infinite words.
Remark: FO^2 has FMP, which we can it prove later if we have time. The proof isn't obvious.

Recall that a decision problem is *decidable* if there is a computer program that solves it in finite time. Note that there is no restriction on the running time and the memory consumption.

▶ **Exercise 2.11.** *In the model checking problem for FO, we ask if an input formula φ is satisfied in an input finite structure $\mathfrak{A}$. Show that this problem is decidable.*
Remark: You may assume w.l.o.g. that φ is in prenex normal form.

▶ **Exercise 2.12.** *Let L be a syntactic fragment of FO and assume that L enjoys FMP. Prove that the satisfiability problem for L ("given $\varphi \in \mathsf{L}$, does it have a model?") is decidable.*

Hint: Do not care about the running time of your program.
If you struggle too much with a solution, ask Gödel for help.