An Introduction to Finite and Classical Model Theory

Week 1 (Wrocław 27.02.2023, Short version)

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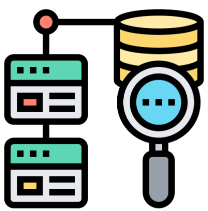
Today's agenda

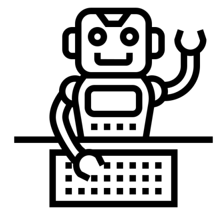
1. Basic information regarding the course.
2. An informal definition of a logic with examples.
3. Potential applications and further research options.

Query languages?



Formal verification?

Formal languages?



Complexity?

4. Recap from BSc studies: Syntax & Semantics of First-Order Logic (FO).
5. Basic notations, provability, and Gödel's theorem "$\models$ equals $\vdash$".
6. Gödel's Compactness theorem with a proof and an application.

Feel free to ask questions and interrupt me!

Please don't be shy! If needed, send me an email (bartek@cs.uni.wroc.pl) or approach me before/after the lecture!

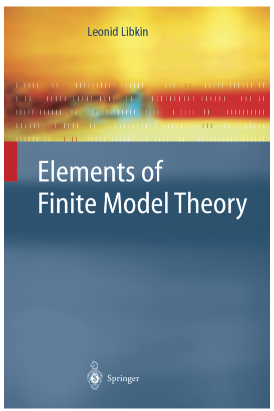
If you would like to do some research project on related topics let me know!

Course Information

https://skos.ii.uni.wroc.pl/course/view.php?id=526

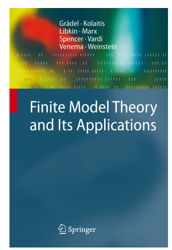
Contact me via email: bartek@cs.uni.wroc.pl

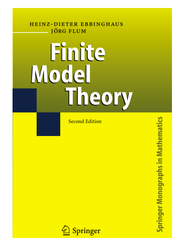
1. Lectures: Monday 10:15-12:00 (119), Exercises: ??? (???)
2. Course website: (at [SKOS]) ← check for slides, notes, and exercise lists.
3. Each week a new exercise list will be published. Do not worry if you can't solve all exercises.
4. Oral exam: question about the basic understanding + selected theorems. Intended to be easy!
5. Goal: understand power/limitations of 1st-order logic and selected fragments (with a bit of complexity).

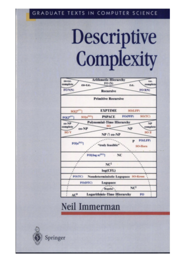


Books and literature.

+ Lecture notes by Martin Otto [HERE] and lecture notes by Erich Grädel [HERE]





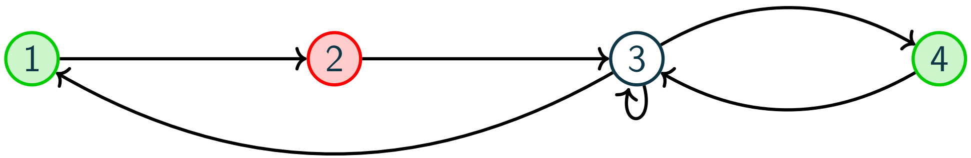


Last but Not Least: I offer MSc/PHD research projects for motivated students!

What is a “logic”? A running example.

Naively: a “formal language” for expressing properties of relational structures ($\approx$ hypergraphs).

Made formal via abstract model theory, c.f. articles at [ncatlab.org] and [Lindström’s theorems].

$\mathfrak{A} :=$ 

over a signature $\tau := \{\mathrm{G}^{(1)}, \mathrm{R}^{(1)}, \mathrm{E}^{(2)}\}$

$\mathrm{G}^{\mathfrak{A}} := \{1, 4\}, \qquad \mathrm{R}^{\mathfrak{A}} := \{2\}$

$\mathrm{E}^{\mathfrak{A}} := \{(1, 2), (2, 3), (3, 1), (3, 3)(3, 4), (4, 3)\}$

A signature contains (at most countably* many) constant and relation symbols (each with a fixed arity).

Structure = Domain + interpretation of symbols, e.g. $\mathfrak{A} := (A, \cdot^{\mathfrak{A}})$ depicted above,

where $A = \{1, 2, 3, 4\}$ and $\cdot^{\mathfrak{A}}(\mathrm{G}), \cdot^{\mathfrak{A}}(\mathrm{R}), \cdot^{\mathfrak{A}}(\mathrm{E})$ are as above.

Constants $\approx$ elements, unary relations $\approx$ colours, binary (resp. higher-arity) relations $\approx$ (hyper)edges

Example (of a First-Order Logic (FO) Formula)

(in a coloured graph:) Any node is either green or red.

$\varphi := \forall x \, (\mathrm{G}(x) \vee \mathrm{R}(x)) \wedge (\mathrm{G}(x) \leftrightarrow \neg \mathrm{R}(x))$

We write $\mathfrak{A} \models \varphi$ to indicate that

$\mathfrak{A}$ satisfies φ or $\mathfrak{A}$ is a model of φ.

Formulae often employ: Variables: $x, y, z, X, Y, \ldots$ Boolean connectives: $\wedge, \vee, \neg, \leftrightarrow, \bigvee_{i=0}^{\infty}, \ldots$

Quantifiers: $\forall, \exists, \exists^{even}, \exists^{=42}, \exists^{35\%}, \exists Set, \Diamond$, Predicates (relational symbols): $\mathrm{P}, \in, =, \sim$, and more?

More examples I.

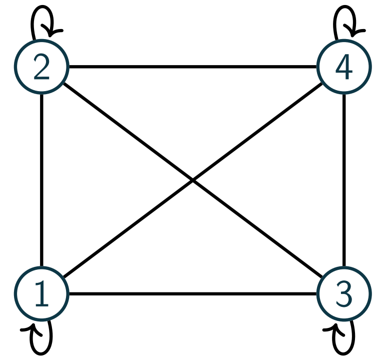
Exercise (An FO[$\{\mathrm{E}^{(2)}\}$] formula/query testing if a graph is a 4-element clique [here E = edge relation].)

1. There are precisely 4 elements . . .

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \Big(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge x_3 \neq x_4$$
$$\wedge \forall x \big[x = x_1 \vee x = x_2 \vee x = x_3 \vee x = x_4 \big] \Big)$$

2. and any two of them are linked by E.

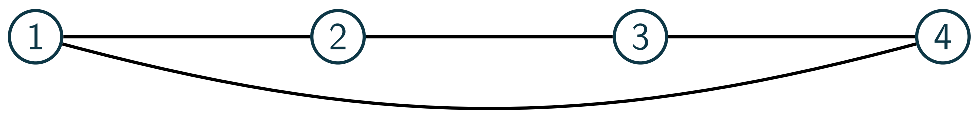
$$\wedge\ \forall x \forall y\ \mathrm{E}(x, y).$$



$\models \varphi_{4\text{clique}}$

Exercise (Write an MSO$\{\mathrm{E}^{(2)}\}$ checking if a graph is two-colourable.)

$\mathfrak{G} :=$

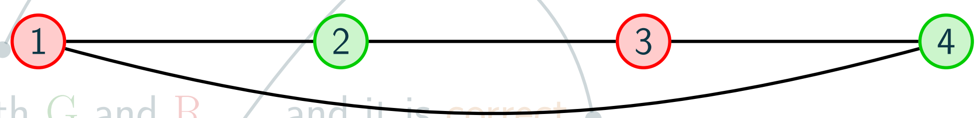


$\models \varphi_{2COL}$

$$\varphi_{2COL} = \exists \mathrm{G} \exists \mathrm{R}\ \forall x\ \big((x \in \mathrm{G} \vee x \in \mathrm{R}) \wedge (x \in \mathrm{G} \leftrightarrow x \notin \mathrm{R})\ \wedge\ \varphi_{ok} \big)$$

$$\varphi_{ok} = \forall x (x \in \mathrm{G} \rightarrow (\forall y\ E(x, y) \rightarrow y \in \mathrm{R})) \wedge \forall x (x \in \mathrm{R} \rightarrow (\forall y\ E(x, y) \rightarrow y \in \mathrm{G}))$$

Quantification over sets: $\mathfrak{G}' :=$



$\models \varphi_{ok}$

There exists a colouring with G and R . . . and it is correct

More examples II.

Exercise (Write an $\mathrm{FO}[\{\mathrm{E}^{(2)}, \mathrm{a}, \mathrm{b}\}]$ formula $\varphi_k^{\mathrm{reach(a,b)}}$ testing if there is a path from a to b of length k.)

1. Case $k = 0$ is trivial: Take $\varphi_0^{\mathrm{reach(a,b)}} := \mathrm{a} = \mathrm{b}$
2. Case $k = 1$ is easy too: Take $\varphi_1^{\mathrm{reach(a,b)}} := \mathrm{E}(\mathrm{a}, \mathrm{b})$
3. Case $k = 2$ is a tiny bit harder: Take $\varphi_2^{\mathrm{reach(a,b)}} := \exists x_1 \mathrm{E}(\mathrm{a}, x_1) \land \mathrm{E}(x_1, \mathrm{b})$
4. Case $k = 3$ is a similar: Take $\varphi_3^{\mathrm{reach(a,b)}} := \exists x_1 \exists x_2 \mathrm{E}(\mathrm{a}, x_1) \land \mathrm{E}(x_1, x_2) \land \mathrm{E}(x_2, \mathrm{b})$
5. So for any $k \geq 2$ just take: Take $\varphi_k^{\mathrm{reach(a,b)}} := \exists x_1 \ldots \exists x_{k-1}\, \mathrm{E}(\mathrm{a}, x_1) \land \bigwedge_{i=1}^{k-2} \mathrm{E}(x_i, x_{i+1}) \land \mathrm{E}(x_{k-1}, \mathrm{b})$

Question (Can we do better in terms of the total number of quantifiers?)

The current state of the art: $\log_2(k) - \mathcal{O}(1) \leq\ ???\ \leq 3\log_3(k) + \mathcal{O}(1)$ by Fagin at al. [MFCS 2022]

Exercise (Write an $\mathrm{FO} + \bigvee_{i=0}^{\infty}$ formula φ^{conn} over $\{\mathrm{E}^{(2)}\}$ that tests if a given graph is E-connected.)

$$\varphi^{\mathrm{reach(a,b)}} := \forall x \forall y\ \bigvee_{i=0}^{\infty} \varphi_k^{\mathrm{reach(a,b)}}[\mathrm{a}/x, \mathrm{b}/y].$$

Is there a chance to get a "plain" FO formula?

No. And we will show it today!

Syntax and Semantics of FO. Signatures (vocabularies)

Signature σ is a (countable) collection of symbols: $(\mathsf{c}_1, \mathsf{c}_2, \ldots, \mathrm{R}_1, \mathrm{R}_2, \ldots)$

Constant symbols, e.g. $\emptyset, 7, \texttt{Bartek}$

Relational symbols, e.g. $\in, \subseteq, \mathrm{isEven}$ with an associated arity, e.g. $\mathsf{ar}(\subseteq) = 2, \mathsf{ar}(\mathrm{isEven}) = 1$

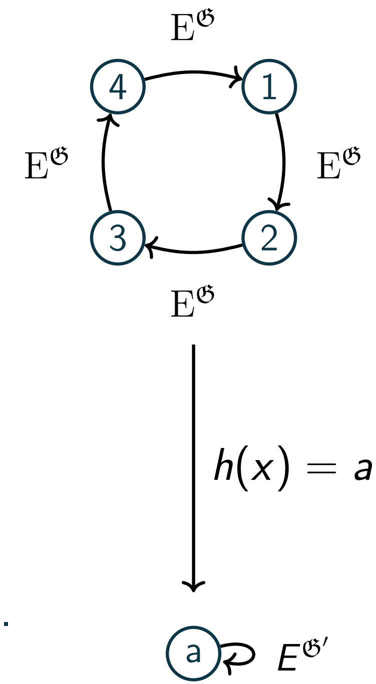
Structures

For a signature σ we define σ-structures $\mathfrak{A}$ as pairs $(A, \cdot^{\mathfrak{A}})$ composed of:

- Non-empty set A called the domain of $\mathfrak{A}$ + Interpretation function $\cdot^{\mathfrak{A}}$ such that:

1. For each constant symbol c, we have $\cdot^{\mathfrak{A}} : \mathsf{c} \mapsto (\mathsf{c}^{\mathfrak{A}} \in A)$

2. For each relational symbol R, we have $\cdot^{\mathfrak{A}} : \mathrm{R} \mapsto (\mathrm{R}^{\mathfrak{A}} \subseteq A^{\mathsf{ar}(\mathrm{R})})$



Morphisms

Let $\mathfrak{A}, \mathfrak{B}$ be σ-structures. A σ-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathfrak{h} : A \to B$ satisfying:

- For all constant symbols $\mathsf{c} \in \sigma$ we have $\mathfrak{h}(\mathsf{c}^{\mathfrak{A}}) = \mathsf{c}^{\mathfrak{B}}$, and
- For all relational symbols $\mathrm{R} \in \sigma$, $\mathrm{R}^{\mathfrak{A}}(a_1, \ldots, a_{\mathsf{ar}(\mathrm{R})})$ implies $\mathrm{R}^{\mathfrak{B}}(\mathfrak{h}(a_1), \ldots, \mathfrak{h}(a_{\mathsf{ar}(\mathrm{R})}))$.

An isomorphism $\mathfrak{h}$ between $\mathfrak{A}$ and $\mathfrak{B}$ is a bijection s.t. $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are homomorphisms.

In this case we write: $\mathfrak{A} \cong \mathfrak{B}$. Important! $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$ for all formulae φ.

Syntax of $\mathsf{FO}[\sigma]$

- Let $\mathrm{Var} := \{x, y, z, u, v, \ldots\}$ be a countably-infinite set of variables.
- The set of terms is $\mathsf{Terms}(\sigma) := \mathrm{Var} \cup \{\mathsf{c} \mid \mathsf{c}$ is a constant symbol from $\sigma\}$.
- The set of atomic formulae $\mathsf{Atoms}(\sigma)$ is the smallest set such that:

1. If t_1, t_2 are terms from $\mathsf{Terms}(\sigma)$ then $t_1 = t_2$ belongs to $\mathsf{Atoms}(\sigma)$.

2. If $t_1, \ldots, t_{\mathsf{ar}(\mathrm{R})} \in \mathsf{Terms}(\sigma)$, and $\mathrm{R} \in \sigma$ is relational implies $\mathrm{R}(t_1, \ldots, t_{\mathsf{ar}(\mathrm{R})}) \in \mathsf{Atoms}(\sigma)$.

- The set $\mathsf{FO}[\sigma]$ of First-Order formulae over σ is the closure of $\mathsf{Atoms}(\sigma)$ under $\land, \lor, \rightarrow, \leftrightarrow, \neg, \exists x, \forall x$ (for all variables $x \in \mathrm{Var}$).

Free variables

$\exists x\ (E(x, y) \land \forall z\ (E(z, y) \rightarrow x = z))$ $\qquad$ $\exists x\ (E(x, y) \land \exists y\ \neg E(y, x))$

Formally, we define the set of free variables of φ, denoted with $\mathsf{FVar}(\varphi)$, as follows:

- $\mathsf{FVar}(x) = \{x\}, \mathsf{FVar}(\mathsf{c}) = \emptyset$ for all $x \in \mathrm{Var}$ and constant symbols c from σ.
- $\mathsf{FVar}(t_1 = t_2) = \mathsf{FVar}(t_1) \cup \mathsf{FVar}(t_2)$ for all $t_1, t_2 \in \mathsf{Terms}(\sigma)$.
- $\mathsf{FVar}(\neg\varphi) = \mathsf{FVar}(\varphi)$ and $\mathsf{FVar}(\varphi \land \psi) = \mathsf{FVar}(\varphi) \cup \mathsf{FVar}(\psi)$. (and similarly for $\rightarrow, \leftrightarrow, \lor, \top, \bot$)
- $\mathsf{FVar}(\exists x\ \varphi) = \mathsf{FVar}(\varphi) \setminus \{x\}$ for all $x \in \mathrm{Var}$.

Notation regarding formulæ

We write $\varphi(x_1, x_2, \ldots, x_k)$ to indicate that the variables $x_1, \ldots, x_k$ are free in φ.

A formula without free-variables is called a sentence.

A formula without occurrences of $\forall, \exists$ is called a quantifier-free.

A (possibly infinite) set of sentences is called a theory.

Semantics of FO

For a σ-structure $\mathfrak{A}$ we define inductively, for each term $t(x_1, x_2, \ldots, x_n)$

the value of $t^{\mathfrak{A}}(a_1, \ldots, a_n)$, where $(a_1, \ldots, a_n) \in A^n$ as follows:

1. For a constant symbol $\mathrm{c} \in \sigma$, the value of c in $\mathfrak{A}$ is $\mathrm{c}^{\mathfrak{A}}$.
2. The value of x_i in $t^{\mathfrak{A}}(a_1, a_2, \ldots, a_n)$ is a_i.

Now we define $\models$ for $\varphi(x_1, x_2, \ldots, x_n)$:

- If $\varphi \equiv t_1 = t_2$, then $\mathfrak{A} \models \varphi(\bar{a})$ iff $t_1^{\mathfrak{A}}(\bar{a}) = t_2^{\mathfrak{A}}(\bar{a})$.
- If $\varphi \equiv \mathrm{R}(t_1, t_2, \ldots, t_n)$, then $\mathfrak{A} \models \varphi(\bar{a})$ iff $(t_1^{\mathfrak{A}}(\bar{a}), \ldots, t_n^{\mathfrak{A}}(\bar{a})) \in \mathrm{R}^{\mathfrak{A}}$.
- $\mathfrak{A} \models \neg\varphi$ iff not $\mathfrak{A} \models \varphi$; $\quad \mathfrak{A} \models \varphi \land \psi$ iff $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \psi$ (similarly for other connectives)
- If $\varphi \equiv \exists x\ \psi(x, \bar{y})$, then $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{A} \models \psi(a', \bar{a})$ for some $a' \in A$ (similarly for the $\forall$ quantifier)

The last bunch of notations. Proof systems.

A formula φ is satisfiable if it has a model (there is a structure $\mathfrak{A}$ s.t. $\mathfrak{A} \models \varphi$).

For a theory $\mathcal{T}$ (set of sentences) we write $\mathfrak{A} \models \mathcal{T}$ instead of $\mathfrak{A} \models \bigwedge_{\varphi \in \mathcal{T}} \varphi$.

φ is a tautology iff every structure satisfies φ (written: $\models \varphi$). Note: φ is a tautology iff $\neg\varphi$ is unsatisfiable.

$\mathcal{T}_1 \models \mathcal{T}_2$ means that every model of $\mathcal{T}_1$ is also a model of $\mathcal{T}_2$. Note: $\mathcal{T} \models \bot$ iff $\mathcal{T}$ is unSAT.

Warning! Models can be of any size: finite, countably-infinite and even larger!

Löwenheim–Skolem 1922

If a countable theory $\mathcal{T}$ has an infinite model then $\mathcal{T}$ has a countably-inf one.

FO has dedicated proof systems, e.g. Gentzen's sequents or Natural Deduction. Check T. Lyon's [Lectures]

$\mathcal{T} \vdash \varphi$ means φ is provable from $\mathcal{T}$ via some proof system.Proofs are finite!

(we treat $\mathcal{T}$ as extra axioms, note that proofs are finite)

Gödel 1929: $\mathcal{T} \models \varphi$ iff $\mathcal{T} \vdash \varphi$

SAT for FO is Recursively Enumerable

$$
\begin{prooftree}
\AxiomC{Ax}
\UnaryInfC{$\forall x[P(x)], \forall x[P(x) \to Q(x)], Q(a) \vdash Q(a)$}
\AxiomC{Ax}
\UnaryInfC{$\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash P(a), Q(a)$}
\RightLabel{$[\neg \vdash]$}
\UnaryInfC{$\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \vdash Q(a)$}
\RightLabel{$[\vee \vdash]$}
\BinaryInfC{$\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \vee Q(a) \vdash Q(a)$}
\RightLabel{$[\to\vdash\ r.w.]$}
\UnaryInfC{$\forall x[P(x)], \forall x[P(x) \to Q(x)], P(a) \to Q(a) \vdash Q(a)$}
\RightLabel{$[\forall \vdash]$}
\UnaryInfC{$\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash Q(a)$}
\RightLabel{$[\vdash \forall]$}
\UnaryInfC{}
\end{prooftree}
$$

Motivations I: why do we care about logic?

Query: Give me IDs of all candidates who applied for "computer science".

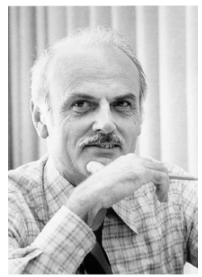
```sql
SELECT CandID
FROM Candidate
WHERE Major = "Computer Science"
```

$\rightsquigarrow \quad \varphi(i)$

$$\varphi(i) = \exists n \exists s \ \textsc{Candidate}(i, n, s) \land \textsc{Appl}(\texttt{"Computer Science"}, i)$$

Theorem (Codd 1971)

Basic SQL $\approx$ First-Order Logic



Other useful logic: Datalog $\approx$ SQL + recursion

1. VLog: a rule engine for querying data graphs
2. Vadalog: querying data graphs based on Datalog

Nice lecture on VadaLog by Gottlob [here], and a course on knowledge graphs by Krötzsch [here].

Description logics: a family of logics for knowledge representation.

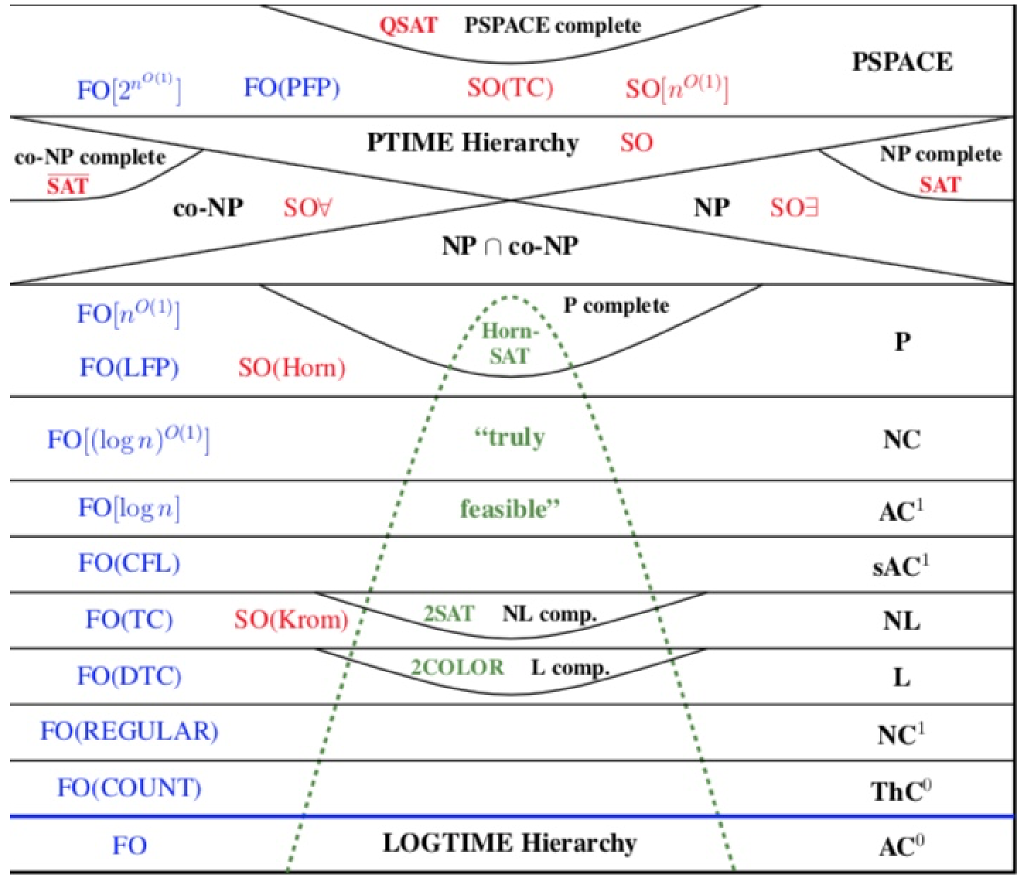
Motivations II: why do we care about logic?

In "standard" computational complexity we measure resources, e.g. space and time.

Descriptive complexity: how strong the language must be to describe the problem?

A logic $\mathcal{L}$ characterises the complexity class $\mathcal{C}$ if for every property of finite structures $\mathcal{P}$:

1. $\mathcal{P}$ is expressible in $\mathcal{L}$ if and only if
2. There is an algorithm in $\mathcal{C}$ deciding $\mathcal{P}$.



Theorem (Fagin'1973)

Existential Second Order Logic characterises NP.

Is there a logic for PTIME?
No idea since 1988.

Motivations III: why do we care about logic?

Meta algorithms: say what you want instead of writing code! Hot topic nowadays!

Is every property of graphs expressible in FO is checkable in linear time for all graphs from class $\mathcal{C}$?

Theorem (Courcelle 1990)

$\mathcal{C} :=$ graphs of bounded-treewidth.

Theorem (Seese 1996)

$\mathcal{C} :=$ graphs of bounded-degree.

Theorem (Dvorák et al. 2010)

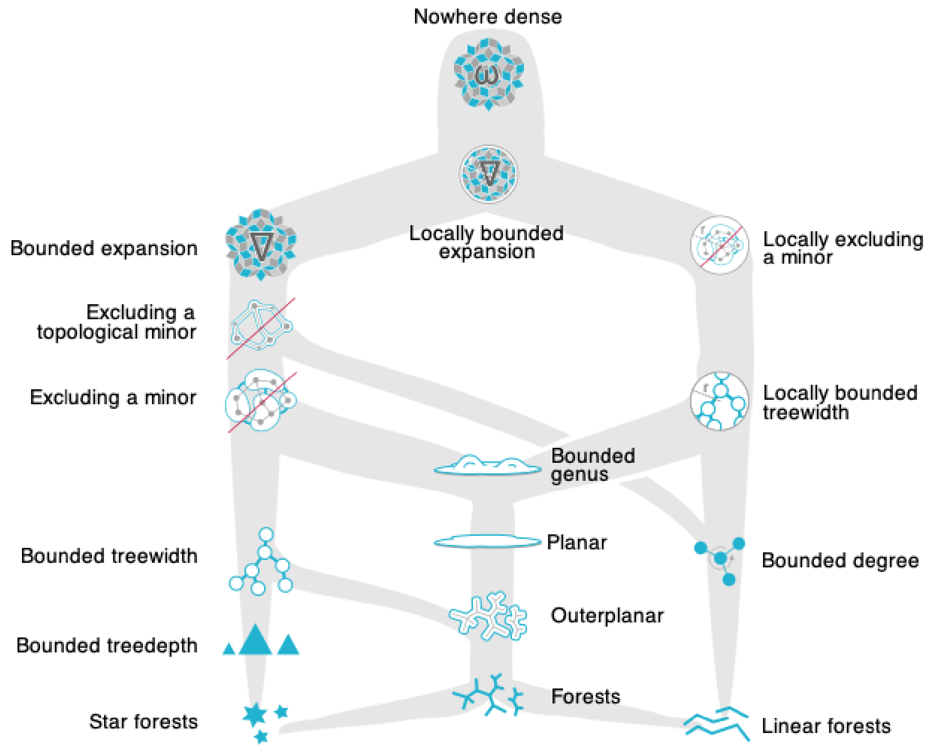
$\mathcal{C} :=$ graphs of bounded-expansion.

Theorem (Bonnet et al. 2022)

$\mathcal{C} :=$ graphs of bounded-twinwidth.

Theorem (Grohe, Kreutzer, Siebertz 2014)

$\mathcal{O}(|\varphi|^{1+\varepsilon})$ for $\mathcal{C} :=$ nowhere-dense graphs.



What kind of problems do we study? Over finite and over unrestricted models.

1. Given a logic L, can we characterise its expressive power?

- Algebra: Relational calculus [Codd'72][Jaakkola&Kuusisto'23], Languages recognizable by monoids [Book]
- Games: Ehrenfeucht–Fraïssé games [LINK], Bisimulations [LINK], Pebble Games [LINK] and more...
- Counting: $\mathfrak{A} \equiv_{\mathsf{L}} \mathfrak{B}$ if for all $\mathfrak{C} \in \mathcal{C}$ the number of homomorphisms $\mathfrak{C} \to \mathfrak{A}$ and $\mathfrak{C} \to \mathfrak{B}$ coincide.
- L $=$ FO and $\mathcal{C}$ is composed of all finite structures [Lovasz'67]
- L $=$ FO of quant. depth $\leq k$ + counting quant. $\exists^{\geq k}$; $\mathcal{C}$ = finite str. of tree-depth at most k [Grohe'20]
- L $=$ k-variable FO + counting quant. $\exists^{\geq k}$; $\mathcal{C}$ = finite str. of tree-width at most k [Dvorák'10]

2. Given a semantic property $\mathscr{P}$, is there a syntactic fragment L $\subseteq$ FO that captures precisely $\mathscr{P}$?

- Preserved under extensions = Existential FO [Łoś-Tarski'54]. Fails in the finite [`Tait 1959`].
- Monotone in a pred. P = eq. to φ that is positive in P [`Lyndon 1959`]. Fails in the finite [Kuperberg'23].
- Preserved under homomorphisms = Positive $\exists^*$FO [Łoś-Tarski-Lyndon'54]. Stays in finite [`Rossmann'05`].

3. Does L $\subseteq$ FO inherit good properties of FO? E.g. Interpolation $\Rightarrow$ good balance btw syntax & semantics

4. Does L $\subseteq$ FO has decidable satisfiability problem? Model checking problem? Query entailment problem?

5. Does L posses the finite model property, i.e. is every satisfiable formula satisfied in some finite model?

The Gödel's Compactness Theorem

Let $\mathcal{T}$ be an FO-theory and let φ be an FO sentence.

1. If $\mathcal{T} \models \varphi$ then there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\mathcal{T}_0 \models \varphi$.

2. If every *finite* $\mathcal{T}_0 \subseteq \mathcal{T}$ is satisfiable then $\mathcal{T}$ is satisfiable.

Use case:

Showing

inexpressivity

"$\models \; = \; \vdash$"

Proofs are finite

Craft $\mathcal{T}_0$

1st excursion: Proving (1)

Assume $\mathcal{T} \models \varphi$. Then by Gödel's completeness theorem $\mathcal{T} \vdash \varphi$. So there is a formal proof $\mathcal{P}$ of $\mathcal{T} \vdash \varphi$. Since proofs are finite the proof $\mathcal{P}$ uses only finitely many axioms of $\mathcal{T}$. Call them $\mathcal{T}_0$. Thus $\mathcal{T}_0 \vdash \varphi$ holds (use the same proof as before!). After asking Gödel about "$\models \; = \; \vdash$" again we are done.

Ad absurdum

$\mathcal{T}$ unSAT iff $\mathcal{T} \models \bot$

Employ (1)

2nd excursion: Proving (2)

Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$.

By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\mathcal{T}_0 \models \bot$.

Thus $\mathcal{T}$ has an unsatisfiable finite subset ($\mathcal{T}_0$). A contradiction!

Employing compactness I: Reachability in $\{\mathrm{E}\}$-structures

The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

Ad absurdum suppose that φ defines $\mathcal{P}$. $\rightsquigarrow$ Manufacture a theory $\mathcal{T}$ containing φ. $\rightsquigarrow$

$\rightsquigarrow$ Prove that $\mathcal{T}$ is unsatisfiable $\rightsquigarrow$ but its every finite subset is satisfiable. $\rightsquigarrow$ Contradict Compactness.

There is no FO$[\{\mathrm{E}\}]$ formula for connectivity over $\{\mathrm{E}\}$-structures.

So there is no formula saying that between any two nodes there is a directed $\{\mathrm{E}\}$-path.

No info about the finite models!

Proof:

Employ reachability!

Assume that there is such φ, and let $\mathcal{T}$ be

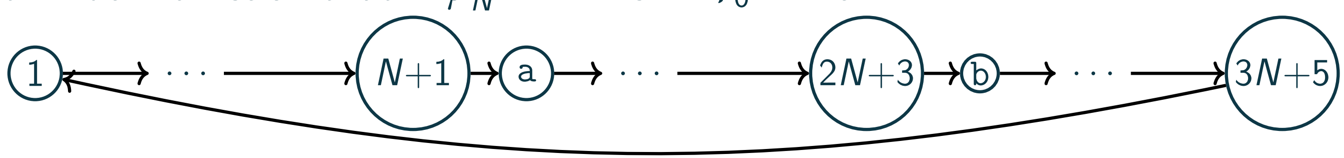
$$\mathcal{T} := \{\varphi\} \cup \{\neg\varphi_k^{\mathsf{reach}(\mathsf{a},\mathsf{b})} \mid k \geq 0\}.$$

Since a and b are disconnected, $\mathcal{T}$ is unSAT.

Let $\mathcal{T}_0$ be any non-empty finite subset of $\mathcal{T}$.

Let N be max such that $\neg\varphi_N^{\mathsf{reach}(\mathsf{a},\mathsf{b})}$ is in $\mathcal{T}_0$. Then:

$\varphi_0^{\mathsf{reach}(\mathsf{a},\mathsf{b})} := \mathsf{a} = \mathsf{b}$, $\varphi_1^{\mathsf{reach}(\mathsf{a},\mathsf{b})} := \mathrm{E}(\mathsf{a},\mathsf{b})$, $\varphi_k^{\mathsf{reach}(\mathsf{a},\mathsf{b})} :=$
$\exists x_1 \ldots \exists x_{k-1}\, \mathrm{E}(\mathsf{a}, x_1) \wedge \wedge_{i=1}^{k-2} \mathrm{E}(x_i, x_{i+1}) \wedge \mathrm{E}(x_{k-1}, \mathsf{b})$



$\models \mathcal{T}_0$. A contradiction!

Employing compactness II: Parity of the domain

The previous proof does not give us any information about the finite domain reasoning.

Even worse, Compactness fails in the finite setting (exercise). Can we use it nevertheless?

There is no FO[$\emptyset$] formula expressing that the domain is even over $\emptyset$-structures.

Proof:

Suppose that such a φ exists. Consider two theories $\mathcal{T}_1$ and $\mathcal{T}_2$:

$\mathcal{T}_1 := \{\varphi\} \cup \{\lambda_k \mid k \geq 0\}, \quad \mathcal{T}_2 := \{\neg\varphi\} \cup \{\lambda_k \mid k \geq 0\}.$

It's easy to see that any finite subset of $\mathcal{T}_1$ and $\mathcal{T}_2$ is satisfiable (WHY?).

So by compactness $\mathcal{T}_1$ and $\mathcal{T}_2$ are also satisfiable (∞ models!).

Thus, by Löwenheim–Skolem, $\mathcal{T}_1, \mathcal{T}_2$ have countably-inf models $\mathfrak{A}$ and $\mathfrak{B}$.

By $\mathfrak{A} \models \mathcal{T}_1$ we get $\mathfrak{A} \models \varphi$, and $\mathfrak{B} \models \mathcal{T}_2$ we get $\mathfrak{B} \models \neg\varphi$.

As there is a bijection between any two countably-inf sets, we get $\mathfrak{A} \cong \mathfrak{B}$.

Formulae are preserved by isomorphisms, so $\mathfrak{B} \models \neg\varphi$ implies $\mathfrak{A} \models \neg\varphi$:

Thus $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \neg\varphi$. A contradiction (with the semantics of $\models$)!

Exploit ∞!

Let λ_k say "there are $\geq k$ elem.".

Löwenheim–Skolem!

$\emptyset$-structures $=$ sets

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