Lista 9

środa, 19 kwietnia 2023 16:26

Zadanie 1.

$AB = BA$, $A\vec{v} = \lambda\vec{v}$

Chcemy $\vec{v} \in V_\lambda \Rightarrow B\vec{v} \in V_\lambda$

Niech $\vec{v} \in V_\lambda$, wtedy $\exists \vec{v}'$ tż, że $\vec{v} = A\vec{v}_1 = \lambda\vec{v}_1$

$$\left.\begin{array}{l} B\vec{v} = B\lambda\vec{v}_1 = \lambda B\vec{v}_1 \\ B\vec{v} = BA\vec{v}_1 = AB\vec{v}_1 \end{array}\right\} \Rightarrow AB\vec{v}_1 = \lambda B\vec{v}_1 \Rightarrow (B\vec{v}_1) \text{ należy do } V_\lambda.$$

Skoro V_λ podprzestrzenią liniową to skoro $B\vec{v}_1 \in V_\lambda$ to $\lambda \cdot B\vec{v}_1 \in V_\lambda \Rightarrow B\vec{v} \in V_\lambda$ □

Zadanie 3.

a)
$$\begin{vmatrix} 7 & -12 & 6 \\ 10 & -13 & 10 \\ 12 & -24 & 13 \end{vmatrix}$$

$$\det\begin{vmatrix} 7-\lambda & -12 & 6 \\ 10 & -13-\lambda & 10 \\ 12 & -24 & 13-\lambda \end{vmatrix} = \left\{\begin{array}{l}(7-\lambda)(-13-\lambda)(13-\lambda) - 1440 - 1440 \\ +12\cdot 6(13+\lambda) + 240(7-\lambda) + 120(13-\lambda)\end{array}\right\} = \left\{\begin{array}{l}(12\cdot 6\cdot 13 + 240\cdot 7 + 120\cdot 13 - 1440 - 1440) + \\ +(72\lambda - 240\lambda - 120\lambda) + (\quad X \quad X)\end{array}\right\}$$

tak wyszło

$$\rightarrow -\lambda^3 + \lambda^2 + \lambda - 1 = -\lambda^2(\lambda - 1) + \lambda - 1 = (1-\lambda^2)(\lambda - 1) = (1-\lambda)(1+\lambda)(\lambda-1) = (-1)(1+\lambda)(1-\lambda)^2$$

$\lambda = 1$, $\lambda = -1$ dwukrotnie

$$\lambda = 1 \quad \det\begin{vmatrix} 6 & -12 & 6 \\ 10 & -20 & 10 \\ 12 & -24 & 12 \end{vmatrix} = \det\begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \det\begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$A\vec{v} = \lambda\vec{v}$

$$\begin{vmatrix} 7 & -12 & 6 \\ 10 & -13 & 10 \\ 12 & -24 & 13 \end{vmatrix}\begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

$$\begin{cases} x = 7x - 12y + 6z \\ y = 10x - 13y + 10z \\ z = 12x - 24y + 13z \end{cases} = \begin{cases} 0 = x - 2y + z \\ 0 = x - 2y + z \\ 0 = x - 2y + z \end{cases} \Leftrightarrow \left(x, \frac{x+z}{2}, z\right) \rightarrow \underline{\underline{LIN\left((1, \tfrac{1}{2}, 0), (0, \tfrac{1}{2}, 1)\right)}}$$

wymiar = 2

b)
$$\begin{vmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{vmatrix}$$

jeszcze krotność geometryczna

$$\det\begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = (-1)\det\begin{vmatrix} -1 & 2 \\ (3-\lambda) & 3 \end{vmatrix} + (-2-\lambda)\det\begin{vmatrix} 2-\lambda & -1 \\ 5 & (3-\lambda) \end{vmatrix} =$$

$$= (-1)\left[(-3+6+2\lambda) + (2+\lambda)(5-(3+\lambda)(2-\lambda))\right] =$$

$$= (-1)\left[2\lambda + 3 + (2+\lambda)(-1+\lambda^2+\lambda)\right] =$$

$$= (-1)(2\lambda + 3 + \lambda^3 + 3\lambda^2 + \lambda - 2) =$$

$$= (-1)(\lambda^3 + 3\lambda^2 + 3\lambda + 1) = (-1)\underline{(\lambda+1)^3}$$

$$\begin{vmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -x_1 \\ -x_2 \\ -x_3 \end{vmatrix}$$

ogólny: $(x_1, x_1, -x_1)$

$LIN((1,1,-1))$

przykład

$x_1 = -2x_1 + x_1 + 2x_1 = x_1$ ✓

$x_2 = -5x_1 + 3x_2 + 3x_1 = 3x_2 - 2x_1 \Rightarrow 2x_2 = 2x_1$

$x_2 = x_1$

$x_3 = x_1 + 2x_3 \Rightarrow x_3 = -x_1$

dla pozostałych podobnie

Zadanie 10.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 1/2 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1/3 & 1/2 & 0 & 1/3 \\ 4 & 1/3 & 1/2 & 0 & 0 \end{array}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} \rightarrow \begin{vmatrix} x_3 + \frac{1}{2}x_4 \\ x_1 \\ \frac{1}{3}x_4 \\ \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 \end{vmatrix}$$

$M\vec{x} = \vec{x}$

$x_1 = x_3 + \frac{1}{2}x_4 = \frac{1}{2}x_1 + \frac{1}{2}x_1 + \frac{1}{2}x_4 = \frac{1}{2}x_1 + x_4$

$x_2 = \underline{\frac{1}{3}x_1}$

$x_3 = \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 = \frac{1}{3}x_1 + \frac{1}{6}x_1 + \frac{1}{2}x_1 = \frac{1}{2}x_1 + \frac{1}{2}x_4 = \underline{\frac{3}{4}x_1}$

$x_4 = \frac{1}{3}x_1 + \frac{1}{2}x_2 = \frac{1}{3}x_1 + \frac{1}{6}x_1 = \underline{\frac{1}{2}x_1}$

$x_1 + \frac{1}{3}x_1 + \frac{3}{4}x_1 + \frac{1}{2}x_1 = 1$

$\frac{12}{12}x_1 + \frac{4}{12}x_1 + \frac{9}{12}x_1 + \frac{6}{12}x_1 = 1$

$\boxed{x_1 = \frac{12}{31}}$

Zadanie 6.

a) trio

b) $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$

$A \times B = C$

$B \times A = D$

$$c_{ij} = \sum_x a_{ix} b_{xj} = \sum_x b_{xj} a_{ix} = \sum_x b_{jx} a_{xi} = d_{ji}$$

$$AB = BA \Leftrightarrow C = D \Leftrightarrow c_{ij} = d_{ij} \Leftrightarrow \underbrace{d_{ji} = d_{ij}}_{\text{symetryczna}}$$

c) A - symetryczna

$A^{-1} = ???$

$AA^{-1} = A^{-1}A = Id$

$AB = C$

$A^T B^T = C^T$

$\cancel{(A^T)^{-1} = (A^{-1})^T}$

$$\boxed{(AB)^T = B^T \cdot A^T}$$

Wiemy: A^{-1} istnieje

$A^T = A$

$A \cdot A^{-1} = Id$

$A^T \cdot (A^{-1})^T = (Id)^T$

sym, sym

$A \cdot (A^{-1})^T = Id$

$A \cdot (A^{-1})^T = Id$

$A \cdot A^{-1} = Id$

$A^{-1} = (A^{-1})^T$ □

symetryczne

suma = 1 w każdej kolumnie

$\sum(A\vec{v}) = \sum\vec{v}$

$\vec{v} = v_1, \dots, v_n$

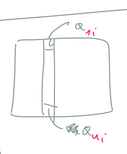
$A = (a_{ij})$

$\sum A\vec{v} = \sum_i^n$

3

Zadanie 9.

A - stochastyczna



$\sum_x a_{xi} = 1$

$$A\vec{v} = \begin{pmatrix} \sum_x a_{1x} v_x \\ \vdots \\ \sum_x a_{ix} v_x \\ \vdots \\ \sum_x a_{nx} v_x \end{pmatrix}$$

$$\|A\vec{v}\|_1 = \sum_i \left|\sum_x a_{ix} v_x\right|$$

$$\|\vec{v}\|_1 = \sum_i |v_i| = \sum_i \left(|v_i| \cdot \sum_x a_{xi}\right) =$$

skoro $\sum_x a_{xi} = 1$

$$= \sum_i \sum_x (a_{xi}|v_i|) = \sum_x \sum_i (a_{xi}|v_i|) = \sum_x \sum_y (a_{xy}|v_y|) = \sum_i \sum_y (a_{iy}|v_y|) = \sum_i \sum_x (a_{ix}|v_x|)$$

bo można zamienić sumy miejscami

Chcemy pokazać, że $\left|\sum_x a_{ix} v_x\right| \leq \sum_x a_{ix}|v_x|$

! Ważne: macierz stochastyczna ma wszystkie wyrazy nieujemne

zatem

$$\left|\sum_x a_{ix} v_x\right| \leq \sum_x a_{ix}|v_x| \Leftrightarrow \left|\sum_x a_{ix} v_x\right| \leq \sum_x |a_{ix} v_x|$$

z nierówności trójkąta

$$\sum_x^n |a_{ix} v_x| \geq \sum_x^{n-2} |a_{ix} v_x| + \left|\sum_{x=n-1}^{n} a_{ix} v_x\right| \geq$$

$$\geq \sum_x^{n-3} |a_{ix} v_x| + |a_{i,n-2} v_{n-2}| + \left|\sum_{x=n-1}^{n} a_{ix} v_x\right| \geq$$

$$\geq \sum_x^{n-3} |a_{ix} v_x| + \left|\sum_{x=n-2}^{n} a_{ix} v_x\right| \geq \dots \text{ (indukcja)}$$

$$\dots \geq \left|\sum_x a_{ix} v_x\right| \quad □$$

Zadanie 2.

λ_1

λ_2

$V_1 = \{\vec{v} \mid M\vec{v} = \lambda_1\vec{v}\}$

$V_2 = \{\vec{v} \mid M\vec{v} = \lambda_2\vec{v}\}$

$LIN(\vec{a}_1, \dots, \vec{a}_n) = V_1 \quad a \in V_1$

$LIN(\vec{b}_1, \dots, \vec{b}_m) = V_2 \quad b \in V_2$

Chcemy pokazać, że są liniowo niezależne.

Załóżmy, że są zależne $c_1 = a_1, \dots, c_n = a_n, c_{n+1} = b_1, \dots, c_{n+m} = b_m$

$\sum_i \gamma_i c_i = 0 \Rightarrow \sum_i \alpha_i a_i + \sum_i \beta_i b_i = 0 \Rightarrow$ istnieje wektor taki, że

i niezerowe α

niezerowe niezerowe

jest w jednym i drugim (bo $-1\vec{v}$ jest bo to przestrzeń liniowa).

Wtedy $M\vec{v} = \lambda_1\vec{v}$ i $M\vec{v} = \lambda_2\vec{v} \Rightarrow$ skoro $\vec{v} \neq \vec{0}$ to $\lambda_1 = \lambda_2$ □

Dla $\underbrace{\lambda_1, \dots, \lambda_k}_{\text{idą niezależne}}, \lambda_{k+1}$

$$\sum \gamma_i \vec{x}_i = 0 \Leftrightarrow \overbrace{\sum \alpha_i \vec{a}_i}^{\lambda_1, \dots, \lambda_k} + \underbrace{\sum \beta_i \vec{b}_i}_{} = 0$$

$\vec{v} \in V_{k+1} \Rightarrow M\vec{v} = \lambda_{k+1}\vec{v}$

$\vec{v} \in V_1 + \dots + V_k$

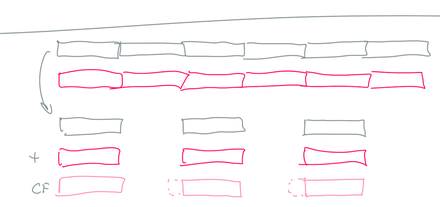
$\vec{v} = \sum \alpha_i \vec{a}_i$ i tylko jeden wektor $\vec{a}_i$

$\vec{w} = M\vec{v} = M\sum \alpha_i \vec{a}_i = \lambda_1 \sum \alpha_{1i} \vec{a}_{1i} + \dots + \lambda_k \sum \alpha_{ki} \vec{a}_{ki}$ niezależne wystąpienia a_i

$\vec{w} = M\vec{v} = \lambda_{k+1}\vec{v} = \lambda_{k+1} \sum \alpha_i \vec{a}_i$

Zatem $\sum(\lambda_1 - \lambda_{k+1})\alpha_{1i}\vec{a}_{1i} + \dots + \sum(\lambda_k - \lambda_{k+1})\alpha_{ki}\vec{a}_{ki} = 0$

bo a liniowo niezależne



1) jeśli dodatnie i ujemne to dzielę

2) jeśli dwie ujemne to $\cdot(-1)$ i jeden dodatnie

jeśli pierwszy się zrobił 1 to overflow

jeśli

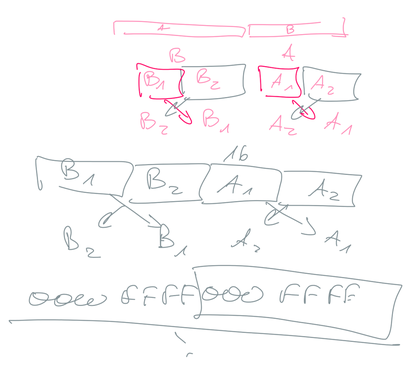
log

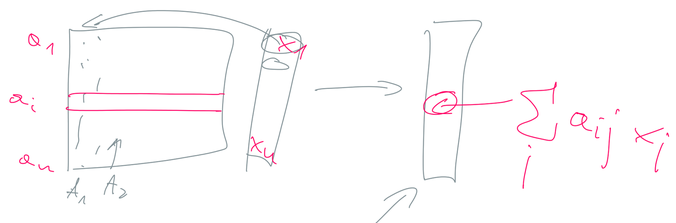
64 : 32, 16, 8, 4, 2, 1 $\boxed{6}$

$\underline{5 \cdot 6 = 30}$

$\boxed{48}$

$\boxed{32}$



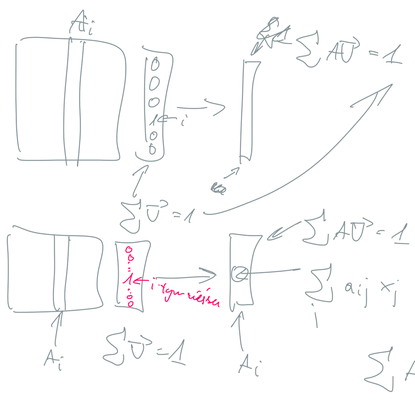


$\sum_i a_{ij} x_j$

$\sum(A\vec{v}) = \sum_i \vec{v}_i \cdot A_i = \sum \vec{v}_i = \sum \vec{v}$

$$\sum(A\vec{v}) = \sum_i \sum_j a_{ij} x_j = \sum_j \sum_i a_{ij} x_j = \sum_j x_j \left(\sum a_{ij}\right) = \sum_j x_j = \sum \vec{v}$$

kolumna $A_i = 1$



$\sum A\vec{v} = 1$

$\sum \vec{v} = 1$

$\sum A\vec{v} = 1$

$\sum_j a_{ij} x_j = \boxed{\begin{array}{l} x_i = 1 \\ x_{\neq i} = 0 \end{array}} = a_{ij} x_i = a_{ij}$

$\sum \vec{v} = 1$

$$\sum A\vec{v} = \sum_i \left(\sum_j a_{ij} x_j\right) = \sum_i a_{ij} = A_i \quad 1$$

$\sum\sum \dots =$