Lista 10

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Zadanie 1

$A\vec{v} = -\vec{v} \Rightarrow A^2\vec{v} = \vec{v}$
$A\vec{v} = \vec{v} \Rightarrow A^2\vec{v} = \vec{v}$
} dim to suma takich (to wielokrotne)

Skoro $\dim(V_1) = 1$ dla wektorów własnych,
to mamy $1 + x = 1$ ↯
V_{1A} V_{-1}

Zadanie 3.

$\vec{u} = (u_1, \dots, u_n)$
$\vec{v} = (v_1, \dots, v_m)$
$\|\vec{u}\| \cdot \|\vec{v}\| = \sqrt{\vec{u}\cdot\vec{u}} \cdot \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\sum u_i^2} \cdot \sqrt{\sum v_i^2} = \sqrt{(\sum u_i^2)\cdot(\sum v_i^2)}$ (I)
$|\vec{u}\cdot\vec{v}| = \sum u_i v_i$ (II)
$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\cdot\|\vec{v}\| \iff$
$\iff \sum u_i v_i \le \sqrt{(\sum u_i^2)(\sum v_i^2)}$ ⇐ (IV)
$\Leftarrow \left(\sum u_i v_i\right)^2 \le \left(\sum u_i^2\right)\left(\sum v_i^2\right)$ ⇐ nierówność Cauchy'ego-Schwarza (V)

Zadanie 2.

$\vec{u}\cdot(M\vec{v}) = \vec{u}\cdot\begin{pmatrix}\sum m_{1i}v_i\\ \vdots \\ \sum m_{ni}v_i\end{pmatrix} = \sum\left(u_i \cdot \sum_x m_{ix}v_x\right) = \sum_i\sum_x u_i m_{ix} v_x = \sum_x\sum_i v_x m_{ix} u_i$

$(M^T\vec{u})\cdot\vec{v} = \vec{v}\begin{pmatrix}\sum t_{1i}u_i\\ \vdots \\ \sum t_{ni}u_i\end{pmatrix} = \sum_i\left(v_i\cdot\sum_x t_{ix}u_x\right) = \sum_i\sum_x v_i t_{ix} u_x = \sum_i\sum_x v_i m_{xi} u_x$

$t_{ix} = m_{xi}$

Dla symetrycznej $\vec{u}(M\vec{v}) = (M\vec{u})\vec{v}$ oczywiste

Udowodnij: Jeśli $\vec{u}(M\vec{v}) = (M\vec{u})\vec{v}$ to M symetryczna

Niech $\vec{u}_i$: $u_i = 1$, $u_x = 0$ dla $x \neq i$
$\vec{v}_i$ = podobnie

$\vec{u}_i(M\vec{v}_j) = \sum_y\sum_x u_{iy} m_{yx} v_x = \sum_y u_y m_{yj} v_i = u_i m_{ij} v_i = m_{ij}$ ($u_{ix}=0$, $v_{jx}=0$, $u_{ii}=1$, $v_{ji}=1$)

$(M\vec{u})\vec{v} = \sum_y\sum_x u_y m_{xy} v_x = \dots = m_{ji}$

Skoro $\forall \vec{u},\vec{v}$ to równość to $\forall_{i,j}$ $m_{ij} = m_{ji}$ ☐

$\lambda_1 \neq \lambda_2$ M symetryczna
$M\vec{v} = \lambda_1\vec{v}$
$M\vec{u} = \lambda_2\vec{u}$ (dowolne skalary)
$\vec{u}\cdot(M\vec{v}) = (M\vec{u})\cdot\vec{v}$
$\vec{u}\cdot(\lambda_1\vec{v}) = (\lambda_2\vec{u})\cdot\vec{v}$
$\sum u_i\lambda_1 v_i = \sum u_i \lambda_2 v_i$
$\lambda_1\sum u_i v_i = \lambda_2 \sum u_i v_i$
$\lambda_1 \neq \lambda_2 \Rightarrow$
$\boxed{\sum u_i v_i = 0}$ ☐

Zadanie 8.

$\langle p, q\rangle = \int_{-1}^{1} p(x)\,q(x)\,dx$

$\langle x^i, x^j\rangle = \int_{-1}^{1} x^i\cdot x^j\,dx = \int_{-1}^{1} x^{i+j}\,dx = \begin{cases} i+j=0: & 0 \\ i+j>0: & \frac{x^{i+j+1}}{i+j+1}\Big|_{-1}^{1} = \frac{1}{i+j+1} - \frac{(-1)^{i+j+1}}{i+j+1}\end{cases}$

$i+j$ (wynik):
0: 0
1: $\frac{1}{2} - \frac{-1}{2} = 0$
2: $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$
3: 0
4: $\frac{2}{5}$
5: 0
6: $\frac{2}{7}$

Zadanie 4.

$\|\vec{u}\| = \|\vec{v}\| \iff \sqrt{\sum u_i^2} = \sqrt{\sum v_i^2} \iff \sum u_i^2 = \sum v_i^2$
$(\vec{u}-\vec{v}) \perp (\vec{u}+\vec{v}) \iff (\vec{u}-\vec{v})\cdot(\vec{u}+\vec{v}) = 0 \iff \sum (u_i - v_i)(u_i+v_i) = 0 \iff$
$\iff \sum u_i^2 - v_i^2 = 0 \iff \sum u_i^2 = \sum v_i^2$

Zadanie 5.

a) $W = LIN((1,0,1),(2,3,1))$
$(a,b,c)(1,0,1) = a + c = 0 \Rightarrow c = -a \Rightarrow c = 36$
$(a,b,c)(2,3,1) = 2a + 3b + c = a + 3b \Rightarrow a = -36$
$W^\perp = LIN((-36,6,36)) = LIN((-3,1,3))$

b) $W = LIN((1,0,1,0),(0,1,0,1))$
$0 = (a,b,c,d)(1,0,1,0) = a + c \overset{\mathbb{Z}_2}{\Rightarrow} a = c$
$0 = (a,b,c,d)(0,1,0,1) = b + d \overset{\mathbb{Z}_2}{\Rightarrow} b = d$
(a,b,a,b)
$W^\perp = LIN((1,0,1,0),(0,1,0,1)) = W$

c) $W = LIN((1,0,2))$ nad $\mathbb{Z}_3$
$0 = (a,b,c)(1,0,2) = a + 2c \overset{\mathbb{Z}_3}{=} a - c \Rightarrow a = c$
(a,b,a)
$W^\perp = LIN((1,0,1),(0,1,0))$

Zadanie 6.

a) Pokaż: $V_1 \subseteq V_2 \iff V_1^\perp \supseteq V_2^\perp$
$V_1 \subseteq V_2$
$\vec{v} \in V_1 \Rightarrow \vec{v} \in V_2$
$\vec{a} \in V_2^\perp \Rightarrow \forall \vec{v} \in V_2 \ \vec{a} \perp \vec{v} \Rightarrow \forall \vec{v} \in V_1 \ \vec{a} \perp \vec{v}$
$\Rightarrow \vec{a} \in V_1^\perp$ ☐

b) $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$
$\vec{a} \in A \iff \forall \vec{v} \in (V_1 + V_2) \Rightarrow \vec{a} \perp \vec{v}) \iff$
$\iff \forall \vec{v}\left(\vec{v} = \vec{v_1} + \vec{v_2} \Rightarrow \vec{a} \perp (\vec{v_1} + \vec{v_2})\right) \iff$
$\iff \forall \vec{v_1}, \vec{v_2} \ \ \vec{a} \perp (\vec{v_1} + \vec{v_2}) \iff$ (bo $\vec{v}+0 = \vec{v}$, $0 \in V_1$ i $\vec{v} \in V_2$)
$\iff \forall \vec{v_1} \ \vec{a} \perp \vec{v_1} \ \wedge \forall \vec{v_2} \ \vec{a} \perp \vec{v_2} \iff$
$\iff \vec{a} \in V_1^\perp \ \wedge \vec{a} \in V_2^\perp \iff \vec{a} \in B$

c) $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$
$(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp \iff V_1 \cap V_2 = (V_1^\perp + V_2^\perp)^\perp \iff V_1 \cap V_2 = (V_1^\perp)^\perp \cap (V_2^\perp)^\perp$ ☐ (U)

$\boxed{A = B \iff A^\perp = B^\perp} \Leftarrow \begin{cases} A = B \Rightarrow A^\perp = B^\perp \\ A^\perp = B^\perp \Rightarrow (A^\perp)^\perp = (B^\perp)^\perp \Rightarrow A = B\end{cases}$

Zadanie 7.

$\vec{u}\times\vec{v} = \sum_i^k u_i v_i$

Zadanie 9.

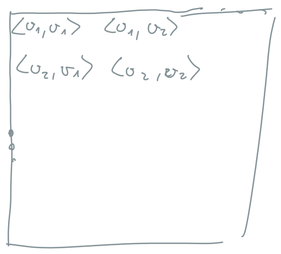
$U^\perp = (LIN(U))^\perp$
Wiemy: $LIN(U) \supseteq U$, więc $U^\perp \supseteq (LIN(U))^\perp$.
Zatem: $\exists \vec{a} \perp \vec{u} \ \wedge \exists u_1,\dots,u_n,\alpha_1,\dots,\alpha_n \ \ \vec{a} \not\perp \sum \alpha_i \vec{u_i}$
Wiemy: $\vec{a} \perp \vec{u_i}$, zatem $\vec{a} \perp \alpha_i \vec{u_i}$.
$\sum a_x u_{ix} = 0$ $\quad \sum a_x \alpha_i u_{ix} = 0$
$\alpha_i \sum a_x u_{ix}$

Skoro $\vec{a} \perp \alpha_i \vec{u_i}$ oraz $\vec{a} \perp \alpha_j \vec{u_j}$ to
$\sum a_x \alpha_i u_{ix} = 0$
$+ \sum a_x \alpha_j u_{jx} = 0$
$\sum a_x(\alpha_i u_{ix} + \alpha_j u_{jx}) = 0 \Rightarrow \vec{a} \perp (\alpha_i u_{in} + \alpha_j u_j)$

zatem $\vec{a} \perp \sum \alpha_i \vec{u_i}$ ☐

II $(U^\perp)^\perp = LIN(U)$
$(U^\perp)^\perp = \left(\left(LIN(U)\right)^\perp\right)^\perp = LIN(U)$ (bo z def postaci)

Zadanie 10.



$\sum_x u_{ix}u_{ix} = \sum_x \left(\sum_y m_{xy}v_y\right)\left(\sum_y m_{xy}w_y\right)$

$B = (\vec{v_1},\dots,\vec{v_n}$ - baza ortonormalna V $\quad u_i = (\vec{v_i})_B$
$A = [(\vec{v_1})_B \mid \dots \mid (\vec{v_n})_B]$
Połóż: $G = A^TA$ $\quad i=v \longleftrightarrow j=u$
$g_{ij} = \langle v_i, v_j\rangle$ $\quad i=u$
$(A^TA)_{ij} = \sum_x t_{ix}a_{xj} = \sum_x (u_i)_x (u_j)_x = \langle u_i, u_j\rangle = \langle (v_i)_B, (v_j)_B\rangle$

Uwaga! Wektory B muszą być długości 1 ⇒ wzajemnie prostopadłe

$M \in$ macierz przekształcenia

$\dim(V) = n$
$G = [\dots]$

Lemat 11.12 - $\langle \vec{u}, \vec{v}\rangle = \langle \vec{u}_B, \vec{v}_B\rangle$
więc $g_{ij} = (A^TA)_{ij}$
$\det(G) = \det(A^T)\det(A) = \det(A)^2 \geq 0$
$a = 0 \iff \det(A) = 0 \iff \vec{v}$ zależny liniowo

$\det(G) = \det(A^TA) =$

Zadanie 11.

$\vec{e_1},\dots,\vec{e_k}$ są układem ortonormalnym

$L = \sum_{i=1}^{k} |\langle \vec{e_i}, \vec{v}\rangle|^2 \leq \|\vec{v}\|^2$ $\quad u_i u_i$

$L = \sum_i |\langle \vec{e_i}, \vec{v}\rangle|^2 = \sum_i \left(\sum_x (e_i)_x v_x\right)^2$ $\quad c \to s$

$P = \left(\sum v_i^2\right)^2$

$\left(\sum_x e_{ix}^2\right) = 1$

$\sum_i \left(\sum_x (e_i)_x^2\right)\left(\sum_x v_x^2\right) = \sum_i^k \left(\sum_x v_x^2\right) = k \cdot \sum_x v_x^2$

$\|\vec{v}_E\|$ wyraźnie dodatnie, więc jak nie wierzę części to będzie mniejsze

$\vec{v}_B = [\langle e_1, \vec{v}\rangle \dots \langle e_k, \vec{v}\rangle, \langle v_1, \vec{v}\rangle, \dots, \langle v_n, \vec{v}\rangle]$

$\sum |\langle e_i, \vec{v}\rangle|^2 \leq \|\vec{v}_B\| = \sum (\ )^2 = \|\vec{v}\|^2$

$\|\vec{v}_E\| = \sqrt{\sum_i^n (\langle \vec{v}, \vec{e_i}\rangle)^2}$

$\|\vec{v}_E\| = \|\vec{v}\|$

$\vec{v}_E = \begin{pmatrix}\langle \vec{v}, \vec{e_1}\rangle \\ \vdots \\ \langle \vec{v}, \vec{e_n}\rangle\end{pmatrix}$

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$\langle x^i, x^j\rangle = \int_{-1}^{1} x^i \cdot x^j\,dx = \int_{-1}^{1} x^{i+j}\,dx = \frac{x^{i+j+1}}{i+j+1}\Big|_{-1}^{1} = \frac{1}{i+j+1} - \frac{(-1)^{i+j+1}}{i+j+1}$

$i+j$:
0: $\frac{1^1}{1} - \frac{(-1)^1}{1} = 2$
1: $\frac{1^2}{2} - \frac{(-1)^2}{2} = 0$
2: $\frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}$
3: $\frac{1^4}{4} - \frac{(-1)^4}{4} = 0$
4: $\frac{1^5}{5} - \frac{(-1)^5}{5} = \frac{2}{5}$
5: $\frac{1^6}{6} - \frac{(-1)^6}{6} = 0$
6: $\frac{1^7}{7} - \frac{(-1)^7}{7} = \frac{2}{7}$