State Space Models

Hidden Markov Models

Dynamic Graphical Models I

Introduction

- Dynamic graphical models extend traditional graphical models to handle data sequences where statistical properties evolve over time.

Dynamic Graphical Models II

State-Observation Models

- Observations $\mathbf{y}_n \in \mathbb{R}^l$ at time n are linked with latent random vectors $\mathbf{x}_n$.
- System dynamics modeled by latent variables with transition and observation models.

$$\mathbf{x}_{n+1} \perp (\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) | \mathbf{x}_n$$

$$\mathbf{y}_n \perp (\mathbf{x}_1, \dots, \mathbf{x}_{n-1}, \mathbf{x}_{n+1}, \dots, \mathbf{x}_N) | x_n$$

Dynamic Graphical Models III

Model Equations

- Transition Model: $p(\mathbf{x}_{n+1}|\mathbf{x}_n)$
- Observation Model: $p(\mathbf{y}_n|\mathbf{x}_n)$
- Linear dynamical systems: $\mathbf{x}_n$ is continuous
- Hidden Markiv Model : $\mathbf{x}_n$ is discrete

$$\mathbf{x}_n = F_n \mathbf{x}_{n-1} + \eta_n$$
$$\mathbf{y}_n = H_n \mathbf{x}_n + \mathbf{v}_n$$

$$p(\mathbf{x}_n|\mathbf{x}_{n-1}) = \mathcal{N}(\mathbf{x}_n|F_n\mathbf{x}_{n-1}, Q_n)$$
$$p(y_n|\mathbf{x}_n) = \mathcal{N}(\mathbf{y}_n|H_n\mathbf{x}_n, R_n)$$

Hidden Markov Models (HMMs) I

- HMMs are used to model sequences where states are not directly observable but influence observations.
- **State Transition Probability:**

$$P(k_n|k_{n-1}) = P_{ij}, \quad \text{for } i, j = 1, 2, \ldots, K$$

  Represents the probability of transitioning from one state to another.
- **Observation Model:**

$$p(\mathbf{y}_n|k_n)$$

  Describes the probability of observing y_n given the current state k_n.

Hidden Markov Models (HMMs) II

- **Full Model Dynamics:**

$$p(Y, X) = P(\mathbf{x}_1)p(\mathbf{y}_1|\mathbf{x}_1)\prod_{n=2}^{N} P(\mathbf{x}_n|\mathbf{x}_{n-1})p(\mathbf{y}_n|\mathbf{x}_n)$$

  Joint distribution of all observations and states.
- **Inference Techniques:**
  - **Sum-Product Algorithm:** Used for computing marginal probabilities by passing messages in the graph.

Learning Parameters in Hidden Markov Models (HMMs) I

- **Expectation (E-step):** Compute expected log-likelihood:

$$\mathcal{Q}(\Theta, \Theta^{(t)}) = \mathbb{E}[\log p(Y, X; \Theta)]$$

- **Maximization (M-step):** Update parameters by maximizing $\mathcal{Q}(\Theta, \Theta^{(t)})$, subject to constraints $\sum_{k=1}^{K} P_k = 1$ and $\sum_{i=1}^{K} P_{ij} = 1$ for all j.

$$P_k^{(t+1)} = \frac{\gamma(x_{1,k} = 1; \Theta^{(t)})}{\sum_{i=1}^{K} \gamma(x_{1,i} = 1; \Theta^{(t)})}$$

$$P_{ij}^{(t+1)} = \frac{\sum_{n=2}^{N} \xi(x_{n-1,j} = 1, x_{n,i} = 1; \Theta^{(t)})}{\sum_{n=2}^{N} \sum_{k=1}^{K} \xi(x_{n-1,j} = 1, x_{n,k} = 1; \Theta^{(t)})}$$

Analysis of COVID-19 Admission Data using HMMs I

1. **Preprocessing:**
   - Apply a moving average filter to smooth the case data.
   - Calculate day-to-day differences to derive the difference sequence and use the signum function to categorize changes.
2. **Model Setup:** Configure and initialize a Multinomial HMM with 3 states and transition and emission probability matrices:

$$P = \begin{bmatrix} 0.90 & 0.10 & 0.00 \\ 0.10 & 0.80 & 0.10 \\ 0.00 & 0.10 & 0.90 \end{bmatrix} \quad E = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.15 & 0.80 \end{bmatrix}$$

3. **Model Fitting:** Fit the HMM to the preprocessed data using one-hot encoding.
4. **State Decoding:** Employ the Viterbi algorithm to decode the state sequence.

Analysis of COVID-19 Admission Data using HMMs II