State Space Models

Kalman Filtering

Kalman Filtering

- Estimation of the state of a linear dynamic system
- Time-varying systems where the statistical properties of the process may change over time

State-Space Representation

The Kalman filter operates under the state-space model, where the system is described by two equations:

- **State Equation:**

$$\mathbf{x}_n = F_n \mathbf{x}_{n-1} + \boldsymbol{\eta}_n$$

- **Output Equation:**

$$\mathbf{y}_n = H_n \mathbf{x}_n + \mathbf{v}_n$$

Assumptions for Kalman Filtering

1. The noise vectors $\boldsymbol{\eta}_n$ (process noise) and $\mathbf{v}_n$ (measurement noise) are assumed to be zero-mean Gaussian noise:

$$\mathbb{E}[\boldsymbol{\eta}_n] = 0, \quad \mathbb{E}[\mathbf{v}_n] = 0$$

2. The covariance matrices of the noise vectors are known and given by Q_n for process noise and R_n for measurement noise:

$$\mathbb{E}[\boldsymbol{\eta}_n \boldsymbol{\eta}_n^T] = Q_n, \quad \mathbb{E}[\mathbf{v}_n \mathbf{v}_n^T] = R_n$$

3. Noise vectors are uncorrelated with each other at different time steps:

$$\mathbb{E}[\boldsymbol{\eta}_n \boldsymbol{\eta}_m^T] = O \text{ for } n \neq m, \quad \mathbb{E}[\mathbf{v}_n \mathbf{v}_m^T] = O \text{ for } n \neq m$$

4. Cross-correlation between process and measurement noise at any time is zero:

$$\mathbb{E}[\boldsymbol{\eta}_n \mathbf{v}_m^T] = \mathbf{0} \text{ for all } n, m$$

Kalman Filter Recursions

The Kalman filter updates estimates via the following recursive steps:

1. **Prediction Step:**

$$\hat{\boldsymbol{x}}_{n|n-1} = F_n \hat{\boldsymbol{x}}_{n-1|n-1},$$
$$P_{n|n-1} = F_n P_{n-1|n-1} F_n^\top + Q_n.$$

2. **Update Step:**

$$K_n = P_{n|n-1} H_n^\top (H_n P_{n|n-1} H_n^\top + R_n)^{-1},$$
$$\hat{\boldsymbol{x}}_{n|n} = \hat{\boldsymbol{x}}_{n|n-1} + K_n(\boldsymbol{y}_n - H_n \hat{\boldsymbol{x}}_{n|n-1}),$$
$$P_{n|n} = (I - K_n H_n) P_{n|n-1}.$$

Kalman Filtering for 1D Object Movement

State Equation:

$$\boldsymbol{x}_n = \begin{bmatrix} p_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_{n-1}$$

Observation Equation:

$$\boldsymbol{y}_n = H\boldsymbol{x}_n + \boldsymbol{v}_n$$
$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad v_n \sim \mathcal{N}(0, R)$$

AR(3) Process Estimation with Kalman Filtering I

Problem Setup:

- We consider an AR process of order 3 represented by the equation:

$$x_n = -a_1 x_{n-1} - a_2 x_{n-2} - a_3 x_{n-3} + \eta_n$$

  where η_n is a white noise sequence with variance σ_η^2.
- Observed data is given by:

$$y_n = x_n + v_n$$

  where v_n is the measurement noise with variance σ_v^2.

AR(3) Process Estimation with Kalman Filtering II

State-Space Representation:

- State vector $\mathbf{x}_n$ and state transition matrix F are defined as:

$$\mathbf{x}_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{bmatrix}, \quad F = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Noise vector η_n and its covariance Q_n are:

$$\eta_n = \begin{bmatrix} \eta_n \\ 0 \\ 0 \end{bmatrix}, \quad Q_n = \begin{bmatrix} \sigma_\eta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Observation matrix H and its noise covariance R_n:

$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad R_n = \sigma_v^2$$

Exercise 11.3: Moving object in 2D I

- **State Transition Matrix** A:

$$A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Process Noise Covariance** Q:

$$Q = \begin{bmatrix} \frac{\Delta t^3}{3} & 0 & \frac{\Delta t^2}{2} & 0 \\ 0 & \frac{\Delta t^3}{3} & 0 & \frac{\Delta t^2}{2} \\ \frac{\Delta t^2}{2} & 0 & \Delta t & 0 \\ 0 & \frac{\Delta t^2}{2} & 0 & \Delta t \end{bmatrix}$$

Exercise 11.3: Moving object in 2D II

- **Observation Matrix** H:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- **Measurement Noise Covariance** R:

$$R = s^2 I_2$$

Simulation:

- Initial state $\mathbf{m}_0 = [0, 0, 1, -1]^T$
- Initial covariance $P_0 = I_4$

Exercise 11.3: Moving object in 2D III