Signal Representations

Sparsity Analysis Models

Sparsity Promoting Algorithms

- **Key Objective:** Efficiently reconstruct signals from limited data by leveraging their inherent sparsity in certain transform domains (e.g., Fourier).
- **Orthogonal Matching Pursuit (OMP):**
  - *Goal:* Solves the ℓ_0 pseudo-norm minimization problem.
- **Iterative Shrinkage-Thresholding (IST):**
  - *Goal:* Addresses the ℓ_1 minimization problem.

The OMP Algorithm I

- Greedy approach
- No guarantee that OMP find the optimal solution.
- LARS and LARS-LASSO

Algorithm The OMP Algorithm

Initialize:
$\theta^{(0)} = \mathbf{0} \in \mathbb{R}^l$
$S^{(0)} = \emptyset$ (Support set)
$e^{(0)} = y$
for $i = 1, \ldots, k$ **do**
    Select the column in X that forms the smallest angle with the error $e^{(i-1)}$
    Update the indices of active vectors, $S^{(i)}$
    Update the parameter vector $\theta^{(i)}$ using least squares with the columns in X indexed by $S^{(i)}$
    Update the error vector $e^{(i)} = y - X_{S^{(i)}}\theta^{(i)}$
end for

The IST Algorithm

- Estimates the LASSO solution

$$\arg\min_{\theta\in\mathbb{R}^l}\left\{\frac{1}{2}\|y-X\theta\|_2^2+\lambda\|\theta\|_1\right\}$$

Algorithm The IST Algorithm

Initialize:
$\theta^{(0)} = 0 \in \mathbb{R}^l$
Select the value of μ
Select the value of λ
for $i = 1, 2, \dots$ **do**
    $e^{(i-1)} = y - X\theta^{(i-1)}$
    $\tilde{\theta} = \theta^{(i-1)} + \mu X^T e^{(i-1)}$
    $\theta^{(i)} = \text{sign}(\tilde{\theta}) \max(|\tilde{\theta}| - \lambda\mu, 0)$
end for

Exercise 7.2.1: Deriving ISTA I

- If we have the cost function

$$J(\boldsymbol{\theta}) = \frac{1}{2}\|\mathbf{y} - X\boldsymbol{\theta}\|_2^2$$

- Gradient Descent Update:

$$\begin{aligned}\boldsymbol{\theta}(i) &= \boldsymbol{\theta}^{(i-1)} - \mu\nabla J(\theta^{(i-1)}) \\ &= \boldsymbol{\theta}^{(i-1)} - \mu X^T\left(X\boldsymbol{\theta}^{(i-1)} - \mathbf{y}\right) \\ &= \boldsymbol{\theta}^{(i-1)} + \mu X^T\mathbf{e}^{(i-1)}\end{aligned}$$

where $\mathbf{e}^{(i-1)} = \mathbf{y} - X\boldsymbol{\theta}^{(i-1)}$.

Exercise 7.2.1: Deriving ISTA II

- Optimization as Solution to:

$$\boldsymbol{\theta}^{(i)} = \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^l} \left\{ J(\boldsymbol{\theta}^{(i-1)}) + (\boldsymbol{\theta} - \boldsymbol{\theta}^{(i-1)})^T \nabla J(\boldsymbol{\theta}^{(i-1)}) + \frac{1}{2\mu} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{(i-1)}\|_2^2 \right\}$$

- Derivative Set to Zero to Find Minimum:

$$0 = \nabla J(\boldsymbol{\theta}^{(i-1)}) + \frac{1}{\mu}(\boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{(i-1)})$$

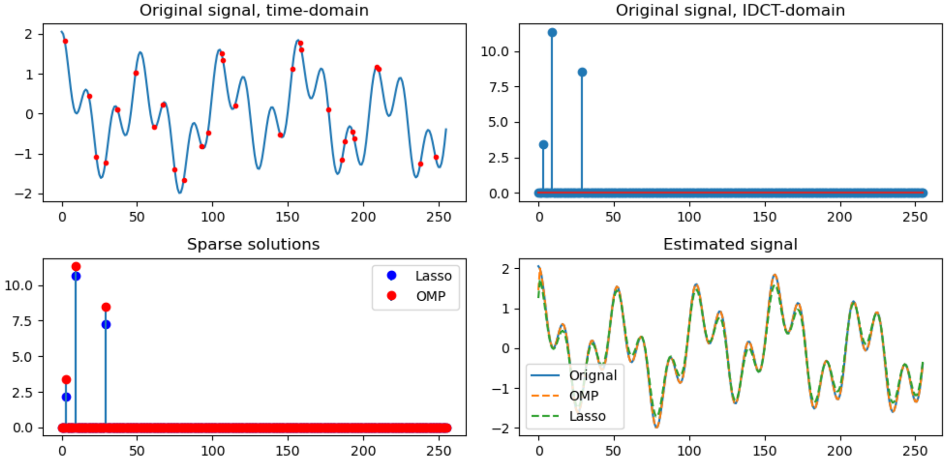
$$\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^{(i-1)} - \mu(\nabla J(\boldsymbol{\theta}^{(i-1)}))$$

- Insert $+\lambda||\boldsymbol{\theta}||_1$ and lots of rewriting, derivative, set equal to zero...
- Soft-thresholding Operator:

$$\boldsymbol{\theta}_j^{(i)} = \text{sign}(\tilde{\boldsymbol{\theta}}_j) \max(|\tilde{\boldsymbol{\theta}}_j| - \lambda\mu, 0)$$

where $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}^{(i-1)} - \mu\nabla J(\boldsymbol{\theta}^{(i-1)})$.

Example form Exercise 7.1.1



Linear Signal Representations I

Concept

Linear signal representation involves transforming a signal into a different space (analysis) and reconstructing it back (synthesis) using a unitary transformation matrix.

- **Signal (s)**: The vector of raw samples in the original space.
- **Transformed Signal ($\tilde{s}$)**: The vector representing the signal in the transformed space.
- **Transformation Matrix (Φ)**: A unitary matrix used for the transformation. It satisfies $\Phi\Phi^H = I$, where Φ^H is the Hermitian transpose (conjugate transpose) of Φ, ensuring that the transformation is orthonormal.

Linear Signal Representations II

Analysis

$$\tilde{s} = \Phi^H s$$

Analysis process transforms the raw signal s into the representation $\tilde{s}$ using the Hermitian transpose of the unitary matrix.

Synthesis

$$s = \Phi\tilde{s}$$

Synthesis process reconstructs the original signal s from its transformed representation $\tilde{s}$ using the unitary matrix.

Linear Signal Representations III

Construction of Transformation Matrix

- **DFT (Discrete Fourier Transform)**: Transforms a signal into its frequency components.
- **DCT (Discrete Cosine Transform)**: Similar to the DFT but uses only cosine functions.

Time-Frequency Analysis I

Introduction

Time-frequency analysis is essential for studying non-stationary data, where the signal's frequency components vary over time.

Analysis Approach

- **Segmented Analysis**: Representation models are applied to smaller, manageable chunks of the data to handle non-stationarity effectively.
- **Short-Time Fourier Transform (STFT):** divides a longer time signal into shorter segments of equal length and computes Fourier Transform separately on each segment

Time-Frequency Analysis II

Resolution Trade-off

- **Key Challenge**: Balancing time and frequency resolution to accurately capture the dynamics of the signal.
- **Windows Role**: Window functions are used to manage these trade-offs:
  - **Rectangular Window**: Offers maximum time resolution but poor frequency resolution and high spectral leakage.
  - **Hamming Window**: Provides a good compromise with reduced spectral leakage and better frequency resolution.
  - **Hann Window**: Similar to Hamming but with improved sidelobe suppression for even less leakage.

Time-Frequency Analysis III

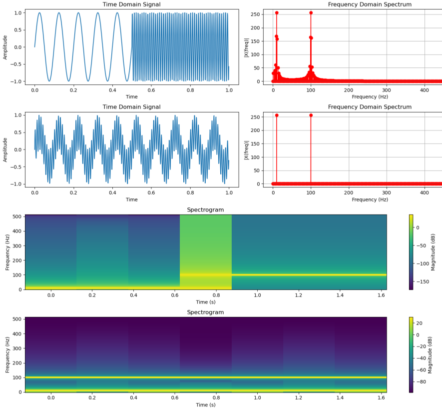


Figure: Frame size: 256, Hop size: 128