Solution to exercise Day 1

1. Show that the 2-sample logrank test (i.e., $K(s) = I(Y_{\cdot}(s) > 0)$) can be written in the form
$$Z_1(t) = \int_0^t \frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)}(\mathrm{d}\widehat{A}_1(s) - \mathrm{d}\widehat{A}_2(s)).$$

2. Show that, under $H_0 : \alpha_1(t) = \alpha_2(t)$, Z_1 is a martingale.

3. Find the predictable variation process $\langle Z_1 \rangle$ and the variance estimate $\widehat{\sigma}_{11}(t)$.

1. The general test statistic, for $k = 2$ and $K(s) = I(Y_{\cdot}(s) > 0)$, is
$$Z_1(t) = \int_0^t I(Y_{\cdot}(s) > 0)Y_1(s)(\frac{\mathrm{d}N_1(s)}{Y_1(s)} - \frac{\mathrm{d}N_1(s) + \mathrm{d}N_2(s)}{Y_1(s)+Y_2(s)})$$
$$= \int_0^t (\mathrm{d}N_1(s)(1 - \frac{Y_1(s)}{Y_1(s)+Y_2(s)}) - \mathrm{d}N_2(s)\frac{Y_1(s)}{Y_1(s)+Y_2(s)})$$
$$= \int_0^t \frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)}(\frac{\mathrm{d}N_1(s)}{Y_1(s)} - \frac{\mathrm{d}N_2(s)}{Y_2(s)})$$

2. Under $H_0 : \alpha_1(t) = \alpha_2(t) = \alpha(t)$:
$$Z_1(t) = \int_0^t \frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)}(\frac{\alpha(s)Y_1(s)\mathrm{d}s + \mathrm{d}M_1(s)}{Y_1(s)} - \frac{\alpha(s)Y_2(s)\mathrm{d}s + \mathrm{d}M_2(s)}{Y_2(s)})$$
$$= \int_0^t \frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)}(\frac{\mathrm{d}M_1(s)}{Y_1(s)} - \frac{\mathrm{d}M_2(s)}{Y_2(s)})$$

3. Using $\langle M_j \rangle(t) = \int_0^t \alpha(s)Y_j(s)\mathrm{d}s, j = 1, 2$ we get:
$$\langle Z_1 \rangle(t) = \int_0^t (\frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)})^2 \left(\frac{\mathrm{d}\langle M_1 \rangle(s)}{(Y_1(s))^2} + \frac{\mathrm{d}\langle M_2 \rangle(s)}{(Y_2(s))^2}\right)$$
$$= \int_0^t (\frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)})^2 \left(\frac{\alpha(s)}{Y_1(s)} + \frac{\alpha(s)}{Y_2(s)}\right) \mathrm{d}s$$

$$= \int_0^t \frac{Y_1(s)Y_2(s)}{Y_1(s)+Y_2(s)}\alpha(s)\mathrm{d}s.$$

Estimating $\alpha(s)\mathrm{d}s$ by $\frac{\mathrm{d}N_{\cdot}(s)}{Y_1(s)+Y_2(s)}$ we get

$$\widehat{\sigma}_{11}(t) = \int_0^t \frac{Y_1(s)Y_2(s)}{(Y_1(s)+Y_2(s))^2}\mathrm{d}N_{\cdot}(s).$$

The same estimator is obtained by using directly the expression for $\widehat{\sigma}_{11}(t)$ from the slides.