Two inequalities

Lemma 1. *For* $|a_i| \leq 1$, $|b_i| \leq 1$, $i = 1, \ldots, n$,

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| \leq \sum |a_i - b_i|$$

Proof. The result is true for $n = 1$. For $n > 1$,

$$\begin{aligned}
\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| &\leq \left| a_n \prod_{i=1}^{n-1} a_i - a_n \prod_{i=1}^{n-1} b_i \right| + \left| a_n \prod_{i=1}^{n-1} b_i - b_n \prod_{i=1}^{n-1} b_i \right| \\
&\leq |a_n| \left| \prod_{i=1}^{n-1} a_i - \prod_{i=1}^{n-1} b_i \right| + |a_n - b_n| \left| \prod_{i=1}^{n-1} b_i \right| \\
&\leq \left| \prod_{i=1}^{n-1} a_i - \prod_{i=1}^{n-1} b_i \right| + |a_n - b_n|
\end{aligned}$$

Now use induction. □

Lemma 2. *For* $|x| \leq 1$

$$|e^x - (1 + x)| \leq x^2$$

Proof.

$$e^x - (1 + x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} - (1 + x) = \sum_{k=2}^{\infty} \frac{x^k}{k!}$$

For $|x| \leq 1$,

$$\left| \sum_{k=2}^{\infty} \frac{x^k}{k!} \right| \leq \frac{x^2}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{x^2}{2} \frac{1}{1 - 1/2} = x^2$$

□