pspipe notes : leakage

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1 Mean leakage

We represent the beam leakage affecting a given array α using two functions $\gamma^{\alpha}_{\mathrm{TE}}$, $\gamma^{\alpha}_{\mathrm{TB}}$ so that

$$\tilde{E^{\alpha}}_{\ell m} = E_{\ell m} + \gamma^{\alpha}_{\ell,\mathrm{TE}} T_{\ell m} \quad (1)$$
$$\tilde{B^{\alpha}}_{\ell m} = B_{\ell m} + \gamma^{\alpha}_{\ell,\mathrm{TB}} T_{\ell m} \quad (2)$$
$$(3)$$

Let's consider the cross spectra between two arrays (α, β), in our notation $C_{\ell}^{T_{\alpha}E_{\beta}}$ is the true cross spectrum between array α and array β and $\tilde{C}_{\ell}^{T_{\alpha}E_{\beta}}$ is the spectrum affected by leakage, the leakage will act the power spectra as follow,

$$\tilde{C}_{\ell}^{T_{\alpha}T_{\beta}} = C_{\ell}^{T_{\alpha}T_{\beta}} \quad (4)$$
$$\tilde{C}_{\ell}^{T_{\alpha}E_{\beta}} = C_{\ell}^{T_{\alpha}E_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TE}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (5)$$
$$\tilde{C}_{\ell}^{T_{\alpha}B_{\beta}} = C_{\ell}^{T_{\alpha}B_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TB}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (6)$$
$$\tilde{C}_{\ell}^{E_{\alpha}T_{\beta}} = C_{\ell}^{E_{\alpha}T_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TE}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (7)$$
$$\tilde{C}_{\ell}^{B_{\alpha}T_{\beta}} = C_{\ell}^{B_{\alpha}T_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TB}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (8)$$
$$\tilde{C}_{\ell}^{E_{\alpha}E_{\beta}} = C_{\ell}^{E_{\alpha}E_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TE}} C_{\ell}^{E_{\alpha}T_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TE}} C_{\ell}^{T_{\alpha}E_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TE}}\gamma^{\alpha}_{\ell,\mathrm{TE}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (9)$$
$$\tilde{C}_{\ell}^{E_{\alpha}B_{\beta}} = C_{\ell}^{E_{\alpha}B_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TE}} C_{\ell}^{T_{\alpha}B_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TB}} C_{\ell}^{E_{\alpha}T_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TB}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (10)$$
$$\tilde{C}_{\ell}^{B_{\alpha}E_{\beta}} = C_{\ell}^{B_{\alpha}E_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TB}} C_{\ell}^{T_{\alpha}E_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TE}} C_{\ell}^{B_{\alpha}T_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TE}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (11)$$
$$\tilde{C}_{\ell}^{B_{\alpha}B_{\beta}} = C_{\ell}^{B_{\alpha}B_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TB}} C_{\ell}^{B_{\alpha}T_{\beta}} + \gamma^{\alpha}_{\ell,\mathrm{TB}} C_{\ell}^{T_{\alpha}B_{\beta}} + \gamma^{\beta}_{\ell,\mathrm{TB}}\gamma^{\alpha}_{\ell,\mathrm{TB}} C_{\ell}^{T_{\alpha}T_{\beta}} \quad (12)$$

We can write this in matricial form as

$$\tilde{\boldsymbol{C}}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} = (\mathbb{I} + \boldsymbol{\Gamma})\boldsymbol{C}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} \quad (13)$$

with

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\beta}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\beta}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\alpha}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\alpha}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\beta}_{\ell,\mathrm{TE}}\gamma^{\alpha}_{\ell,\mathrm{TE}} & \gamma^{\alpha}_{\ell,\mathrm{TE}} & 0 & \gamma^{\beta}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TB}} & 0 & \gamma^{\alpha}_{\ell,\mathrm{TE}} & \gamma^{\beta}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 \\ \gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TE}} & \gamma^{\alpha}_{\ell,\mathrm{TB}} & 0 & 0 & \gamma^{\beta}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 \\ \gamma^{\beta}_{\ell,\mathrm{TB}}\gamma^{\alpha}_{\ell,\mathrm{TB}} & 0 & \gamma^{\alpha}_{\ell,\mathrm{TB}} & 0 & \gamma^{\beta}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (14)$$

Another potential useful form is the symbolic form

$$\tilde{C}_{\ell}^{X_{\alpha}Y_{\beta}} = C_{\ell}^{X_{\alpha}Y_{\beta}} + (\delta_{XE}\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\gamma^{\alpha}_{\ell,\mathrm{TB}}) C_{\ell}^{T_{\alpha}Y_{\beta}} + (\delta_{YE}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\gamma^{\beta}_{\ell,\mathrm{TB}}) C_{\ell}^{X_{\alpha}T_{\beta}}$$
$$+ (\delta_{XE}\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\gamma^{\alpha}_{\ell,\mathrm{TB}})(\delta_{YE}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\gamma^{\beta}_{\ell,\mathrm{TB}}) C_{\ell}^{T_{\alpha}T_{\beta}} \quad (15)$$

Note that we keep the second order term in γ because it multiples $C_\ell^{T_\alpha T_\beta}$ which is large.

2 Propagating uncertainties due to leakage

In reality the γ^α are noisy measurements of the true beam leakage

$$\gamma^\alpha_{\ell,X} = \bar{\gamma}^\alpha_{\ell,X} + \Delta\gamma^\alpha_{\ell,X} \tag{16}$$

2.0.1 Bias

An estimator for beam leakage corrected spectra can be written

$$\begin{aligned}
\hat{\boldsymbol{C}}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} = \tilde{\boldsymbol{C}}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} - \bar{\boldsymbol{\Gamma}}\boldsymbol{C}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} &= \left[\mathbb{I} + (\boldsymbol{\Gamma} - \bar{\boldsymbol{\Gamma}})\right]\boldsymbol{C}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} & (17)\\
&= \left[\mathbb{I} + \Delta\boldsymbol{\Gamma}\right]\boldsymbol{C}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} & (18)
\end{aligned}$$

Where we would use the average measurement for constructing $\bar{\boldsymbol{\Gamma}}$

$$\Delta\boldsymbol{\Gamma} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\beta_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\beta_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\alpha_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\alpha_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\beta_{\ell,\mathrm{TE}}\bar{\gamma}^\alpha_{\ell,\mathrm{TE}} + \bar{\gamma}^\beta_{\ell,\mathrm{TE}}\Delta\gamma^\alpha_{\ell,\mathrm{TE}} + \Delta\gamma^\beta_{\ell,\mathrm{TE}}\Delta\gamma^\alpha_{\ell,\mathrm{TE}} & \Delta\gamma^\alpha_{\ell,\mathrm{TE}} & 0 & \Delta\gamma^\beta_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\alpha_{\ell,\mathrm{TE}}\bar{\gamma}^\beta_{\ell,\mathrm{TB}} + \bar{\gamma}^\alpha_{\ell,\mathrm{TE}}\Delta\gamma^\beta_{\ell,\mathrm{TB}} + \Delta\gamma^\alpha_{\ell,\mathrm{TE}}\Delta\gamma^\beta_{\ell,\mathrm{TB}} & 0 & \Delta\gamma^\alpha_{\ell,\mathrm{TE}} & \Delta\gamma^\beta_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0\\
\Delta\gamma^\alpha_{\ell,\mathrm{TB}}\bar{\gamma}^\beta_{\ell,\mathrm{TE}} + \bar{\gamma}^\alpha_{\ell,\mathrm{TB}}\Delta\gamma^\beta_{\ell,\mathrm{TE}} + \Delta\gamma^\alpha_{\ell,\mathrm{TB}}\Delta\gamma^\beta_{\ell,\mathrm{TE}} & \Delta\gamma^\alpha_{\ell,\mathrm{TB}} & 0 & 0 & \Delta\gamma^\beta_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0\\
\Delta\gamma^\beta_{\ell,\mathrm{TB}}\bar{\gamma}^\alpha_{\ell,\mathrm{TB}} + \bar{\gamma}^\beta_{\ell,\mathrm{TB}}\Delta\gamma^\alpha_{\ell,\mathrm{TB}} + \Delta\gamma^\beta_{\ell,\mathrm{TB}}\Delta\gamma^\alpha_{\ell,\mathrm{TB}} & 0 & \Delta\gamma^\alpha_{\ell,\mathrm{TB}} & 0 & \Delta\gamma^\beta_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0
\end{pmatrix} \tag{19}$$

Note that the estimator is biased since $\langle\Delta\boldsymbol{\Gamma}\rangle$ is non zero for some of the elements

$$\begin{aligned}
\langle\Delta\boldsymbol{\Gamma}\rangle &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\langle\Delta\gamma^\beta_{\ell,\mathrm{TE}}\Delta\gamma^\alpha_{\ell,\mathrm{TE}}\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\langle\Delta\gamma^\alpha_{\ell,\mathrm{TE}}\Delta\gamma^\beta_{\ell,\mathrm{TB}}\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\langle\Delta\gamma^\alpha_{\ell,\mathrm{TB}}\Delta\gamma^\beta_{\ell,\mathrm{TE}}\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\langle\Delta\gamma^\beta_{\ell,\mathrm{TB}}\Delta\gamma^\alpha_{\ell,\mathrm{TB}}\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} & (20)\\
&= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Sigma^{\beta\alpha}_{\ell,\mathrm{TETE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Sigma^{\alpha\beta}_{\ell,\mathrm{TETB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Sigma^{\alpha\beta}_{\ell,\mathrm{TBTE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Sigma^{\beta\alpha}_{\ell,\mathrm{TBTB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} & (21)
\end{aligned}$$

An unbiased estimator is given by

$$\hat{\boldsymbol{C}}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} = \tilde{\boldsymbol{C}}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} - \boldsymbol{\Lambda}\boldsymbol{C}_{\boldsymbol{\ell}}^{\boldsymbol{\alpha\beta}} \tag{22}$$

$$
\mathbf{\Lambda} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}}\bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}} + \Sigma^{\beta\alpha}_{\ell,\mathrm{TETE}} & \bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}} & 0 & \bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} + \Sigma^{\alpha\beta}_{\ell,\mathrm{TETB}} & 0 & \bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}} & \bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} + \Sigma^{\alpha\beta}_{\ell,\mathrm{TBTE}} & \bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}} & 0 & 0 & \bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} & 0 & 0 & 0 & 0 \\
\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}}\bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}} + \Sigma^{\beta\alpha}_{\ell,\mathrm{TBTB}} & 0 & \bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}} & 0 & \bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} & 0 & 0 & 0 & 0
\end{pmatrix} \tag{23}
$$

Let's go back to the symbolic form

$$
\begin{aligned}
\tilde{C}^{X_\alpha Y_\beta}_\ell &= C^{X_\alpha Y_\beta}_\ell + (\delta_{XE}\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\gamma^{\alpha}_{\ell,\mathrm{TB}})C^{T_\alpha Y_\beta}_\ell + (\delta_{YE}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\gamma^{\beta}_{\ell,\mathrm{TB}})C^{X_\alpha T_\beta}_\ell \\
&+ (\delta_{XE}\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\gamma^{\alpha}_{\ell,\mathrm{TB}})(\delta_{YE}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\gamma^{\beta}_{\ell,\mathrm{TB}})C^{T_\alpha T_\beta}_\ell
\end{aligned} \tag{24}
$$

and expand the last term

$$
\begin{aligned}
\tilde{C}^{X_\alpha Y_\beta}_\ell &= C^{X_\alpha Y_\beta}_\ell + (\delta_{XE}\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\gamma^{\alpha}_{\ell,\mathrm{TB}})C^{T_\alpha Y_\beta}_\ell + (\delta_{YE}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\gamma^{\beta}_{\ell,\mathrm{TB}})C^{X_\alpha T_\beta}_\ell \\
&+ (\delta_{XE}\delta_{YE}\gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{XB}\delta_{YB}\gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TB}} + \delta_{XB}\delta_{YE}\gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{XE}\delta_{YB}\gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TB}})C^{T_\alpha T_\beta}_\ell
\end{aligned} \tag{25}
$$

The leakage beam corrected spectra can be written

$$
\begin{aligned}
\hat{C}^{X_\alpha Y_\beta}_\ell &= \tilde{C}^{X_\alpha Y_\beta}_\ell - \Lambda^{X_\alpha Y_\beta}_{W_\mu Z_\nu} C^{W_\mu Z_\nu}_\ell & (26) \\
&= C^{X_\alpha Y_\beta}_\ell + (\delta_{XE}\Delta\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\Delta\gamma^{\alpha}_{\ell,\mathrm{TB}})C^{T_\alpha Y_\beta}_\ell + (\delta_{YE}\Delta\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\Delta\gamma^{\beta}_{\ell,\mathrm{TB}})C^{X_\alpha T_\beta}_\ell \\
&+ [\delta_{XE}\delta_{YE}(\gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TE}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} - \Sigma^{\beta\alpha}_{\ell,\mathrm{TETE}}) + \delta_{XB}\delta_{YB}(\gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TB}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} - \Sigma^{\beta\alpha}_{\ell,\mathrm{TBTB}}]C^{T_\alpha T_\beta}_\ell \\
&+ [\delta_{XB}\delta_{YE}(\gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TE}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} - \Sigma^{\alpha\beta}_{\ell,\mathrm{TBTE}}) + \delta_{XE}\delta_{YB}(\gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TB}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} - \Sigma^{\alpha\beta}_{\ell,\mathrm{TETB}})]C^{T_\alpha T_\beta}_\ell & (27)
\end{aligned}
$$

2.0.2 Covariance

The covariance of the leakage corrected spectrum will be given by

$$
\begin{aligned}
&\mathrm{Cov}(\hat{\mathrm{C}}^{\mathrm{X}_\alpha \mathrm{Y}_\beta}_\ell, \hat{\mathrm{C}}^{\mathrm{W}_\mu \mathrm{Z}_\nu}_{\ell'}) = \langle(\hat{\mathrm{C}}^{\mathrm{X}_\alpha \mathrm{Y}_\beta}_\ell - \mathrm{C}^{\mathrm{X}_\alpha \mathrm{Y}_\beta}_\ell)(\hat{\mathrm{C}}^{\mathrm{W}_\mu \mathrm{Z}_\nu}_{\ell'} - \mathrm{C}^{\mathrm{W}_\mu \mathrm{Z}_\nu}_{\ell'})\rangle & (28) \\
= \; &\langle[[(\delta_{XE}\Delta\gamma^{\alpha}_{\ell,\mathrm{TE}} + \delta_{XB}\Delta\gamma^{\alpha}_{\ell,\mathrm{TB}})C^{T_\alpha Y_\beta}_\ell + (\delta_{YE}\Delta\gamma^{\beta}_{\ell,\mathrm{TE}} + \delta_{YB}\Delta\gamma^{\beta}_{\ell,\mathrm{TB}})C^{X_\alpha T_\beta}_\ell \\
&+[\delta_{XE}\delta_{YE}(\gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TE}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} - \Sigma^{\beta\alpha}_{\ell,\mathrm{TETE}}) + \delta_{XB}\delta_{YB}(\gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TB}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} - \Sigma^{\beta\alpha}_{\ell,\mathrm{TBTB}}]C^{T_\alpha T_\beta}_\ell \\
&+[\delta_{XB}\delta_{YE}(\gamma^{\alpha}_{\ell,\mathrm{TB}}\gamma^{\beta}_{\ell,\mathrm{TE}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TB}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TE}} - \Sigma^{\alpha\beta}_{\ell,\mathrm{TBTE}}) + \delta_{XE}\delta_{YB}(\gamma^{\alpha}_{\ell,\mathrm{TE}}\gamma^{\beta}_{\ell,\mathrm{TB}} - \bar{\gamma}^{\alpha}_{\ell,\mathrm{TE}}\bar{\gamma}^{\beta}_{\ell,\mathrm{TB}} - \Sigma^{\alpha\beta}_{\ell,\mathrm{TETB}})]C^{T_\alpha T_\beta}_\ell] \\
&[(\delta_{WE}\Delta\gamma^{\mu}_{\ell',\mathrm{TE}} + \delta_{WB}\Delta\gamma^{\mu}_{\ell',\mathrm{TB}})C^{T_\mu Z_\nu}_{\ell'} + (\delta_{ZE}\Delta\gamma^{\nu}_{\ell',\mathrm{TE}} + \delta_{ZB}\Delta\gamma^{\nu}_{\ell',\mathrm{TB}})C^{W_\mu T_\nu}_{\ell'} \\
&+[\delta_{WE}\delta_{ZE}(\gamma^{\mu}_{\ell',\mathrm{TE}}\gamma^{\nu}_{\ell',\mathrm{TE}} - \bar{\gamma}^{\mu}_{\ell',\mathrm{TE}}\bar{\gamma}^{\nu}_{\ell',\mathrm{TE}} - \Sigma^{\nu\mu}_{\ell',\mathrm{TETE}}) + \delta_{WB}\delta_{ZB}(\gamma^{\mu}_{\ell',\mathrm{TB}}\gamma^{\nu}_{\ell',\mathrm{TB}} - \bar{\gamma}^{\mu}_{\ell',\mathrm{TB}}\bar{\gamma}^{\nu}_{\ell',\mathrm{TB}} - \Sigma^{\nu\mu}_{\ell',\mathrm{TBTB}}]C^{T_\mu T_\nu}_{\ell'} \\
&+[\delta_{WB}\delta_{ZE}(\gamma^{\mu}_{\ell',\mathrm{TB}}\gamma^{\nu}_{\ell',\mathrm{TE}} - \bar{\gamma}^{\mu}_{\ell',\mathrm{TB}}\bar{\gamma}^{\nu}_{\ell',\mathrm{TE}} - \Sigma^{\mu\nu}_{\ell',\mathrm{TBTE}}) + \delta_{WE}\delta_{ZB}(\gamma^{\mu}_{\ell',\mathrm{TE}}\gamma^{\nu}_{\ell',\mathrm{TB}} - \bar{\gamma}^{\mu}_{\ell',\mathrm{TE}}\bar{\gamma}^{\nu}_{\ell',\mathrm{TB}} - \Sigma^{\mu\nu}_{\ell',\mathrm{TETB}})]C^{T_\mu T_\nu}_{\ell'}]\rangle & (29)
\end{aligned}
$$