Out[1]: (Show / Hide code)

Euler Angle Object Rotation

CSE 4303 / CSE 5365 Computer Graphics

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There are many, many ways to represent and describe rotations in three dimensions. Different references have different orientations, descriptions, and notations and it is easy to get quite confused. Be especially careful about terminology and notation when you cross-reference this document with any other references.

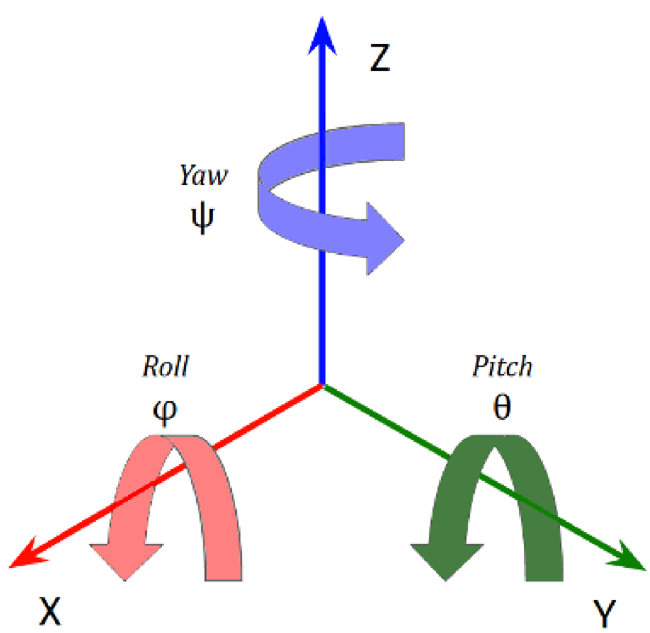
Euler Angle Rotation of an Object

In three-dimensional space, to *orient* a *vector* rooted at the origin in any *direction* requires only two angles. One way to do this orientation is to turn the vector about the z axis and then raise or lower the vector's and point until the vector is pointing in the desired direction. Even though this is happening in a three-dimensional space, two angles suffice since we are identifying only a *direction* for the vector and not a specific *point* $\in \mathfrak{R}^3$.

However to *orient* a *rigid body* with respect to a *fixed coordinate system* in three-dimensional space requires *three* angles, one for each axis. Leonhard Euler created a systematic way to do this (in 1776) and thus we generally call these angles the *Euler Angles*. Peter Tait and George Bryan later extended Euler's work (in 1911) and therefore some refer to certain classes of these angles and their application as *Tait-Bryan Angles*. We will stick with the generic name Euler angles to avoid having to switch back and forth between two designations.

For the purposes of this class we will denote the three Euler angles as ϕ, θ, and ψ.

- ϕ represents the rotation about the x axis and is know as *Roll*.
- θ represents the rotation about the y axis and is known as *Pitch*.
- ψ represents the rotation about the z axis and is known as *Yaw*.



The names *Roll*, *Pitch*, and *Yaw* come from the aeronautical world, where they represent the aircraft's bearing, elevation, and bank angle.

Doing the Rotation

Each of these rotations is an *affine* three-dimensional transformation. We express them in terms of homogeneous coordinates so the matrix is 4×4.

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are twelve ways in which the angles can be applied,

- The *(Proper) Euler* orders: *z-x-z*, *z-y-z*, *y-x-y*, *y-z-y*, *x-y-x*, *x-z-x*.
- The *Tait-Bryan* orders: *x-y-z*, *x-z-y*, *y-x-z*, *y-z-x*, *z-x-y*, *z-y-x*.

We will use the *Tait-Bryan* order *z-y-x* and will therefore do the rotations starting with the application of the ψ (z axis) rotation, then apply the θ (y axis) rotation, and finally apply the ϕ (x axis) rotation. The composition therefore appears thus,

$$R_\phi R_\theta R_\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representing these transforms can get quite lengthy. We will use some abbreviations to make the expressions more concise. The *cosine* of an angle will be represented by the letter *c_ subscripted with the angle, thus* $\cos\phi$ *will be written* c_ϕ*,* $\cos\theta$ *as* c_θ*, and* $\cos\psi$ *as* c_ψ *. Similarly, the _sine* of an angle will be represented by the letter *s* subscripted with the angle, thus $\sin\phi$ as s_ϕ, $\sin\theta$ as s_θ, and $\sin\psi$ as s_ψ. This might not seem like much of an abbreviation, but it will help in some of the more complex expressions that follow.

Rewriting the composition $R_\phi R_\theta R_\psi$ using this abbreviated notation results in the following,

Out[3]: $\mathbf{R}_{\phi\theta\psi} = \mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\phi & -s_\phi & 0 \\ 0 & s_\phi & c_\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 & 0 \\ s_\psi & c_\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Out[4]: By multiplying these matrices together, we get this composite transformation matrix,

$$= \begin{bmatrix} c_\psi c_\theta & -c_\theta s_\psi & s_\theta & 0 \\ c_\phi s_\psi + c_\psi s_\phi s_\theta & c_\phi c_\psi - s_\phi s_\psi s_\theta & -c_\theta s_\phi & 0 \\ -c_\phi c_\psi s_\theta + s_\phi s_\psi & c_\phi s_\psi s_\theta + c_\psi s_\phi & c_\phi c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since all three of the rotations are affine transformations, their composition is still an affine transformation, giving the bottom row of the composite matrix the values $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$. As the transformations did not include any translation, the first three elements of the right-most column are all 0.

This representation of that product of matrices is kind of complicated. Let's try to make it a bit simpler by creating nine scalars, named r_{00} through r_{22}, where r_{mn} means the scalar from the m^{th} row and n^{th} column. (Remember rows and columns are numbered from 0.)

We give these scalars the following definitions,

Out[5]:

$$\begin{array}{lll} r_{00} = c_\psi c_\theta & r_{01} = -c_\theta s_\psi & r_{02} = s_\theta \\ r_{10} = c_\phi s_\psi + c_\psi s_\phi s_\theta & r_{11} = c_\phi c_\psi - s_\phi s_\psi s_\theta & r_{12} = -c_\theta s_\phi \\ r_{20} = -c_\phi c_\psi s_\theta + s_\phi s_\psi & r_{21} = c_\phi s_\psi s_\theta + c_\psi s_\phi & r_{22} = c_\phi c_\theta \end{array}$$

Each one of these scalars is some combination of sine and cosine values of the Euler angles and is easy to compute if one is careful about the various negations, orders, etc. Having made these definitions, we can then represent the composite Euler rotation matrix as,

$$\mathbf{R}_{\phi\theta\psi} = \begin{bmatrix} r_{00} & r_{01} & r_{02} & 0 \\ r_{10} & r_{11} & r_{12} & 0 \\ r_{20} & r_{21} & r_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We don't bother to create any scalar names for the bottom row or the right-most column since as we said above the composite rotation transformation matrix is an affine transformation that does not include any translation. Therefore, the values of those seven entries are fixed: all are 0 except for the w element, which has the value 1.

One may wonder why we bother to make this representation as it doesn't seem to gain us anything. However, remember that the rotation that we just described is with respect to the *origin*. Since we want to rotate the object about *its own center*, we first have to *translate* it to the origin, then do the rotation, and then *translate* it back to where it came from. Thus we have to include two more transformations with the rotation. The use of r_{00} through r_{22} in this representation will save us considerable complexity when this composition is made.

Adding the Translation

Let t_x, t_y, and t_z represent the coordinates of the *center* of the object. (See the Appendix for info on how to obtain the center of an object.) We then have the complete transform to rotate an object about its own center according to the Euler angles ϕ, θ, and ψ represented thus,

Out[6]:

$$\mathbf{T_{inv}}\mathbf{R}_{\phi\theta\psi}\mathbf{T} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{00} & r_{01} & r_{02} & 0 \\ r_{10} & r_{11} & r_{12} & 0 \\ r_{20} & r_{21} & r_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{00} & r_{01} & r_{02} & -r_{00}t_x - r_{01}t_y - r_{02}t_z + t_x \\ r_{10} & r_{11} & r_{12} & -r_{10}t_x - r_{11}t_y - r_{12}t_z + t_y \\ r_{20} & r_{21} & r_{22} & -r_{20}t_x - r_{21}t_y - r_{22}t_z + t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The translation on the right uses the *negated* values of the object's center coordinates. This is because it represents a translation *to* the origin. The translation on the left uses the *non-negated* values of the object's center coordinates as it represents a translation *from* the origin.

Since the additional two transformations are translations only, neither of them affects the rotation represented by rows 0 through 2, columns 0 through 2 in the matrix. Those nine values stayed the same, r_{00} through r_{22}. Only the translation portion of the matrix (the right-most column, rows 0-2) was affected.

Given this representation of the complete transformation, we can apply it to a point $\mathbf{p}$ and get the transformed point $\mathbf{p}'$ in return.

Out[7]:

$$\mathbf{p}' = (\mathbf{T_{inv}}\mathbf{R}_{\phi\theta\psi}\mathbf{T})\mathbf{p} =$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} r_{00} & r_{01} & r_{02} & -r_{00}t_x - r_{01}t_y - r_{02}t_z + t_x \\ r_{10} & r_{11} & r_{12} & -r_{10}t_x - r_{11}t_y - r_{12}t_z + t_y \\ r_{20} & r_{21} & r_{22} & -r_{20}t_x - r_{21}t_y - r_{22}t_z + t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} -r_{00}t_x + r_{00}x - r_{01}t_y + r_{01}y - r_{02}t_z + r_{02}z + t_x \\ -r_{10}t_x + r_{10}x - r_{11}t_y + r_{11}y - r_{12}t_z + r_{12}z + t_y \\ -r_{20}t_x + r_{20}x - r_{21}t_y + r_{21}y - r_{22}t_z + r_{22}z + t_z \\ 1 \end{bmatrix}$$

Since this is an affine transformation, w' will always be 1. What we are really interested in are the expressions for x', y', and z'. They can be extracted as thus,

Out[8]:

$$x' = -r_{00}t_x + r_{00}x - r_{01}t_y + r_{01}y - r_{02}t_z + r_{02}z + t_x$$

$$y' = -r_{10}t_x + r_{10}x - r_{11}t_y + r_{11}y - r_{12}t_z + r_{12}z + t_y$$

$$z' = -r_{20}t_x + r_{20}x - r_{21}t_y + r_{21}y - r_{22}t_z + r_{22}z + t_z$$

Or, in a more useful format obtained by doing a bit of algebraic reordering,

Out[9]:

$$x' = r_{00}x + r_{01}y + r_{02}z + (-r_{00}t_x - r_{01}t_y - r_{02}t_z + t_x)$$

$$y' = r_{10}x + r_{11}y + r_{12}z + (-r_{10}t_x - r_{11}t_y - r_{12}t_z + t_y)$$

$$z' = r_{20}x + r_{21}y + r_{22}z + (-r_{20}t_x - r_{21}t_y - r_{22}t_z + t_z)$$

As a further simplification, we can make three definitions for the final parenthesized terms in these expressions for x', y', and z'. Those terms do not include x, y, or z.

Out[10]:

$$e_x = -r_{00}t_x - r_{01}t_y - r_{02}t_z + t_x$$

$$e_y = -r_{10}t_x - r_{11}t_y - r_{12}t_z + t_y$$

$$e_z = -r_{20}t_x - r_{21}t_y - r_{22}t_z + t_z$$

We can now express the values of the transformed coordinates quite succinctly as,

$$x' = r_{00}x + r_{01}y + r_{02}z + e_x$$

$$y' = r_{10}x + r_{11}y + r_{12}z + e_y$$

$$z' = r_{20}x + r_{21}y + r_{22}z + e_z$$

Note the regularity in these final expressions. Each of the x', y', and z' coordinates is the sum of a constant (e_x, e_y, or e_z) and the original x, y, and z coordinates each multiplied by a constant coefficient (the r_{00} through r_{22} values). When we say 'constant' here, we mean that those scalars can be computed without knowing the point's specific x, y, or z coordinate values. We will make use of this fact when we discuss efficiency below.

It's also interesting that in the final expressions the cost of computing each of the transformed coordinates is now only three multiplications and three additions, for a grand total of nine multiplications and nine additions for the entire set of three. That's pretty cheap considering all the work that went into computing r_{00} through r_{22} and e_x, e_y, and e_z.

Computing Transformed Points

The derivation above gave us the precise definition of what the transformed value of a point will be after it has been rotated about its object's center by the three Euler angles.

Let's now consider how to compute these transformed points efficiently.

Required Inputs

Three sets of inputs are required to compute the transformed value of a point.

- The Euler angles, ϕ, θ, and ψ. These are generally obtained from the user.
- The center of the object, t_x, t_y, and t_z. These are calculated by examining the all of the vertices of the object. (See the Appendix for info on how to obtain the center of an object.)
- The point's coordinates, x, y, and z. These are obtained from each of the vertices of the object.

Step One: Compute the r Values

We first use the Euler angles and compute the nine values r_{00} through r_{22}. Computing these values is easy, if tedious. After computing the sine and cosine of each of the three Euler angles ϕ, θ, and ψ, we combine them according to the definitions for r_{00} through r_{22} given above. Care must be taken that these values are computed accurately, as any mistake will severely screw up the results and will likely be tricky to track down. Don't confuse $\sin$ and $\cos$ or ϕ, θ, or ψ. Remember that some trigonometry packages express angles in radians and some express angles in degrees. Ensure that you know how to use your trigonometry package correctly.

Step Two: Compute the a Values

Next, we use the values r_{00} through r_{22} and the center-of-object values t_x, t_y, and t_z to compute the three values e_x, e_y, and e_z according to their definitions given above. Again, care must be taken to compute these values accurately as any mistake will severely screw up the results and will likley be tricky to track down. This is simpler than *Step One*, but still be careful.

Step Three: Compute the Transformed Point

Finally, we use the values r_{00} through r_{22}, e_x, e_y, and e_z, and the point's coordinates x, y, and z to compute the x', y', and z' values according to the definitions given above.

Et voila, c'est fait!

Being Efficient

Important points to notice about the above steps are,

- *Step One* has to be performed any time any of the ϕ, θ, or ψ Euler angles change. This is usually due to user input.
- *Step Two* has to be performed any time *Step One* is performed or when any of t_x, t_y, or t_z values change. The object's center changes whenever the object is translated to a new position.
- *Step Three* has to be performed for every point that is to be transformed any time *Step One* or *Step Two* is performed. All points have to be transformed every time the object is drawn.

Being efficient means recognizing that once *Step One* and *Step Two* have been performed, they do *not* have to be performed again until one of the specified changes occurs. So when drawing an object, one performs *Step One* and *Step Two* exactly *once* and then one transforms *all* of the points without performing *Step One* or *Step Two* again.

Even though there's a bunch of work in *Step One* (three sines, three cosines, sixteen multiplies, four additions/subtractions, and two negations), it has to be done *only* when the Euler angles change. *Step Two* costs twelve additions/subtractions, but it has to be done *only* when *Step One* is done or when the object's center changes.

Step Three will be done tens, hundreds, thousands of times — it has to be done for each point of the object. It is therefore worth some effort to make *Step Three* as efficient as possible, which is why we went through this lengthy derivation. Thus we got the cost of *Step Three* down to only ***nine multiplies*** and ***nine additions*** per point. That's a heck of a lot less work than if one did all of those trigonometric functions calls and matrix multiplies for each point.

Note

> *A careful examination of the definitions of r_{00} through r_{22} reveals that four of the factors ($c_\phi s_\psi$, $c_\psi s_\phi$, $c_\phi c_\psi$, and $s_\phi s_\psi$) appear twice, so the number of multiplications in* Step One *can be reduced from sixteen to fourteen by computing those as partial products and using them twice each.*

Numerical Examples

Let's consider some simple numerical examples.

Simplest Example

First, the simple case of ϕ, θ, and ψ all equal to 0.

Step One: Since all of the angles are 0, all of the cosine values are 1 and all of the sine values are 0. Examining the definitions for r_{00} through r_{22},

Out[11]:

$$\begin{array}{lll} r_{00} = c_\psi c_\theta & r_{01} = -c_\theta s_\psi & r_{02} = s_\theta \\ r_{10} = c_\phi s_\psi + c_\psi s_\phi s_\theta & r_{11} = c_\phi c_\psi - s_\phi s_\psi s_\theta & r_{12} = -c_\theta s_\phi \\ r_{20} = -c_\phi c_\psi s_\theta + s_\phi s_\psi & r_{21} = c_\phi s_\psi s_\theta + c_\psi s_\phi & r_{22} = c_\phi c_\theta \end{array}$$

we see that they reduce to,

$$\begin{array}{lll} r_{00} = 1 \cdot 1 & r_{01} = -1 \cdot 0 & r_{02} = 0 \\ r_{10} = 1 \cdot 0 + 1 \cdot 0 \cdot 0 & r_{11} = 1 \cdot 1 + 0 \cdot 0 \cdot 0 & r_{12} = -1 \cdot 0 \\ r_{20} = -1 \cdot 1 \cdot 0 + 0 \cdot 0 & r_{21} = 1 \cdot 0 \cdot 0 + 0 & r_{22} = 1 \cdot 1 \end{array}$$

or,

$$\begin{array}{lll} r_{00} = 1 & r_{01} = 0 & r_{02} = 0 \\ r_{10} = 0 & r_{11} = 1 & r_{12} = 0 \\ r_{20} = 0 & r_{21} = 0 & r_{22} = 1 \end{array}$$

Step Two: Next we consider the definitions of e_x, e_y, and e_z,

Out[12]:

$$e_x = -r_{00}t_x - r_{01}t_y - r_{02}t_z + t_x$$

$$e_y = -r_{10}t_x - r_{11}t_y - r_{12}t_z + t_y$$

$$e_z = -r_{20}t_x - r_{21}t_y - r_{22}t_z + t_z$$

Plugging in the values we have for a_{00} through a_{22}, these expressions reduce to,

$$e_x = -1 \cdot t_x - 0 \cdot t_y - 0 \cdot t_z + t_x \quad = \quad -t_x + t_x \quad = \quad 0$$

$$e_y = -0 \cdot t_x - 1 \cdot t_y - 0 \cdot t_z + t_y \quad = \quad -t_y + t_y \quad = \quad 0$$

$$e_z = -0 \cdot t_x - 0 \cdot t_y - 1 \cdot t_z + t_z \quad = \quad -t_z + t_z \quad = \quad 0$$

Now that we have all of the r_{00} through r_{22} and the e_x, e_y, and e_z values, we can build the expressions for x', y', and z'. These are defined as,

$$x' = r_{00}x + r_{01}y + r_{02}z + e_x$$

$$y' = r_{10}x + r_{11}y + r_{12}z + e_y$$

$$z' = r_{20}x + r_{21}y + r_{22}z + e_z$$

Plugging in r_{00} through r_{22} and e_x, e_y, and e_z, we obtain,

$$x' = 1 \cdot x + 0 \cdot y + 0 \cdot z + 0 \quad = \quad x$$

$$y' = 0 \cdot x + 1 \cdot y + 0 \cdot z + 0 \quad = \quad y$$

$$z' = 0 \cdot x + 0 \cdot y + 1 \cdot z + 0 \quad = \quad z$$

So, as we ought to have expected, no matter where an object's center is located, when the Euler angles are all zero, the "transformed" points are the same as the original points.

A Bit More Complex Example

We now consider what happens if the Euler angles are $\phi = \frac{\pi}{4}$ (45°), $\theta = \frac{\pi}{6}$ (30°), and $\psi = 0$ and the center of the object is located at $\mathbf{t} = \begin{bmatrix} 4 & 8 & 12 \end{bmatrix}^\intercal$.

Step One: The cosine and sine values for the angles are as follows,

$$\begin{aligned} c_\phi &= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} & s_\phi &= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ c_\theta &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} & s_\theta &= \sin\frac{\pi}{6} = \frac{1}{2} \\ c_\psi &= \cos 0 = 1 & s_\psi &= \sin 0 = 0 \end{aligned}$$

Using the definitions for r_{00} through r_{22},

Out[13]:

$$
\begin{array}{lll}
r_{00} = c_\psi c_\theta & r_{01} = -c_\theta s_\psi & r_{02} = s_\theta \\
r_{10} = c_\phi s_\psi + c_\psi s_\phi s_\theta & r_{11} = c_\phi c_\psi - s_\phi s_\psi s_\theta & r_{12} = -c_\theta s_\phi \\
r_{20} = -c_\phi c_\psi s_\theta + s_\phi s_\psi & r_{21} = c_\phi s_\psi s_\theta + c_\psi s_\phi & r_{22} = c_\phi c_\theta
\end{array}
$$

we have,

$$
\begin{array}{lll}
r_{00} = 1 \cdot \frac{\sqrt{3}}{2} & r_{01} = -\frac{\sqrt{3}}{2} \cdot 0 & r_{02} = \frac{1}{2} \\
r_{10} = \frac{\sqrt{2}}{2} \cdot 0 + 1 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} & r_{11} = \frac{\sqrt{2}}{2} \cdot 1 - \frac{\sqrt{2}}{2} \cdot 0 \cdot \frac{1}{2} & r_{12} = -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
r_{20} = -\frac{\sqrt{2}}{2} \cdot 1 \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot 0 & r_{21} = \frac{\sqrt{2}}{2} \cdot 0 \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{2}}{2} & r_{22} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}
\end{array}
$$

Out[14]: and, simplifying,

$$
\begin{array}{lll}
r_{00} = \frac{\sqrt{3}}{2} & r_{01} = 0 & r_{02} = \frac{1}{2} \\
r_{10} = \frac{\sqrt{2}}{4} & r_{11} = \frac{\sqrt{2}}{2} & r_{12} = -\frac{\sqrt{6}}{4} \\
r_{20} = -\frac{\sqrt{2}}{4} & r_{21} = \frac{\sqrt{2}}{2} & r_{22} = \frac{\sqrt{6}}{4}
\end{array}
$$

Step 2: Next we consider the definitions of e_x, e_y, and e_z,

Out[15]:

$$e_x = -2\sqrt{3} - 2$$

$$e_y = -5\sqrt{2} + 3\sqrt{6} + 8$$

$$e_z = -3\sqrt{6} - 3\sqrt{2} + 12$$

Out[16]: Plugging in the values we have for r_{00} through r_{22} and t_x, t_y, and t_z, we obtain,

$$e_x = -(\frac{\sqrt{3}}{2}) \cdot 4 - (0) \cdot 8 - (\frac{1}{2}) \cdot 12 + 4$$

$$e_y = -(\frac{\sqrt{2}}{4}) \cdot 4 - (\frac{\sqrt{2}}{2}) \cdot 8 - (-\frac{\sqrt{6}}{4}) \cdot 12 + 8$$

$$e_z = -(-\frac{\sqrt{2}}{4}) \cdot 4 - (\frac{\sqrt{2}}{2}) \cdot 8 - (\frac{\sqrt{6}}{4}) \cdot 12 + 12$$

Out[17]: or, simplifying,

$$e_x = -2\sqrt{3} - 2$$

$$e_y = -5\sqrt{2} + 3\sqrt{6} + 8$$

$$e_z = -3\sqrt{6} - 3\sqrt{2} + 12$$

Now that we have the r_{00} through r_{22} and the e_x, e_y, and e_z values, we can build the expressions for x', y', and z'. These are defined as,

$$x' = r_{00}x + r_{01}y + r_{02}z + e_x$$

$$y' = r_{10}x + r_{11}y + r_{12}z + e_y$$

$$z' = r_{20}x + r_{21}y + r_{22}z + e_z$$

Plugging in r_{00} through r_{22} and e_x, e_y, and e_z, we obtain,

Out[18]:

$$x' = (\frac{\sqrt{3}}{2}) \cdot x + (0) \cdot y + (\frac{1}{2}) \cdot z + (-2\sqrt{3} - 2)$$

$$y' = (\frac{\sqrt{2}}{4}) \cdot x + (\frac{\sqrt{2}}{2}) \cdot y + (-\frac{\sqrt{6}}{4}) \cdot z + (-5\sqrt{2} + 3\sqrt{6} + 8)$$

$$z' = (-\frac{\sqrt{2}}{4}) \cdot x + (\frac{\sqrt{2}}{2}) \cdot y + (\frac{\sqrt{6}}{4}) \cdot z + (-3\sqrt{6} - 3\sqrt{2} + 12)$$

or, finally, when rounded to *three decimal places*,

$$x' = (0.866) \cdot x + (0.000) \cdot y + (0.500) \cdot z + (-5.464)$$

$$y' = (0.354) \cdot x + (0.707) \cdot y + (-0.612) \cdot z + (8.277)$$

$$z' = (-0.354) \cdot x + (0.707) \cdot y + (0.612) \cdot z + (0.409)$$

Step 3: Now that we have found numerical values for all of the required coefficients, we can transform any point by performing only nine multiplications and nine additions, three of each for each of x', y', and z'.

For instance, the point $\mathbf{p} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\intercal$ becomes,

Out[19]:

$$x' = (0.866) \cdot 1.000 + (0.000) \cdot 2.000 + (0.500) \cdot 3.000 + (-5.464)$$

$$y' = (0.354) \cdot 1.000 + (0.707) \cdot 2.000 + (-0.612) \cdot 3.000 + (8.277)$$

$$z' = (-0.354) \cdot 1.000 + (0.707) \cdot 2.000 + (0.612) \cdot 3.000 + (0.409)$$

Out[20]: or,

$$x' = (0.866) + (0.000) + (1.500) + (-5.464)$$

$$y' = (0.354) + (1.414) + (-1.837) + (8.277)$$

$$z' = (-0.354) + (1.414) + (1.837) + (0.409)$$

Out[21]: and, finally,

$$x' = -3.098$$

$$y' = 8.208$$

$$z' = 3.307$$

That was the transformation of just one point. Any number of points may be transformed using the r_{00} through r_{22} and e_x, e_y, and e_z values that we have calculated using the Euler angles $\phi = \frac{\pi}{4}$ (45°), $\theta = \frac{\pi}{6}$ (30°), and $\psi = 0$ and object center $\mathbf{t} = \begin{bmatrix} 4 & 8 & 12 \end{bmatrix}^\intercal$.

For example, let's now transform point $\mathbf{q} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}^\intercal$. Again, we start with the definition of x', y', and z' with the r_{00} through r_{22} and e_x, e_y, and e_z values plugged in. We can reuse the ones we have already calculated as only the *point* changed, and the Euler angles and object center stayed the same.

Out[22]:

$$x' = (0.866) \cdot x + (0.000) \cdot y + (0.500) \cdot z + (-5.464)$$

$$y' = (0.354) \cdot x + (0.707) \cdot y + (-0.612) \cdot z + (8.277)$$

$$z' = (-0.354) \cdot x + (0.707) \cdot y + (0.612) \cdot z + (0.409)$$

Out[23]: Plugging in the values of x, y, and z for the point $\mathbf{q}$ we have,

$$x' = (0.866) \cdot 4.000 + (0.000) \cdot 5.000 + (0.500) \cdot 6.000 + (-5.464)$$

$$y' = (0.354) \cdot 4.000 + (0.707) \cdot 5.000 + (-0.612) \cdot 6.000 + (8.277)$$

$$z' = (-0.354) \cdot 4.000 + (0.707) \cdot 5.000 + (0.612) \cdot 6.000 + (0.409)$$

Out[24]: or,

$$x' = (3.464) + (0.000) + (3.000) + (-5.464)$$

$$y' = (1.414) + (3.536) + (-3.674) + (8.277)$$

$$z' = (-1.414) + (3.536) + (3.674) + (0.409)$$

Out[25]: and, finally,

$$x' = 1.000$$

$$y' = 9.553$$

$$z' = 6.204$$

In transforming point $\mathbf{q}$, we just reused the values for r_{00} through r_{22} that we had since the Euler angles had not changed. We also were able to reuse the values for e_x, e_y, and e_z since not only the Euler angles had not changed but also the object center had not changed either.

We would have to go back to *Step One* to recalculate the r_{00} through r_{22} values and *Step Two* to recalculate the e_x, e_y, and e_z values only if the Euler angle or object center values changed. Otherwise we can continue in *Step Three* for as many more points as we wish.

A Very Specific Example

Let's keep the the Euler angles are $\phi = \frac{\pi}{4}$ (45°), $\theta = \frac{\pi}{6}$ (30°), and $\psi = 0$ for our next example.

Step One: Since we are using the same Euler angles, the values of r_{00} through r_{22} will be the same as in the previous example,

Out[26]:

$$\begin{array}{lll} r_{00} = \dfrac{\sqrt{3}}{2} & r_{01} = 0 & r_{02} = \dfrac{1}{2} \\ r_{10} = \dfrac{\sqrt{2}}{4} & r_{11} = \dfrac{\sqrt{2}}{2} & r_{12} = -\dfrac{\sqrt{6}}{4} \\ r_{20} = -\dfrac{\sqrt{2}}{4} & r_{21} = \dfrac{\sqrt{2}}{2} & r_{22} = \dfrac{\sqrt{6}}{4} \end{array}$$

Out[27]: *Step Two*: We will use the *pyramid* as the object of interest. which has an object center of $\mathbf{t} = \begin{bmatrix} 0.400 & 0.400 & 0.350 \end{bmatrix}^{\mathsf{T}}$. (See the Appendix for info on how to obtain the center of an object.)

Using our definitions for e_x, e_y, and e_z,

$$e_x = -r_{00}t_x - r_{01}t_y - r_{02}t_z + t_x$$

$$e_y = -r_{10}t_x - r_{11}t_y - r_{12}t_z + t_y$$

$$e_z = -r_{20}t_x - r_{21}t_y - r_{22}t_z + t_z$$

Out[28]: we can plug in the values we have already computed for r_{00} through r_{22} and the t_x, t_y, and t_z values, obtaining,

$$e_x = -(\frac{\sqrt{3}}{2}) \cdot 0.4 - (0) \cdot 0.4 - (\frac{1}{2}) \cdot 0.35 + 0.4$$

$$e_y = -(\frac{\sqrt{2}}{4}) \cdot 0.4 - (\frac{\sqrt{2}}{2}) \cdot 0.4 - (-\frac{\sqrt{6}}{4}) \cdot 0.35 + 0.4$$

$$e_z = -(-\frac{\sqrt{2}}{4}) \cdot 0.4 - (\frac{\sqrt{2}}{2}) \cdot 0.4 - (\frac{\sqrt{6}}{4}) \cdot 0.35 + 0.35$$

Out[29]: or, simplifying,

$$e_x = -0.2\sqrt{3} + 0.225$$

$$e_y = -0.3\sqrt{2} + 0.0875\sqrt{6} + 0.4$$

$$e_z = -0.0875\sqrt{6} - 0.1\sqrt{2} + 0.35$$

Out[30]: Now that we have r_{00} through r_{22} and the e_x, e_y, and e_z values, we can construct the expressions for x', y', and z' as follows,

$$x' = (0.866) \cdot x + (0.000) \cdot y + (0.500) \cdot z + (-0.121)$$

$$y' = (0.354) \cdot x + (0.707) \cdot y + (-0.612) \cdot z + (0.190)$$

$$z' = (-0.354) \cdot x + (0.707) \cdot y + (0.612) \cdot z + (-0.006)$$

Out[31]: *Step 3*: Plugging in the values of x, y, and z for the first vertex of the $pyramid$ object ($\begin{bmatrix} 0.000 & 0.000 & 0.000 \end{bmatrix}^\intercal$), we have,

$$x' = (0.866) \cdot 0.000 + (0.000) \cdot 0.000 + (0.500) \cdot 0.000 + (-0.121)$$

$$y' = (0.354) \cdot 0.000 + (0.707) \cdot 0.000 + (-0.612) \cdot 0.000 + (0.190)$$

$$z' = (-0.354) \cdot 0.000 + (0.707) \cdot 0.000 + (0.612) \cdot 0.000 + (-0.006)$$

Out[32]: or,

$$x' = (0.000) + (0.000) + (0.000) + (-0.121)$$

$$y' = (0.000) + (0.000) + (-0.000) + (0.190)$$

$$z' = (-0.000) + (0.000) + (0.000) + (-0.006)$$

Out[33]: and, finally,

$$x' = -0.121$$

$$y' = 0.190$$

$$z' = -0.006$$

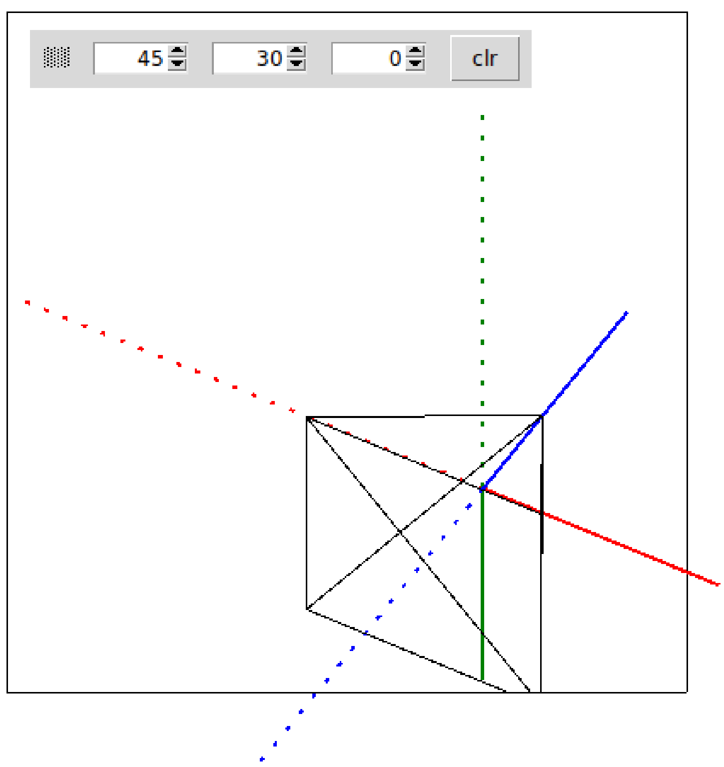
Using the `EulerRotation.py` tool (available along with this handout), we can transform all of the *pyramid* object points and obtain these results,

```
For ϕ   0.785 (  45.000°), θ   0.524 (  30.000°), ψ   0.000 (   0.000°),
  r00   0.866, r01  -0.000, r02   0.500
  r10   0.354, r11   0.707, r12  -0.612
  r20  -0.354, r21   0.707, r22   0.612
```

```
For t = (   0.400,   0.400,   0.350 ),
  ex   -0.121, ey    0.190, ez   -0.006
```

```
Some point transformations,
(   0.000,   0.000,   0.000 ) -> (  -0.121,    0.190,   -0.006 )
(   0.800,   0.000,   0.000 ) -> (   0.571,    0.473,   -0.289 )
(   0.000,   0.800,   0.000 ) -> (  -0.121,    0.756,    0.560 )
(   0.800,   0.800,   0.000 ) -> (   0.571,    1.039,    0.277 )
(   0.400,   0.400,   0.700 ) -> (   0.575,    0.186,    0.564 )
```

This rotation can be seen in the following picture. The canvas size is $width = 500$, $height = 500$. The displayed axes are *local* to the center of the *pyramid* object.



Appendix—Finding the Object Center

The $\mathbf{t} = \begin{bmatrix} t_x & t_y & t_z \end{bmatrix}^\mathsf{T}$ value used above as the *center* of the object must be determined by examining all of the vertices that make up the object. Consider all of vertices and determine the *minimum* and *maximum* values for each of the x, y, and z coordinates across all of the vertices. When all of the vertices have been considered, you will have x_{max}, x_{min}, y_{max}, y_{min}, z_{max}, and z_{min} values for the object. These values define the *bounding box* within which all vertices of the object exist. The center of this bounding box is the center of the object.

When the minimum and maximum values are known, the center of the object may be computed as,

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} x_{min} + \frac{x_{max}-x_{min}}{2} & y_{min} + \frac{y_{max}-y_{min}}{2} & z_{min} + \frac{z_{max}-z_{min}}{2} \end{bmatrix}^\mathsf{T}$$

As a numeric example, let's consider the *pyramid* object mentioned above. This object's vertices are given as,

```
v 0.0 0.0 0.0
v 0.8 0.0 0.0
v 0.0 0.8 0.0
v 0.8 0.8 0.0
v 0.4 0.4 0.7
```

Simple inspection gives us these values,

$$\begin{aligned} x_{min} &= 0.0 & x_{max} &= 0.8 \\ y_{min} &= 0.0 & y_{max} &= 0.8 \\ z_{min} &= 0.0 & z_{max} &= 0.7 \end{aligned}$$

We then compute the object center as,

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$= \begin{bmatrix} 0.0 + \frac{0.8-0.0}{2} & 0.0 + \frac{0.8-0.0}{2} & 0.0 + \frac{0.7-0.0}{2} \end{bmatrix}^\mathsf{T}$$

$$= \begin{bmatrix} \frac{0.8}{2} & \frac{0.8}{2} & \frac{0.7}{2} \end{bmatrix}^\mathsf{T}$$

$$= \begin{bmatrix} 0.400 & 0.400 & 0.350 \end{bmatrix}^\mathsf{T}$$