Out[1]: (Show / Hide code)

Cross Product

CSE 4303 / CSE 5365 Computer Graphics

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The *cross product* of two vectors is a third vector perpendicular to the plane containing the first two and with a magnitude equal to the area of the parallelogram bounded by the first two vectors.

The easiest way to compute the cross product is to use the determinant representation. We therefore begin with a review of the determinant and how to calculate it.

Determinant

The 2×2 determinant is defined as

Out[2]:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv ad - bc$$

The 2×2 determinant is the product of the down-to-the-right (main) diagonal $(a \cdot d)$ minus the product of the down-to-the-left (reverse) diagonal $(b \cdot c)$.

The 3×3 determinant at first appears to be a bit more intricate, but fundamentally it's the same process, the adding of the products along the down-to-the-right diagonals $(a \cdot e \cdot k, b \cdot f \cdot g, c \cdot d \cdot h)$ minus the products along the down-to-the-left diagonals $(c \cdot e \cdot g, a \cdot f \cdot h, b \cdot d \cdot k)$. We have, therefore,

Out[3]:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \equiv aek - afh - bdk + bfg + cdh - ceg$$

The diagonals are a bit easier to see if we augment the matrix by copying the first two columns to the end. Here are the down-to-the-right diagonals.

$$\left[\begin{array}{ccccccccc} a & & b & & c & \vdots & a & & b \\ & \searrow & & \searrow & & \searrow & & & \\ d & & e & & f & \vdots & d & & e \\ & & & \searrow & & \searrow & & \searrow & \\ g & & h & & k & \vdots & g & & h \end{array}\right]$$

And here are the down-to-the-left diagonals.

$$\left[\begin{array}{ccccccccc} a & & b & & c & \vdots & a & & b \\ & & & \swarrow & & \swarrow & & \swarrow & \\ d & & e & & f & \vdots & d & & e \\ & \swarrow & & \swarrow & & \swarrow & & & \\ g & & h & & k & \vdots & g & & h \end{array}\right]$$

It's perhaps easier to think of the 3×3 determinant recursively in terms of the 2×2 determinant. To compute the 3×3 determinant this way, we evaluate the following.

Out[4]:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= a \cdot (ek - fh) - b \cdot (dk - fg) + c \cdot (dh - eg)$$

$$= aek - afh - bdk + bfg + cdh - ceg \qquad \blacksquare$$

The decomposition of a 3×3 determinant to multiple 2×2 determinants follows a simple pattern. Each of the top row values a, b, and c is multiplied by the determinant of the 2×2 submatrix formed by eliminating the first row of the 3×3 matrix as well as its corresponding column. Further, the signs of the factors alternate between $+$ and $-$.

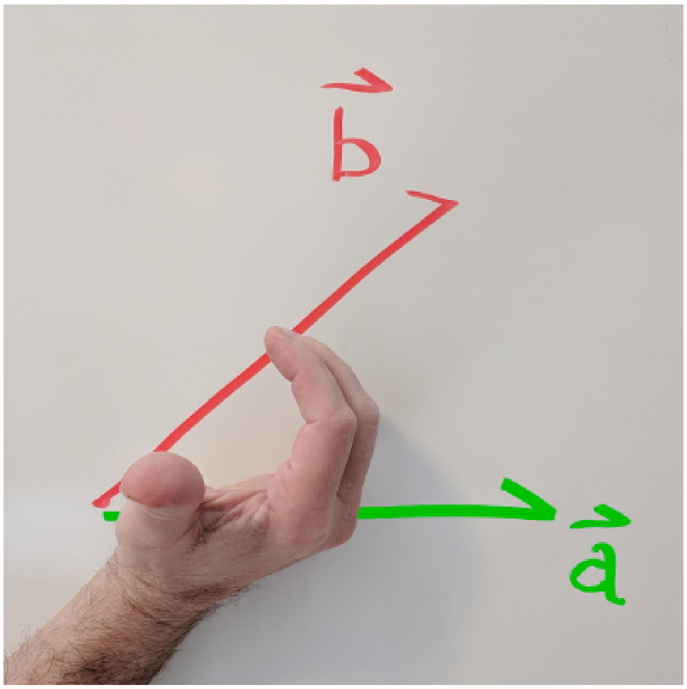
For larger matrices, this recursive process is easily extended,

$$\det \mathbf{M} \equiv \sum_{i=0}^{n-1} (-1^i) \cdot m_{0i} \cdot \det \mathbf{M}_{0,i}$$

In this definition, m_{0i} denotes the ith element in the top row of the matrix $\mathbf{M}$ and $\mathbf{M}_{0,i}$ denotes the submatrix obtained by removing row 0 and column i from $\mathbf{M}$.

Cross Product

The *cross product* of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is defined to be the vector $\mathbf{p}$ that is *right-hand* perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. The magnitude of $\mathbf{p}$ is the area of the parallelogram bounded by $\mathbf{a}$ and $\mathbf{b}$. *Right-hand* perpendicular means if you imagine using the fingers of your right hand to 'turn' vector $\mathbf{a}$ towards vector $\mathbf{b}$, your thumb will point in the direction of $\mathbf{p}$. In the image, $\mathbf{p} = \mathbf{a} \times \mathbf{b}$ points out of the board, towards the camera.



The cross product is *anti-commutative*; that is, if $\mathbf{u} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{v} = \mathbf{b} \times \mathbf{a}$, then $\mathbf{u} = -\mathbf{v}$. This derives from the *handedness* of the cross product. Reversing the arguments causes the resultant vector to point in the opposite direction, though it has the same magnitude. Therefore reversing the arguments *negates* the resultant vector.

The cross product is *linear*; that is

① If $\mathbf{u} = \mathbf{a} \times \mathbf{b}$, then $(cd) \cdot \mathbf{u} = (c \cdot \mathbf{a}) \times (d \cdot \mathbf{b})$, where c, d are any scalars, and,

② $\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = (\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b})$

Both can be derived from the definition given above of the cross product. ① If $\mathbf{a}$ and $\mathbf{b}$ are scaled, then the area of the parallelogram they bound is similarly scaled. ② An area argument can be made in this case as well though it's a bit more involved so we won't bother to show it here.

Given the definition of the magnitude of the cross product, the cross product of any vector with itself must be $\mathbf{0}$, the zero (or null) vector. If the sides of a parallelogram coincide, its area is zero.

The cross product of any two vectors that are linear combinations of each other is also $\mathbf{0}$. This follows from the linearity of the cross product and the fact that the cross product of any vector with itself is zero,
$(c \cdot \mathbf{a}) \times (d \cdot \mathbf{a}) \rightarrow (cd) \cdot (\mathbf{a} \times \mathbf{a}) \rightarrow (cd) \cdot \mathbf{0} \rightarrow \mathbf{0}$

All of the vectors in this discussion are $\in \mathfrak{R}^3$, though cross products can be calculated in higher dimensional spaces.

Basis Vectors

We represent the *standard basis vectors* by $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. Each is a *unit* vector along one of the axes, with $\mathbf{i}$ along the x axis, $\mathbf{j}$ along the y axis, and $\mathbf{k}$ along the z axis.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Any vector $\mathbf{p} \in \mathfrak{R}^3$ can be represented as a linear combination of these three basis vectors. The coefficients of the linear combination are the magnitudes of the projections of $\mathbf{p}$ on $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$, and can be determined by simple inspection of $\mathbf{p}$. Since $\mathbf{i}$ is aligned with the positive x axis, the coefficient of $\mathbf{i}$ is $\mathbf{p}$'s x coordinate. Since $\mathbf{j}$ is aligned with the positive y axis, the coefficient of $\mathbf{j}$ is $\mathbf{p}$'s y coordinate. And finally, since $\mathbf{k}$ is aligned with the positive z axis, the coefficient of $\mathbf{k}$ is $\mathbf{p}$'s z coordinate. Thus we have for any $\mathbf{p} \in \mathfrak{R}^3$,

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = p_x \cdot \mathbf{i} + p_y \cdot \mathbf{j} + p_z \cdot \mathbf{k}$$

In the derivation of the cross product below, we will make use of the cross products of basis vectors. Since these vectors are orthogonal to each other, any two of them define a plane to which the third is perpendicular. Since these vectors are all unit vectors (that is, have a magnitude of 1), the magnitude of the cross product of any two of them is also one.

Given the definition of cross product described above and the orthogonality and unit-vector status of the basis vectors, a complete table of their cross products is easy to construct.

$$\begin{array}{ccc} \mathbf{i} \times \mathbf{i} \rightarrow \mathbf{0} & \mathbf{i} \times \mathbf{j} \rightarrow \mathbf{k} & \mathbf{i} \times \mathbf{k} \rightarrow -\mathbf{j} \\ \mathbf{j} \times \mathbf{i} \rightarrow -\mathbf{k} & \mathbf{j} \times \mathbf{j} \rightarrow \mathbf{0} & \mathbf{j} \times \mathbf{k} \rightarrow \mathbf{i} \\ \mathbf{k} \times \mathbf{i} \rightarrow \mathbf{j} & \mathbf{k} \times \mathbf{j} \rightarrow -\mathbf{i} & \mathbf{k} \times \mathbf{k} \rightarrow \mathbf{0} \end{array}$$

The cross product of any vector with itself is $\mathbf{0}$, so the elements of the diagonal of this table are all $\mathbf{0}$. By its definition, the cross product of any two vectors is perpendicular to the plane containing them. The standard basis vectors are all orthogonal to each other and aligned to the axes. Any two of them define a plane which is perpendicular to the third axis. Therefore the cross product of any two different basis vectors is the third basis vector, possibly negated. This negation will happen depending on the order of the arguments to the cross product and is determined based on the *right-hand rule* since the standard definition of cross product is right-handed.

As an example, $\mathbf{i}$ and $\mathbf{j}$ are different basis vectors so $\mathbf{i} \times \mathbf{j}$ must be $\mathbf{k}$, possibly negated. $\mathbf{i}$ is aligned with the positive x axis and $\mathbf{j}$ is aligned with the positive y axis. 'Turning' the positive x axis into the positive y axis in a right-handed system causes the 'thumb' to point along the *positive* z axis. Thus negation is *not* required for this cross product and therefore $\mathbf{i} \times \mathbf{j} \rightarrow \mathbf{k}$.

As another example, $\mathbf{j}$ and $\mathbf{i}$ are different basis vectors so $\mathbf{j} \times \mathbf{i}$ must be $\mathbf{k}$, possibly negated. But we have just determined $\mathbf{i} \times \mathbf{j} \rightarrow \mathbf{k}$ and we know that the cross product is *anti-commutative*, as stated above. Therefore, $\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}$.

As a final example, $\mathbf{k}$ and $\mathbf{j}$ are different basis vectors so $\mathbf{k} \times \mathbf{j}$ must be $\mathbf{i}$, possibly negated. $\mathbf{k}$ is aligned with the positive z axis and $\mathbf{j}$ is aligned with the positive y axis. 'Turning' the positive z axis into the positive y axis in a right-handed system causes the 'thumb' to point along the *negative* x axis. Thus negation *is* required for this cross product and therefore $\mathbf{k} \times \mathbf{j} \rightarrow -\mathbf{i}$. Once we have determined this, we instantly know that $\mathbf{j} \times \mathbf{k}$ *must* be $-(\mathbf{k} \times \mathbf{j}) = -(-\mathbf{i}) = \mathbf{i}$ because the cross product is anti-commutative.

The rest of the table can be similarly derived.

Derivation of the Cross Product

Deriving the cross product is straightforward, if tedious. Here we will take the *distributiveness* of the cross product as given; this property is showable from the definition already given but requires a somewhat longer derivation.

We represent the vectors $\mathbf{a}$ and $\mathbf{b}$ as linear combinations of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ thus,

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

Using the distribution property of the cross product, we have,

$$\mathbf{a} \times \mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})$$

$$= \begin{matrix} a_x\mathbf{i} \times b_x\mathbf{i} & + & a_x\mathbf{i} \times b_y\mathbf{j} & + & a_x\mathbf{i} \times b_z\mathbf{k} & + \\ a_y\mathbf{j} \times b_x\mathbf{i} & + & a_y\mathbf{j} \times b_y\mathbf{j} & + & a_y\mathbf{j} \times b_z\mathbf{k} & + \\ a_z\mathbf{k} \times b_x\mathbf{i} & + & a_z\mathbf{k} \times b_y\mathbf{j} & + & a_z\mathbf{k} \times b_z\mathbf{k} & \end{matrix}$$

Using the linearity property of the cross product, this reduces to,

$$= \begin{matrix} a_x \cdot b_x \cdot (\mathbf{i} \times \mathbf{i}) & + & a_x \cdot b_y \cdot (\mathbf{i} \times \mathbf{j}) & + & a_x \cdot b_z \cdot (\mathbf{i} \times \mathbf{k}) & + \\ a_y \cdot b_x \cdot (\mathbf{j} \times \mathbf{i}) & + & a_y \cdot b_y \cdot (\mathbf{j} \times \mathbf{j}) & + & a_y \cdot b_z \cdot (\mathbf{j} \times \mathbf{k}) & + \\ a_z \cdot b_x \cdot (\mathbf{k} \times \mathbf{i}) & + & a_z \cdot b_y \cdot (\mathbf{k} \times \mathbf{j}) & + & a_z \cdot b_z \cdot (\mathbf{k} \times \mathbf{k}) & \end{matrix}$$

Given that any vector's cross product with itself is $\mathbf{0}$, this further reduces to,

$$= \begin{matrix} \mathbf{0} & + & a_x \cdot b_y \cdot (\mathbf{i} \times \mathbf{j}) & + & a_x \cdot b_z \cdot (\mathbf{i} \times \mathbf{k}) & + \\ a_y \cdot b_x \cdot (\mathbf{j} \times \mathbf{i}) & + & \mathbf{0} & + & a_y \cdot b_z \cdot (\mathbf{j} \times \mathbf{k}) & + \\ a_z \cdot b_x \cdot (\mathbf{k} \times \mathbf{i}) & + & a_z \cdot b_y \cdot (\mathbf{k} \times \mathbf{j}) & + & \mathbf{0} & \end{matrix}$$

Using the table of basis vector cross products, we can reduce this to,

$$= \begin{matrix} a_x \cdot b_y \cdot \mathbf{k} & + & a_x \cdot b_z \cdot (-\mathbf{j}) & + \\ a_y \cdot b_x \cdot (-\mathbf{k}) & + & a_y \cdot b_z \cdot \mathbf{i} & + \\ a_z \cdot b_x \cdot \mathbf{j} & + & a_z \cdot b_y \cdot (-\mathbf{i}) & \end{matrix}$$

And rearranging and combining like terms gives us,

$$= \mathbf{i} \cdot (a_y b_z - a_z b_y) + \mathbf{j} \cdot (a_z b_x - a_x b_z) + \mathbf{k} \cdot (a_x b_y - a_y b_x)$$

The coefficients of the basis vectors in this last representation look suspiciously like what one would get when evaluating the determinants of 2×2 matrices populated with the proper elements from the $\mathbf{a}$ and $\mathbf{b}$ vectors. If we rearrange the coefficient of the $\mathbf{j}$ basis vector so that $\mathbf{j}$ itself is negative, then $(a_z b_x - a_x b_z)$ becomes $(a_x b_z - a_z b_x)$. We can put the representation in this final form,

$$= \mathbf{i} \cdot (a_y b_z - a_z b_y) - \mathbf{j} \cdot (a_x b_z - a_z b_x) + \mathbf{k} \cdot (a_x b_y - a_y b_x)$$

Now we have not only what looks like determinants of 2×2 matrices but also the alternating signs of the factors as we would expect in the determinant of a 3×3 matrix.

Cross Product as a Determinant

Given what we have just derived, construction of a 3×3 matrix whose determinant is the cross product of vectors $\mathbf{a}$ and $\mathbf{b}$ is straightforward. We put the basis vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ as the first row, the elements of $\mathbf{a}$ as the second row, and the elements of $\mathbf{b}$ as the third row.

Out[5]:

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot \det \begin{bmatrix} a_y & a_z \\ b_y & b_z \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} a_x & a_z \\ b_x & b_z \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot (a_y b_z - a_z b_y) - \mathbf{j} \cdot (a_x b_z - a_z b_x) + \mathbf{k} \cdot (a_x b_y - a_y b_x) \qquad \blacksquare$$

We previously stated that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. This is easily seen using the determinant representation of the cross product.

Out[6]:

$$\mathbf{b} \times \mathbf{a} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ a_x & a_y & a_z \end{bmatrix}$$

$$\mathbf{b} \times \mathbf{a} = \mathbf{i} \cdot \det \begin{bmatrix} b_y & b_z \\ a_y & a_z \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} b_x & b_z \\ a_x & a_z \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} b_x & b_y \\ a_x & a_y \end{bmatrix}$$

$$\mathbf{b} \times \mathbf{a} = \mathbf{i} \cdot (-a_y b_z + a_z b_y) - \mathbf{j} \cdot (-a_x b_z + a_z b_x) + \mathbf{k} \cdot (-a_x b_y + a_y b_x)$$

$$-(\mathbf{b} \times \mathbf{a}) = \mathbf{i} \cdot (a_y b_z - a_z b_y) - \mathbf{j} \cdot (a_x b_z - a_z b_x) + \mathbf{k} \cdot (a_x b_y - a_y b_x)$$

$$-(\mathbf{b} \times \mathbf{a}) = \mathbf{a} \times \mathbf{b} \qquad \blacksquare$$

Out[7]:

Some Numerical Examples

For the standard basis vectors $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we will here show numerically what was stated above in the table of basis vector cross products.

$$\mathbf{i} \times \mathbf{i} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{i} \cdot \det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 0$$

$$\mathbf{i} \times \mathbf{i} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The cross product $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, as expected for any vector. The following two examples show $\mathbf{j} \times \mathbf{i}$ and $\mathbf{k} \times \mathbf{i}$

$$\mathbf{j} \times \mathbf{i} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{j} \times \mathbf{i} = \mathbf{i} \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{j} \times \mathbf{i} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot -1$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Similarly for $\mathbf{k} \times \mathbf{i}$,

$$\mathbf{k} \times \mathbf{i} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{i} \cdot \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot -1 + \mathbf{k} \cdot 0$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The remaining six entries in the basis vector cross product table can be computed the same way.

Moving away from basis vectors, we will show numerically some of the properties of the cross product that were mentioned above.

If we let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, we have,

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

If we reverse the order of the arguments to the cross product, we have,

$$\mathbf{b} \times \mathbf{a} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{b} \times \mathbf{a} = \mathbf{i} \cdot \det \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{b} \times \mathbf{a} = 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{b} \times \mathbf{a} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

The result of reversing the arguments to the cross products is the negation of the original resultant vector.

If we cross product a vector with itself, we obtain,

$$\mathbf{a} \times \mathbf{a} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{i} \cdot \det \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 0$$

$$\mathbf{a} \times \mathbf{a} = 0$$

Taking the cross product of a vector and a multiple of itself also gives the zero vector as a result. Here we have

$$\mathbf{c} = -1.5 \cdot \mathbf{a} = -1.5 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -3.0 \\ -4.5 \end{bmatrix}$$

and therefore find

$$\mathbf{a} \times \mathbf{c} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1.5 & -3.0 & -4.5 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{c} = \mathbf{i} \cdot \det \begin{bmatrix} 2 & 3 \\ -3.0 & -4.5 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 1 & 3 \\ -1.5 & -4.5 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 1 & 2 \\ -1.5 & -3.0 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{c} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 0$$

$$\mathbf{a} \times \mathbf{c} = 0$$

Geometrical Interpretation

In the definition above, we stated that the magnitude of the cross product of two vectors is the area of the parallelogram bounded by those two vectors. This is easy to demonstrate if we construct an example with the vectors aligned with the axes. By doing so, the parallelogram bounded by the vectors is a rectangle and its area is simply the product of the magnitudes of the two vectors.

Out[8]:

For $\mathbf{a} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$,

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot \det \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 35$$

$$\mathbf{a} \times \mathbf{b} = 35\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 35 \end{bmatrix}$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{0^2 + 0^2 + 35^2} = \sqrt{1225} = 35$$

In this example, the $\mathbf{a}$ and $\mathbf{b}$ vectors were in the XY plane, so their cross product was purely in the z dimension, that is, its x and y coordinates are zero. The magnitude of this vector is therefore just its z coordinate.

We did not need the cross product to compute such a simple case, but it gets more interesting when the vectors are *not* aligned with the axes or when the angle between them is not $\frac{\pi}{2}$ (90°).

The parallelogram area property of the cross product leads naturally to the equation

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta$$

where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$. This is easy to show as the area of a parallelogram is $base \times height$. If we think of $\mathbf{a}$ as the $base$ of the parallelogram and of $\mathbf{b}$ as the other side, the $height$ of this parallelogram is $\|\mathbf{b}\| \sin\theta$ so its area is $\|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta$.

In the numerical example just given, θ was $\frac{\pi}{2}$ (90°) as the two vectors are perpendicular to each other. Therefore we have $\sin\theta = 1$ and the bounded parallelogram is actually a rectangle. Its area may be computed simply as $\|\mathbf{a}\|\,\|\mathbf{b}\|$.

Angle between $\mathbf{a}$ *and* $\mathbf{b}$

Unfortunately we cannot simply solve that equation for $\sin\theta$ and use that to determine the angle θ between the vectors $\mathbf{a}$ and $\mathbf{b}$. This angle θ will always be in the range $0 \le \theta \le \pi$ $(0^\circ \le \theta \le 180^\circ)$ but in this range, $\sin\theta$ is ambiguous, going from 0 (at 0) to 1 (at $\frac{\pi}{2}$) and back down to 0 (at π). There are therefore two values of θ for every value of $\sin\theta$ in the range $0 \ldots 1$. When $\frac{\pi}{2} < \theta \le \pi$, $\arcsin \sin\theta$ returns the *supplement* of θ (that is, $\pi - \theta$) instead of the actual value.

Compare this to the previously stated property of the *dot product*,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$$

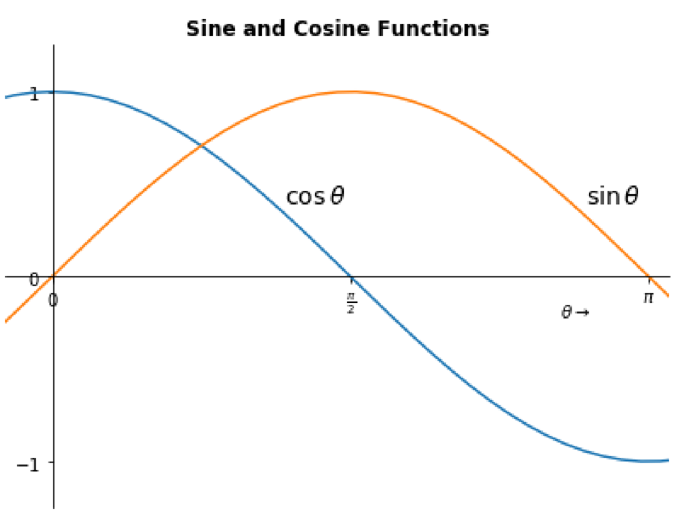
which can be solved for $\cos\theta$ as follows,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

Because the dot product can be negative, this computation returns a value in the range $-1 \ldots 1$ for $\cos\theta$, going from 1 at $\theta = 0$ to 0 at $\frac{\pi}{2}$ to -1 at π. Thus, $\cos\theta$ is *not* ambiguous for θ in the range $0 \ldots \pi$ and an accurate value for the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be computed by,

$$\theta = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

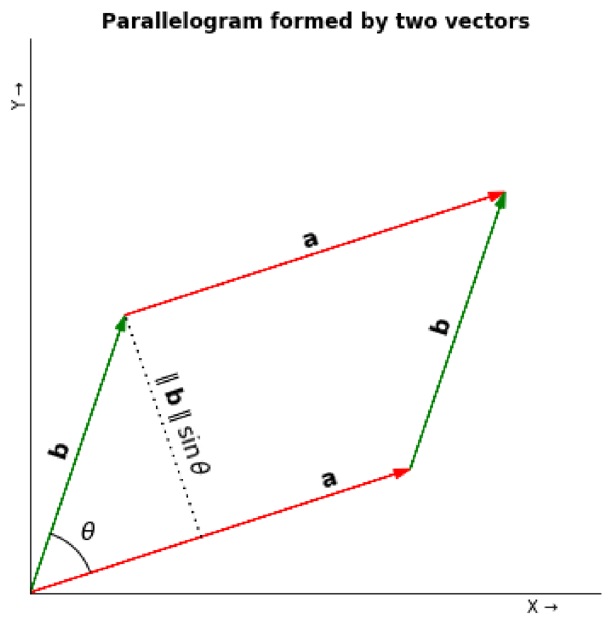
Graphs of the $\sin$ and $\cos$ functions in the range $0 \ldots \pi$ are shown in the following figure.



Be careful to remember always that the dot product of two vectors is always a *scalar* whereas the cross product of two vectors is always a *vector*.

Parallelogram General Case

The more general case of the parallelogram formed by two vectors is shown in the following diagram. $\mathbf{a}$ forms the *base* of the parallelogram and $\mathbf{b}$ is the other side. The area of this parallelogram is the *length* of the base times the height. The length of the base is $\|\mathbf{a}\|$ and the height is $\|\mathbf{b}\| \sin\,\theta$. (We say *length* of the base since we have to compute the magnitude of the vector $\mathbf{a}$.)



Out[11]: Numerically, in this example we have

$$\mathbf{a} = \begin{bmatrix} 6.0 \\ 2.0 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1.5 \\ 4.5 \\ 0 \end{bmatrix}$$

Thus,

$$\|\mathbf{a}\| = 6.325, \|\mathbf{b}\| = 4.743$$

We get the angle θ between the two vectors using our previous $\cos\theta$ property of the dot product,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{6 \cdot 1 + 2 \cdot 4}{6.325 \cdot 4.743} = \frac{18}{30.000} = 0.600$$

$$\theta = \arccos 0.600 = 0.927 \text{ rad, or } 53.130^{\circ}$$

We compute the height of the parallelogram using θ,

$$\|\mathbf{b}\| \sin\theta = 4.743 \sin 0.927 = 4.743 \cdot 0.800 = 3.795$$

And we get the area by multiplying the length of the base $\|\mathbf{a}\|$ by the height.

$$area = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta = 6.325 \cdot 4.743 \cdot \sin 0.927 = 6.325 \cdot 3.795 = 24.000$$

Whew!

Out[12]: Or, we could have just computed the cross product of $\mathbf{a}$ and $\mathbf{b}$ and avoided all that trigonometry. The magnitude of $\mathbf{a} \times \mathbf{b}$ is defined to be the area of the parallelogram bounded by the $\mathbf{a}$ and $\mathbf{b}$ vectors. Let's see if it is.

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6.0 & 2.0 & 0 \\ 1.5 & 4.5 & 0 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot \det \begin{bmatrix} 2.0 & 0 \\ 4.5 & 0 \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} 6.0 & 0 \\ 1.5 & 0 \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} 6.0 & 2.0 \\ 1.5 & 4.5 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 24.0$$

$$\mathbf{a} \times \mathbf{b} = 24.0\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 24.0 \end{bmatrix}$$

Since $\mathbf{a}$ and $\mathbf{b}$ were both in the XY plane (their z coordinates were zero), their cross product is purely in the z dimension, that is, its x and y coordinates are both zero. We can get the magnitude via simple inspection (it's the value of the z coordinate). Or, to go the long way around for the most general answer, we can compute it as

$$area = \|\mathbf{a} \times \mathbf{b}\| = \sqrt{0.000^2 + 0.000^2 + 24.000^2} = \sqrt{576.000} = 24.000 \qquad \blacksquare$$

That was a much easier process than what we went through using trigonometry. The complete process from beginning to end required calculating only three 2×2 determinants and a norm. If both of the two vectors are in the same standard plane (XY, XZ, or YZ), we have to calculate only one of the determinants. The other two determinants will be zero and the norm won't be necessary. No trigonometric functions such as $\arccos$ or $\sin$ are required in either case.

A 2×2 determinant costs two multiplications and one subtraction. A norm in $\Re^3$ requires three multiplies, two additions, and a square root. In the general case where the vectors are *not* both in the same standard plane, the total operation count is therefore two additions, three subtractions, nine multiplications, and one square root. In the case where the two vectors *are* in the same standard plane, the total operation count is only two multiplications and one subtraction (the cost of calculating one 2×2 determinant). Wow!

Appendix — Deriving the Value of the 2×2 Determinant

In the review of the determinant, we began with a definition of the 2×2 case and later presented the general representation,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv ad - bc$$

$$\det \mathbf{M} \equiv \sum_{i=0}^{n-1} (-1^i) \cdot m_{0i} \cdot \det \mathbf{M}_{0,i}$$

Instead of *defining* the meaning of the 2×2 determinant, we could have started with a simpler determinant and used the general representation to *derive* the value of the 2×2 determinant. The only case simpler than 2×2 is 1×1. Therefore we will start with the definition of the determinant of a 1×1 matrix and derive the 2×2 and all larger cases from the general representation.

We make the definition that the determinant of a 1×1 matrix is the single element of that matrix,

$$\det \begin{bmatrix} a \end{bmatrix} \equiv a$$

For the 2×2 case, we follow the general rule and obtain,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{i=0}^{n-1} (-1^i) \cdot m_{0i} \cdot \det \mathbf{M}_{0,i}, \text{ where } n = 2.$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{i=0}^{1} (-1^i) \cdot m_{0i} \cdot \det \mathbf{M}_{0,i}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1^0) \cdot m_{00} \cdot \det \mathbf{M}_{0,0} + (-1^1) \cdot m_{01} \cdot \det \mathbf{M}_{0,1}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = m_{00} \cdot \det \mathbf{M}_{0,0} - m_{01} \cdot \det \mathbf{M}_{0,1}$$

Recall that in the definition of the general case, m_{0i} denotes the ith element in the top row of the matrix $\mathbf{M}$ and $\mathbf{M}_{0,i}$ denotes the submatrix obtained by removing row 0 and column i from $\mathbf{M}$. In this case, $\mathbf{M}$ is a 2×2 matrix, so when we eliminate a row and a column, we end up with a 1×1 matrix. Thus,

$$m_{00} = a, \;\; m_{01} = b$$

$$\mathbf{M}_{0,0} = \begin{bmatrix} - & - \\ - & d \end{bmatrix} = \begin{bmatrix} d \end{bmatrix}, \text{ first row and first column removed.}$$

$$\mathbf{M}_{0,1} = \begin{bmatrix} - & - \\ c & - \end{bmatrix} = \begin{bmatrix} c \end{bmatrix}, \text{ first row and second column removed.}$$

Making these substitutions for $m_{00}, m_{01}, \mathbf{M}_{0,0}$, and $\mathbf{M}_{0,1}$, we obtain,

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot \det\begin{bmatrix} d \end{bmatrix} - b \cdot \det\begin{bmatrix} c \end{bmatrix}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \blacksquare$$

The 3×3 case follows from this 2×2 case and the general representation as we showed above in the review of the determinant.

Appendix — Linearity of the Cross Product

Now that we have a way to calculate the cross product of any two vectors in $\mathfrak{R}^3$, we can show the two properties of linearity that we skipped over above. We first consider the requirement that $(cd) \cdot (\mathbf{a} \times \mathbf{b}) = (c \cdot \mathbf{a}) \times (d \cdot \mathbf{b})$, where c, d are any scalars.

Out[13]: Using the determinant method of calculating the cross product, we have for any two vectors in $\mathfrak{R}^3$,

$$\mathbf{a} \times \mathbf{b} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot \det\begin{bmatrix} a_y & a_z \\ b_y & b_z \end{bmatrix} - \mathbf{j} \cdot \det\begin{bmatrix} a_x & a_z \\ b_x & b_z \end{bmatrix} + \mathbf{k} \cdot \det\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \cdot (a_y b_z - a_z b_y) - \mathbf{j} \cdot (a_x b_z - a_z b_x) + \mathbf{k} \cdot (a_x b_y - a_y b_x)$$

If we now multiply $\mathbf{a}$ by the scalar c and $\mathbf{b}$ by the scalar d, we have,

$$c\mathbf{a} \times d\mathbf{b} = \begin{bmatrix} a_x c \\ a_y c \\ a_z c \end{bmatrix} \times \begin{bmatrix} b_x d \\ b_y d \\ b_z d \end{bmatrix} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x c & a_y c & a_z c \\ b_x d & b_y d & b_z d \end{bmatrix}$$

$$c\mathbf{a} \times d\mathbf{b} = \mathbf{i} \cdot \det\begin{bmatrix} a_y c & a_z c \\ b_y d & b_z d \end{bmatrix} - \mathbf{j} \cdot \det\begin{bmatrix} a_x c & a_z c \\ b_x d & b_z d \end{bmatrix} + \mathbf{k} \cdot \det\begin{bmatrix} a_x c & a_y c \\ b_x d & b_y d \end{bmatrix}$$

$$c\mathbf{a} \times d\mathbf{b} = \mathbf{i} \cdot (a_y b_z cd - a_z b_y cd) - \mathbf{j} \cdot (a_x b_z cd - a_z b_x cd) + \mathbf{k} \cdot (a_x b_y cd - a_y b_x cd)$$

$$c\mathbf{a} \times d\mathbf{b} = cd \cdot \mathbf{i} \cdot (a_y b_z - a_z b_y) - cd \cdot \mathbf{j} \cdot (a_x b_z - a_z b_x) + cd \cdot \mathbf{k} \cdot (a_x b_y - a_y b_x)$$

$$c\mathbf{a} \times d\mathbf{b} = (cd) \cdot (\mathbf{i} \cdot (a_y b_z - a_z b_y) - \mathbf{j} \cdot (a_x b_z - a_z b_x) + \mathbf{k} \cdot (a_x b_y - a_y b_x))$$

$$c\mathbf{a} \times d\mathbf{b} = (cd) \cdot (\mathbf{a} \times \mathbf{b}) \qquad \blacksquare$$

Out[14]: The second property required for linearity is that $\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = (\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b})$. For $\mathbf{c} \times \mathbf{a}$, we have,

$$\mathbf{c} \times \mathbf{a} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{bmatrix}$$

$$\mathbf{c} \times \mathbf{a} = \mathbf{i} \cdot \det \begin{bmatrix} c_y & c_z \\ a_y & a_z \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} c_x & c_z \\ a_x & a_z \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} c_x & c_y \\ a_x & a_y \end{bmatrix}$$

$$\mathbf{c} \times \mathbf{a} = \mathbf{i} \cdot (-a_y c_z + a_z c_y) - \mathbf{j} \cdot (-a_x c_z + a_z c_x) + \mathbf{k} \cdot (-a_x c_y + a_y c_x)$$

Similarly for $\mathbf{c} \times \mathbf{b}$, we have,

$$\mathbf{c} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{bmatrix}$$

$$\mathbf{c} \times \mathbf{b} = \mathbf{i} \cdot \det \begin{bmatrix} c_y & c_z \\ b_y & b_z \end{bmatrix} - \mathbf{j} \cdot \det \begin{bmatrix} c_x & c_z \\ b_x & b_z \end{bmatrix} + \mathbf{k} \cdot \det \begin{bmatrix} c_x & c_y \\ b_x & b_y \end{bmatrix}$$

$$\mathbf{c} \times \mathbf{b} = \mathbf{i} \cdot (-b_y c_z + b_z c_y) - \mathbf{j} \cdot (-b_x c_z + b_z c_x) + \mathbf{k} \cdot (-b_x c_y + b_y c_x)$$

Summing these two, we obtain,

$$(\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b})$$

$$= \mathbf{i} \cdot (-a_y c_z + a_z c_y - b_y c_z + b_z c_y)$$

$$-\mathbf{j} \cdot (-a_x c_z + a_z c_x - b_x c_z + b_z c_x)$$

$$+\mathbf{k} \cdot (-a_x c_y + a_y c_x - b_x c_y + b_y c_x)$$

or, by doing a bit of algebraic factoring,

$$= \mathbf{i} \cdot (c_y (a_z + b_z) - c_z \left(a_y + b_y\right))$$

$$-\mathbf{j} \cdot (c_x (a_z + b_z) - c_z (a_x + b_x))$$

$$+\mathbf{k} \cdot (c_x \left(a_y + b_y\right) - c_y (a_x + b_x))$$

If we now compute the cross product of $\mathbf{c}$ with the sum of $\mathbf{a}$ and $\mathbf{b}$, we have,

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \times (\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix})$$

$$= \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \times \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_x & c_y & c_z \\ a_x + b_x & a_y + b_y & a_z + b_z \end{bmatrix}$$

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{i} \cdot \det \begin{bmatrix} c_y & c_z \\ a_y + b_y & a_z + b_z \end{bmatrix}$$

$$-\mathbf{j} \cdot \det \begin{bmatrix} c_x & c_z \\ a_x + b_x & a_z + b_z \end{bmatrix}$$

$$+\mathbf{k} \cdot \det \begin{bmatrix} c_x & c_y \\ a_x + b_x & a_y + b_y \end{bmatrix}$$

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{i} \cdot (c_y (a_z + b_z) - c_z \left(a_y + b_y\right))$$

$$-\mathbf{j} \cdot (c_x (a_z + b_z) - c_z (a_x + b_x))$$

$$+\mathbf{k} \cdot (c_x \left(a_y + b_y\right) - c_y (a_x + b_x))$$

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = (\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b}) \qquad \blacksquare$$