1 Exercise 1

Describe clearly but briefly the meaning of each term below, including advantages and disadvantages, number of operations, etc., in your own words (not anyone else's words, so not copied from any book, Wikipedia, internet, etc.,) and without using any equations whatsoever.

(a) Gram-Schmidt

(b) QR factorization

(c) Orthogonal subspaces, bases, and matrices

(d) least squares approximations

(e) Projection vector

2 Exercise 2

Completely and clearly answer the following questions showing all relevant steps:

(a) Suppose A is 3 by 4 and B us 4 by 5 and $AB = 0$. So $N(A)$ contains $C(B)$. Prove from the dimensions of $N(A)$ and $C(B)$ that $\text{rank}(A) + \text{rank}(B) \leq 4$.

(b) For four non-zero vectors r, n, c, l, in $\mathbf{R}^2$

(1) What are the conditions for those to be bases for the four fundamental subspaces for a 2×2 matrix?

(2) What is one possible matrix A?

(c) Project the vector b onto the line through a. Verify that e is perpendicular to a.

(1)

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ and } a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

(2)

$$b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \tag{2}$$

(d) Project b onto the column space of A by solving $A^T A\widehat{x} = A^T b$ and $p = A\widehat{x}$

(1)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \tag{3}$$

(2)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \tag{4}$$

3 Exercise 3

(a) With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ set up and solve the normal equations $A^T A\widehat{x} = A^T b$. What are the heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

(b) Project $b = (0, 8, 8, 20)$ onto the line $a = (1, 1, 1, 1)$. Find $\widehat{x} = a^T b/a^T a$ and the projection $p = \widehat{x}a$. check that $e = b - p$ is perpendicular to a, and find the shortest distance $\|e\|$ from b to the line through a.

(c) Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{x} = (C, D)$ and draw the closest line.

(d) Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

4 Exercise 4

(a) (1) Find the orthonormal vectors q_1, q_2, q_3 such that q_1 and q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \tag{5}$$

(2) Which of the four fundamental subspaces contains q_3?

(3) Solve $Ax = (4, 5, 2, 2)$ by least squares.

(b) Find an orthonormal basis for the column space of A:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \tag{6}$$

Then compute the projection of b onto that column space.

(c) Find orthogonal vectors A, B, C, by Gram-Schmidt from

$$a = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \tag{7}$$

(d) Find q_1, q_2, q_3 (orthonormal) as combinations of a, b, c (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} \tag{8}$$