MAE 501 *Fall 2020*

Linear Algebra in Engineering

Module 2: Vector Spaces & Subspaces

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1 Introduction

In this module, we will characterize the system $A\boldsymbol{x} = \boldsymbol{b}$ in terms of *vector spaces* and *subspaces* of the matrix A.

So what's a vector space? Let's start with some examples:

- all 2-dimensional vectors: $\mathbb{R}^2$: two-dimensional space.
- all 3-dimensional vectors: $\mathbb{R}^3$: three-dimensional space.
- all n-dimensional vectors: $\mathbb{R}^n$: n-dimensional space.

At least two operations must be possible in a vector space:

1) Addition of vectors
2) Multiplication by scalars

} Linear combinations

Vector spaces obey 8 rules:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ commutative "+" operation
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ associative "+"
3. There is a zero! $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$
4. There is a one! $1\boldsymbol{x} = \boldsymbol{x}$

5. For each $\boldsymbol{x}$ there is a unique $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = 0$

6. $(c_1 c_2)\boldsymbol{x} = c_1 (c_2 \boldsymbol{x})$ (Associative scalar multiplication)

7. $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$ (Distributive scalar multiplication)

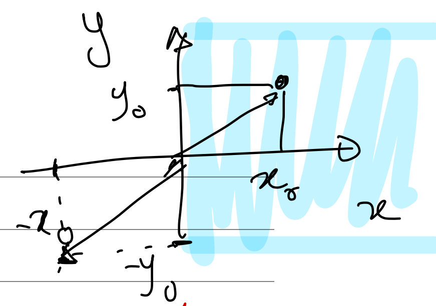
8. $(c_1 + c_2)\boldsymbol{x} = c_1 \boldsymbol{x} + c_2 \boldsymbol{x}$

Definition A set of vectors together with the 8 rules for vector addition and multiplication by real numbers. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are any vectors in the vector space and c is a real number then

1. $\boldsymbol{x} + \boldsymbol{y}$ stays in the vector space,
2. $c\boldsymbol{x}$ stays in the vector space.

$-1\boldsymbol{x} = (-x_0, -y_0)$ $x > 0$

Example 1



not a vector space

- Three dimensional space $\mathbb{R}^3$
- Infinite dimensional space $\mathbb{R}^\infty$
- All 2×3 matrices : $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$
- All functions $f(x)$ defined for $0 \le x \le 1$

example: $f(x) = e^x$

$g(x) = \sqrt{x}$

$\alpha f + \beta g = \alpha e^x + \beta\sqrt{x}$

is also a function for $0 \le x \le 1$

Definition A subspace of a vector space is a *nonempty* subset that satisfies the requirements for a vector space

Comments

- In a subspace all 8 rules are automatically satisfied, since they are by definition satisfied in the host space.
- The zero vector must belong to every subspace.
  if $\boldsymbol{v} \in$ subspace, $c\boldsymbol{v} \in$ subspace and in particular for $c=0$.
- Examples of subspaces of $\mathbb{R}^2$:
  - $\mathbb{R}^2$
  - $\{\boldsymbol{0}\}$
  - A line going through the origin
- Examples of subspaces of $\mathbb{R}^3$:
  - $\mathbb{R}^3$ and $\{\boldsymbol{0}\}$
  - A plane going through the origin $(\boldsymbol{0})$
  - A line going through the origin
- Examples of subspaces of the vector space of square matrices:
  - Symmetric matrices: $S_1^T = S_1$ and $S_2^T = S_2$

    $(\alpha S_1 + \beta S_2)^T = \alpha S_1^T + \beta S_2^T = \alpha S_1 + \beta S_2$ is also symmetric!!
  - Diagonal matrices and upper triangular matrices
  - What about invertible matrices? The zero matrix doesn't have an inverse, $\Rightarrow$ this can't be a subspace!!

2 Column space of A

We will now focus on subspaces of a matrix A. If A is not invertible, solving $A\boldsymbol{x} = \boldsymbol{b}$ will be possible only for some $\boldsymbol{b}$ and not others.

Definition The column space of A, denoted $\boldsymbol{C}(A)$, contains all linear combinations of the column vectors of A.

Consider the system of linear equations

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{linear combination of the column vectors}}$$

When is this system solvable?

Only if $\mathbf{b}$ is a linear combination of the columns!

$\Rightarrow$ $\mathbf{b}$ must be contained in $\boldsymbol{C}(A)$!!

What are some vectors in $\boldsymbol{C}(A)$ such that the system is solvable?

Take $\boldsymbol{x} = (1,0) \longrightarrow \mathbf{b} = (1,2,1)$

$\boldsymbol{x} = (0,1) \longrightarrow \mathbf{b} = (0,1,1)$

$\boldsymbol{x} = (0,0) \longrightarrow \mathbf{b} = (0,0,0)$

What does the column space of A look like?

All possible linear combinations of the two column vectors form a plane going through the origin.

$\mathbf{b}$ must be this in this plane to yield a solvable system.

Comments

- The column space of A is a subspace.

  Proof: if $\mathbf{b}$ and $\mathbf{b}' \in \boldsymbol{C}(A)$, ie $A\boldsymbol{x} = \mathbf{b}$ and $A\boldsymbol{x}' = \mathbf{b}'$

  Take a linear combination $\alpha A\boldsymbol{x} + \beta A\boldsymbol{x}' = \alpha\mathbf{b} + \beta\mathbf{b}'$

  $\Rightarrow A\boxed{(\alpha\boldsymbol{x} + \beta\boldsymbol{x}')} = \alpha\mathbf{b} + \beta\mathbf{b}' \in \boldsymbol{C}(A)$

- The smallest column space is

  if A is the zero matrix then $\boldsymbol{C}(A) = \{\mathbf{0}\}$

- The largest column space is

  if $A = I$ with A $m \times n$ matrix then $\boldsymbol{C}(A) = \mathbb{R}^m$

- The column space of A defines all possible linear combinations of the columns of A, i.e., all possible vectors $A\boldsymbol{x}$.
- If $\boldsymbol{b}$ is not in $\boldsymbol{C}(A)$, then there is no solution to $A\boldsymbol{x} = \boldsymbol{b}$!!
- For a $m \times n$ matrix, the system $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable when

  $C(A) = \mathbb{R}^m$

Example 2 Describe the column space of

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \& \quad A = \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \end{array}$$

For I: $C(I) = \mathbb{R}^2$ the two columns $(1,0)$ and $(0,1)$ fill up $\mathbb{R}^2$

For A: $c_2 = 2c_1 \Rightarrow$ only one independent column $\Rightarrow$ the column space is a line oriented along $c_1 = (1,2)$ and going through the origin.

Definition The column space of a matrix derives from a more general concept called the *span*. For a set of vectors $S = (\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n)$ in the vector space $\boldsymbol{V}$, the span of S is the subspace consisting of all linear combinations of $\boldsymbol{v}_i$.

3 Nullspace of A

In the previous section, we introduced the column space of A, which is all possible vectors $A\boldsymbol{x}$. We did this to determine the possible right hand sides $\boldsymbol{b}$ for which there is a solution (they must belong to $\boldsymbol{C}(A)$). Now, we focus on the special case where the right hand side is identically $\boldsymbol{0}$.

Definition The nullspace of A, denoted $\boldsymbol{N}(A)$, consists of all vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{0}$. The nullspace is a subspace of $\mathbb{R}^n$.

Consider the previous example

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 2x + y \\ x + y \end{bmatrix}$$

What's the nullspace of A?

$$\begin{bmatrix} x \\ 2x+y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$ In this case there is only one $\boldsymbol{x} \in \boldsymbol{N}(A)$ which is $\boldsymbol{x} = (0,0)$

Let's consider a slightly different example

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Add an extra column

* Does this change the column space?

Notice that $c_3 = c_1 + c_2$

The column space does not change!

* What about the nullspace?

We have $c_3 = c_1 + c_2 \Rightarrow 1 \times c_1 + 1 \times c_2 - 1 \times c_3 = 0$

So $\boldsymbol{x} = (1,1,-1)$ is also solution to $A\boldsymbol{x} = 0$

the Nullspace is a line oriented along $(1,1,-1)$ and going through the origin.

Comments

- For an invertible matrix A, the nullspace is $\boldsymbol{N}(A) = \boldsymbol{0}$.

  A invertible $A\boldsymbol{x} = 0 \Rightarrow A^{-1}A\boldsymbol{x} = 0$

  $\Rightarrow I\boldsymbol{x} = 0 \Rightarrow \boldsymbol{x} = 0$

- if $\boldsymbol{x}_n \in \boldsymbol{N}(A)$ and $\boldsymbol{x}_p$ is a particular solution ($A\boldsymbol{x}_p = \boldsymbol{b}$), then $\boldsymbol{x}_p + \boldsymbol{x}_n$ is a solution.

  $A\boldsymbol{x}_p = \boldsymbol{b}$ and $A\boldsymbol{x}_n = 0 \Rightarrow A(\boldsymbol{x}_p + \boldsymbol{x}_n) = \boldsymbol{b}$

  $\Rightarrow \boldsymbol{x}_p + \boldsymbol{x}_n$ is also a solution!! Characterizing the nullspace and one particular solution allow us to find all possible solutions!!!

3.1 The reduced row echelon form R

How can we describe the nullspace of A? Before we go further, let's introduce the reduced row echelon form of A, as it will help us find all $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$.

Let's start with an example of a 3×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 6 & 5 \\ -1 & -2 & 2 & 5 \end{bmatrix}$$

rectangular shape

And apply eliminations

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 2 & 4 & 6 & 5 \\ -1 & -2 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

and

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Comments

- What do you notice about the pivots of U?

  The pivots of U are not on the main diagonal!

- U has a staircase pattern. It's called an **echelon** form.

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Check that $A = LU$

Note

For any $m \times n$ matrix A, there is a permutation P, a lower triangular L with unit diagonal, and an $m \times n$ echelon matrix U, such that $PA = LU$.

We can go further and clean up U to obtain the *reduced row echelon form R.* To do so,

1. make pivots equal to 1.

2. produce 0 above all pivots using back substitutions

$$U = \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{1} & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Comments

- The reduced row echelon form is denoted $R = \text{rref}(A)$. All linear packages have an implementation for rref.

- What is the reduced row form of an invertible, square matrix?
  The identity matrix !!

- R and A have the same nullspace!
  if $A\boldsymbol{x}=0$, then $R\boldsymbol{x}=0$ (and also $U\boldsymbol{x}=0$) because R (and U) are obtained by linear combinations of the rows!

3.2 Pivot variables, free variables and special solutions

Continuing with the previous example

$$R\boldsymbol{x} = 0 \iff \begin{array}{c} \begin{array}{cccc} P & F & P & F \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The **pivot variables** are those that multiply the pivot columns
  Here, the pivot variables are : x and z

- The **free variables** are those that multiply the other columns
  Here, the free variables are : y and t.

- The free variables can be assigned any values. Pivot variables are completely determined by the free variables

$R\boldsymbol{x} = \mathbf{0} \to$ First equation $x + 2y - 2t = 0 \Rightarrow x = -2y + 2t$

Second equation $z + 3/2\, t = 0 \Rightarrow z = -3/2\, t$

Third equation : $0 = 0$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2y+2t \\ y \\ -3/2\, t \\ t \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -3/2 \\ 1 \end{bmatrix}$$

- The two vectors $(-2, 1, 0, 0)$ and $(2, 0, -3/2, 1)$ are called **special solutions**.

Note

The nullspace contains all linear combinations of the special solutions.

Steps to get the special solutions:

1. Reduce A to its reduced row echelon form R.
2. Identify free and pivot variables.
3. Set one free variable to 1 and the remaining to 0. Solve $R\boldsymbol{x} = \mathbf{0}$ for the pivot variables. The resulting is a special solution.
4. Repeat the previous step for every free variable.
5. The linear combination of all special solutions form the nullspace of A.

$N(A)$ is spanned by the special solutions

$$= \text{span}\left(\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_s : \boldsymbol{x}_s : \text{special solutions}\}\right)$$

$$= \{\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \dots + \alpha_s \boldsymbol{x}_s\}$$

3.3 Rank

Definition The rank r of the matrix A is equal to the number of pivot.

The rank r gives a true size of the linear system. Rows that are linear combinations of previous rows, reduce to $0 = 0$ and thus should not count. The rank r is the number of non-trivial rows.

Comments

- What if there are more unknowns than equations $n > m$? (A is $m \times n$)

  $\Leftrightarrow$ A has more columns than rows

  m rows can hold at most m pivots

  $\Leftrightarrow$ there must $n-m$ free variables

  $\Leftrightarrow$ there are more solutions to $A\boldsymbol{x} = \mathbf{0}$ than just $\boldsymbol{x} = \mathbf{0}$!

- What is a relationship between rank and dimension of the nullspace?

  $\dim(\boldsymbol{N}(A)) =$ number of free variables

  $=$ number of special solutions

  $= n - r$

- What is the dimension of the column space?

  $\dim(\boldsymbol{C}(A)) =$ number of pivot variables $= r$

- If a matrix A has full column rank ($r = n$) then it has these properties: ($n \leq m$)
  1. All columns of A are pivot columns
  2. There are no free variables or special solutions.
  3. The nullspace $\boldsymbol{N}(A)$ contains only the zero vector $\boldsymbol{x} = 0$.
  4. If $A\boldsymbol{x} = \boldsymbol{b}$ is solvable, then there is only one solution.

Example 3 What's the rank of

$$\begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix}$$

Obviously rank $r = 1$ because rows 2 and 3 are multiple of row 1 !!

Let's count the number of pivots

$$\begin{bmatrix} \boxed{1} & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 3 & 10 \\ 0 & 0 & 0 \\ 3 & 9 & 30 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

only one pivot!! $\Rightarrow r = 1$

Example 4 Show that this matrix can be written as $A = \boldsymbol{u}\boldsymbol{v}^T$.

$\left.\begin{array}{l} col_2 = \tfrac{3}{2}\, col_1 \\ col_3 = 2\, col_1 \end{array}\right\} \Rightarrow$ rank $= 1$

$$\begin{array}{cl} \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix} & \begin{array}{l} \leftarrow row_1 \\ \leftarrow row_2 = 2 \times row_1 \\ \leftarrow row_3 = 3 \times row_1 \end{array} \end{array}$$

The rows are linearly dependent

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix} = \begin{bmatrix} 1\times 2 & 1\times 3 & 1\times 4 \\ 2\times 2 & 2\times 3 & 2\times 4 \\ 3\times 2 & 3\times 3 & 3\times 4 \end{bmatrix}$$

coefs first row

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \boldsymbol{u}\boldsymbol{v}^T \text{ where } \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Note

Every rank 1 matrix can be written as $A = \boldsymbol{u}\boldsymbol{v}^T$.

4 The complete solution $A\mathbf{x} = \mathbf{b}$

We will apply eliminations while keeping track of the right hand side.

Augmented form of A

$$\begin{array}{c} \begin{array}{ccccc} P & & P & & \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 1 & b_1 \\ 2 & 4 & 6 & 5 & b_2 \\ -1 & -2 & 2 & 5 & b_3 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 1 & b_1 \\ 0 & 0 & \boxed{2} & 3 & b_2 - 2b_1 \\ 0 & 0 & 4 & 6 & b_3 + b_1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 1 & b_1 \\ 0 & 0 & \boxed{2} & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + 5b_1 - b_2 \end{array}\right]$$

linearly independent columns

↳ gives a condition on **b**

The new system is

$$U\boldsymbol{x} = \boldsymbol{c}; \quad \begin{array}{c} \begin{array}{cccc} P & & P & \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + 5b_1 - b_2 \end{bmatrix}$$

Here again $\boldsymbol{c} = L^{-1}\boldsymbol{b}$. Condition on $\boldsymbol{b}$ to be a solution:

The zero row imposes the condution

$0 = b_3 + 5b_1 - b_2$

Although there are more unknowns than equations, a solution does not exist unless $\boldsymbol{b} \in \boldsymbol{C}(A)$.

What's the column space of A, i.e, $\boldsymbol{C}(A)$?

The column vectors $(1,2,-1)$, $(2,4,-2)$, $(2,6,2)$ and $(1,5,5)$ span the column space of A.
However they are not independent
eg: $col_2 = (2,4,-2) = 2 \times col_1$

By looking at the echelon form U, we can deduce a basis of *linearly independent* columns spanning $\boldsymbol{C}(A)$. These are

Pivot columns of A: columns of A that lead to a pivot in U (or R).

In our example: $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$

Note

The dependent columns are those without a pivot!

Let's pick one $\boldsymbol{b}$ where there is a solution, for example $\boldsymbol{b} = (1, 2-1)$. Elimination step lead to

satisfies $b_3 + 5b_1 - b_2 = 0$
$\Leftrightarrow$ $\boldsymbol{b} \in \boldsymbol{C}(A)$

$$\begin{array}{l} \text{eq 2} \rightarrow \\ \text{eq 1} \rightarrow \\ \\ \end{array} \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now let's perform back substitutions:

eq 1: $2z + 3t = 0 \Rightarrow z = -3/2\, t$
eq 2: $x + 2y + 2z + t = 1 \Rightarrow x = 1 - 2y - 2(-3/2\, t) - t$
$= 1 - 2y + 2t$

which we can write as:

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{x_p} + y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -3/2 \\ 1 \end{bmatrix}$$

Free variables

So now you can see that the complete solution is the particular solution $\boldsymbol{x}_p = (1, 0, 0, 0)$ plus all of $\boldsymbol{N}(A)$.

Comments

- To find a particular solution, set all free variables to 0 and solve the system!

Rather than stopping at $U\boldsymbol{x} = \boldsymbol{c}$, let's compute the reduced row echelon form $R\boldsymbol{x} = \boldsymbol{d}$:

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Again, to find the particular solution $\boldsymbol{x}_p$ set all free variables to 0 and solve for $\boldsymbol{x}_p$.

Note

Follow these steps to get all solutions to $A\boldsymbol{x} = \boldsymbol{b}$:

1. Reduce $A\boldsymbol{x} = \boldsymbol{b}$ to $R\boldsymbol{x} = \boldsymbol{d}$.
2. Set free variables to 0 and find the particular solution $R\boldsymbol{x}_p = \boldsymbol{d}$.
3. Find the special solutions to $A\boldsymbol{x} = \boldsymbol{0}$ by solving $R\boldsymbol{x} = 0$ (make each free variable in turn 1, the others 0 to find each special solution).
4. The complete solution is then $\boldsymbol{x} = \boldsymbol{x}_p +$ all linear combinations of the special solutions.

5 Linear Independence, Basis and Dimension

A matrix may have n columns, but the number that characterizes its true size is the rank r. Thus, a linear combination of the columns of A may span a subspace with dimension lower than the number of columns n. This section will provide definitions to help formalize some of the concepts related to the size of a subspace.

5.1 Linear independence

Definition Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\cdots$, $\boldsymbol{v}_k$ be vectors and c_1, c_2, $\cdots$, c_k be scalars. If $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = 0$ has a solution only when all $c_i = 0$, then the vectors are *linearly independent*. If any $c_i \neq 0$, they are *linearly dependent*.

In other words, a family of vectors of linearly dependent if any one vector can be expressed as a linear combination of the other vectors.

Example 5 Consider these two matrices

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 6 & 5 \end{bmatrix}; \quad U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Note

1. The columns of A are linearly independent exactly when $\boldsymbol{N}(A) = \{\mathbf{0}\}$.
2. The r nonzero rows of an echelon matrix U and a row reduced echelon matrix R are linearly independent.
3. The r pivot columns are linearly independent

Comments

- To check the linear dependence, put all vectors in a matrix A and find the nullspace of A. If $\boldsymbol{N}(A) = \{\mathbf{0}\}$, then the vectors are linearly independent.
- Alternatively, if A has a full rank $r = n$, then the vectors are linearly independent.
- if A is $m \times n$ and $n > m$, then there are at most $m < n$ pivots which means that the vectors must be linearly dependent.

Example 6 Check the linear dependence of the following vectors: $\boldsymbol{v}_1 = (1, 2)$, $\boldsymbol{v}_2 = (3, 2)$ and $\boldsymbol{v}_3 = (5/2, 1)$.

5.2 Basis and dimension of a vector space

Definition A basis for a vector space V is a sequence of vectors that are 1) linearly independent, and 2) span the space V.

Comments

- Any vectors in V can be expressed as a linear combination of the basis.
- There exists only one unique combination of the basis.
- The basis is not unique.
- The columns of a square invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

Although there are infinitely many basis for a vector space, the number of basis vectors is always the same!

Definition Any two bases for a vector space $\boldsymbol{V}$ contain the same number of vectors. This number is the dimension of the vector space $\boldsymbol{V}$.

If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_n$ are both bases for the same vector space, then $m = n$ (c.f. textbook for proof). This theorem has the following consequences:

1. In a subspace of dimension k, no set of more than k vectors can be independent, and no set of few than k vectors can span the space.
2. Any linearly independent set in V can be extended to a basis by adding more vectors if necessary. Any spanning set in $\boldsymbol{V}$ can be reduced to a basis, by discarding vectors if necessary.
3. A basis is a maximal independent set and a minimal spanning set.

6 The Four Fundamental Subspaces of a Matrix

All that we have seen so far, brings us to the final conclusion of this module. A matrix A of size $m \times n$, has 4 fundamental subspaces associated with it:

1. The **column space** $\boldsymbol{C}(A)$, a subspace of $\mathbb{R}^m$.

2. The **nullspace** $\boldsymbol{N}(A)$, a subspace of $\mathbb{R}^n$.

3. The **row space** of A. This subspace of $\mathbb{R}^n$ is spanned by the rows of A. It is just the column space of A^T, i.e, $\boldsymbol{C}(A^T)$.

4. The **left nullspace** of A is the nullspace of the transpose: $\boldsymbol{N}(A^T)$. It's a subspace of $\mathbb{R}^m$ that contains all vectors such that $A^T\boldsymbol{x} = \boldsymbol{0}$.

Let's go back to our previous example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 6 & 5 \\ -1 & -2 & 2 & 5 \end{bmatrix};\ U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Column space

1. The pivot columns of A form a basis for $\boldsymbol{C}(A)$

2. The dimension of $\boldsymbol{C}(A)$ is equal to

3. The column space is sometimes called the range of A.

4. Be careful, $\boldsymbol{C}(A) \neq \boldsymbol{C}(U)$.

Nullspace

1. The special solutions form a basis for $\boldsymbol{N}(A)$

2. The dimension of $\boldsymbol{N}(A)$ is equal to

3. The nullspace is sometimes called the kernel of A.
4. A, U and R have the same nullspace.

Row space

1. The nonzero rows of U form a basis of the row space, i.e., $\boldsymbol{C}(A^T)$.

2. The dimension of $\boldsymbol{C}(A^T)$ is equal to $\boldsymbol{C}(A)$

3. A, U and R have the same row space because the elementary operations are just linear combinations of the rows.

Left nullspace

1. The left nullspace represents combinations of the rows that produce zero: $A^T\boldsymbol{y} = \boldsymbol{0}$, or alternatively $\boldsymbol{y}^T R = 0$.
2. The dimension of $\boldsymbol{N}(A^T)$ is

3. A, U and R have the same row space because the elementary operations are just linear combinations of the rows.
4. There are two methods to find a basis of the left nullspace:
   (a) First method: using the special solutions of A^T

(b) Second method: using the augmented version of A