1 Exercise 1

No "solution" are provided for these types of questions, since the whole point of them is to encourage you to express *briefly* but *clearly* and *in your own words* what you understand. As explained in the directions, definitions taken from text books or the internet do not reflect a good understanding if these therms, nor do extremely long explanations. Equations do not express the meaning of these, nor do literal word translations of equations show that you know what they mean. Instead, we are looking for clear evidence that you understand what each term means.

2 Exercise 2

Solve the following system of equations, showing all relevant steps and stating the method used to solve the system (e.g., LU factorization, direct numerical solution, matrix elimination, etc.,). Furthermore, if a system has no solution state it has no solution and briefly describe why it has no solution.

(a) The following system of equations can be solved using the inverse matrix approach, namely

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{1a}$$

therefore we need to obtain A^{-1}. We can determine the inverse of matrix $\mathbf{A}$ using the augmented matrix method. Where the matrix $\mathbf{A}$ is augmented with the identity matrix $\mathbf{I}$, and row operations are performed until the left hand side of the augmented matrix is the identity matrix. Starting with the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \tag{1b}$$

Then we can take $4r_1 - r_2$ to zero out the a_{21} term giving

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 12 & 2 & 4 & -1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \tag{1c}$$

Next, the a_{31} is zeroed out by performing $r_1 - r_3$ giving

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 12 & 2 & 4 & -1 & 0 \\ 0 & 0 & -3 & 1 & 0 & -1 \end{array}\right] \tag{1d}$$

Now we continue zeroing out entries of $\mathbf{A}$ moving up the matrix starting with a_{23} by performing $-\frac{2}{3}r_3 - r_2$ giving

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -12 & 0 & -\frac{14}{3} & 1 & \frac{2}{3} \\ 0 & 0 & -3 & 1 & 0 & -1 \end{array}\right] \tag{1e}$$

Moving up the third column we zero out a_{13} using $-\frac{1}{3}r_3 - r1$ giving

$$\left[\begin{array}{ccc|ccc} -1 & -3 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -12 & 0 & -\frac{14}{3} & 1 & \frac{2}{3} \\ 0 & 0 & -3 & 1 & 0 & -1 \end{array}\right] \tag{1f}$$

Next we can zero a_{12} with $\frac{1}{4}r_2 - r_1$ giving

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{4} & -\frac{1}{6} \\ 0 & -12 & 0 & -\frac{14}{3} & 1 & \frac{2}{3} \\ 0 & 0 & -3 & 1 & 0 & -1 \end{array}\right] \tag{1g}$$

Lastly we can multiply each trace entry by its reciprocal, i.e., $r_2 = -\frac{r_2}{12}$ and $r_3 = -\frac{r_3}{3}$, to finally get the inverse of matrix $\mathbf{A}$,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{4} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{7}{18} & -\frac{1}{12} & -\frac{1}{18} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right] \tag{1h}$$

Thus

$$\boxed{\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} & -\frac{1}{6} \\ \frac{7}{18} & -\frac{1}{12} & -\frac{1}{18} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{23}{18} \\ \frac{44}{24} \\ -\frac{2}{3} \end{bmatrix}} \tag{1i}$$

(b) From the system of equations

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \tag{2}$$

it is obvious that the second row is a linear combination of the first row, i.e. $r_2 = 2r_1$. Therefore the second equation in the system adds no new information, that we do not already have from the first row. Thus we have a system with three unknowns, x, y, z, and only two equations and we cannot obtained an unique solution.

(c) ***Still need to add***

(d) The easiest way to solve this system of equations

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 5 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{3a}$$

is to swap the rows so that all the entries along the trace are non-zero giving

$$\begin{bmatrix} 5 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \\ 4 \end{bmatrix} \tag{3b}$$

Now the system of equations can be solved quite easily using backwards substitution. Thus

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 16/5 \\ 15 \\ -7 \\ -1 \\ 4 \end{bmatrix}} \tag{4}$$

3 Exercise 3

The following system of equations

$$2x + 3y = 4 \tag{5a}$$

$$x + 2y = 5 \tag{5b}$$

can easily be solve by hand. First solve for x in terms of y in Eq. (5b),

$$x = 5 - 2y \tag{5c}$$

Next substitute x into Eq. (5a) to obtain a value for y, namely

$$2(-5 - 2y) + 3y = 4 \rightarrow y = 6 \tag{5d}$$

Thus

$$\boxed{x = -7} \tag{5e}$$

$$\boxed{y = 6} \tag{5f}$$

Furthermore, the solution can be verified by solving for y in both equations and plotting both curves and observing their intersection point, as shown in Fig (1).

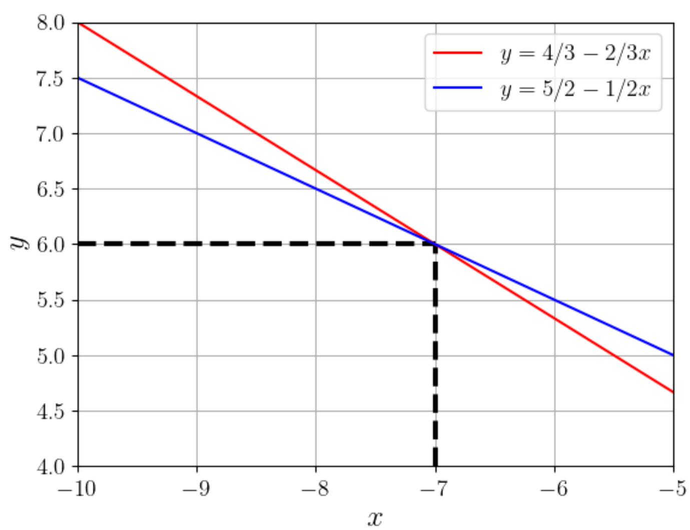

Figure 1: Verification plot for Exercise 3

The following python commands were used in to verify the solution.

```
#!/usr/bin/env python3
"""=========================================================================
Purpose:
    Verification of exercise 3

Author:
    Emilio Torres
========================================================================="""
#==========================================================================#
# Preamble                                                                 #
#==========================================================================#
#--------------------------------------------------------------------------#
# Python packages                                                          #
#--------------------------------------------------------------------------#
import sys
import os
from subprocess import call
from numpy import *
import matplotlib.pyplot as plt
#--------------------------------------------------------------------------#
# User packages                                                            #
#--------------------------------------------------------------------------#
from ales_post.plot_settings import plot_setting
#==========================================================================#
# Main preamble                                                            #
#==========================================================================#
if __name__ == '__main__':
    #----------------------------------------------------------------------#
    # Main preamble                                                        #
    #----------------------------------------------------------------------#
    call(['clear'])
    sep         = os.sep
    pwd         = os.getcwd()
    media_path  = pwd + '%c..%cmedia%c'         %(sep, sep, sep)
    #----------------------------------------------------------------------#
    # Domain variables                                                     #
    #----------------------------------------------------------------------#
    x   = linspace(-10, -5, 100)
    y1  = 4./3. - 2./3*x
    y2  = 5./2.-x/2.
    #----------------------------------------------------------------------#
    # Plotting solution                                                    #
    #----------------------------------------------------------------------#
    plot_setting()
    plt.plot([-10,-7], [6,6], 'k--', lw = 3.0)
    plt.plot([-7,-7], [4,6], 'k--', lw = 3.0)
    plt.plot(x,y1,'r', lw=1.5, label = '$y = 4/3 - 2/3 x$')
    plt.plot(x,y2,'b', lw=1.5, label = '$y = 5/2 - 1/2 x$')
    #----------------------------------------------------------------------#
    # Plot settings                                                        #
    #----------------------------------------------------------------------#
    plt.legend(loc=0)
    plt.ylabel('$y$')
    plt.xlabel('$x$')
```

```
    plt.grid()
    plt.xlim([-10,-5])
    plt.ylim([4,8])
    plt.savefig(media_path + 'exercise-3.png')
    plt.close()

    print('**** Successful run ****')
    sys.exit(0)
```

4 Exercise 4

Prove the following matrix properties:

(a) Prove the following:

$$(AB)^T = B^T A^T \tag{6a}$$

Start be defining two $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{A} = A_{ij} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n_2} & \cdots & a_{n,n} \end{bmatrix} \tag{6b}$$

$$\mathbf{B} = B_{ij} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n_2} & \cdots & b_{n,n} \end{bmatrix} \tag{6c}$$

which gives the following for the LHS of Eq. (6a)

$$AB = \underbrace{\sum_{k=1}^{N} A_{ik} B_{kj}}_{\equiv C_{ij}} = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} & \cdots \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix} \tag{6d}$$