

MAT 394, Spring 2020 Project: Making a Forecast

Steffen E. Eikenberry

Assigned: 3/30/2020. **Due:** 5/8/2020. **Max Points:** 50 (60 with extra credit).

1 Overview

This is a guided project, with the fundamental idea that we can couple a simple, but realistic model of the short-term carbon cycle (including the CO₂ buffering chemistry of the ocean, CO₂ fertilization of the biota, and the clearing of land/deforestation that is a significant source of atmospheric emissions), the response of this cycle to anthropogenic CO₂ emissions, and the resulting radiative forcing and temperature response. Moreover, we will include a *climate-carbon feedback* in the form of increasing temperatures increasing the decay rate of standing biomass. Note that we will (mercifully!) neglect non-CO₂ emissions in this project, though they could be added in straightforward fashion.

This is essentially an extended elaboration of what we did in Homework #2, but we use more sophisticated representations of each system component.

Your basic task is to implement each sub-component of the model as described (there are a few sub-exercises for each component, and some of these partially duplicate work from earlier homework sets), and then bring everything together to explore the response of the climate to different future emissions scenarios.

2 Data

Data for driving the model/comparing results is included in three Excel spreadsheets:

1. `Emissions_Data.xlsx`: Yearly *emissions* data, from 1751 through 2014.
2. `C02_Data.xlsx`: Data from 1750 through 2018 on atmospheric CO₂ *concentration*.
3. `Temp_Data.xlsx`: Data from 1850 through 2018 on the *global surface temperature anomaly*.

3 Building/Running the base model: Atm-Ocean-Temp

For the base model, we will couple the follow components: (1) A calculation of the *equilibrium climate sensitivity* (ECS) based on radiative forcing for CO₂, the Planck sensitivity, and our four canonical fast feedbacks (water vapor, lapse rate, ice-albedo, and clouds); (2) A model for the temperature response of Earth with an upper and lower ocean (already implemented in HW #5); and (3) A carbon cycle model that includes an atmosphere, upper and lower ocean, and CO₂ buffering chemistry, but that *omits* the terrestrial biota. For the extended model considered at the end of project, we will finally add in the terrestrial biota.

3.1 Equilibrium Climate Sensitivity & Feedbacks (5 pts)

Consult the previously posted notes for more details on the mathematics of climate sensitivity. An outline follows.

To calculate the temperature perturbation from a perturbation in CO₂ in the model below, we must have some estimate of *equilibrium climate sensitivity* (ECS), again defined as the temperature change, at equilibrium, from a doubling of atmospheric CO₂. More generally, we have the surface temperature change, ΔT_s , in response any radiative forcing, ΔN , as

$$\Delta T_s = \frac{-\Delta N}{\lambda}, \quad (3.1)$$

where the feedback parameter λ can be expressed as the sum,

$$\lambda = \lambda_0 + \sum_{i=1}^M \lambda_i, \quad (3.2)$$

where $\lambda_0 \approx -3.3 \text{ W m}^{-2} \text{ K}^{-1}$ is the “Planck feedback parameter,” and characterizes the temperature response if the terrestrial sphere acts purely as a blackbody. Feedback i has associated feedback parameter λ_i . If $\lambda_i > 0$, then the feedback is *destabilizing*, and increases the magnitude of the temperature response to radiative forcing.

3.1.1 Calculate ECS and λ from λ_i s

Recall that the radiative forcing from a change in atmospheric CO₂ concentration can be expressed as

$$\Delta N = k \ln \left(\frac{C}{C_0} \right), \quad (3.3)$$

where $k = 5.35 \text{ W m}^{-2}$, C is our current atmospheric CO₂ concentration, and C_0 is the initial (pre-industrial) atmospheric concentration (about 277 ppm).

Recall our four canonical fast feedbacks: (1). Across global climate models, spread in the predicted shortwave cloud feedback is a major source of intermodal spread and uncertainty regarding the climate response to doubling CO₂. Recent results from the coupled model inter-comparison project 6 (CMIP6) indicate ongoing uncertainty, but with some models predicting an appreciably larger positive feedback (i.e. warming) in the shortwave cloud component, suggesting that higher ECS values cannot be ruled out!

Suppose the following values for λ_i s:

1. $\lambda_0 = -3.3$
2. $\lambda_{\text{Ice-albedo}} = -0.2 \times \lambda_0$
3. $\lambda_{\text{Water vapor}} = -0.55 \times \lambda_0$
4. $\lambda_{\text{Lapse rate}} = 0.2 \times \lambda_0$
5. $\lambda_{\text{Clouds}} = 0.1 \times \lambda_0$

What, then, is the overall climate sensitivity parameter, λ ? (this is just an arithmetic problem!). What is the radiative forcing that results from doubling CO₂? Based on these numbers, what is the equilibrium climate sensitivity?

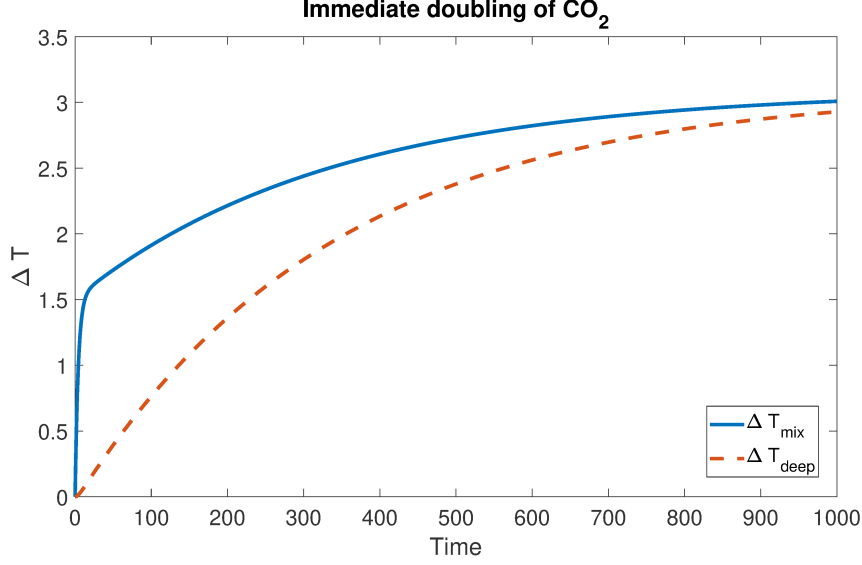


Figure 1: Temperature anomaly model run starting with $T'_{mix}(0) = T'_{deep}(0) = 0$, with $C = 2C_0$ at time zero.

3.2 Temperature-response model

3.2.1 Model description

Here we will simply re-implement the temperature response model previously considered in HW #5. In HW #5, however, we used a very simple model for the atmospheric CO_2 anomaly. Here we will use the more complex carbon cycle model presented below (and discussed in class/notes) to determine the CO_2 anomaly and hence the radiative forcing perturbation, $\Delta N(t)$.

Now, recall that the two-compartment temperature response model describes the temperature *perturbations* in the mixed (upper) and deep ocean layers, dT'_{mix} , and dT'_{deep} , respectively. Model equations are given as

$$\mu_{mix} \frac{dT'_{mix}}{dt} = -\hat{\lambda} T'_{mix} - \gamma(T'_{mix} - T'_{deep}) + \Delta N(t), \quad (3.4)$$

$$\mu_{deep} \frac{dT'_{deep}}{dt} = \gamma(T'_{mix} - T'_{deep}), \quad (3.5)$$

where μ_{mix} and μ_{deep} are the heat capacities of the mixed and deep ocean layers (with $\mu_{deep} \gg \mu_{mix}$), γ is a heat-transfer coefficient for heat transfer from the upper to deep ocean (approximately equal to $\hat{\lambda}$), and $\hat{\lambda} = -\lambda$ is just the negation of the climate sensitivity parameter (as determined above). The $\Delta N(t)$ term, again, represents the (time-varying) *radiative forcing perturbation* (in W m^{-2}) due to anthropogenic CO_2 gases, and is again given as

$$\Delta N = k \ln \left(\frac{C}{C_0} \right), \quad (3.6)$$

where $k = 5.35 \text{ W m}^{-2}$, C is our current atmospheric CO_2 concentration, and C_0 is the initial (pre-industrial) atmospheric concentration (about 277 ppm).

3.2.2 Running the basic model (5 pts)

To make sure everything is running correctly, suppose we start with $T'_{mix}(0) = T'_{deep}(0) = 0$. At time 0, set $C = 2C_0$ to get (constant) ΔN , and run the temperature model for 1,000 years to reproduce the temperature course in Figure 1. Use parameter values $\mu_{mix} = 3.154 \times 10^8 \text{ J m}^{-2} \text{ K}^{-1}$, $\mu_{deep} = 6.307 \times 10^9 \text{ J m}^{-2} \text{ K}^{-1}$, $\hat{\lambda} = 1.2 \text{ W m}^{-2} \text{ K}^{-1}$, $\gamma = 1.2 \text{ W m}^{-2} \text{ K}^{-1}$, and $k = 5.35 \text{ W m}^{-2}$.

3.3 Carbon cycle model: Atmosphere-ocean-temperature dynamics

3.3.1 Atm-Ocean Model

Following the posted notes and our in-class work, please implement the so-called BEAM model (based on [1]), which determines the fluxes of carbon through the atmosphere, upper ocean, and lower ocean. based upon the turnover times and relative sizes of these compartments, in addition to the CO_2 buffering chemistry.

The model variables are the *amounts* of carbon in the atmosphere, upper ocean, and lower ocean, denoted $Q_A(t)$, $Q_U(t)$, and $Q_L(t)$, respectively, and we will use *moles* as our fundamental unit. *It is essential to keep straight that we are using moles and not some other unit of mass or concentration as our unit.* The model equations follow as

$$\frac{dQ_A}{dt} = -k_a Q_A + k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U + E(t), \quad (3.7)$$

$$\frac{dQ_U}{dt} = k_a Q_U - k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U - k_d Q_U + \frac{k_d}{\delta_d} Q_L, \quad (3.8)$$

$$\frac{dQ_L}{dt} = k_d Q_U - \frac{k_d}{\delta_d} Q_L, \quad (3.9)$$

where

$$\Lambda(t) = 1 + \frac{K_1}{[H^+]} + \frac{K_1 K_2}{[H^+]^2}, \quad (3.10)$$

and k_a and k_d (year^{-1}) are time-constants for carbon turnover between the atmosphere and upper ocean, and upper ocean and lower ocean, respectively, δ_d is the ratio of moles of deep ocean to upper ocean, δ_a is the ratio of moles of upper ocean to atmosphere, and $\Lambda = \Lambda(t, [H^+])$ varies with time according to pH, with $[H^+] = [H^+](t)$ the current concentration of hydrogen ions, in terms of *mole fraction*. The parameters K_1 and K_2 are (given) equilibrium constants. The function $E(t)$ is an imposed emissions time-course, which you/we must specify.

To calculate $[H^+]$, we approximate the titration alkalinity, Alk , as

$$\begin{aligned} Alk &= [HCO_3^-] + 2[CO_3^{2-}] = \left(\frac{K_1}{[H^+]} + \frac{2K_1 K_2}{[H^+]^2} \right) [CO_2(aq)] \\ &= Q_U \frac{1}{\Lambda} \left(\frac{K_1}{[H^+]} + \frac{2K_1 K_2}{[H^+]^2} \right). \end{aligned} \quad (3.11)$$

By doing some algebra, we eventually arrive at

$$[H^+]^2 + [H^+] K_1 \left(1 - \frac{Q_U}{Alk} \right) + K_1 K_2 \left(1 - \frac{2Q_U}{Alk} \right) = 0, \quad (3.12)$$

and we take $[H^+]$ as the positive root of this quadratic equation. So note, you now have alkalinity prescribed as a constant, and thus for any amount of upper ocean carbon, you can

determine $[H^+]$. From this, you get Λ , and now you have everything you need to numerically run the system of differential equations above for a *single time-step*. At the next step, we simply re-solve for $[H^+]$ and Λ , and so on.

Notes on units. We have our baseline equilibrium constants, K_1 and K_2 given in units of molality (moles kg^{-1}). To convert from molality to mole fraction, we assume an average molecular weight of 18 for sea water, and use the fact that there are 1000 g per kg. That is, we simply multiply by the factor 18/1000. To convert from GtC to moles, we simply use carbon's molecular weight of 12 (and the fact that 1 GtC = 10^{15} moles). If we are interested in tracking pH from $[H^+]$, we have pH as

$$pH = -\log_{10} \left([H^+] \frac{1000}{18} \right), \quad (3.13)$$

where $[H^+]$ is in mole fraction.

Parameter values and initial conditions. We have the following parameter values from [1]: $k_a = 0.2 \text{ yrs}^{-1}$, $k_d = 0.05 \text{ yrs}^{-1}$, $\delta_d = 50$, $AM = 1.77 \times 10^{20}$ moles (moles in the atmosphere), $OM = 7.8 \times 10^{22}$ moles (moles in ocean), $K_H = 1.23 \times 10^3$ (unitless), $K_1 = 8 \times 10^{-7}$ moles $\text{kg}^{-1} = 1.44 \times 10^{-8}$ mole fraction, $K_2 = 4.53 \times 10^{-10}$ moles $\text{kg}^{-1} = 8.154 \times 10^{-12}$ mole fraction, and $Alk = 767 \text{ GtC} = 767 \times 10^{15}/12$ moles. Note that we can calculate $\delta_a = OM/(AM(1 + \delta))$.

We can use the following for *pre-industrial* initial conditions: $Q_A(0) = 590 \text{ GtC}$, $Q_U(0) = 713 \text{ GtC}$, and $Q_L(0) = 35,658 \text{ GtC}$. Be sure to convert to these values from GtC to moles (multiply by $10^{15}/12$).

3.3.2 Atm-Ocean-Temp Model

The model above can be coupled with the temperature-response model in a very straightforward manner. Our coupled model equations will simply be

$$\frac{dQ_A}{dt} = -k_a Q_A + k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U + E(t), \quad (3.14)$$

$$\frac{dQ_U}{dt} = k_a Q_U - k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U - k_d Q_U + \frac{k_d}{\delta_d} Q_L, \quad (3.15)$$

$$\frac{dQ_L}{dt} = k_d Q_U - \frac{k_d}{\delta_d} Q_L, \quad (3.16)$$

$$\mu_{mix} \frac{dT'_{mix}}{dt} = -\hat{\lambda} T'_{mix} - \gamma(T'_{mix} - T'_{deep}) + \Delta N(t), \quad (3.17)$$

$$\mu_{deep} \frac{dT'_{deep}}{dt} = \gamma(T'_{mix} - T'_{deep}), \quad (3.18)$$

where

$$\Lambda(t) = 1 + \frac{K_1}{[H^+]} + \frac{K_1 K_2}{[H^+]^2}, \quad (3.19)$$

$$\Delta N(t) = k \ln \left(\frac{C}{C_0} \right), \quad (3.20)$$

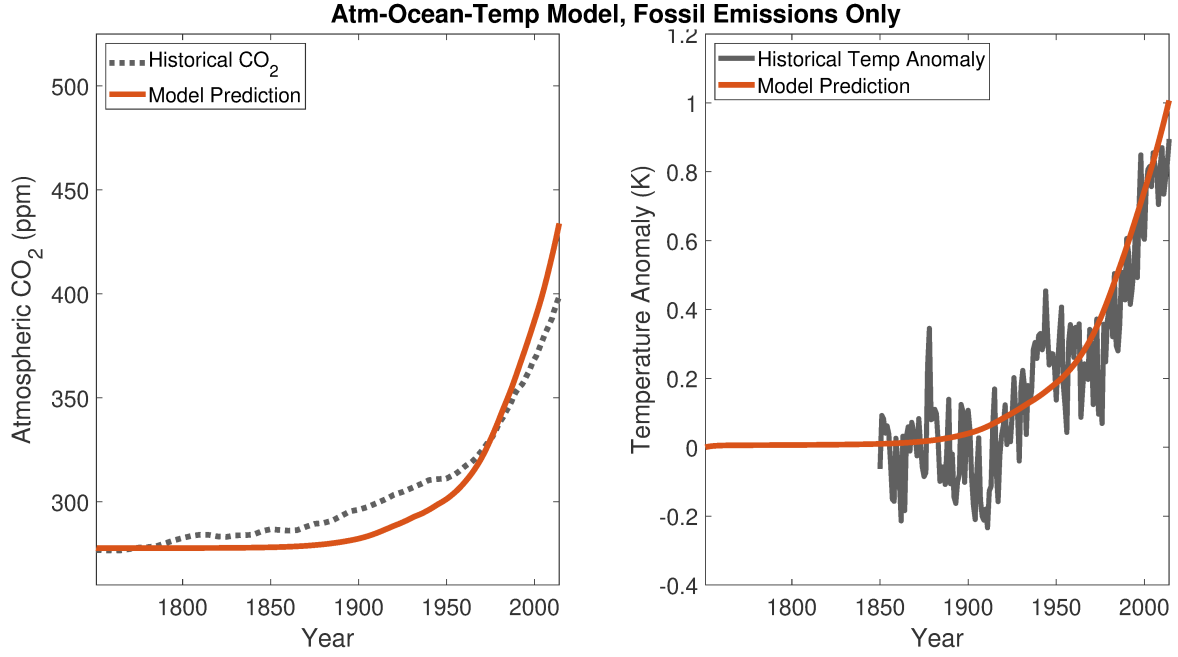


Figure 2: Atm-Ocean-Temp model driven by historical fossil emissions only.

C is the current atmospheric CO_2 concentration, and $[H^+]$ is the positive root of

$$[H^+]^2 + [H^+]K_1 \left(1 - \frac{Q_U}{Alk}\right) + K_1K_2 \left(1 - \frac{2Q_U}{Alk}\right) = 0. \quad (3.21)$$

Other parameter values and initial conditions are as above. The above model can be straightforwardly run using MATLAB's `ode45` or other built-in solver. We need only specify our initial conditions, parameter values, and impose the emissions course, $E(t)$.

3.3.3 Historical emissions and atmosphere-ocean only model (10 pts)

Use the emissions data contained in `Emissions_Data.xlsx` to drive the integrated model with either **(1) fossil emissions only, or (2) with land use plus fossil emissions, from 1751 through 2014**. Note that the data gives yearly emissions in GtC, which you must convert to moles C. Use the pre-industrial initial conditions and parameter values detailed above.

Compare the model-predicted atmospheric CO_2 concentration (in ppm) to the data provided in AR5 (used previously in this course). Additionally, compare the predicted T'_{mix} to the temperature anomaly data provided in `Temp_Data.xlsx` (data begins in 1850); the anomaly is given *approximately* relative the pre-industrial baseline.

Your results should mimic those in Figures 2 and 3. Note that using only fossil emissions underestimates atmospheric CO_2 early in time and overestimates late in time, while using all emissions overestimates atmospheric CO_2 at almost all times. This is because we have not accounted for the biomass carbon sink (the so-called “missing carbon sink”). However, for future projections, it is unclear to what degree the biomass will continue to take up anthropogenic emissions, and we can (with relative safety) ignore the biota for future emissions (vs. historical emissions) scenarios, as we do in the following section.

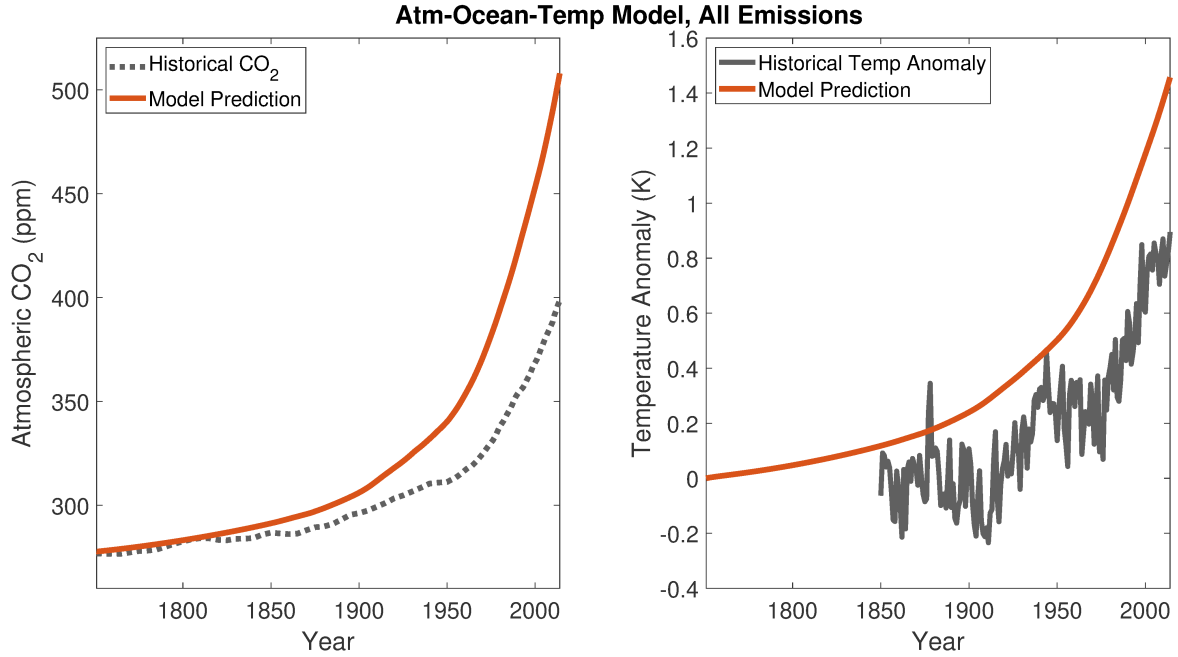


Figure 3: Atm-Ocean-Temp model driven by historical emissions, including both fossil and land use emissions.

4 Future scenarios: *Atm-Ocean-Temp Model Only* (10 pts)

In this section, you need only use the Atm-Ocean-Temp model (i.e. ignore the biomass components of the model described later in the document, and do not distinguish between fossil and land use emissions).

4.1 Getting initial conditions

In the following, we are using the Atm-Ocean-Temp model to make projections into the future, using 2020 as our baseline year. To do this, we need initial conditions corresponding, obviously, to conditions in 2020. Since we get systematic errors running the Atm-Ocean model under historical emissions (due to the omission of biomass), it would not work well to simply drive the system with emissions data from the pre-industrial state. Instead, we impose atmospheric CO_2 , and run the rest of the model as is. Doing so yields the following initial conditions for 2020: $Q_A(0) = 870 \text{ GtC}$, $Q_U(0) = 730 \text{ GtC}$, $Q_L(0) = 35,707 \text{ GtC}$, $T'_{mix}(0) = 0.94 \text{ K}$, and $T'_{deep}(0) = 0.25 \text{ K}$.

To be clear, you don't need to do this yourself: This is just an explanation of where these numbers come from!

4.2 Idealized future emissions scenarios

As above, use the following initial conditions to represent the present-day, 2020. We will impose various emissions streams (i.e. define $E(t)$) and explore the model response. **Suppose baseline (total) CO_2 emissions of 12 GtC/year in 2020.**

4.2.1 Limiting warming

Carbon emissions have continued to increase by roughly 1.5% per year over the last few decades, and, until the onset of the coronavirus pandemic, showed little sign of peaking or even appreciably slowing. Emissions reductions via nationally determined contribution (NDCs) under the Paris Climate Agreement are woefully inadequate to global limit warming either to 1.5 or 2 K, and even these are not being met by many countries. Let us explore the effect of peak emission years. In all cases, run the simulation starting in 2020 and make your final year 2300.

To get started, first experiment with an annualized decrease in CO₂ emissions: Starting with the baseline emissions of 12 GtC/year in 2020, by what annualized rate must we decrease emissions to limit ultimate warming to 1.5 K? What about 2 K?

Suppose emissions increase at an annualized rate of 1.5% until some peak year, and then decline thereafter. Experiment with the model:

1. If emissions peak at 2030, by approximately what annual rate must they decrease thereafter to limit *peak* warming to 1.5 K? What about 2 K? *Include a figure* showing your emissions time series, the atmospheric CO₂ levels, and the temperature anomaly (T'_{mix}) for a case where warming is successfully limited to ≤ 1.5 K.
2. What if emissions don't peak until 2050 (after increasing at 1.5% annualized), is it even possible to limit peak warming to 1.5K? To 2K?
3. What if emissions only increase at a 0.5% rate until 2030? How rapidly must they decline thereafter to limit warming to 1.5 K?

Support your conclusions above with at least one figure demonstrating your imposed emissions course, and resulting changes in CO₂ and temperature.

Finally, consider a biphasic phasing out of emissions immediately, as commonly urged. Using linear changes in $E(t)$, (1) Halve all CO₂ emissions from 2020 to 2030, and (2) decrease $E(t)$ (again, linearly) to 0 by 2050. *Run the simulation until 2300 to see that this does, indeed, hold both peak and ultimate global warming to less than 1.5 K. Include a figure demonstrating your results.*

5 Geoengineering: *Atm-Ocean-Temp Model Only* (5 pts)

Geoengineering is a lumped concept for large-scale efforts to (at least partially) mitigate global warming, and can range from albedo modification schemes to various methods to enhance carbon capture and storage. Geoengineering is controversial, to say the least, and albedo modification, in particular, is a fraught approach. I suggest reading the editorial by Pierrehumbert on the trouble with “hacking the planet,” at

<https://thebulletin.org/2017/06/the-trouble-with-geoengineers-hacking-the-planet/>.

Your task here is more open-ended: Try playing with various emissions scenarios (construct these yourself) and the albedo modification geoengineering scheme, which is described as follows.

5.1 Albedo engineering

Albedo modification at the planetary scale, e.g. via injection of sulfate aerosols into the high atmosphere, has the potential to offset global warming, but has at least three major issues: (1)

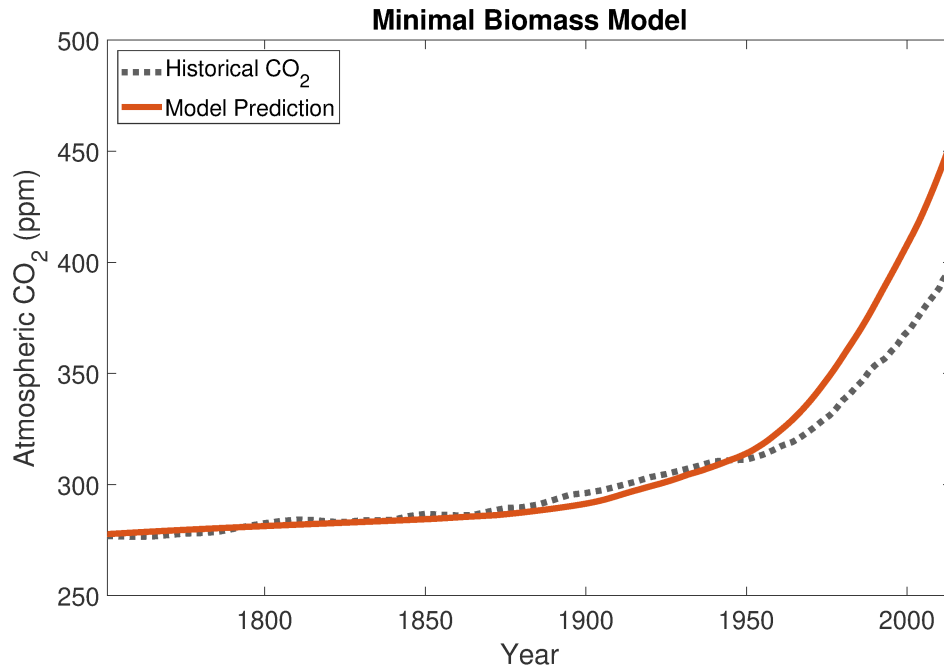


Figure 4: Extended model using the very minimal representation of the biomass, as explained in the text, driven by historical emissions.

Possible large-scale regional to continental effects on climate; (2) does not alter ocean acidification from ongoing CO₂ emissions, and (3) albedo modification affects energy balance on a yearly time-scale, while CO₂ perturbations last up to 100,000 years: If albedo modification ever ceases over the next millennia, a huge amount of latent warming could be unleashed on the planet.

Experiment with albedo modification under the Atm-Ocean-Temp model: Try running possible future emissions scenarios (e.g. ongoing exponential growth, constant emissions, a transient increase followed by decline, etc.). How does ocean pH and temperature evolve with or without albedo modification? What happens if albedo modification suddenly ceases at some point in the future? Experiment with the model to see if *transient* (say, for the next 50 years) albedo modification make senses (to you) as a stop-gap measure while CO₂ emissions are simultaneously being phased out.

To actually model the effect of albedo modification, I suggest simply adding a radiative forcing term to (partially? transiently? It's up to you!) counteract CO₂-induced radiative forcing. Briefly explain your methods, your results, and present your conclusions. Your goal here is to understand the potential and possible pitfalls of albedo modification, based on experimenting with the model, and then explain and present these.

6 Building the extended model: Ocean-atmosphere plus biota dynamics (5 pts)

The following extensions add biomass dynamics (the “missing carbon sink”) to the atmosphere-ocean model, and these are more complex to implement. *Therefore, you will only be required to run this version of the model under historical emissions*; we will just use the atmosphere-ocean model for projections, as already done above. The enhanced biomass sink and CO₂ fertilization is essential to properly representing the historical carbon and temperature response to CO₂

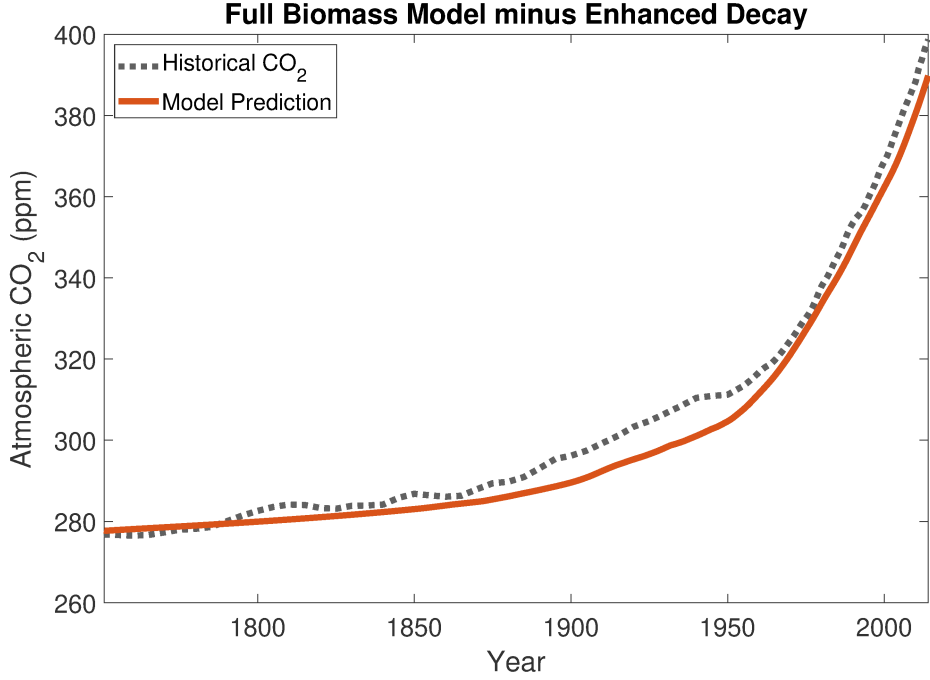


Figure 5: Extended model using the more complete representation of the biomass, except for enhanced decay, driven by historical emissions.

emissions. However, because increasing global temperatures enhance biomass decay (as well lead to potential nonlinear increases in forest dieback, permafrost melting, etc.), this effect and CO₂ fertilization may largely cancel out in the future, and indeed even the sign of the overall biomass response to future emissions is uncertain.

Basic idea. We will loosely follow Walker and Kasting [2] and Tomizuka [3] to add in a minimal extension that accounts for biomass accumulation as the balance of growth and decay, where growth is a function of atmospheric CO₂ (CO₂ fertilization), while decay increases in proportion to temperature. Let $M(t)$ be total biomass expressed in moles of carbon C^1 .

Now, suppose we have some baseline biomass *potential*, which we will denote C_{bio} , and set equal to 2,400 GtC = 2×10^{17} moles. Suppose that growth and decay follow first-order kinetics, characterized by a time constant for biomass growth, τ_G , and a time constant for decay, τ_D . We will assume initially that $\tau_G = \tau_D = 50$ years, and that the biomass grows toward the potential C_{bio} ; we will also include a CO₂ fertilization factor, f to be defined further below. Simply set $f = 1$ at baseline. This yields the following baseline differential equation for dM/dt ,

$$\frac{dM}{dt} = f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D}. \quad (6.1)$$

Now, any *change* in the biomass carbon pool implies an opposite change in the atmospheric carbon pool! Thus, to incorporate this biomass dynamic into the atmosphere-ocean model considered above, we simply subtract dM/dt from dQ_A/dt . Moreover, we should now distinguish between carbon emissions from land use change and fossil/industrial emissions, which we will denote $E_{land}(t)$ and $E_{fossil}(t)$, respectively. It is reasonable to assume that any land use emissions come from the biomass (and thus are a loss term for dM/dt , and these considerations give

¹Note that trees are roughly 50% carbon by *dry* weight, and about 50% water by *wet* weight. Thus, 1 GtC as tree biomass corresponds to about 2 Gt of dry tree mass and 4 Gt of wet tree mass. We needed worry about all this: Everything is just in units of C!

the following modified model

$$\frac{dQ_A}{dt} = -k_a Q_A + k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U + E_{fossil}(t) + E_{land}(t) - \left(f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D} \right), \quad (6.2)$$

$$\frac{dQ_U}{dt} = k_a Q_U - k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U - k_d Q_U + \frac{k_d}{\delta_d} Q_L, \quad (6.3)$$

$$\frac{dQ_L}{dt} = k_d Q_U - \frac{k_d}{\delta_d} Q_L, \quad (6.4)$$

$$\frac{dM}{dt} = \left(f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D} \right) - E_{land}(t), \quad (6.5)$$

where $f = 1$.

If we drive the model with our historical fossil and land use emissions data (from 1751 through 2014), we will get the results shown in Figure 4. **Reproduce this figure to make sure you're on the right track.** Now, such a model is a clear improvement over the atmosphere-ocean model, but remains lacking. In particular, we can add three essential components:

1. CO₂ fertilization, yielding an enhanced biomass growth rate (reflected in f).
2. Increased biomass decay from warming temperatures (reflected in τ_D).
3. Decreasing biomass potential, C_{bio} , from land use/deforestation emissions (we will need a new, simple ODE for C_{bio}).

CO₂ fertilization. We will use the logarithmic relationship in Tomizuka [3], and have the fertilization factor, f , scale as

$$f(t) = \left(1 + \beta \ln \left(\frac{C(t)}{C_0} \right) \right), \quad (6.6)$$

where $C(t)$ is simply the atmospheric CO₂ level at time t , and C_0 is the pre-industrial CO₂ concentration. We will use $\beta = 0.75$ as a baseline value, although this may be a bit of an overestimate.

Decreasing biomass potential. Land use emissions are not just a one-time emission, but often represent deforestation, etc. that undermines the ability of the biota to recover fully. Let us suppose that biomass potential decreases in direct proportion to land use emissions (basically representing permanent deforestation). This yields

$$\frac{dC_{bio}}{dt} = -E_{land}(t), \quad (6.7)$$

and our initial condition is simply $C_{bio} = 2,400 \text{ GtC} = 2 \times 10^{17} \text{ moles}$.

Enhanced biomass decay. Finally, to incorporate increased biomass decay under warming temperatures, we can use the method of Walker and Kasting [2], who modeled enhanced biomass decay by having

$$\tau_D = \tau_{D0} \left(1 - \frac{\Delta T_s}{20} \right), \quad (6.8)$$

where ΔT_s is the surface temperature perturbation, and we may take $\Delta T_s = T'_{mix}$. This expression was considered reasonable for ΔT_s up to 10 K, and uses $\tau_{D0} = 50 \text{ yrs}$, as before. Note that,

to implement this component, we must couple our carbon cycle model with the temperature perturbation model (see further below)!

Final model equations for ocean-atmosphere-biota dynamics. Model equations are given in sum as

$$\frac{dQ_A}{dt} = -k_a Q_A + k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U + E_{land}(t) + E_{fossil}(t) - \left(f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D} \right), \quad (6.9)$$

$$\frac{dQ_U}{dt} = k_a Q_U - k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U - k_d Q_U + \frac{k_d}{\delta_d} Q_L, \quad (6.10)$$

$$\frac{dQ_L}{dt} = k_d Q_U - \frac{k_d}{\delta_d} Q_L, \quad (6.11)$$

$$\frac{dM}{dt} = f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D} - E_{fossil}(t), \quad (6.12)$$

$$\frac{dC_{bio}}{dt} = -E_{land}(t), \quad (6.13)$$

where

$$f(t) = \left(1 + \beta \ln \left(\frac{C(t)}{C_0} \right) \right), \quad (6.14)$$

$$\tau_D = \tau_{D0} \left(1 - \frac{\Delta T_s}{20} \right), \quad (6.15)$$

and we take $\tau_{D0} = 50$ yrs, $\beta = 0.75$, $\Delta T_s = T'_{mix}$, and $C_0 \approx 277$ ppm; $C(t)$ is simply the atmospheric CO₂ value in ppm (convert from Q_U). Pre-industrial initial conditions are $Q_A(0) = 590$ GtC, $Q_U(0) = 713$ GtC, $Q_L(0) = 35,658$ GtC, $M(0) = 2,400$ GtC, and $C_{bio}(0) = 2,400$ GtC (convert to moles in the implementation).

Figure 5 gives model dynamics when enhanced biomass decay is omitted, i.e. $\tau_D = \tau_{D0}$. You may wish to make sure you can replicate this, before coupling this model with temperature dynamics and enhanced biomass decay.

6.0.1 All together now!

Now, putting everything from the above together, we arrive at a final, master set of differential equations for the extended model! We have the following model variables:

And we have the following system of differential equations:

$$\frac{dQ_A}{dt} = -k_a Q_A + k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U + E_{land}(t) + E_{fossil}(t) - \left(f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D} \right), \quad (6.16)$$

$$\frac{dQ_U}{dt} = k_a Q_U - k_a \frac{k_H}{\delta_a \Lambda(t)} Q_U - k_d Q_U + \frac{k_d}{\delta_d} Q_L, \quad (6.17)$$

$$\frac{dQ_L}{dt} = k_d Q_U - \frac{k_d}{\delta_d} Q_L, \quad (6.18)$$

$$\frac{dM}{dt} = f \frac{C_{bio}}{\tau_G} - \frac{M}{\tau_D} - E_{fossil}(t), \quad (6.19)$$

$$\frac{dC_{bio}}{dt} = -E_{land}(t), \quad (6.20)$$

$$\mu_{mix} \frac{dT'_{mix}}{dt} = -\hat{\lambda} T'_{mix} - \gamma(T'_{mix} - T'_{deep}) + \Delta N(t), \quad (6.21)$$

$$\mu_{deep} \frac{dT'_{deep}}{dt} = \gamma(T'_{mix} - T'_{deep}), \quad (6.22)$$

where

$$\Lambda(t) = 1 + \frac{K_1}{[H^+]} + \frac{K_1 K_2}{[H^+]^2}, \quad (6.23)$$

$$\Delta N(t) = k \ln \left(\frac{C}{C_0} \right), \quad (6.24)$$

$$f(t) = \left(1 + \beta \ln \left(\frac{C(t)}{C_0} \right) \right), \quad (6.25)$$

$$\tau_D = \tau_{D0} \left(1 - \frac{\Delta T_s}{20} \right), \quad (6.26)$$

and $[H^+]$ is the positive root of

$$[H^+]^2 + [H^+] K_1 \left(1 - \frac{Q_U}{Alk} \right) + K_1 K_2 \left(1 - \frac{2Q_U}{Alk} \right) = 0. \quad (6.27)$$

In addition, we must impose two emissions courses, $E_{fossil}(t)$ and $E_{land}(t)$. Baseline parameter values are given in the prior section (and in the temperature response model section).

7 Driving the extended model with historical emissions (10 pts)

Finally, let us drive the complete, integrated model (described in the prior section) with historical emissions (1751 through 2014), and compare our predicted to actual atmospheric CO₂ and temperature anomaly, as shown in Figure 6. Try running the model for a few hundred more years, assuming emissions suddenly cease after 2014, just to see what the carbon removal and temperature anomaly dynamics look like. Play around with the model, look at phase out schedules, etc., the world is now your mathematical oyster.

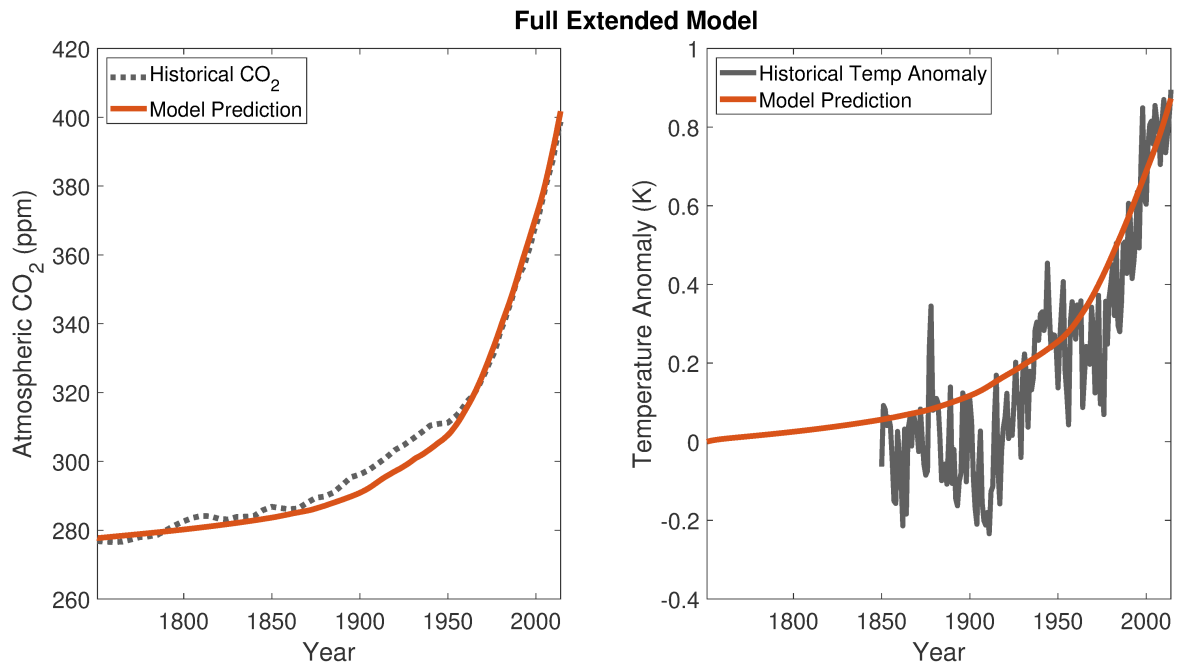


Figure 6: Results for the complete, extended model driven by historical emissions.

8 Extra Credit: Reforestation/Aforestation as “geoengineering” (5 pts EC)

Explore how reforestation or aforestation (growth of forests into previously unforested land) might affect the carbon budget for fossil emissions. It is up to you to decide how to approach this problem: Please explain your methods and results (a figure or two is always nice for results), for extra credit! (Note: to get initial conditions, in this case it is reasonable to simply run the model from 1751 to 2014 as a spin up)

9 Extra Credit: Albedo modification under the extended model (5 pts EC)

Explore albedo modification schemes under the extended model. Present your methods and results. Does including the biota modify any of your conclusions? You could further experiment with combining such a schemes with reforestation/aforestation and/or emissions phase-outs to explore possible futures. This is very open-ended!

References

- [1] Glotter M.J., R.T. Pierrehumbert, J.W. Elliott, N.J. Matteson, & E.J. Moyer. (2014). A simple carbon cycle representation for economic and policy analyses. *Climatic Change*, 126(3–4), 319–335.
- [2] Walker, J., & Kasting, J. (1992). Effects of fuel and forest conservation on future levels of atmospheric carbon dioxide. *Global and planetary change*, 5(3), 151–189.

- [3] Tomizuka, A. (2009). Is a box model effective for understanding the carbon cycle?. American Journal of Physics, 77(2), 156-163.