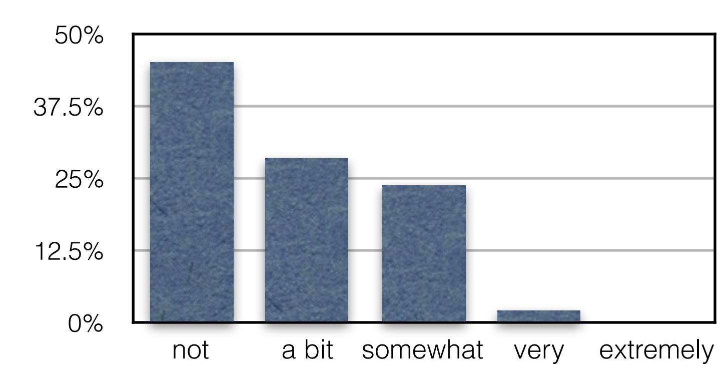
• Muddiest Points from Class 02/22

- *"Are there real world applications associated with the particular explicit methods derived in class. For example, would a certain fluid rate problem require a specific method over another?"*
- *"Will there ever be a point where one method is the best for a majority of the problems […]"*
  - any method that's consistent and stable will work
  - which method is better suited is a tradeoff between cost/complexity to run/code a method and accuracy/order
  - that being said, there's one method that is very popular for parabolic PDEs: Crank-Nicholson (covered today)

- *"Aside from the increase in computation cost, can the requirement that the time step scale down with the space step as a factor of 4 quickly force the time step to be so small that machine precision error becomes the dominant error? Therefore for very small space steps do implicit methods become both more accurate and computationally cheaper?"*
  - theoretically yes, but in practice no
  - in practice one struggles to reach the "asymptotic convergence regime", i.e., where the log/log plot of error vs mesh spacing is a line; precision limit is typically far far away (for double precision)

- *"You kept mentioning about the equation being linear. How far in nature do we see the extent of non linear type flows ?"*
  - heat conduction in fluids is linear process
  - the flow of fluids is non-linear and we will address this in hyperbolic PDEs

- *"The notion that the error spreads everywhere in the mesh seemed unusual to me since we saw in the FTCS method that error diffuses in the next step, n+1, forwards in the mesh. How does the error then propagate back to the nth locations? Or neighboring nth locations?"*
- *"So just to be clear, we will not be using the perturbation method? That just for demonstration?"*
- *"How is it possible to state that In the end, the error will approximately be the same magnitude?"*
- *[…]*
  - Example: Discrete Perturbation analysis for FTCS
    - periodic domain, single error in center of domain, varying time steps



$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Consistency of Du-Fort Frankel

Recap: Consistency ≜ numerical approximation approaches PDE

Du-Fort Frankel

$$\left(1 + 2\alpha \frac{\Delta t}{\Delta x^2}\right) \varphi_i^{n+1} = \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right) \varphi_i^{n-1} + 2\alpha \frac{\Delta t}{\Delta x^2} \left(\varphi_{i+1}^n + \varphi_{i-1}^n\right)$$

- Question: Does this approach the PDE, as $\Delta x, \Delta t \to 0$?

Substitute Taylor series into finite difference form

$$\left(1+2\alpha\frac{\Delta t}{\Delta x^2}\right)\varphi_i^{n+1}=\left(1-2\alpha\frac{\Delta t}{\Delta x^2}\right)\varphi_i^{n-1}+2\alpha\frac{\Delta t}{\Delta x^2}\left(\varphi_{i+1}^n+\varphi_{i-1}^n\right)$$

$$\frac{\partial\varphi}{\partial t}=\alpha\frac{\partial^2\varphi}{\partial x^2}$$

Write Taylor series for each term in the finite difference form

$$\varphi_i^{n+1}=\varphi_i^n+\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+\frac{\Delta t^2}{2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n+\frac{\Delta t^3}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)$$

$$\varphi_i^{n-1}=\varphi_i^n-\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+\frac{\Delta t^2}{2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n-\frac{\Delta t^3}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)$$

$$\varphi_{i+1}^n=\varphi_i^n+\Delta x\left.\frac{\partial\varphi}{\partial x}\right|_i+\frac{\Delta x^2}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i+\frac{\Delta x^3}{6}\left.\frac{\partial^3\varphi}{\partial x^3}\right|_i+\frac{\Delta x^4}{24}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+O(\Delta x^5)$$

$$\varphi_{i-1}^n=\varphi_i^n-\Delta x\left.\frac{\partial\varphi}{\partial x}\right|_i+\frac{\Delta x^2}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i-\frac{\Delta x^3}{6}\left.\frac{\partial^3\varphi}{\partial x^3}\right|_i+\frac{\Delta x^4}{24}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+O(\Delta x^5)$$

Substitute Taylor series into FTCS

$$\left(1+2\alpha\frac{\Delta t}{\Delta x^2}\right)\left(\varphi_i^n+\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+\frac{\Delta t^2}{2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n+\frac{\Delta t^3}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)\right)=$$

$$\left(1-2\alpha\frac{\Delta t}{\Delta x^2}\right)\left(\varphi_i^n-\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+\frac{\Delta t^2}{2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n-\frac{\Delta t^3}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)\right)$$

$$+2\alpha\frac{\Delta t}{\Delta x^2}\left(\varphi_i^n+\Delta x\left.\frac{\partial\varphi}{\partial x}\right|_i+\frac{\Delta x^2}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i+\frac{\Delta x^3}{6}\left.\frac{\partial^3\varphi}{\partial x^3}\right|_i+\frac{\Delta x^4}{24}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+O(\Delta x^5)\right.$$

$$\left.+\varphi_i^n-\Delta x\left.\frac{\partial\varphi}{\partial x}\right|_i+\frac{\Delta x^2}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i-\frac{\Delta x^3}{6}\left.\frac{\partial^3\varphi}{\partial x^3}\right|_i+\frac{\Delta x^4}{24}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+O(\Delta x^5)\right)$$

$$\left(1+2\alpha\frac{\Delta t}{\Delta x^2}\right)\left(\varphi_i^n+\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+\frac{\Delta t^2}{2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n+\frac{\Delta t^3}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)\right)=$$

$$\left(1-2\alpha\frac{\Delta t}{\Delta x^2}\right)\left(\varphi_i^n-\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+\frac{\Delta t^2}{2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n-\frac{\Delta t^3}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)\right)$$
$$+2\alpha\frac{\Delta t}{\Delta x^2}\left(\varphi_i^n+\Delta x\left.\frac{\partial\varphi}{\partial x}\right|_i+\frac{\Delta x^2}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i+\frac{\Delta x^3}{6}\left.\frac{\partial^3\varphi}{\partial x^3}\right|_i+\frac{\Delta x^4}{24}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+O(\Delta x^5)\right.$$
$$\left.+\varphi_i^n-\Delta x\left.\frac{\partial\varphi}{\partial x}\right|_i+\frac{\Delta x^2}{2}\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i-\frac{\Delta x^3}{6}\left.\frac{\partial^3\varphi}{\partial x^3}\right|_i+\frac{\Delta x^4}{24}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+O(\Delta x^5)\right)$$

$$2\Delta t\left.\frac{\partial\varphi}{\partial t}\right|^n+2\alpha\frac{\Delta t^3}{\Delta x^2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n+\frac{\Delta t^3}{3}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^4)=\ 2\alpha\Delta t\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i+2\alpha\Delta t\frac{\Delta x^2}{12}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+2\alpha\frac{\Delta t}{\Delta x^2}O(\Delta x^5)$$

$$\left.\frac{\partial\varphi}{\partial t}\right|^n+\alpha\frac{\Delta t^2}{\Delta x^2}\left.\frac{\partial^2\varphi}{\partial t^2}\right|^n+\frac{\Delta t^2}{6}\left.\frac{\partial^3\varphi}{\partial t^3}\right|^n+O(\Delta t^3)=\alpha\left.\frac{\partial^2\varphi}{\partial x^2}\right|_i+\alpha\frac{\Delta x^2}{12}\left.\frac{\partial^4\varphi}{\partial x^4}\right|_i+\alpha\frac{1}{\Delta x^2}O(\Delta x^5)$$

$$\frac{\partial\varphi}{\partial t}=\alpha\frac{\partial^2\varphi}{\partial x^2}-\alpha\frac{\Delta t^2}{\Delta x^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{\Delta t^2}{6}\frac{\partial^3\varphi}{\partial t^3}+\alpha\frac{\Delta x^2}{12}\frac{\partial^4\varphi}{\partial x^4}+O(\Delta x^3)+O(\Delta t^3)$$

Consistency of Du-Fort Frankel

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Recap: Consistency ≙ numerical approximation approaches PDE

Du-Fort Frankel

$$\left(1 + 2\alpha \frac{\Delta t}{\Delta x^2}\right) \varphi_i^{n+1} = \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right) \varphi_i^{n-1} + 2\alpha \frac{\Delta t}{\Delta x^2} \left(\varphi_{i+1}^n + \varphi_{i-1}^n\right)$$

Modified equation:

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2} - \alpha \frac{\Delta t^2}{\Delta x^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\Delta t^2}{6} \frac{\partial^3 \varphi}{\partial t^3} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 \varphi}{\partial x^4} + O(\Delta x^3) + O(\Delta t^3)$$

- Question: Does this approach the PDE, as $\Delta x, \Delta t \rightarrow 0$?

⇒ consistent only if as $\Delta x \rightarrow 0, \Delta t \rightarrow 0,$ **AND** $\Delta t/\Delta x \rightarrow 0$

if $\Delta t/\Delta x = const = C,$ then the PDE that is solved is

$$\frac{\partial \varphi}{\partial t} + \alpha C \frac{\partial^2 \varphi}{\partial t^2} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Parabolic Equations - Implicit Methods

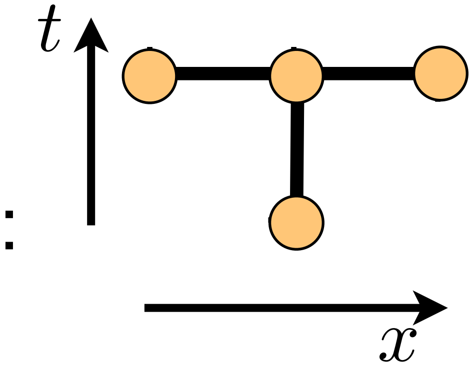
$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Common **implicit** methods (usually preferred for parabolic equations)

Backward time / Laasonen Method (BTCS)

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^{n+1} - 2\varphi_i^{n+1} + \varphi_{i-1}^{n+1}\right)$$

stencil:

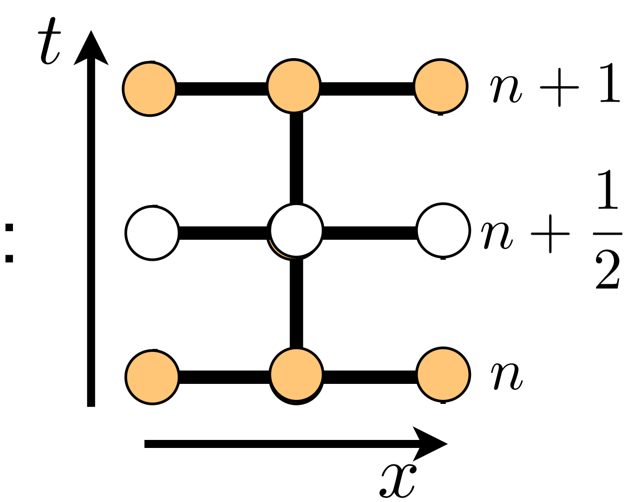


- truncation errors: $O(\Delta t)$ in time, $O(\Delta x^2)$ in space
- always stable

Crank-Nicolson

(very common)

stencil:



Idea: average the right hand side in time to $n+1/2$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{1}{2}\left(f_i^{n+1} + f_i^n\right) = \frac{\alpha}{2}\left(\frac{\varphi_{i+1}^{n+1} - 2\varphi_i^{n+1} + \varphi_{i-1}^{n+1}}{h^2} + \frac{\varphi_{i+1}^{n} - 2\varphi_i^{n} + \varphi_{i-1}^{n}}{h^2}\right)$$

- same as average of FTCS and BTCS
- truncation errors: $O(\Delta t^2)$ in time, $O(\Delta x^2)$ in space

lhs: central difference @ n+1/2
rhs: midpoint average

Crank-Nicolson

(very common)

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Idea: average the right hand side in time to *n+1/2*

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{2}\left(\frac{\varphi_{i+1}^{n+1} - 2\varphi_i^{n+1} + \varphi_{i-1}^{n+1}}{h^2} + \frac{\varphi_{i+1}^{n} - 2\varphi_i^{n} + \varphi_{i-1}^{n}}{h^2} \right)$$

How to code Crank-Nicolson:

- gather all n+1 terms to one side and all other terms to the other

$$a_i \varphi_{i-1}^{n+1} + b_i \varphi_i^{n+1} + c_i \varphi_{i+1}^{n+1} = d_i$$

- incorporate boundary conditions using ghost cell formulas at n and n+1
- solve the resulting tridiagonal system **for interior points** using Gaussian elimination
- use 2 variables: `phi` and `phinew`

Parabolic Equations - Implicit Methods

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Common **implicit** methods (usually preferred for parabolic equations)

Beta-Formulation

Idea: don't just do arithmetic average of n and $n+1$, but use weighted average

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \beta f_i^{n+1} + (1-\beta) f_i^n$$

$\beta = 0$: FTCS
$\beta = 0.5$: Crank-Nicholson
$\beta = 1$: BTCS

$0 \leq \beta \leq 0.5$
conditionally stable

$0.5 \leq \beta \leq 1$
always stable

Parabolic Equations with Source Terms

What changes when we add a source term q to the PDE?

$$\frac{\partial \varphi}{\partial t} = \alpha \, \frac{\partial^2 \varphi}{\partial x^2} \, + q(x, t)$$

- need to evaluate source term q at time t consistent with the chosen method, e.g.,
  - FTCS: evaluate q at t^n
  - BTCS: evaluate q at t^{n+1}
  - Crank-Nicholson: evaluate q at $t^{n+1/2}$ or as average of q evaluated at t^n and t^{n+1}
  - Beta-Formulation: evaluate q as beta average of q evaluate at t^n and t^{n+1}