- Comment on GCI analysis for steady state solutions
  - Common way to reach a steady state solution for time dependent PDEs is to time advance the solution until in discrete form

$$\frac{\partial \phi}{\partial t} < \epsilon$$

  - To perform GCI analysis for spatial discretization errors, must make sure that "non-steady-state" error ϵ is much smaller than spatial errors

- Comment on GCI analysis for unsteady solutions
  - superposition of time and spatial errors
  - GCI analysis as presented can deal with one type of error at a time only
  - for spatial error GCI
    - make temporal errors much smaller than spatial errors (very small Δt)
    - vary mesh spacing h only
  - for temporal error GCI
    - make spatial errors much smaller than temporal errors (very small h)
    - vary time step size only

- Comments on coding parabolic equation solvers
  - Explicit methods have a stable time step limitation

1D: $\Delta t \leq \frac{1}{2}\frac{h^2}{\alpha}$ 2D: $\Delta t \leq \frac{1}{4}\frac{h^2}{\alpha}$

- implement this with a security factor CFL, typically CFL = 0.5 … 0.9

1D: $\Delta t = CFL \cdot \frac{1}{2}\frac{h^2}{\alpha}$ 2D: $\Delta t = CFL \cdot \frac{1}{4}\frac{h^2}{\alpha}$

  - If time step is set by above equations, how to “hit” exactly requested output times?

```
if (time < outputTime) .and. (time + dt >= outputTime) then
    dt = outputTime - time;
    setFlagforOuput;
end if
```

  - code layout for parabolic solver

```
setInitialConditions;
applyBoundaryConditions;
time = intialTime;
while (time < endTime)
    dt = calculateStableTimeStep;
    calculateNewTimeStepSolution;
    applyBoundaryConditions;
    time = time+dt;
    doOutputIfRequired;
end if
```

Hyperbolic Equations

Current status:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu^2\nabla^2 u$$

next

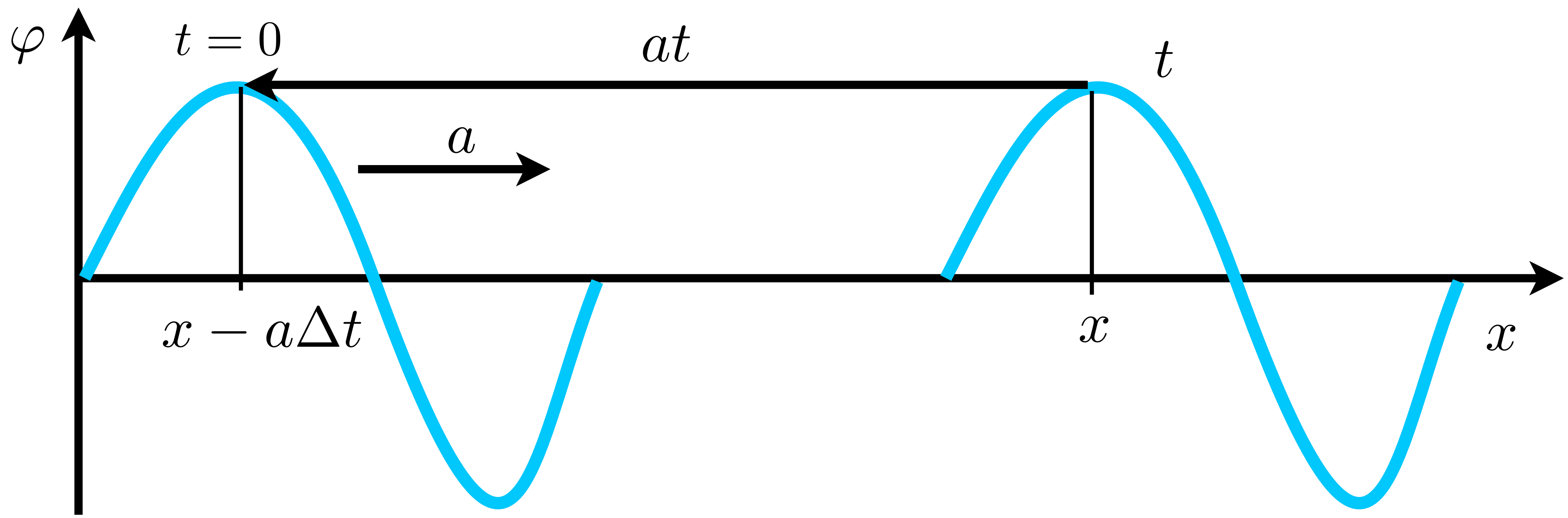
- Convection/Advection:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 \qquad \text{(2D)}$$

- Simplified model equations:
  - 1D non-linear: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$
  - 1D linear: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ with $a \neq f(u)$ (1D wave equation)
- Plan: look at different methods, analyze accuracy, consistency, and stability

Let's start with 1D wave equation

$$\frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} \qquad \text{with} \quad \varphi(x, t=0) = \varphi_0(x) \quad \text{and} \quad a = \text{const.}$$



- signal propagates with constant speed a
- solution: $\varphi(x,t) = \varphi(x - at, t = 0) = \varphi_0(x - at)$

  $\Rightarrow$ Lagrangian or Semi-Lagrangian methods

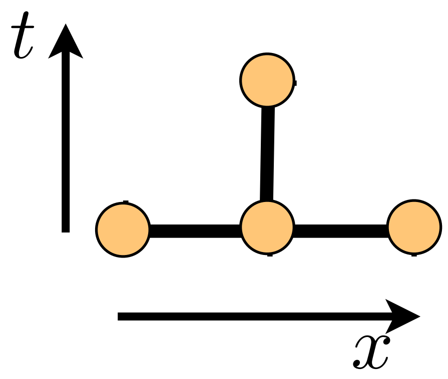
  $\Rightarrow$ Method of Characteristics (see Appendix A of Hoffmann & Chiang)

Linear 1D Wave Equation

Let's follow our standard procedure to go from PDE to FDE

<u>FTCS</u>

stencil:



$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \quad \Rightarrow \quad \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = -a\left(\frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2\Delta x}\right)$$

- Accuracy:
  - from Taylor series: $O(\Delta t)$ and $O(\Delta x^2)$

$$\varphi_i^{n+1} = \varphi_i^n - \frac{a\Delta t}{2\Delta x}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) \qquad \frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x}$$

- Consistency:

Write Taylor series for each term in the finite difference equation

$$\varphi_i^{n+1} = \varphi_i^n + \Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3)$$

$$\varphi_{i+1}^n = \varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + O(\Delta x^4)$$

$$\varphi_{i-1}^n = \varphi_i^n - \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i - \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + O(\Delta x^4)$$

Substitute Taylor series into FTCS

$$\varphi_i^n + \Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3) = \varphi_i^n - \frac{a\Delta t}{2\Delta x}\Bigg(\varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i$$
$$-\varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i - \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + O(\Delta x^4)\Bigg)$$

$$\Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3) = -\frac{a\Delta t}{2\Delta x}\left(2\Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{2\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + O(\Delta x^4)\right) \qquad | : \Delta t$$

$$\boxed{\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x}} - \frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2} - a\frac{\Delta x^2}{6}\frac{\partial^3 \varphi}{\partial x^3} + O(\Delta t^2) + O(\Delta x^3) \qquad \text{as } \Delta t \to 0 \wedge \Delta x \to 0$$

consistent

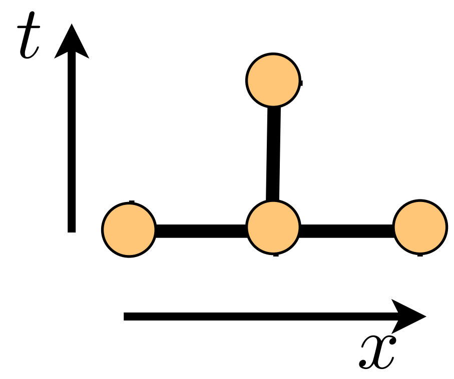
Linear 1D Wave Equation

Let's follow our standard procedure to go from PDE to FDE

FTCS

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \quad \Rightarrow \quad \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = -a\left(\frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2\Delta x}\right)$$

stencil:



- Accuracy:
  - from Taylor series: $O(\Delta t)$ and $O(\Delta x^2)$
- Consistency: consistent
- Stability: Board always unstable

Stability: $\varphi_j^n = \rho^n e^{ikx_j}$

$$\Rightarrow \rho^{n+1} e^{ikx_j} = \rho^n e^{ikx_j} - \frac{a\Delta t}{2\Delta x}\left(\rho^n e^{ik(x_j+\Delta x)} - \rho^n e^{ik(x_j-\Delta x)}\right) \qquad | : e^{ikx_j}$$

$$\Leftrightarrow \rho^{n+1} = \rho^n \left[1 - \frac{a\Delta t}{2\Delta x}\underbrace{\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}_{2i\sin(k\Delta x)}\right]$$

$$\Leftrightarrow \rho^{n+1} = \rho^n \left(1 - \frac{a\Delta t}{\Delta x} i\sin(k\Delta x)\right)$$

$\Rightarrow G = 1 - i\frac{a\Delta t}{2\Delta x}\sin(k\Delta x)$ -> complex number!

complex conjugat

stable if $|G| \leq 1$ -> worst case: $|G| = 1$ => $|G|^2 = 1$ => for complex numbers: $|G|^2 = G \cdot \overline{G}$

let $C = \frac{a\Delta t}{\Delta x}$ => $|G|^2 = (1 - iC\sin(k\Delta x))(1 + iC\sin(k\Delta x)) = 1 + C^2\sin^2(k\Delta x)$

ohoh! since $\Delta x, \Delta t > 0$ => $C > 0$ => $|G|^2 > 1$ => scheme is always <u>unstable</u>

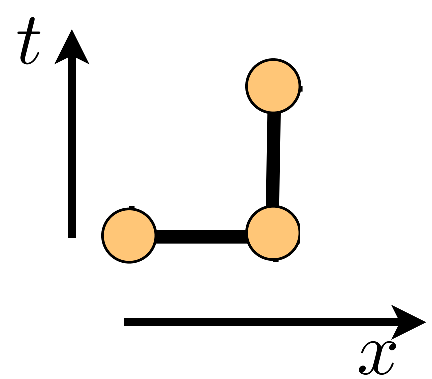
Linear 1D Wave Equation

let's try something else: assume $a > 0$

- use one-sided spatial difference (backwards)

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = -a\left(\frac{\varphi_i^n - \varphi_{i-1}^n}{\Delta x}\right)$$

stencil:

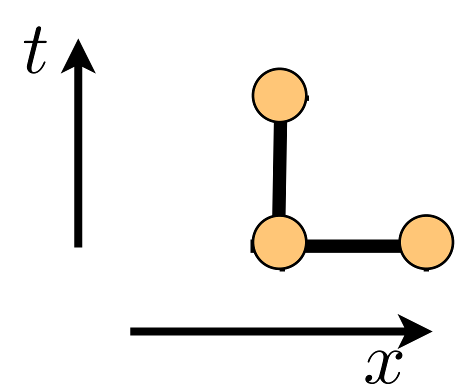


- Accuracy:
  - from Taylor series: $O(\Delta t)$ and $O(\Delta x)$

- Stability: Board $\boxed{C \leq 1 \quad \text{or} \quad \frac{a\Delta t}{\Delta x} \leq 1}$ C : Courant number
  - need to respect direction of information travel!
  - for $a > 0$: information travels from left to right
    $\Rightarrow$ spatial difference must be "upstream" or **upwind**
  - for $a < 0$: information travels from right to left

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = -a\left(\frac{\varphi_{i+1}^n - \varphi_i^n}{\Delta x}\right)$$

stencil:



- Stability: $\varphi_i^{n+1} = \varphi_i^n - C(\varphi_i^n - \varphi_{i-1}^n)$

$$g^{n+1} e^{ikx_j} = g^n e^{ikx_j} - C\left(g^n e^{ikx_j} - g^n e^{ik(x_j-\Delta x)}\right) \quad \Big| : e^{ikx_j}$$

$$g^{n+1} = g^n\left(1 - C\left(1 - e^{-ik\Delta x}\right)\right)$$

$$\Rightarrow G = 1 - C(1 - e^{-ik\Delta x}) = 1 - C + C(\cos(k\Delta x) - i\sin(k\Delta x))$$

$$= 1 + C[\cos(k\Delta x) - 1] - iC\sin(k\Delta x)$$

worst case: $|G|^2 = 1 \Rightarrow \left(1 + C[\cos(k\Delta x) - 1]\right)^2 + C^2\sin^2(k\Delta x) = 1$

$$\Rightarrow 1 + 2C(\cos(k\Delta x) - 1) + C^2\left(\underbrace{\cos^2(k\Delta x) - 2\cos(k\Delta x) + 1) + C^2\sin^2(k\Delta x)}_{=1}\right) = 1$$

$$\Leftrightarrow 1 + 2C\cos(k\Delta x) - 2C + C^2 - 2C^2\cos(k\Delta x) + C^2 = 1$$

$$\Leftrightarrow 1 - 2C(1 - \cos(k\Delta x)) + 2C^2(1 - \cos(k\Delta x)) = 1 \qquad \Big| -1 \;,\; \Big| : (1 - \cos(k\Delta x)) \;,\; \Big| : C$$

$$-2 + 2C = 0 \quad \Rightarrow \quad C = 1 \quad \Rightarrow \text{worst case!}$$

stable if: $C \leq 1 \quad : \quad \frac{a\Delta t}{\Delta x} < 1$

Courant-number, CFL.