Second Model Problem: Parabolic Equations

- 1D heat equation

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{with} \qquad \varphi = \varphi(x,t)$$

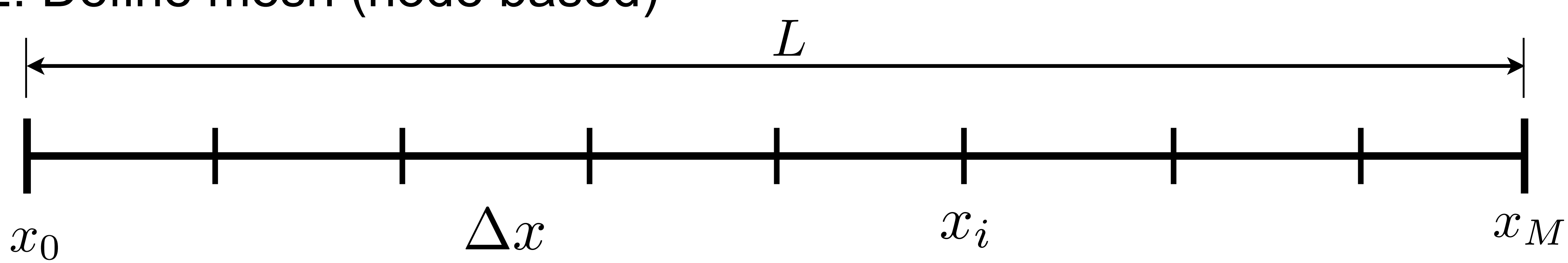
boundary conditions: $\varphi(x=0,t) = \varphi(x=L,t) = 0$

initial condition: $\varphi(x,t=0) = g(x)$

Step 1: Define solution domain

$$0 \leq x \leq L$$

Step 2: Define mesh (node based)



$$\Delta x = h = \frac{L}{M} \qquad\qquad x_i = ih, \quad i = 0 \ldots M$$

Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 3: Approximate spatial derivatives

for example: 2nd-order central:

$$\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2} + O(\Delta h^2)$$

Step 4: Substitute into PDE

$$\left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right)$$ ⇒ now an ODE!

Step 5: Incorporate boundary conditions

$$\varphi(x=0,t) = \varphi(x=L,t) = 0 \quad \Rightarrow \quad \varphi_0 = \varphi_M = 0$$

Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 6: Matrix form (only for illustration, never code!)

$$\left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right)$$

$$\frac{d}{dt}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \frac{\alpha}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix}$$

$$\frac{d\vec{\varphi}}{dt} = A\vec{\varphi}$$

⇒ semi-discrete form ⇒ many ODEs

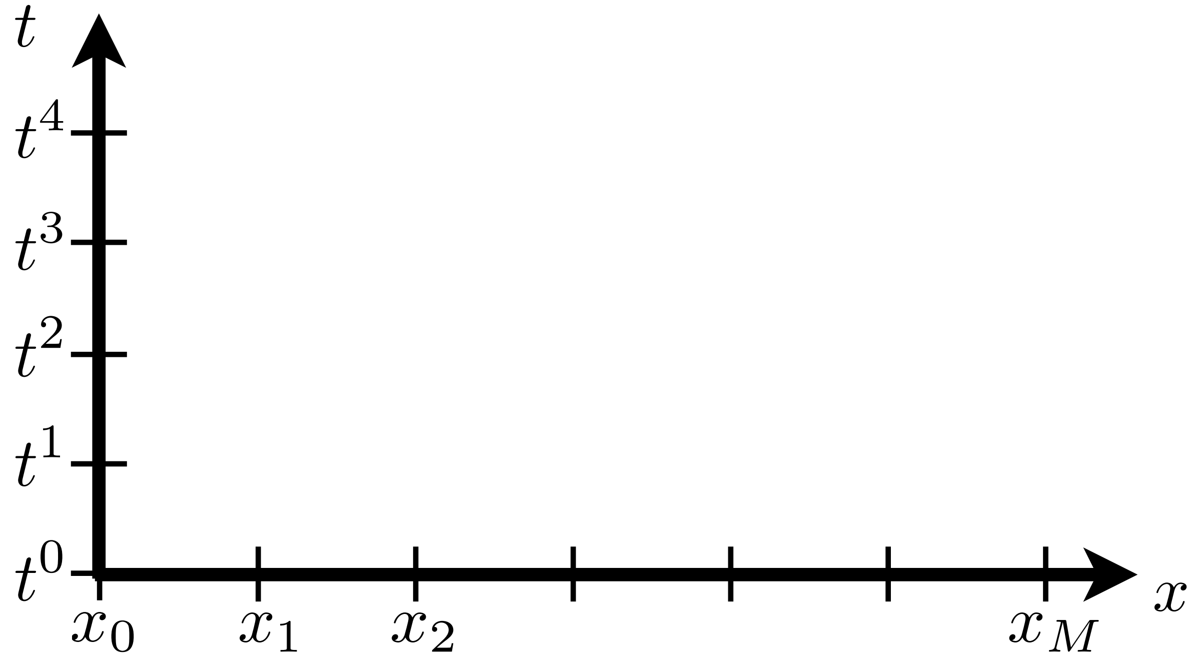
never solve this directly

Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

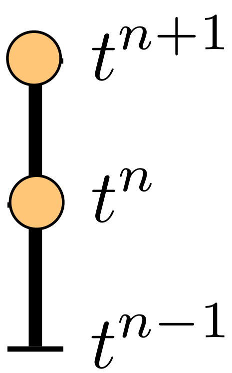
- discretize in time: $t^n = n\Delta t\,, \qquad n = 0, 1, 2, \ldots$



- use finite difference approximation for $\left.\dfrac{d\varphi}{dt}\right|_i$
  - for example: 1st-order forward

$$\left.\frac{d\varphi}{dt}\right|_i^n = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

stencil:



Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

- substitute into ODE

$$\left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right) \qquad \left.\frac{d\varphi}{dt}\right|_i^{n} = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

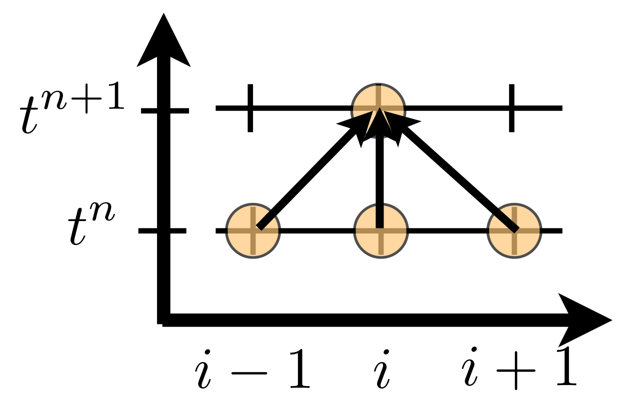
$$\boxed{\varphi_i^{n+1} = \varphi_i^n + \frac{\alpha \Delta t}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)}$$

FTCS <u>F</u>orward <u>T</u>ime <u>C</u>entral <u>S</u>pace

- FTCS expresses a single unknown, φ_i^{n+1}, as a function of only knowns!

  ⇒ feature of **explicit** methods

  solution at t^{n+1} depends only on solution at t^n

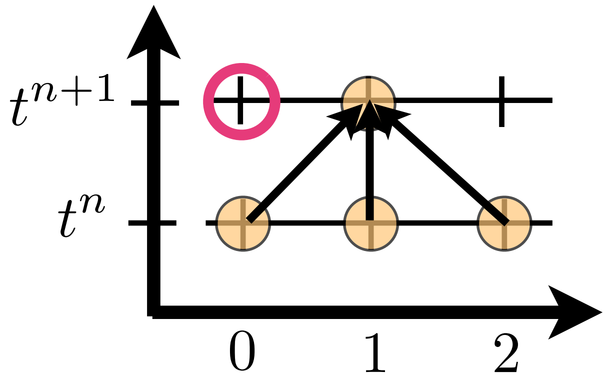


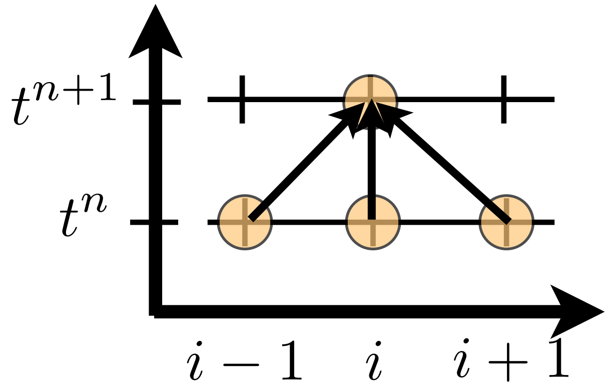
Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

- BUT: problem at boundary





boundary point (bc) does not influence the solution at same t !

- boundaries lag by one time step
- this violates characteristics of parabolic equations

Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

- Alternative: use backwards time difference: **Laasonen Method (BTCS)**

$$\left.\frac{d\varphi}{dt}\right|_i^{\boxed{n+1}} = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) \qquad \left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right)$$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^{\boxed{n+1}} - 2\varphi_i^{\boxed{n+1}} + \varphi_{i-1}^{\boxed{n+1}}\right)$$

Problem: no longer explicit, but now implicit

- gather all $n+1$ terms on left hand side

$$\boxed{\frac{\alpha \Delta t}{h^2}\varphi_{i-1}^{n+1} - \left(1 + 2\frac{\alpha \Delta t}{h^2}\right)\varphi_i^{n+1} + \frac{\alpha \Delta t}{h^2}\varphi_{i+1}^{n+1} = -\varphi_i^n}$$

$$\Rightarrow \quad a_i^n \varphi_{i-1}^{n+1} + b_i^n \varphi_i^{n+1} + c_i^n \varphi_{i+1}^{n+1} = d_i^n$$

⇒ tri-diagonal system

⇒ solve directly using Gauss (see Class 5)

⇒ much more work than FTCS! So, what's the benefit?

⇒ need to discuss accuracy, stability, and consistency

- Definitions:

1. **Consistency**: numerical approximation approaches PDE as $\Delta x, \Delta y, \Delta t \to 0$

2. **Stability**: numerical solution remains bounded

3. **Convergence**: numerical solution approaches PDE solution as $\Delta x, \Delta y, \Delta t \to 0$

turns out if 1. and 2. are true, then 3. is true for linear, well posed initial value problems

Accuracy

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

- both FTCS and BTCS use

$$\left.\frac{d\varphi}{dt}\right|_i^n \approx \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

$$\left.\frac{d\varphi}{dt}\right|_i^{n+1} \approx \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

Taylor series:

$$\varphi_i^{n+1} = \varphi_i^n + \Delta t \left.\frac{\partial \varphi_i}{\partial t}\right|^n + O(\Delta t^2)$$

$$\varphi_i^n = \varphi_i^{n+1} - \Delta t \left.\frac{\partial \varphi_i}{\partial t}\right|^{n+1} + O(\Delta t^2)$$

$$\left.\frac{d\varphi}{dt}\right|_i^n = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) + O(\Delta t)$$

$$\left.\frac{d\varphi}{dt}\right|_i^{n+1} = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) + O(\Delta t)$$

both are first order in time

Consistency

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Consistency ≜ numerical approximation approaches PDE

Example: FTCS

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

- Question: Does this approach the PDE, as $\Delta x, \Delta t \rightarrow 0$?

Consistency

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

Need Taylor series for each term in FTCS

$$\varphi_i^{n+1} = \varphi_i^n + \Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3)$$

$$\varphi_{i+1}^n = \varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5)$$

$$\varphi_{i-1}^n = \varphi_i^n - \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i - \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5)$$

Substitute Taylor series into FTCS

$$\frac{1}{\Delta t}\left(\varphi_i^n + \Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3) - \varphi_i^n\right) = \frac{\alpha}{\Delta x^2}\left(\varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i\right.$$

$$\left. + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5) - 2\varphi_i^n + \varphi_i^n - \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i - \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5)\right)$$

$$\frac{\partial \varphi}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2} + O(\Delta t^2) = \alpha\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\Delta x^2}{12}\frac{\partial^4 \varphi}{\partial x^4} + O(\Delta x)^3\right)$$

Consistency

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Consistency ≜ numerical approximation approaches PDE

<u>Example: FTCS</u>

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

Modified equation:

$$\frac{\partial \varphi}{\partial t} + \cancel{\frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2}} + \cancel{O(\Delta t^2)} = \alpha\left(\frac{\partial^2 \varphi}{\partial x^2} + \cancel{\frac{\Delta x^2}{12}\frac{\partial^4 \varphi}{\partial x^4}} + \cancel{O(\Delta x)^3}\right)$$

- Question: Does this approach the PDE, as $\Delta x, \Delta t \rightarrow 0$?

as $\Delta x, \Delta t \rightarrow 0$: $\dfrac{\partial \varphi}{\partial t} = \alpha \dfrac{\partial^2 \varphi}{\partial x^2}$ ⇒ original PDE ⇒ FTCS is consistent

- Similar analysis shows that BTCS is consistent, too

- Definitions:

1. **Consistency**: numerical approximation approaches PDE as $\Delta x, \Delta y, \Delta t \to 0$

2. **Stability**: numerical solution remains bounded

3. **Convergence**: numerical solution approaches PDE solution as $\Delta x, \Delta y, \Delta t \to 0$

turns out if 1. and 2. are true, then 3. is true for linear, well posed initial value problems

Stability

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Consistency is not enough. We also need stability!

Stability:

- all methods introduce errors! What type of errors?
  - round-off errors (finite precision, ...)
  - discretization errors (truncation errors: Taylor series, method, ...)
- Question is what happens with these error?
  - if the errors grow ⇒ **unstable solution**

⇒ We need to understand and control errors ⇒ **stability analysis**

- Two options:
  - discrete perturbation analysis
  - von Neumann stability analys
- these are doable for model equations, but for full equations often too cumbersome ⇒ numerical experiments

Discrete Perturbation Analysis

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Idea: Introduce a perturbation (error) at one point and study its effect on its neighbors

- if perturbation decreases ⇒ stable
- if perturbation increases ⇒ unstable

Example: FTCS

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

- introduce at point *i* a perturbation (error) *ε* at t^n: $\varphi_i^n \rightarrow \varphi_i^n + \varepsilon$

$$\frac{\varphi_i^{n+1} - (\varphi_i^n + \varepsilon)}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^n - 2(\varphi_i^n + \varepsilon) + \varphi_{i-1}^n\right)$$

- for simplicity, let's assume all $\varphi^n = 0$

$$\frac{\varphi_i^{n+1} - \varepsilon}{\Delta t} = -2\frac{\alpha}{h^2}\varepsilon \quad \Leftrightarrow \quad \varphi_i^{n+1} = \varepsilon\left(1 - 2\frac{\alpha \Delta t}{h^2}\right) \quad \Leftrightarrow \quad \boxed{\frac{\varphi_i^{n+1}}{\varepsilon} = (1 - 2B)}$$

$$B = \frac{\alpha \Delta t}{h^2}$$

Discrete Perturbation Analysis

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

This was of course a bit simplistic, because in general $\varphi^n \neq 0$

- let's drop this simplification
- Question: if we introduce a perturbation ε_i^n, what's the perturbation ε_i^{n+1}?

$$\frac{\varphi_i^{n+1} + \varepsilon_i^{n+1} - (\varphi_i^n + \varepsilon_i^n)}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^n - 2(\varphi_i^n + \varepsilon_i^n) + \varphi_{i-1}^n\right)$$

subtract FTCS from this

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = -2\frac{\alpha}{h^2}\varepsilon_i^n \Leftrightarrow \quad \varepsilon_i^{n+1} = \varepsilon_i^n \left(1 - 2\frac{\alpha \Delta t}{h^2}\right) \Leftrightarrow \boxed{\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = (1 - 2B)}$$

- exact same as before! $\dfrac{\varphi_i^{n+1}}{\varepsilon} = (1 - 2B)$

Discrete Perturbation Analysis

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = (1-2B)$$

For stability, the error must not grow $\Rightarrow \quad \left|\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}\right| \leq 1$

$$\Leftrightarrow \quad |1-2B| \leq 1 \quad \Leftrightarrow \quad -1 \ \leq \ 1-2B \ \leq \ 1$$

right limit:

$$1-2B \leq 1 \quad |-1 \quad \Leftrightarrow \quad -2B \leq 0 \quad \Leftrightarrow \quad B \geq 0 \quad \Leftrightarrow \quad \frac{\alpha \Delta t}{h^2} \geq 0 \quad \checkmark$$

left limit:

$$-1 \leq 1-2B \quad |-1 \quad \Leftrightarrow \quad -2 \leq -2B \quad |:(-2) \quad \Leftrightarrow \quad \boxed{B \leq 1} \quad \Leftrightarrow \quad \frac{\alpha \Delta t}{h^2} \leq 1$$

$$\boxed{\Delta t \leq \frac{h^2}{\alpha}}$$

Discrete Perturbation Analysis

$$\varphi_i^{n+1} = (1-2B)\varepsilon \qquad B = \frac{\alpha \Delta t}{h^2}$$

But this tells us only how the perturbation propagates in time.
What about space?

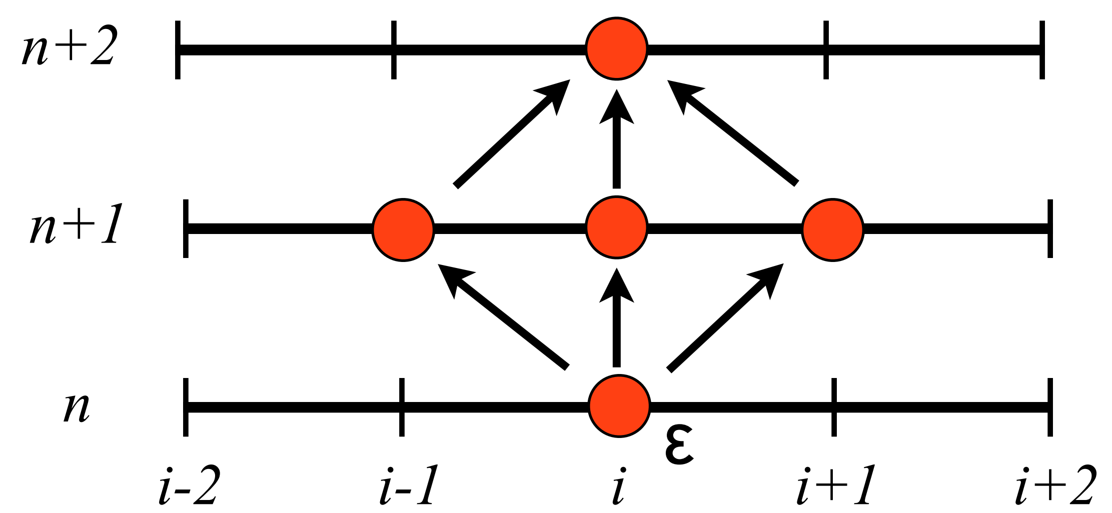
Q: What's the impact on i+1 (at n+1)?

$$\frac{\varphi_{i+1}^{n+1} - \cancel{\varphi_{i+1}^{n}}}{\Delta t} = \frac{\alpha}{h^2}\left(\cancel{\varphi_{i+2}^{n}} - 2\cancel{\varphi_{i+1}^{n}} + (\cancel{\varphi_i^{n}} + \varepsilon_i^n)\right)$$

for simplicity, again $\varphi^n = 0$

$$\varphi_{i+1}^{n+1} = B\varepsilon \qquad \varphi_{i-1}^{n+1} = B\varepsilon$$

error spreads in space!



Q: What's the impact on i (at n+2)?

$$\frac{\varphi_i^{n+2} - \varphi_i^{n+1}}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^{n+1} - 2\varphi_i^{n+1} + \varphi_{i-1}^{n+1}\right) \Leftrightarrow \varphi_i^{n+2} = \varphi_i^{n+1} + B\left(\varphi_{i+1}^{n+1} - 2\varphi_i^{n+1} + \varphi_{i-1}^{n+1}\right)$$

$$\varphi_i^{n+2} = (1-2B)\varepsilon + B\left(B\varepsilon - 2(1-2B)\varepsilon + B\varepsilon\right)$$

$$\varphi_i^{n+2} = \varepsilon\left(1 - 4B + 6B^2\right)$$

Discrete Perturbation Analysis

$$\varphi_i^{n+1} = (1-2B)\varepsilon \qquad B = \frac{\alpha \Delta t}{h^2}$$

$$\varphi_{i+1}^{n+1} = B\varepsilon \qquad \varphi_{i-1}^{n+1} = B\varepsilon$$

$$B \le 1$$

Q: What's the impact on i (at n+2)?

$$\varphi_i^{n+2} = \varepsilon\left(1 - 4B + 6B^2\right)$$

for stability, we must have $\left|\frac{\varphi_i^{n+2}}{\varepsilon}\right| \le 1$

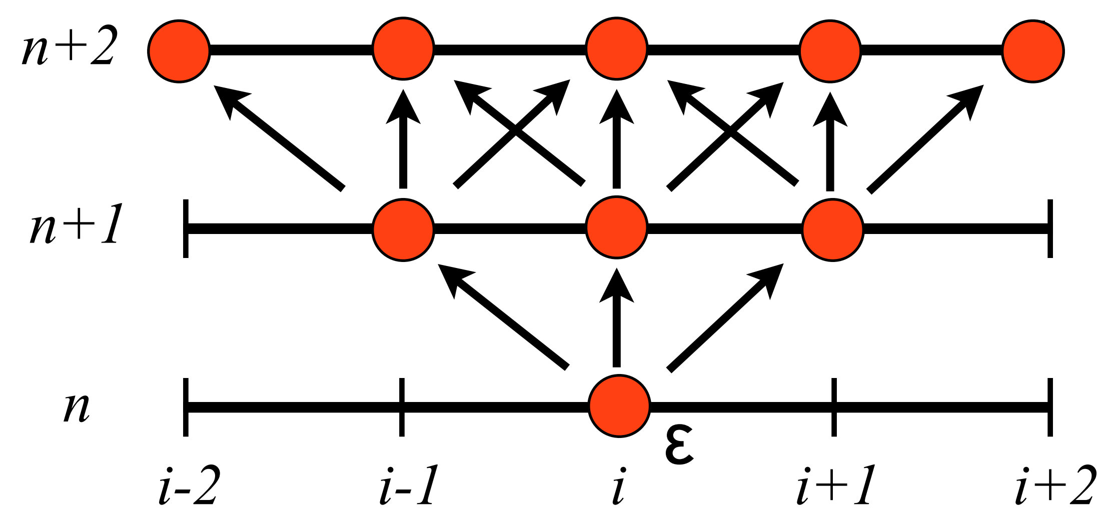
$$\left|1 - 4B + 6B^2\right| \le 1$$

$$1 - 4B + 6B^2 \le 1 \quad \wedge \quad 1 - 4B + 6B^2 \ge -1$$

$$-4B + 6B^2 \le 0 \quad \wedge \quad -4B + 6B^2 \ge -2$$

$$-4 + 6B \le 0 \quad \wedge \quad 2B - 3B^2 \le 1$$

$$\boxed{B \le \frac{2}{3}} \quad \wedge \quad B(2-3B) \le 1 \quad ✓$$

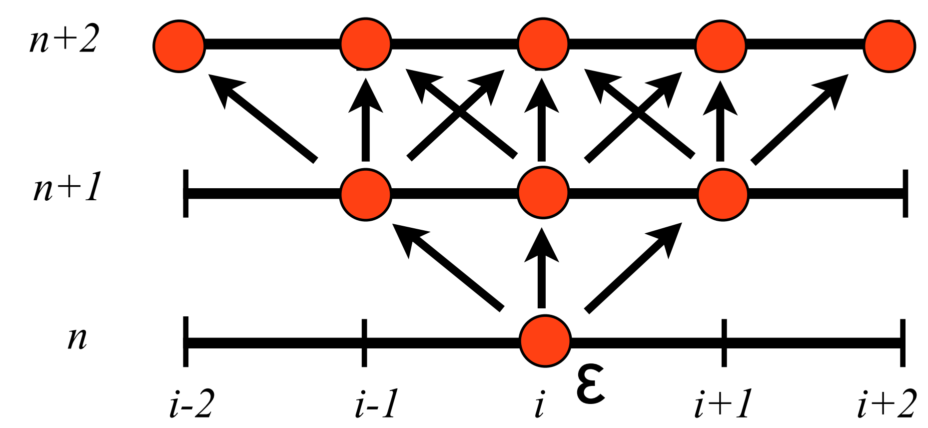


Q: What's the impact on i+1 (at n+2)? $\qquad \varphi_{i+1}^{n+2} = 2B(1-2B)\varepsilon$

Q: What's the impact on i+2 (at n+2)? $\qquad \varphi_{i+2}^{n+2} = B^2\varepsilon$

etc.

Discrete Perturbation Analysis



In the end, the error will reach every grid point with approximately the **same magnitude**

⇒ 2 cases:

- case 1: error has same sign everywhere

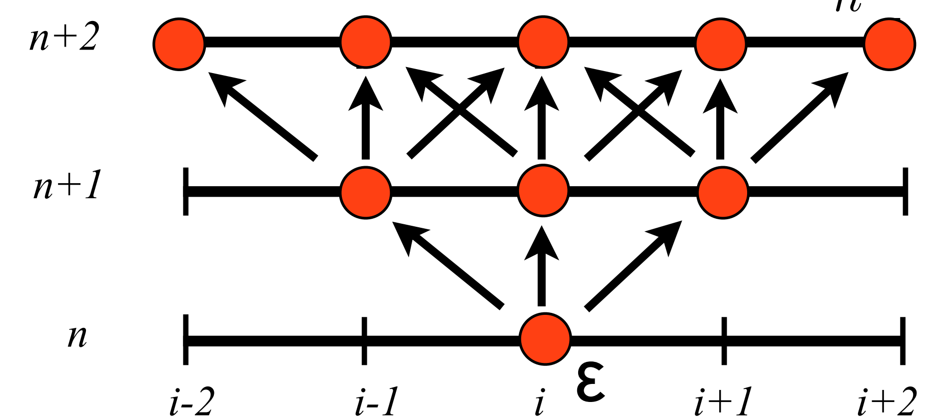
$$\varphi_{i-1}^m = \varepsilon^m \qquad \varphi_i^m = \varepsilon^m \qquad \varphi_{i+1}^m = \varepsilon^m$$

substitute into FTCS: $\dfrac{\varphi_i^{m+1} - \varphi_i^m}{\Delta t} = \dfrac{\alpha}{h^2}\left(\varphi_{i+1}^m - 2\varphi_i^m + \varphi_{i-1}^m\right)$

$$\frac{\varphi_i^{m+1} - \varepsilon^m}{\Delta t} = \frac{\alpha}{h^2}\left(\varepsilon^m - 2\varepsilon^m + \varepsilon^m\right) \quad \Rightarrow \quad \varphi_i^{m+1} = \varepsilon^m$$

Discrete Perturbation Analysis

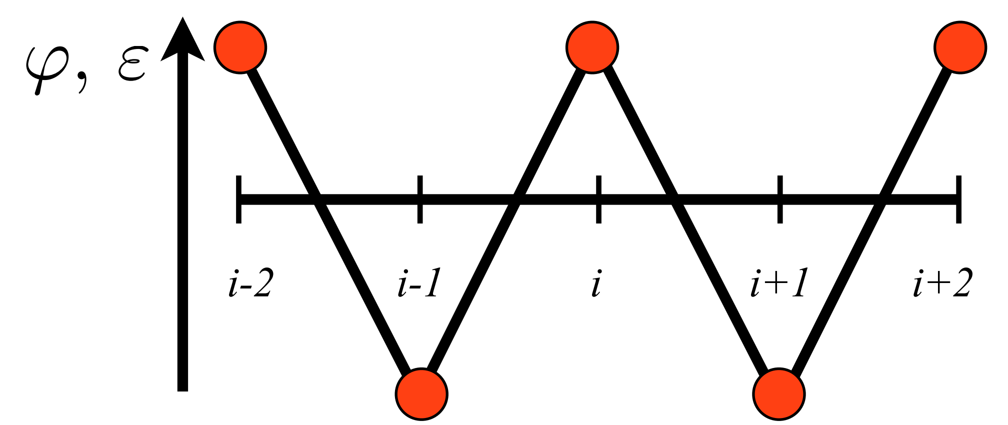
$$B = \frac{\alpha \Delta t}{h^2}$$



In the end, the error will reach every grid point with approximately the **same magnitude**

⇒ 2 cases:

- case 2: error alternates sign



$$\varphi_{i-1}^m = -\varepsilon^m \qquad \varphi_i^m = \varepsilon^m \qquad \varphi_{i+1}^m = -\varepsilon^m$$

substitute into FTCS:

$$\frac{\varphi_i^{m+1} - \varphi_i^m}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^m - 2\varphi_i^m + \varphi_{i-1}^m\right)$$

$$\frac{\varphi_i^{m+1} - \varepsilon^m}{\Delta t} = \frac{\alpha}{h^2}\left(-\varepsilon^m - 2\varepsilon^m - \varepsilon^m\right)$$

$$\varphi_i^{m+1} = \varepsilon^m - 4B\varepsilon^m = \varepsilon^m(1-4B)$$

stable, if $\left|\frac{\varphi_i^{m+1}}{\varepsilon^m}\right| \leq 1$ $\qquad |1-4B| \leq 1$

$1 - 4B \leq 1$ ✓ $\wedge$ $\qquad 1 - 4B \geq -1$

$-4B \leq 0$ $\wedge$ $\qquad B \leq \frac{1}{2}$

$$\boxed{\Delta t \leq \frac{1}{2}\frac{h^2}{\alpha}}$$

but quite tedious on paper.
Useful numerically!
Alternative: von Neumann