von Neumann Stability Analysis

FDE: Finite Difference Equation

Limitations:

- influence of boundary conditions is ignored
- valid only for linear FDEs (if non-linear → linearize locally ⇒ results valid locally only)

- How does FDE respond to a certain type of solution?
  - types of solutions to consider?
    - sinusoidals
      (solutions can be decomposed into sum of sinusoidals by Fourier transform)
    - since we analyze linear FDEs only ⇒ superposition
      ⇒ analysis of single mode is sufficient

$$\varphi_j^n = \rho^n e^{ikx_j}$$

$i = \sqrt{-1}$: imaginary number

k : wave number

ρ : amplitude

$$\Rightarrow \quad \varphi_j^{n+1} = \rho^{n+1} e^{ikx_j} \quad \text{and} \quad \varphi_{j\pm 1}^n = \rho^n e^{ikx_{j\pm 1}}$$

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Example: FTCS

$$\varphi_j^{n+1} = \varphi_j^n + B\left(\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n\right) \qquad B = \frac{\alpha \Delta t}{\Delta x^2}$$

- Substitute in sinusoidal solution:

$$\varphi_j^{n+1} = \rho^{n+1} e^{ikx_j} \qquad \varphi_j^n = \rho^n e^{ikx_j} \qquad \varphi_{j\pm1}^n = \rho^n e^{ikx_{j\pm1}}$$

Board

$$\boxed{\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{\alpha}}$$

- unfortunately it's not always this easy to solve for |G| ≤ 1
  ➙ can use graphical and/or numerical approaches

Example : • FTCS : $$\varphi_j^{n+1} = \varphi_j^n + B(\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n)$$

Substitute in: $$\rho^{n+1} e^{ikx_j} = \rho^n e^{ikx_j} + B(\rho^n e^{ikx_{j+1}} - 2\rho^n e^{ikx_j} + \rho^n e^{ikx_{j-1}}) \quad \Big| : e^{ik}$$

$$\Leftrightarrow \quad \rho^{n+1} = \rho^n + B(\rho^n e^{ik\Delta x} - 2\rho^n + \rho^n e^{-ik\Delta x})$$

$$\Leftrightarrow \quad \rho^{n+1} = \rho^n \left(1 - 2B + B(\underbrace{e^{ik\Delta x} + e^{-ik\Delta x}}_{2\cos(k\Delta x)})\right)$$

$$\Leftrightarrow \quad \rho^{n+1} = \rho^n \left(\underbrace{1 + 2B(\cos(k\Delta x) - 1)}_{G}\right)$$

G: amplification factor : $G = 1 + 2B(\cos(k\Delta x) - 1)$

$$\Leftrightarrow \quad \rho^{n+1} = \rho^n G$$

Stable if $|G| \leq 1$!

$$1 + 2B(\cos(k\Delta x) - 1) \leq 1 \quad \wedge \quad 1 + 2B(\cos(k\Delta x) - 1) \geq -1$$

$$\Leftrightarrow \quad \cos(k\Delta x) \leq 1 \quad (\checkmark) \quad \wedge \quad B(\cos(k\Delta x) - 1) \geq -1 \quad | \cdot (-1)$$

$$B(\underbrace{1 - \cos(k\Delta x)}_{0..2}) \leq 1$$

worst case: 2 $\quad : B(1 - \cos(k\Delta x)) \leq B \cdot 2 \leq 1$

$\Rightarrow \quad B \leq \frac{1}{2}$ $\quad$ same as discrete perturbation method

$\Rightarrow \quad \Delta t \leq \frac{1}{2} \frac{\Delta x^2}{\alpha}$

Unfortunately it's not always this easy to solve for $|G| \leq 1$. → can use graphical and/or numerical approa

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Example: Laasonen (BTCS)

$$\varphi_j^{n+1} = \varphi_j^n + B\left(\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1}\right) \qquad B = \frac{\alpha \Delta t}{\Delta x^2}$$

- Substitute in sinusoidal solution:

$$\varphi_j^{n+1} = \rho^{n+1} e^{ikx_j} \qquad \varphi_j^n = \rho^n e^{ikx_j} \qquad \varphi_{j\pm1}^{n+1} = \rho^{n+1} e^{ikx_{j\pm1}}$$

Board

unconditionally stable

- no time step limit due to stability
- typical of implicit methods

<u>Laasonen</u> : (BTCS):
$$\varphi_j^{n+1} = \varphi_j^n + B\left(\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1}\right)$$

$$\Leftrightarrow \xi^{n+1} e^{ikx_j} = \xi^n e^{ikx_j} + B\left(\xi^{n+1} e^{ikx_{j+1}} - 2\xi^{n+1} e^{ikx_j} + \xi^{n+1} e^{ikx_{j-1}}\right) \quad \Big| : e^{ikx_j}$$

$$\Leftrightarrow \xi^{n+1} = \xi^n + B\xi^{n+1}\Big(\underbrace{e^{ik\Delta x} + e^{-ik\Delta x}}_{2\cos(k\Delta x)} - 2\Big) = \xi^n + 2B\xi^{n+1}\left(\cos(k\Delta x) - 1\right)$$

$$\Leftrightarrow \xi^{n+1}\Big(1 + 2B\big(1 - \cos(k\Delta x)\big)\Big) = \xi^n \qquad \Rightarrow \quad G = \frac{1}{1 + 2B[1 - \cos(k\Delta x)]}$$

Stable if $|G| \le 1$:
$$\frac{1}{1 + 2B[1-\cos(k\Delta x)]} \le 1$$

$$\frac{1}{1 + 2B\underbrace{[1 - \underbrace{\cos(k\Delta x)}_{-1 \dots 1}]}_{0 \dots 2}} \le \frac{1}{1 + 2B \cdot 0} \le \frac{1}{1} \le 1$$
: always true, no matter the B!

$\Rightarrow$ no time step limit due to stability!

$\Rightarrow$ unconditionally stable $\rightarrow$ typical of implicit methods.

von Neumann Stability Analysis

What about 2D?

$$\frac{\partial \varphi}{\partial t} = \alpha \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right)$$

<u>Example: FTCS</u>

$$\varphi_{j,k}^{n+1} = \varphi_{j,k}^{n} + B_x \left(\varphi_{j+1,k}^{n} - 2\varphi_{j,k}^{n} + \varphi_{j-1,k}^{n} \right) + B_y \left(\varphi_{j,k+1}^{n} - 2\varphi_{j,k}^{n} + \varphi_{j,k-1}^{n} \right)$$

$$B_x = \frac{\alpha \Delta t}{\Delta x^2} \qquad B_y = \frac{\alpha \Delta t}{\Delta y^2}$$

- assume solution to be:

$$\varphi_{j,k}^{n} = \rho^n e^{ik_x x_j} e^{ik_y y_k} \qquad \varphi_{j,k}^{n+1} = \rho^{n+1} e^{ik_x x_j} e^{ik_y y_k}$$

$$\varphi_{j\pm1,k}^{n} = \rho^n e^{ik_x x_{j\pm1}} e^{ik_y y_k} \qquad \varphi_{j,k\pm1}^{n} = \rho^n e^{ik_x x_j} e^{ik_y y_{k\pm1}}$$

- substitute in sinusoidal solution: Board

$$\boxed{\Delta t \leq \frac{1}{2\alpha \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right)}}$$

if $\Delta x = \Delta y = h$: $\boxed{\Delta t \leq \frac{1}{4} \frac{h^2}{\alpha}}$

Q: What about 2D? $\frac{\partial \varphi}{\partial t} = \alpha\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)$

FTCS: $$\varphi_{j,\ell}^{n+1} - \varphi_{j,\ell}^{n} = \underbrace{\frac{\alpha \Delta t}{\Delta x^2}}_{B_x}\left(\varphi_{j+1,\ell}^{n} - 2\varphi_{j,\ell}^{n} + \varphi_{j-1,\ell}^{n}\right) + \underbrace{\frac{\alpha \Delta t}{\Delta y^2}}_{B_y}\left(\varphi_{j,\ell+1}^{n} - 2\varphi_{j,\ell}^{n} + \varphi_{j,\ell-1}^{n}\right)$$

assume solution: $\varphi_{j,\ell}^{n} = \rho^n e^{ik_x x_j} e^{ik_y y_\ell}$

Substitute:

$$\rho^{n+1} e^{ik_x x_j} e^{ik_y y_\ell} - \rho^n e^{ik_x x_j} e^{ik_y y_\ell} = B_x\left[\rho^n e^{ik_x x_{j+1}} e^{ik_y y_\ell} - 2\rho^n e^{ik_x x_j} e^{ik_y y_\ell} + \rho^n e^{ik_x x_{j-1}} e^{ik_y y_\ell}\right]$$
$$+ B_y\left[\rho^n e^{ik_x x_j} e^{ik_y y_{\ell+1}} - 2\rho^n e^{ik_x x_j} e^{ik_y y_\ell} + \rho^n e^{ik_x x_j} e^{ik_y y_{\ell-1}}\right] \Big| : e^{ik_x x_j} e^{ik_y y_\ell}$$

$$\Leftrightarrow \rho^{n+1} - \rho^n = B_x\Big(\underbrace{e^{ik_x \Delta x} + e^{-ik_x \Delta x}}_{2\cos(k_x \Delta x)} - 2\Big) + B_y\Big(\underbrace{e^{ik_y \Delta y} + e^{-ik_y \Delta y}}_{2\cos(k_y \Delta y)} - 2\Big)$$

$$\Leftrightarrow \rho^{n+1} = \rho^n \underbrace{\left[1 + 2B_x(\cos(k_x \Delta x) - 1) + 2B_y(\cos(k_y \Delta y) - 1)\right]}_{G}$$

Stable if $|G| \leq 1$:

$$1 + 2B_x(\cos(k_x\Delta x) - 1) + 2B_y(\cos(k_y \Delta y) - 1) \leq 1 \quad \wedge \quad 1 + 2B_x(\cos(k_x \Delta x) - 1) + 2B_y(\cos(k_y \Delta y) - 1) \geq -1$$

$|-1, |k_{x}|:2$ $|+$

$$\underbrace{B_x(\cos(k_x \Delta x) - 1)}_{\leq 0} + \underbrace{B_y(\cos(k_y \Delta y) - 1)}_{\leq 0} \leq 0 \quad \wedge \quad B_x\big(\underbrace{1 - \cos(k_x \Delta x)}_{0..2}\big) + B_y\big(\underbrace{1 - \cos(k_y \Delta y)}_{0..2}\big) \leq 1$$

always true

worst case: $2B_x + 2B_y \leq 1$

$$\Rightarrow B_x + B_y \leq \frac{1}{2} \quad \Leftrightarrow \quad \Delta t \leq \frac{1}{2} \frac{1}{\alpha\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)}$$

if $\Delta x = \Delta y$: $\frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{4}$ or $\Delta t \leq \frac{1}{4}\frac{\Delta x^2}{\alpha}$ $\Rightarrow$ twice as restrictive as 1D! (3D: $B_x + B_y + B_z \leq \frac{1}{2}$