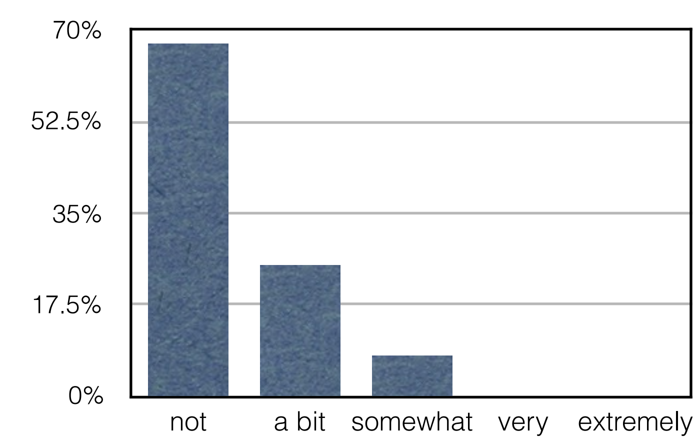
Muddiest Points from Class 04/05

- *“Why do we need different boundaries for our U and V vectors. When allocating the arrays, one goes to M+1, N+2, and the other is M+2,N+1.”*
  - in the x-direction, u coincides with the boundary, while in the y-direction it doesn’t, requiring ghost cells
  - in the y-direction, v coincides with the boundary, while in the x-direction it doesn’t, requiring ghost cells

- *“If we are to visualize the flow using the streamfunction and vorticity, why arent we going to use that techniques to solve the Navier stokes equation?”*
  - we could, however we will learn techniques that are more general and easily applicable to 3D flows as well

- *“How computationally expensive is the vorticity-streamfunction formulation compared to the method we are going to be learning?”*
  - In 2D, the two are quite similar. In 3D, the new methods will always win

- *“Are there any major suggestions for making our code in Homework 10 as applicable to the final project as possible?”*
  - Not really, just use good coding practices: use functions/subroutines for repetitive tasks; don’t hard-code size of the mesh; etc.



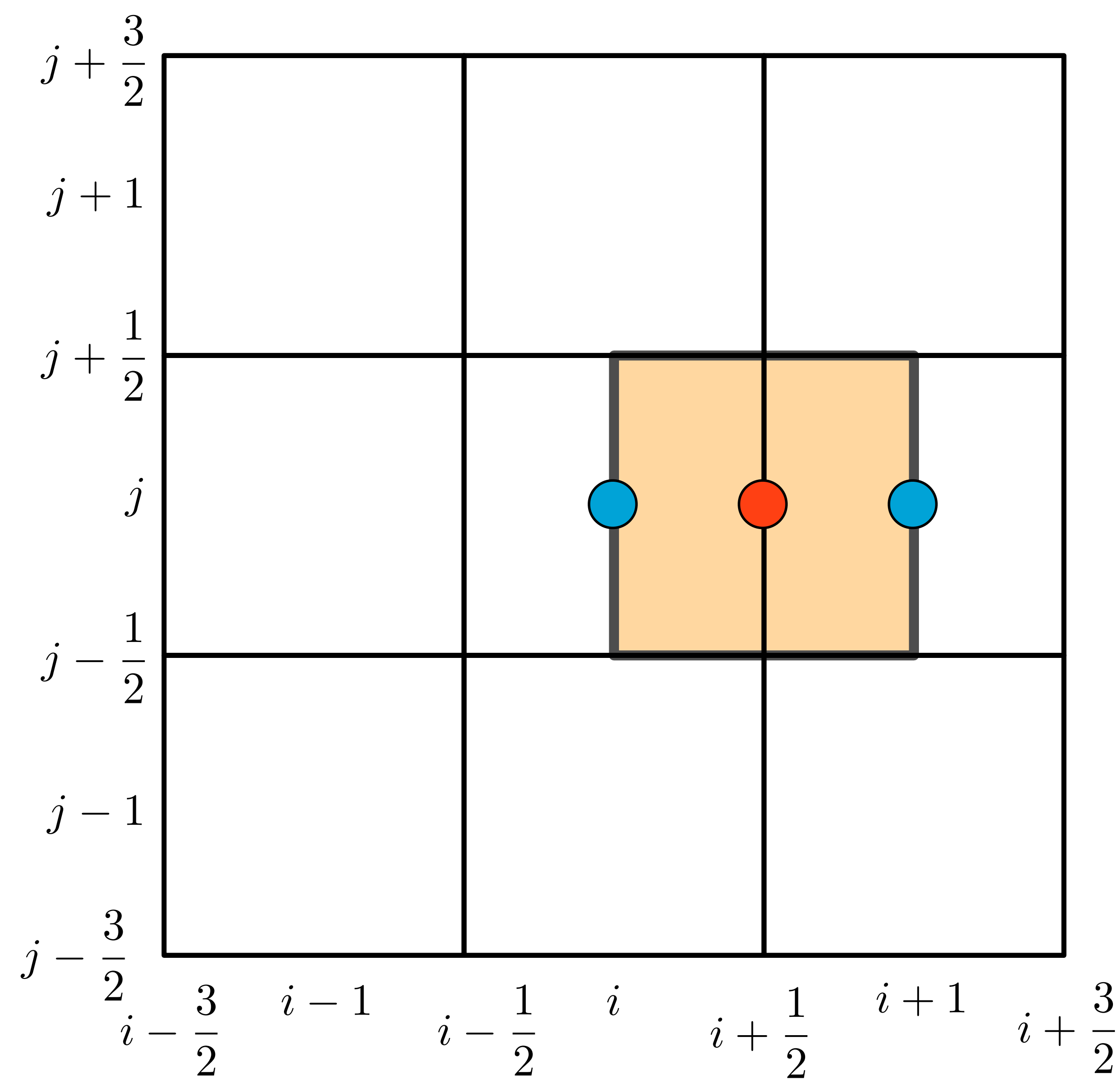
Navier-Stokes on Staggered Meshes

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

solve for $u_{i+1/2,j}$ as before. New term:

$$\left.\frac{\partial p}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x} + O(\Delta x^2)$$

on staggered mesh: define p at cell center!



What's the pressure?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

← solenoidal velocity field

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

← need equation for p

in vector form:

$$\frac{\partial \vec{v}}{\partial t} = \underbrace{-\nabla\cdot(\vec{v}\vec{v})}_{N(\vec{v})} - \nabla p + \frac{1}{\mathrm{Re}}\underbrace{\nabla^2\vec{v}}_{L(\vec{v})}$$

$$\frac{\partial \vec{v}}{\partial t} = N(\vec{v}) - \nabla p + \frac{1}{\mathrm{Re}}L(\vec{v})$$

| take $\nabla\cdot$

$$\frac{\partial}{\partial t}(\nabla\cdot\vec{v}) = \nabla\cdot N(\vec{v}) - \nabla^2 p + \frac{1}{\mathrm{Re}}\nabla\cdot L(\vec{v})$$

use continuity: $\nabla\cdot\vec{v} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}(\nabla\cdot\vec{v}) = 0$

$$\boxed{\nabla^2 p = \nabla\cdot N(\vec{v}) + \frac{1}{\mathrm{Re}}\nabla\cdot L(\vec{v})}$$

pressure Poisson equation!

(in principle $\nabla\cdot L(\vec{v}) = 0$, since L is a linear operator, and $\nabla\cdot$ and L commute) let's substitute everything in: MAC: Marker and Cell Method (Harlow & Welch 1965)

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

use 2nd-order central on staggered mesh

$$\left.\frac{\partial u}{\partial t}\right|_{i+\frac{1}{2},j} = \frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n}_{i+\frac{1}{2},j}}{\Delta t} + O(\Delta t)$$

$$\left.\frac{\partial p}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} + O(\Delta y^2)$$

$$\frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n}_{i+\frac{1}{2},j}}{\Delta t} + \frac{(u^n_{i+1,j})^2 - (u^n_{i,j})^2}{\Delta x} + \frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} = -\frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x} + \\ + \frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j} - 2u^n_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{u^n_{i+\frac{1}{2},j+1} - 2u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\frac{v^{n+1}_{i,j+\frac{1}{2}} - v^{n}_{i,j+\frac{1}{2}}}{\Delta t} + \frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{(v^n_{i,j+1})^2 - (v^n_{i,j})^2}{\Delta y} = -\frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y} + \\ + \frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2} + \frac{v^n_{i,j+\frac{3}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

MAC: Marker and Cell Method

(Harlow & Welch 1965)

$$\frac{u^{n+1}_{i+\frac{1}{2},j} - u^n_{i+\frac{1}{2},j}}{\Delta t} + \frac{(u^n_{i+1,j})^2 - (u^n_{i,j})^2}{\Delta x} + \frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} = -\frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x} + \\ + \frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j} - 2u^n_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{u^n_{i+\frac{1}{2},j+1} - 2u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\frac{v^{n+1}_{i,j+\frac{1}{2}} - v^n_{i,j+\frac{1}{2}}}{\Delta t} + \frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{(v^n_{i,j+1})^2 - (v^n_{i,j})^2}{\Delta y} = -\frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y} + \\ + \frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2} + \frac{v^n_{i,j+\frac{3}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

$$u^{n+1}_{i+\frac{1}{2},j} = u^n_{i+\frac{1}{2},j} + \Delta t\left(\Delta u^n_{i+\frac{1}{2},j} - \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x}\right) \qquad v^{n+1}_{i,j+\frac{1}{2}} = v^n_{i,j+\frac{1}{2}} + \Delta t\left(\Delta v^n_{i,j+\frac{1}{2}} - \frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y}\right)$$

$$\Delta u^n_{i+\frac{1}{2},j} = -\frac{(u^n_{i+1,j})^2 - (u^n_{i,j})^2}{\Delta x} - \frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + \frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j} - 2u^n_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{u^n_{i+\frac{1}{2},j+1} - 2u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\Delta v^n_{i,j+\frac{1}{2}} = -\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} - \frac{(v^n_{i,j+1})^2 - (v^n_{i,j})^2}{\Delta y} + \frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2} + \frac{v^n_{i,j+\frac{3}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

MAC: Marker and Cell Method

(Harlow & Welch 1965)

$$u^{n+1}_{i+\frac{1}{2},j} = u^n_{i+\frac{1}{2},j} + \Delta t\left(\Delta u^n_{i+\frac{1}{2},j} - \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x}\right)$$

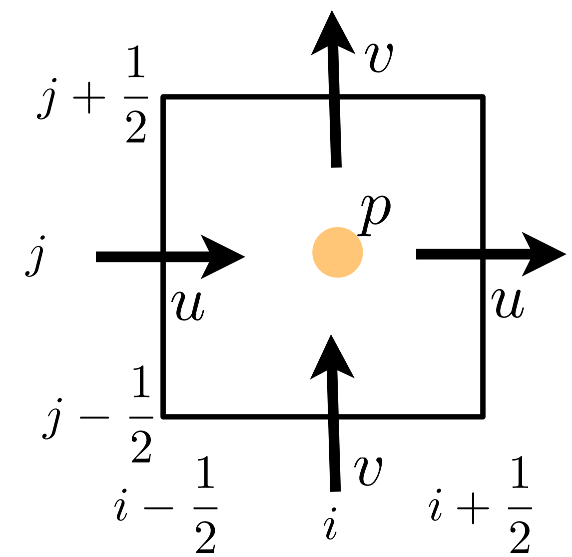
$$v^{n+1}_{i,j+\frac{1}{2}} = v^n_{i,j+\frac{1}{2}} + \Delta t\left(\Delta v^n_{i,j+\frac{1}{2}} - \frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y}\right)$$

but what about pressure? we need it to solve the equations!

- let's enforce the continuity equation at t^{n+1}

$$(\nabla \cdot \vec{v})^{n+1} = 0$$

- but at what spatial location? @ location of pressure!



$$(\nabla \cdot \vec{v})^{n+1}_{i,j} = 0 \qquad \frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n+1}_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^{n+1}_{i,j+\frac{1}{2}} - v^{n+1}_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

$$\frac{u^n_{i+\frac{1}{2},j} + \Delta t\left(\Delta u^n_{i+\frac{1}{2},j} - \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x}\right) - u^n_{i-\frac{1}{2},j} - \Delta t\left(\Delta u^n_{i-\frac{1}{2},j} - \frac{p^n_{i,j} - p^n_{i-1,j}}{\Delta x}\right)}{\Delta x} +$$

$$\frac{v^n_{i,j+\frac{1}{2}} + \Delta t\left(\Delta v^n_{i,j+\frac{1}{2}} - \frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y}\right) - v^n_{i,j-\frac{1}{2}} - \Delta t\left(\Delta v^n_{i,j-\frac{1}{2}} - \frac{p^n_{i,j} - p^n_{i,j-1}}{\Delta y}\right)}{\Delta y} = 0$$

$$\frac{u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^n_{i,j+\frac{1}{2}} - v^n_{i,j-\frac{1}{2}}}{\Delta y} + \Delta t\left(\frac{\Delta u^n_{i+\frac{1}{2},j} - \Delta u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{\Delta v^n_{i,j+\frac{1}{2}} - \Delta v^n_{i,j-\frac{1}{2}}}{\Delta y}\right)$$

$$-\Delta t\left(\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2}\right) = 0$$

MAC: Marker and Cell Method

(Harlow & Welch 1965)

$$\frac{u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^n_{i,j+\frac{1}{2}} - v^n_{i,j-\frac{1}{2}}}{\Delta y} + \Delta t\left(\frac{\Delta u^n_{i+\frac{1}{2},j} - \Delta u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{\Delta v^n_{i,j+\frac{1}{2}} - \Delta v^n_{i,j-\frac{1}{2}}}{\Delta y}\right)$$

$$-\Delta t\left(\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2}\right) = 0$$

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = \frac{\delta_x^2 p^n_{i,j}}{\Delta x^2} + \frac{\delta_y^2 p^n_{i,j}}{\Delta y^2} =$$

$$\frac{1}{\Delta t}\left(\frac{u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^n_{i,j+\frac{1}{2}} - v^n_{i,j-\frac{1}{2}}}{\Delta y}\right) + \frac{\Delta u^n_{i+\frac{1}{2},j} - \Delta u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{\Delta v^n_{i,j+\frac{1}{2}} - \Delta v^n_{i,j-\frac{1}{2}}}{\Delta y}$$

Poisson equation for p with known right hand side!

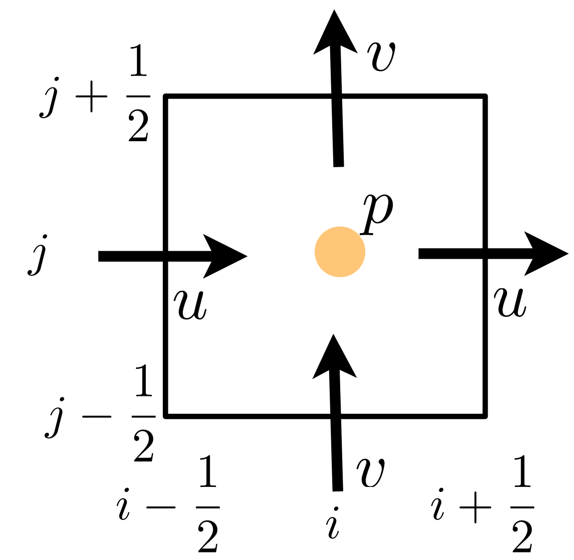
However, right hand side is quite tedious:

$$\Delta u^n_{i+\frac{1}{2},j} = -\frac{(u^n_{i+1,j})^2 - (u^n_{i,j})^2}{\Delta x} - \frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + \frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j} - 2u^n_{i+\frac{1}{2},j} + u^n_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{u^n_{i+\frac{1}{2},j+1} - 2u^n_{i+\frac{1}{2},j} + u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\Delta v^n_{i,j+\frac{1}{2}} = -\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} - \frac{(v^n_{i,j+1})^2 - (v^n_{i,j})^2}{\Delta y} + \frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2} + \frac{v^n_{i,j+\frac{3}{2}} - 2v^n_{i,j+\frac{1}{2}} + v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

MAC: Marker and Cell Method

(Harlow & Welch 1965)



but what about pressure? we need it to solve the equations!

- let's enforce the continuity equation at t^{n+1}

$$(\nabla \cdot \vec{v})^{n+1} = 0$$

- but at what spatial location? @ location of pressure!

$$(\nabla \cdot \vec{v})^{n+1}_{i,j} = 0$$

$$\frac{\delta_x^2 p_{i,j}^n}{\Delta x^2} + \frac{\delta_y^2 p_{i,j}^n}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u^n_{i+\frac{1}{2},j} - u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^n_{i,j+\frac{1}{2}} - v^n_{i,j-\frac{1}{2}}}{\Delta y}\right) + \frac{\Delta u^n_{i+\frac{1}{2},j} - \Delta u^n_{i-\frac{1}{2},j}}{\Delta x} + \frac{\Delta v^n_{i,j+\frac{1}{2}} - \Delta v^n_{i,j-\frac{1}{2}}}{\Delta y}$$

Solution Procedure

- initialize velocity fields
- solve Poisson equation in interior $\Rightarrow p^n$
- solve momentum equations in interior (viscous Burger's + pressure term) $\Rightarrow u^{n+1}, v^{n+1}$

But what about boundary conditions? for velocity: staggered mesh (see Class 23)

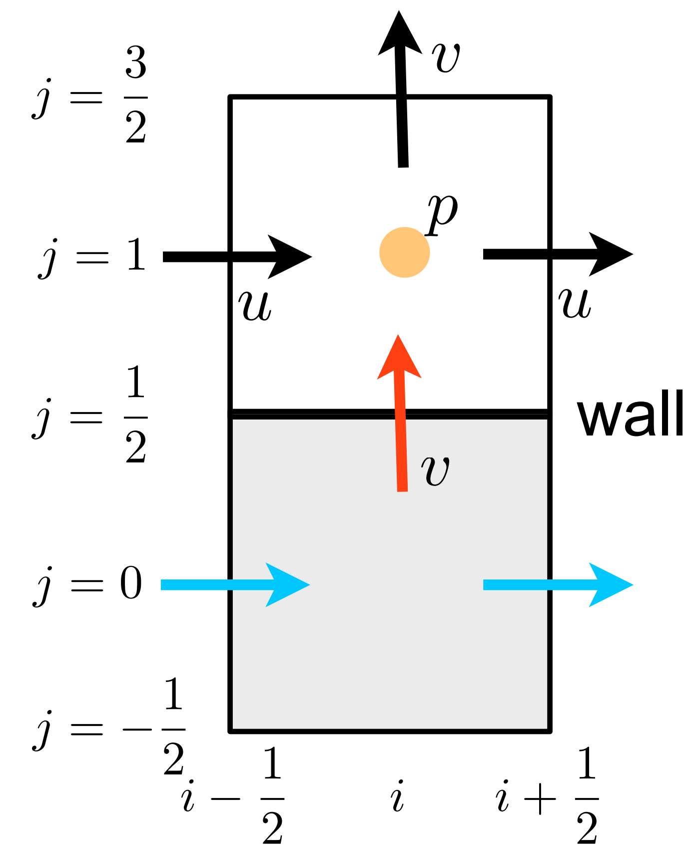
MAC: Marker and Cell Method (Harlow & Welch 1965)

boundary conditions for pressure Poisson equation?

General approach:

1) write divergence constraint at t^{n+1} for all boundary adjacent interior cells in discrete form
2) for all velocities on the boundary, use boundary conditions
3) for all other velocities use momentum equations

Example: bottom wall



- we want $(\nabla\cdot\vec{v})^{n+1}_{i,1}=0$

$$\frac{u^{n+1}_{i+\frac{1}{2},1}-u^{n+1}_{i-\frac{1}{2},1}}{\Delta x}+\frac{v^{n+1}_{i,\frac{3}{2}}-v^{n+1}_{i,\frac{1}{2}}}{\Delta y}=0$$

- use boundary condition for velocity and momentum eqs.

$$\frac{u^{n}_{i+\frac{1}{2},1}-u^{n}_{i-\frac{1}{2},1}}{\Delta x}+\frac{v^{n}_{i,\frac{3}{2}}-v^{n+1}_{i,\frac{1}{2}}}{\Delta y}+$$
$$\Delta t\left[\left(\frac{\Delta u^{n}_{i+\frac{1}{2},1}-\Delta u^{n}_{i-\frac{1}{2},1}}{\Delta x}+\frac{\Delta v^{n}_{i,\frac{3}{2}}}{\Delta y}\right)-\frac{\delta_x^2 p^n_{i,j}}{\Delta x^2}-\frac{p^n_{i,2}-p^n_{i,1}}{\Delta y^2}\right]=0$$

⇒ never need to use ghost cell pressures!

MAC: Marker and Cell Method (Harlow & Welch 1965)

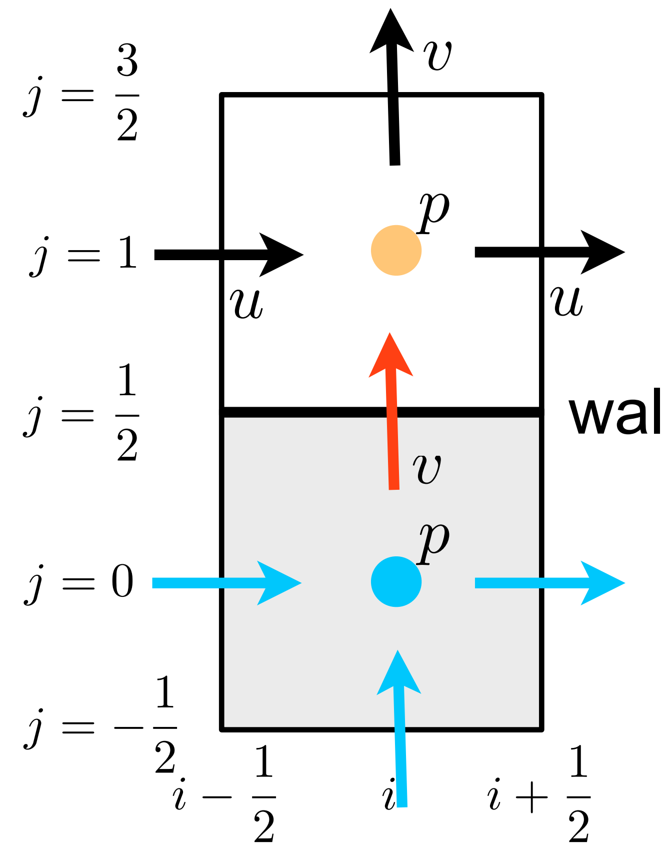
BUT: this requires a change in the code depending on the location of the cell to be updated

⇒ UGLY ⇒ let's try to avoid this by introducing ghost cell pressures

Idea: project momentum equations onto boundary normal direction ***n***

$$[\ .....\ ] \quad \nabla p \cdot \vec{n} = \frac{1}{\mathrm{Re}} \frac{\partial^2 \vec{v}}{\partial l_n^2} \cdot \vec{n}$$

l_n : normal arclength



Example: bottom wall

$$(\nabla p \cdot \vec{n})_{i,\frac{1}{2}} = \left.\frac{\partial p}{\partial y}\right|_{i,\frac{1}{2}} = \frac{p_{i,1} - p_{i,0}}{\Delta y}$$

$$\frac{1}{\mathrm{Re}} \left(\frac{\partial^2 \vec{v}}{\partial l_n^2} \cdot \vec{n} \right)_{i,\frac{1}{2}} = \frac{1}{\mathrm{Re}} \left.\frac{\partial^2 v}{\partial y^2}\right|_{i,\frac{1}{2}} = \frac{1}{\mathrm{Re}} \frac{v_{i,\frac{3}{2}} - 2v_{i,\frac{1}{2}} + v_{i,-\frac{1}{2}}}{\Delta y^2}$$

- we also know the velocity on the boundary and the divergence free constraint!

$v_{i,\frac{1}{2}} = 0$ and $(\nabla \cdot \vec{v})_{i,\frac{1}{2}} = 0$ $\quad u$ @ *wall = const.*

$$\left(\frac{\partial u}{\partial x}\right)_{i,\frac{1}{2}} + \left(\frac{\partial v}{\partial y}\right)_{i,\frac{1}{2}} = 0 \quad \Rightarrow \quad \frac{v_{i,\frac{3}{2}} - v_{i,-\frac{1}{2}}}{2\Delta y} = 0 \quad \Rightarrow \quad v_{i,-\frac{1}{2}} = v_{i,\frac{3}{2}}$$

$$\frac{p_{i,1} - p_{i,0}}{\Delta y} = \frac{1}{\mathrm{Re}} \left(\frac{2v_{i,\frac{3}{2}}}{\Delta y^2} \right) \quad \Rightarrow \quad p_{i,0} = p_{i,1} - \frac{2}{\mathrm{Re}} \frac{v_{i,\frac{3}{2}}}{\Delta y}$$

similar for other boundaries

Fractional Step Method

(Chorin 1965)

Idea: Split momentum equations into separate parts

$\Rightarrow$ since we have no pressure evolution equation, split pressure from the rest:

$$\frac{\partial \vec{v}}{\partial t} = N(\vec{v}) + \frac{1}{\mathrm{Re}} L(\vec{v}) \qquad : \quad \vec{v}^n \rightarrow \vec{v}^*$$

$$\frac{\partial \vec{v}}{\partial t} = -\nabla p \qquad : \quad \vec{v}^* \rightarrow \vec{v}^{n+1}$$

What are the implications of this split?

- Excursion into Linear Algebra:
  - let's introduce the space of all vector functions
  - for incompressible, unsteady flow, the velocity must evolve in the **subspace** of solenoidal functions ($\nabla \cdot \vec{v} = 0$)
  - when we update $\vec{v}^n \rightarrow \vec{v}^*$, then $\vec{v}^*$ is not necessarily in that subspace anymore, even if $\vec{v}^n$ was!
  - How do we get it back into the subspace? **projection**
  - that's exactly what $\vec{v}^* \rightarrow \vec{v}^{n+1}$ does!

Fractional Step Method

Some very important consequences of this thinking:

- p is not determined by a convection/diffusion equation!
- p is solely used to project $\vec{v}^*$ into the subspace of solenoidal functions!

⇒ thus p is not really a pressure, but rather a Lagrange multiplier

⇒ to make this distinction clear, let's call it φ

$$\frac{\partial \vec{v}}{\partial t} = N(\vec{v}) + \frac{1}{\mathrm{Re}} L(\vec{v}) \qquad : \quad \vec{v}^n \to \vec{v}^*$$

$$\frac{\partial \vec{v}}{\partial t} = -\nabla \varphi \qquad : \quad \vec{v}^* \to \vec{v}^{n+1}$$

Fractional Step Method by Kim & Moin (1985)

- step 1: use Adams Bashforth for nonlinear terms and Crank-Nicholson for linear terms

$$\frac{\vec{v}^*_{i,j} - \vec{v}^n_{i,j}}{\Delta t} = \frac{3}{2}\vec{H}^n_{i,j} - \frac{1}{2}\vec{H}^{n-1}_{i,j} + \frac{1}{2}\frac{1}{\mathrm{Re}}\left(\frac{\partial_x^2}{\Delta x^2} + \frac{\partial_y^2}{\Delta y^2}\right)\left(\vec{v}^*_{i,j} + \vec{v}^n_{i,j}\right)$$

⇒ use ADI to solve

- step 2: pressure Poisson equation

$$\frac{\partial \vec{v}}{\partial t} = -\nabla \varphi^{n+1} \qquad \frac{\vec{v}^{n+1} - \vec{v}^*}{\Delta t} = -\mathrm{grad}\ \varphi^{n+1} \qquad |\mathrm{div}(\ldots)$$

$$\frac{\mathrm{div}\ \vec{v}^{n+1} - \mathrm{div}\ \vec{v}^*}{\Delta t} = -\mathrm{div}\left(\mathrm{grad}\ \varphi^{n+1}\right) \qquad \boxed{\mathrm{div}\left(\mathrm{grad}\ \varphi^{n+1}\right) = \frac{1}{\Delta t}\mathrm{div}\left(\vec{v}^*\right)}$$

- step 3: project into solenoidal subspace

$$\boxed{\vec{v}^{n+1} = \vec{v}^* - \Delta t\ \mathrm{grad}\ \varphi^{n+1}}$$

CRUCIAL : You **MUST** use the same discrete grad and div operators in step 2 & 3!

⇒ ∇^2 must be build as div(grad), for example

$$\delta_x^2 = (\varphi_{i+1,j} - \varphi_{i,j}) - (\varphi_{i,j} - \varphi_{i-1,j})$$

Fractional Step Method by Kim & Moin (1985)

But still need to define boundary conditions!

- for step 1: use velocity boundary conditions on staggered mesh
- for step 2: use zero gradient Neumann condition for φ

Neumann for φ: $\varphi_{i,0}^{n+1} = \varphi_{i,1}^{n+1}$

velocity: $v_{i,\frac{1}{2}}^{*} = v_{i,\frac{1}{2}}^{n+1}$