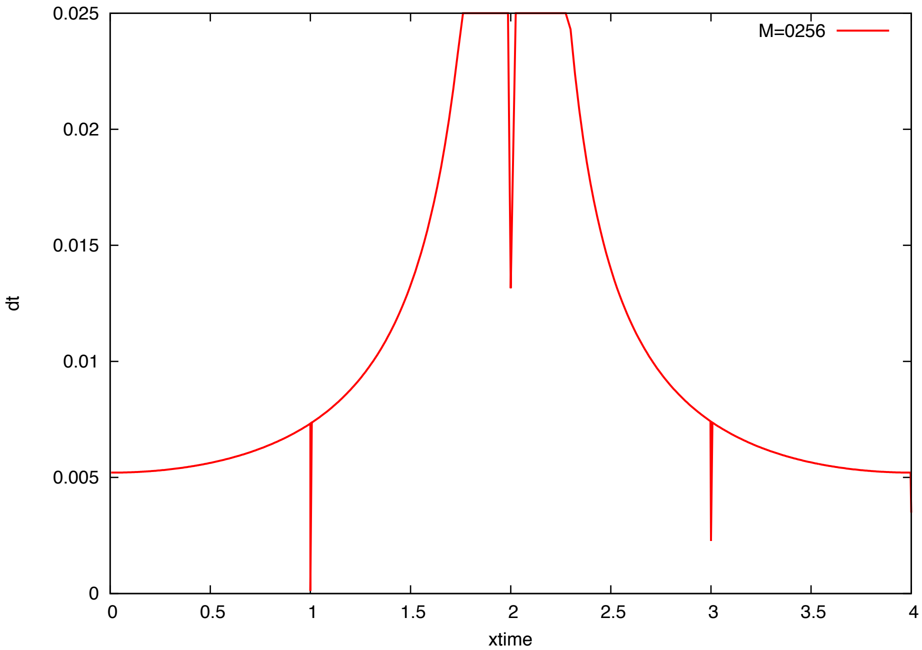
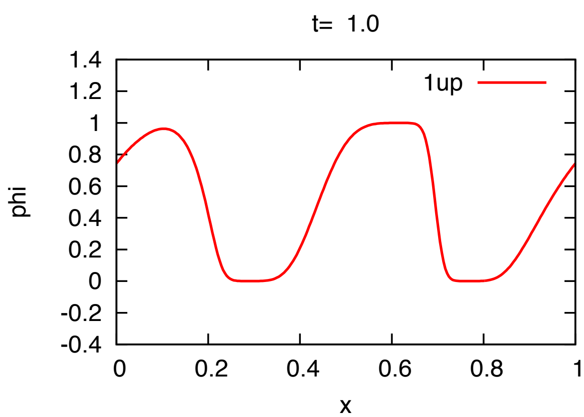
Homework 8 Solution

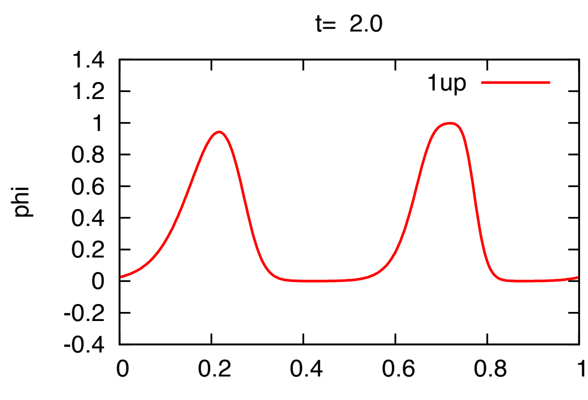
Task 1 (20 points / 8 points) see scan

Task 2 (25 points / 14 points)

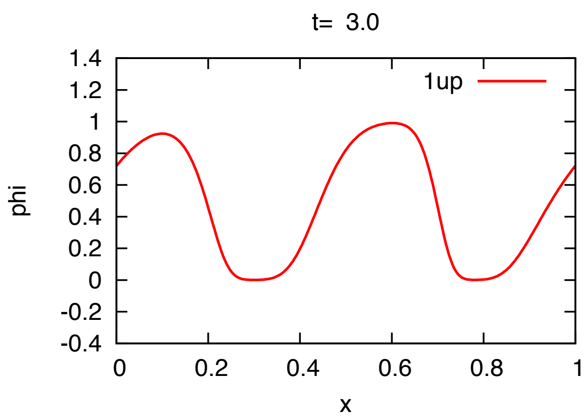


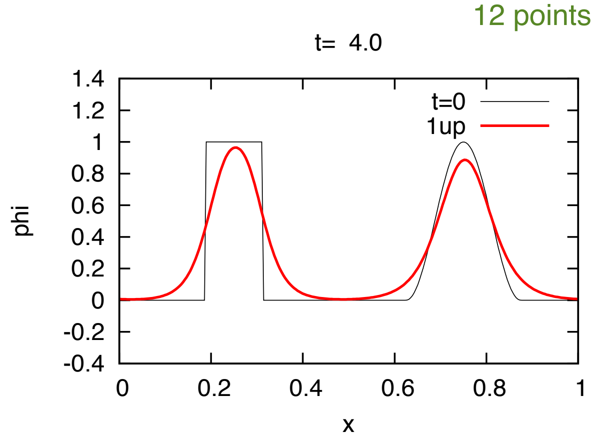
3 points / 2 points





12 points / 8 points





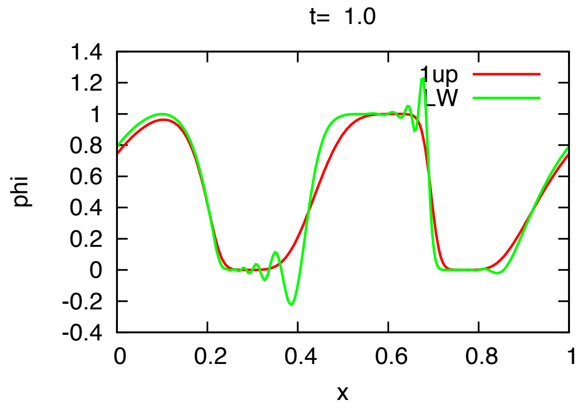
Code: 10 points / 4 points

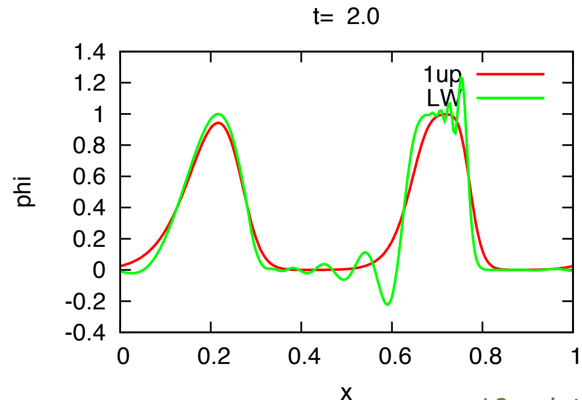
Homework 8 Solution

Task 3 (6 points / 3 points) see scan

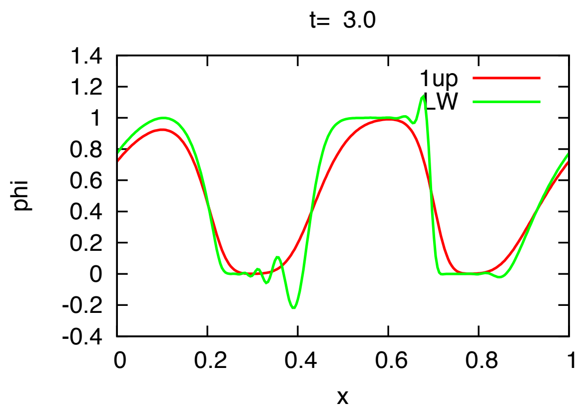
Task 4 (4 points / 2 points) see scan

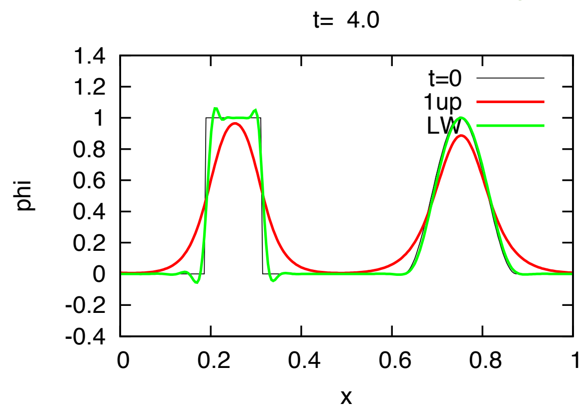
Task 5 (25 points / 12 points) see scan





16 points / 8 points





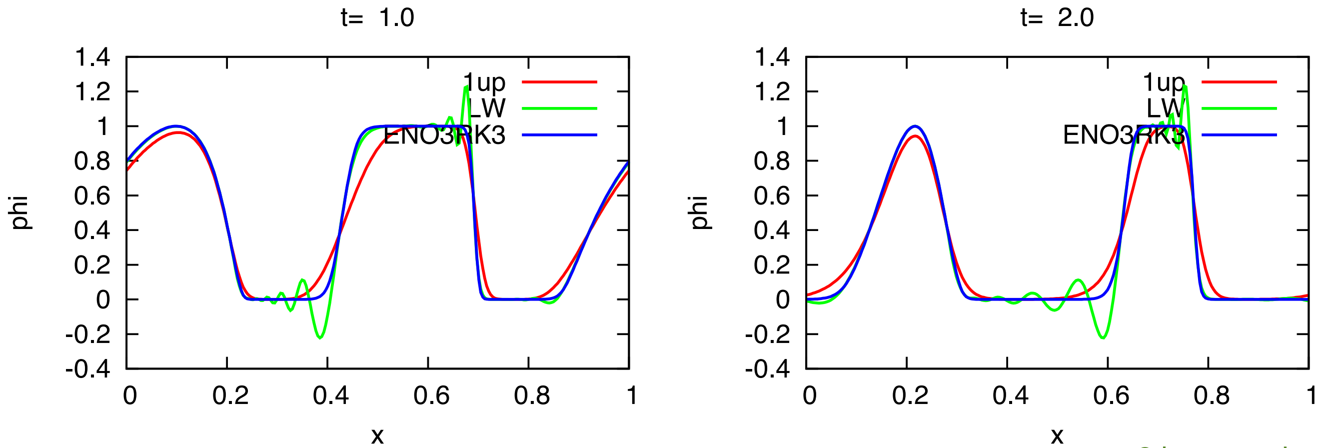
Code: 9 points / 4 points

Task 6 (20 points / 10 points) see scan

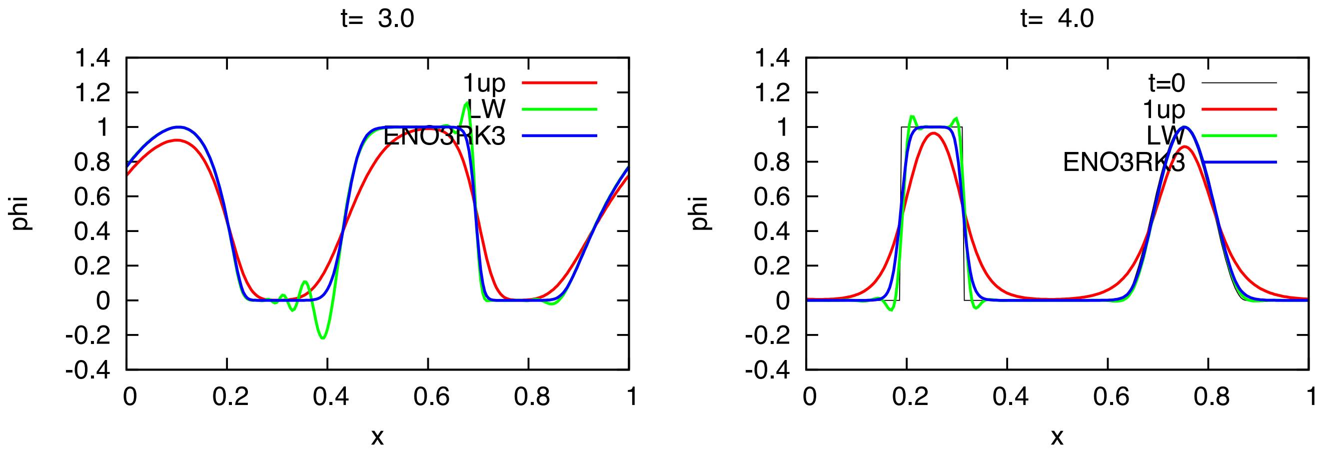
Task 7 (10 bonus points / 25 points) see scan

Homework 8 Solution

Task 8 (13 bonus points / 26 points) see scan



8 bonus points / 16 points



Code: 2 bonus points / 10 points

1) $\boxed{\Sigma 20/8}$

$$\phi_i^{n+1} = \phi_i^n - \frac{a_i^n \Delta t}{\Delta x} \cdot \begin{cases} (\phi_{i+1}^n - \phi_i^n) & \text{if } a_i^n < 0 \\ (\phi_i^n - \phi_{i-1}^n) & \text{if } a_i^n \geq 0 \end{cases}$$ 10/4

$\phi_0^n = \phi_M^n$ 2/1 ; $\phi_{M+1}^n = \phi_1^n$ 2/1 ; $\Delta t_{max} = \dfrac{\Delta x}{\max\limits_{i=1..M} (|a_i^n|)}$ 6/2

$\boxed{\Sigma 6/3}$

3) The solution shows dissipative behaviour, consistent with the modified equation 2/1

from notes: $\dfrac{\partial \phi}{\partial t} = -a \dfrac{\partial \phi}{\partial x} + \dfrac{a \Delta x}{2} (1-c) \dfrac{\partial^2 \phi}{\partial x^2} + \ldots$ 2/1 which has as a leading order error term an even derivative and thus dissipative behaviour. 2/1

$\boxed{\Sigma 4/2}$

4) $\phi_i^{n+1} = \phi_i^n - \dfrac{1}{2} \dfrac{a_i^n \Delta t}{\Delta x} (\phi_{i+1}^n - \phi_{i-1}^n) + \dfrac{1}{2} \left(\dfrac{a_i^n \Delta t}{\Delta x} \right)^2 (\phi_{i+1}^{n+1} - 2\phi_i^n + \phi_{i-1}^n)$ 4/2

$\boxed{\Sigma 20/10}$

6) Lax-Wendroff - modified equation:

T.S.:

$$\cancel{\phi_i^n} + \Delta t \frac{\partial \phi}{\partial t}\Big|_i^n + \frac{\Delta t^2}{2} \frac{\partial^2 \phi}{\partial t^2}\Big|_i^n + \frac{\Delta t^3}{6} \frac{\partial^3 \phi}{\partial t^3}\Big|_i^n + O(\Delta t^4) = \cancel{\phi_i^n} - \frac{a \Delta t}{2 \Delta x} \Big(\cancel{\phi_i^n} + \Delta x \frac{\partial \phi}{\partial x}\Big|_i^n + \cancel{\frac{\Delta x^2}{2} \frac{\partial^2 \phi}{\partial x^2}} - \frac{\Delta x^3}{6} \frac{\partial^3 \phi}{\partial x^3}$$

$$- \cancel{\phi_i^n} + \Delta x \frac{\partial \phi}{\partial x}\Big|_i^n - \cancel{\frac{\Delta x^2}{2} \frac{\partial^2 \phi}{\partial x^2}\Big|_i^n} + \frac{\Delta x^3}{6} \frac{\partial^3 \phi}{\partial x^3}\Big|_i^n + O(\Delta x^4) \Big)$$

$$+ \frac{1}{2} \left(\frac{a^2 \Delta t^2}{\Delta x^2} \right) \Big[\cancel{\phi_i^n} + \cancel{\Delta x \frac{\partial \phi}{\partial x}\Big|_i^n} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi}{\partial x^2}\Big|_i^n + \cancel{\frac{\Delta x^3}{6} \frac{\partial^3 \phi}{\partial x^3}\Big|_i^n} + \frac{\Delta x^4}{24} \frac{\partial^4 \phi}{\partial x^4}\Big|_i^n - \cancel{2\phi_i^n} + \cancel{\phi_i^n} - \cancel{\Delta x \frac{\partial \phi}{\partial x}\Big|_i^n} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi}{\partial x^2}\Big|_i^n -$$

$$\cancel{\frac{\Delta x^3}{6} \frac{\partial^3 \phi}{\partial x^3}\Big|_i^n} - \frac{\Delta x^4}{24} \frac{\partial^4 \phi}{\partial x^4}\Big|_i^n + O(\Delta x^5) \Big] \qquad | : (\Delta t)$$ 8/4

$$\frac{\partial \phi}{\partial t} + a \frac{\partial \phi}{\partial x} = -\frac{\Delta t}{2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\Delta t^2}{6} \frac{\partial^3 \phi}{\partial t^3} - \frac{a \Delta x^2}{6} \frac{\partial^3 \phi}{\partial x^3} + \frac{1}{2} (a^2 \Delta t) \frac{\partial^2 \phi}{\partial x^2} + O(\Delta x^3)$$

use PDE: $\dfrac{\partial^2 \phi}{\partial t^2} = a^2 \dfrac{\partial^2 \phi}{\partial x^2}$ and $\dfrac{\partial^3 \phi}{\partial t^3} = -a^3 \dfrac{\partial^3 \phi}{\partial x^3}$ 2/1

$$\Rightarrow \frac{\partial \phi}{\partial t} + a \frac{\partial \phi}{\partial x} = \left(\frac{a^2 \Delta t}{2} - \frac{a^2 \Delta t}{2} \right) \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{a^3 \Delta t^2}{6} - \frac{a \Delta x^2}{6} \right) \frac{\partial^3 \phi}{\partial x^3} + O(\Delta x^3)$$

$$\Leftrightarrow \frac{\partial \phi}{\partial t} + a \frac{\partial \phi}{\partial x} = \frac{a \Delta x^2}{6}\left(C^2 - 1\right) \frac{\partial^3 \phi}{\partial x^3} + O(\Delta x^3)$$ 6/3

2/1

The solution show dispersive behaviour, consistent with the modified equation which has a leading order error term with an odd-derivative and thus dispersive behaviour. 2/1

$\Sigma: 10/25$

7.)

1/3

$$\frac{\partial \phi}{\partial x}\bigg|_i^- = \begin{cases} \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\phi_{i-3} - \frac{7}{6}\Delta^+\phi_{i-2} + \frac{11}{6}\Delta^+\phi_{i-1}\right) & \text{if } |\Delta^-\Delta^+\phi_{i-1}| < |\Delta^-\Delta^+\phi_i| \,\&\, |\Delta^-\Delta^-\Delta^+\phi_{i-1}| < |\Delta^+\Delta^-\Delta^+\phi \\ \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\phi_{i-1} + \frac{5}{6}\Delta^+\phi_i - \frac{1}{6}\Delta^+\phi_{i+1}\right) & \text{if } |\Delta^-\Delta^+\phi_{i-1}| > |\Delta^-\Delta^+\phi_i| \,\&\, |\Delta^-\Delta^-\Delta^+\phi_i| > |\Delta^+\Delta^-\Delta^+\phi_i| \\ \frac{1}{\Delta x}\left(-\frac{1}{6}\Delta^+\phi_{i-2} + \frac{5}{6}\Delta^+\phi_{i-1} + \frac{1}{3}\Delta^+\phi_i\right) & \text{otherwise} \end{cases}$$

$$\frac{\partial \phi}{\partial x}\bigg|_i^+ = \begin{cases} \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^-\phi_{i+3} - \frac{7}{6}\Delta^-\phi_{i+2} + \frac{11}{6}\Delta^-\phi_{i+1}\right) & \text{if } |\Delta^+\Delta^-\phi_{i+1}| < |\Delta^+\Delta^-\phi_i| \,\&\, |\Delta^+\Delta^+\Delta^-\phi_{i+1}| < |\Delta^-\Delta^+\Delta^-\phi_{i+1}| \\ \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^-\phi_{i+1} + \frac{5}{6}\Delta^-\phi_i - \frac{1}{6}\Delta^-\phi_{i-1}\right) & \text{if } |\Delta^+\Delta^-\phi_{i+1}| > |\Delta^+\Delta^-\phi_i| \,\&\, |\Delta^+\Delta^+\Delta^-\phi_i| > |\Delta^-\Delta^+\Delta^-\phi_i| \\ \frac{1}{\Delta x}\left(-\frac{1}{6}\Delta^-\phi_{i+2} + \frac{5}{6}\Delta^-\phi_{i+1} + \frac{1}{3}\Delta^-\phi_i\right) & \text{otherwise} \end{cases}$$

2/6

RK-3 TVD:

if $a_i^n > 0$: $\frac{\partial \phi}{\partial x}\Big|_i = \frac{\partial \phi}{\partial x}\Big|_i^-$ else $\frac{\partial \phi}{\partial x}\Big|_i = \frac{\partial \phi}{\partial x}\Big|_i^+$ 1/2

1/2

start: $\phi_i^{(0)} = \phi_i^{(n)}$ 1/2

step 1: $\phi_i^{(1)} = \phi_i^{(0)} - a_i^n \Delta t \frac{\partial \phi}{\partial x}\Big|_i^{(0)}$ 1/2

step 2: $\phi_i^{(2)} = \phi_i^{(1)} + \frac{3}{4} a_i^n \Delta t \frac{\partial \phi}{\partial x}\Big|_i^{(0)} - \frac{1}{4} a_i^n \Delta t \frac{\partial \phi}{\partial x}\Big|_i^{(1)}$ 1/2

step 3: $\phi_i^{n+1} = \phi_i^{(2)} + \frac{1}{12} a_i^n \Delta t \frac{\partial \phi}{\partial x}\Big|_i^{(0)} + \frac{1}{12} a_i^n \Delta t \frac{\partial \phi}{\partial x}\Big|_i^{(1)} - \frac{2}{3} a_i^n \Delta t \frac{\partial \phi}{\partial x}\Big|_i^{(2)}$

Ghostcells:

$\phi_{-2} = \phi_{M-2}$; $\phi_{-1} = \phi_{M-1}$; $\phi_0 = \phi_M$ 1/3

$\phi_{M+3} = \phi_3$; $\phi_{M+2} = \phi_2$; $\phi_{M+1} = \phi_1$ 1/3