AEE 471 / MAE 561

Homework #5 - Due: Wednesday, March 4th, at the beginning of class

Combine all code you used to solve the problems into a single text file, and upload the text file to Blackboard using the SafeAssign mechanism. No credit will be given, if your code is not uploaded as a text file using SafeAssign. Also ensure that your code contains adequate comments.

Problem 1 (40 points, AEE471: 10 bonus points for full V-cycle) (AEE471: Core Course Outcomes #2) Program a function/subroutine that solves the two-dimensional Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y) \tag{1}$$

using a dual grid Multigrid method (AEE 471), respective full V-cycle Multigrid (MAE561). AEE471 students may opt for a full V-cycle instead of dual Multigrid for bonus points. On each mesh level, perform one Gauss-Seidel iteration. Use an equidistant **cell-centered mesh** with M interior elements in the x-direction and M interior elements in the y-direction. The mesh spacing in the x- and y-direction is h and ghost cells must be used to enforce the following Neumann boundary condition on all boundaries,

$$\frac{\partial \phi}{\partial n} = 0\,. \tag{2}$$

Your function/subroutine must be able to handle all 3 of the following convergence criteria:

1. fixed number N_{iter} of full V-cycle, respective dual multigrid iterations;
2. infinity norm of the finest mesh residual below a given threshold α (absolute convergence);
3. ratio of the infinity norm of the finest mesh residual to the infinity norm of the initial guess residual below a given threshold α (relative convergence)

In the case of convergence criteria 2 & 3, the function/subroutine should provide an error/warning if a set number of iterations (N_{iter}) were performed without satisfying either criterium 2 or 3. Finally, after convergence, or the maximum number of iterations has been reached, the mean value of ϕ over the interior solution domain should be set to zero to prevent drift.

Your function/subroutine should take as input:

- `phi`: 2D array of cell centered solution variable including ghost cells, containing upon call the initial guess of the solution
- `f`: 2D array of cell centered PDE right hand side including ghost cells (even though are not necessarily defined)
- `M`: integer number of elements in the x and y direction
- `h`: mesh spacing
- `ConvergenceOption`: integer value indicating requested convergence criterium; 1: fixed number of iterations, 2: absolute convergence, 3: relative convergence
- `nIterMax`: integer number of maximum V-cycle/dualgrid iterations to be performed
- `alpha`: convergence threshold for residual

Your function/subroutine should provide as output:

- `phi`: 2D array of cell centered solution variable including ghost cells, containing the solution
- `err`: error code. Set to 0 if solution converged and no errors occurred, set to 1 if the solution did not converge, set to 2 if the number of elements is not divisible by 2 for a dual grid method, or a power of 2 (V-cycle).

Required submission: printout of function/subroutine; fully commented code of function/subroutine uploaded to Safe-Assign.

Problem 2 (60 points) (Core Course Outcome #2)
Using the function/subroutine developed in problem 1, solve the following second-order differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-\cos(\pi y)\left(4\pi\sin(2\pi x^2)+16\pi^2x^2\cos(2\pi x^2)+\cos(2\pi x^2)\pi^2\right), \tag{3}$$

on a square domain ($-1\le x\le 1$, $-1\le y\le 1$) with Neumann boundary conditions

$$\frac{\partial\phi}{\partial n}=0\,. \tag{4}$$

on all boundaries. Use a cell centered mesh with $M = 256$ elements in the x- and y-direction. As initial guess $\phi^{(0)}(x, y)$ use

$$\phi^{(0)}(x,y)=\frac{1}{2}\sin(\pi x)\sin(4\pi\mathbf{y})\,. \tag{5}$$

On Blackboard, you will find text files *variables_dual.txt* and *variables_vcycle.txt* containing the results of each operation for the first two iterations of the dualgrid respective V-cycle Multigrid method using a mesh with $M = 16$. Use these files to help you debug your code.

Note for AEE471: If you coded a full V-cycle in problem 1, do the MAE561 tasks instead of the AEE471 tasks in the following.

a) Plot the initial guess as a surface plot using a range for ϕ from -1 to 1.
a) Plot as a surface plot the solutions after 2, 5, 10, and 20 iterations using a single grid Gauss-Seidel method using a range for ϕ from -1 to 1.
b) Plot as a surface plot the solutions after 2, 5, 10, and 20 iterations using a dual grid (AEE471), respective V-cycle (MAE561) method, using a range for ϕ from -1 to 1.
c) Plot the infinity norm of the residual for both Gauss Seidel and dual grid (AEE471) respective V-cycle (MAE561) vs the iteration number (0-**50**). Use a log scale for the residual and a linear scale for the iteration number. Compare the results and briefly discuss/explain them.

Required submission:
- 1 clearly annotated surface plot containing the initial guess of ϕ;
- 4 clearly annotated surface plots containing the Gauss Seidel solution after 2, 5, 10, and 20 iterations;
- 4 clearly annotated surface plots containing the dual grid (AEE471) respective V-cycle (MAE561) Multigrid solution after 2, 5, 10, and 20 iterations;
- 1 clearly annotated log-linear plot containing L_∞ of residual vs iteration number for Gauss Seidel and dual grid (AEE471) respective V-cycle (MAE561) Multigrid methods for 0-50 iterations;
- discussion of results incl. comparison of Gauss Seidel to Multigrid;
- printout of code used;
- SafeAssign upload of all used, well commented code.

Bonus Problem 3 (10 bonus points; Core Course Outcomes #1 & #2)
For the PDE, domain, and boundary conditions given in problem 2, calculate the partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ using second order central differences at each cell center, making use of ghost cells where necessary. If the exact derivatives are given by

$$\frac{\partial\phi}{\partial x} = -4\pi x \sin(2\pi x^2)\cos(\pi y) \tag{6}$$

and

$$\frac{\partial\phi}{\partial y} = -\pi \cos(2\pi x^2)\sin(\pi y) \tag{7}$$

demonstrate the formal second order of the entire finite difference method in the interior of the domain for both derivatives, by performing a mesh refinement study and reporting the L_∞, L_1, and L_2 norms and their observed order of convergence.

Required submission:
- documentation of solution method (chosen method, number of iterations, convergence criterium);
- table containing number of elements M, L_∞, L_1, and L_2 norms, and their observed order of convergence;
- SafeAssign upload of all used, well commented code.