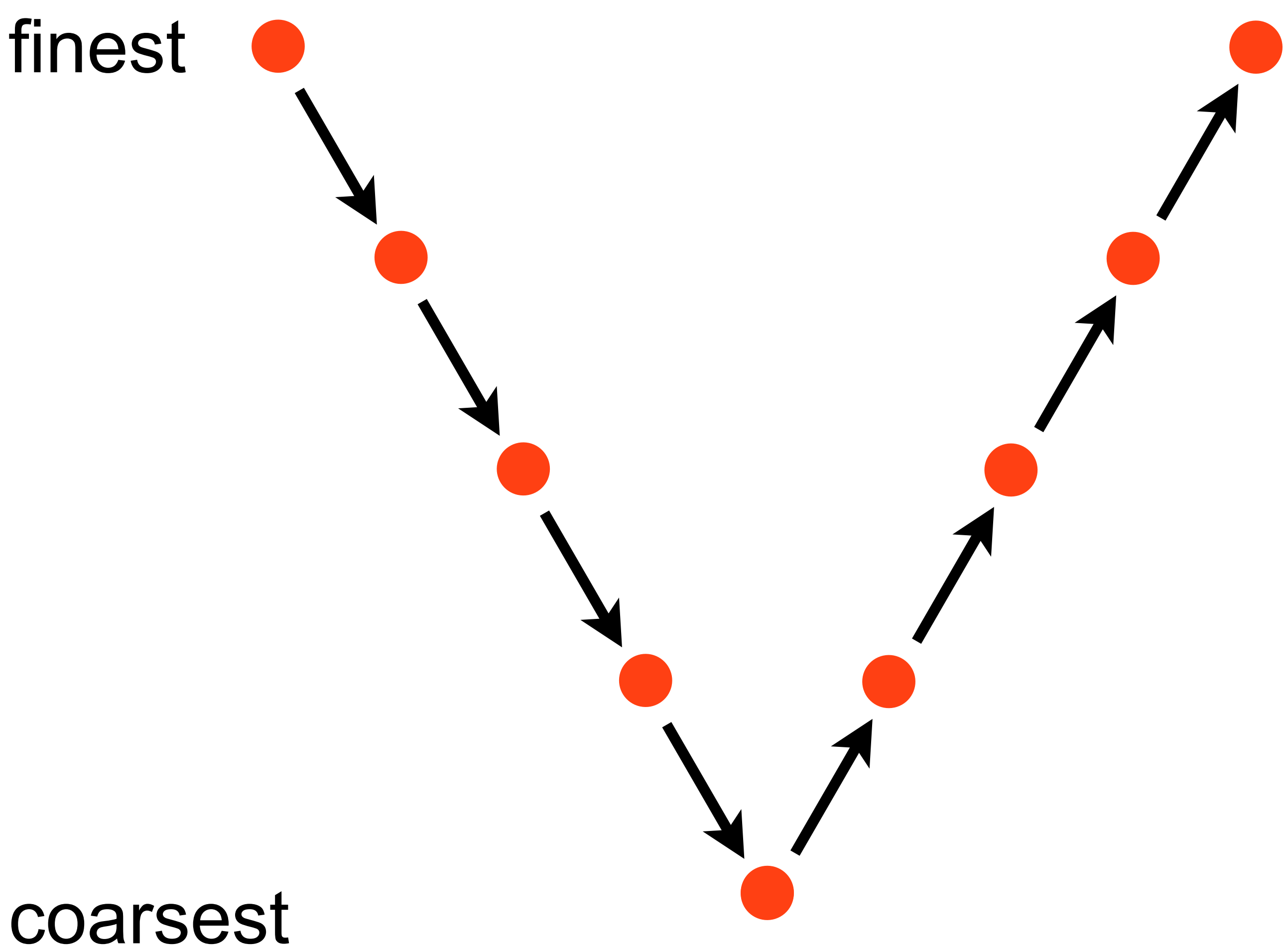
# Multigrid: V-cycle

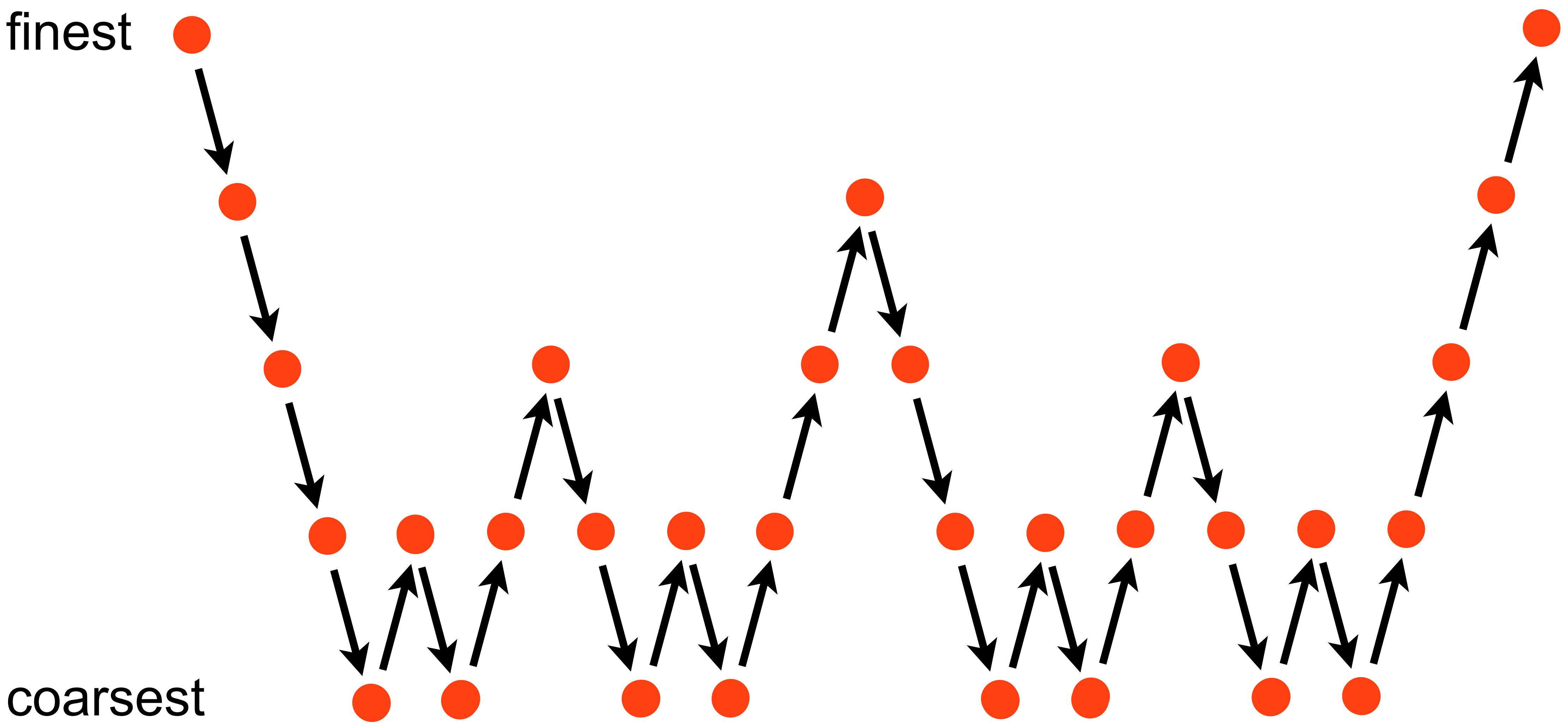
- How to traverse the different grid levels?
  - Many options!




V-cycle
1 iteration

# Multigrid: W-Cycle

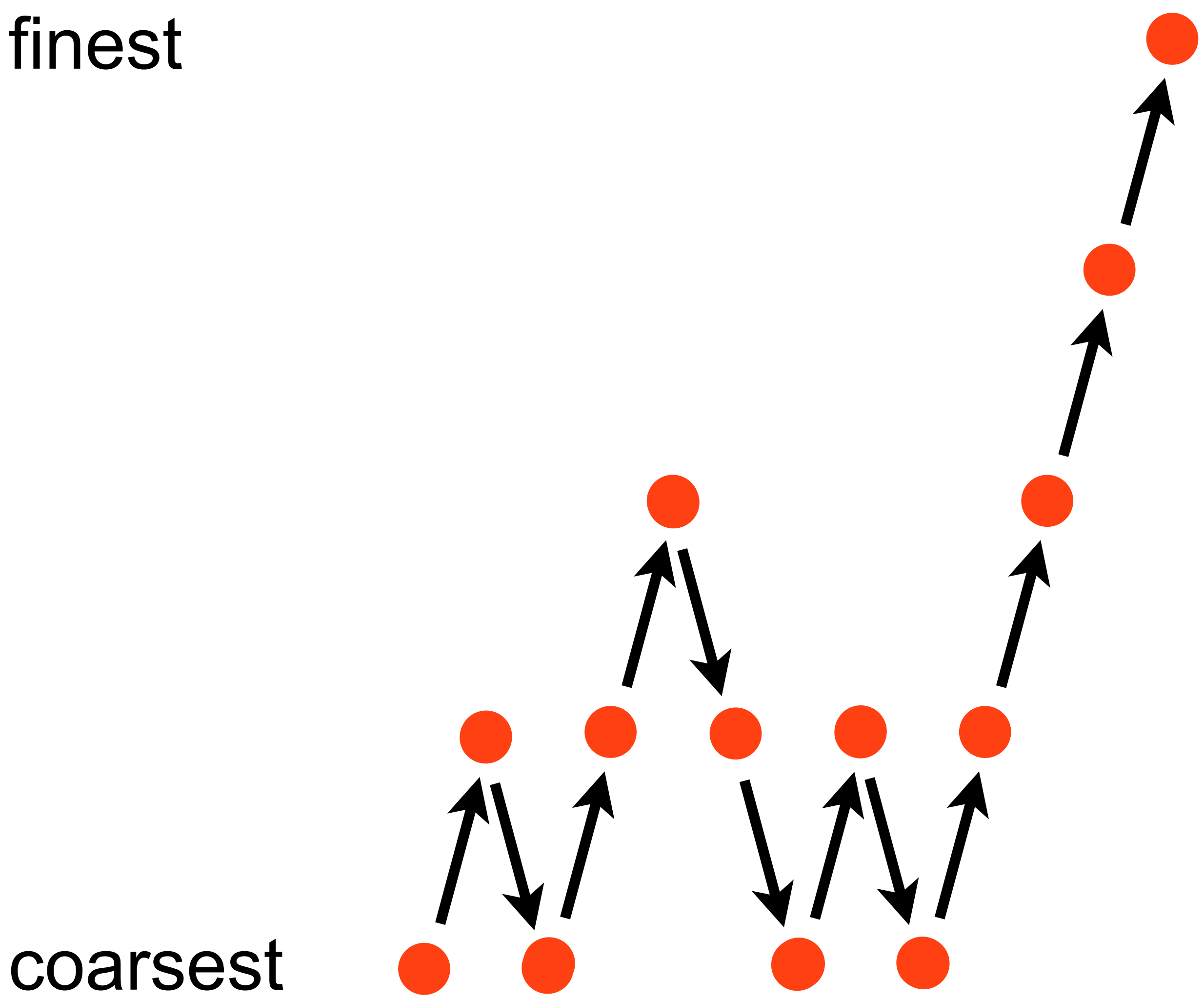
- How to traverse the different grid levels?




W-cycle
1 iteration

# Multigrid: FMC

- Full Multi Grid cycle:
  - start at coarsest grid level

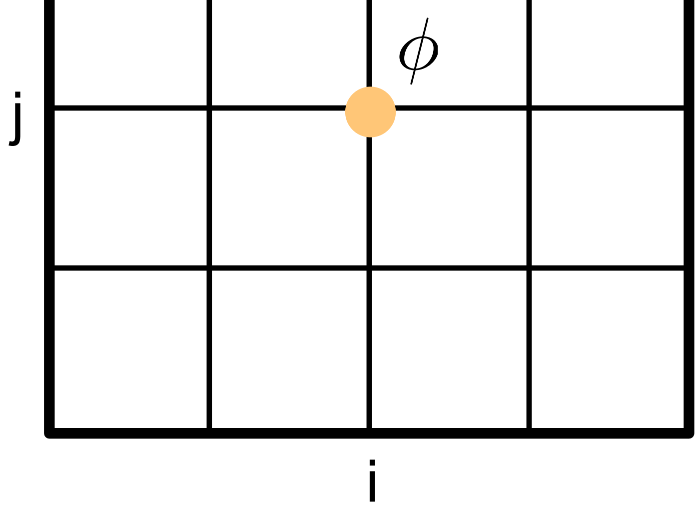



FMC-cycle
1 iteration

Next: need to revisit meshing

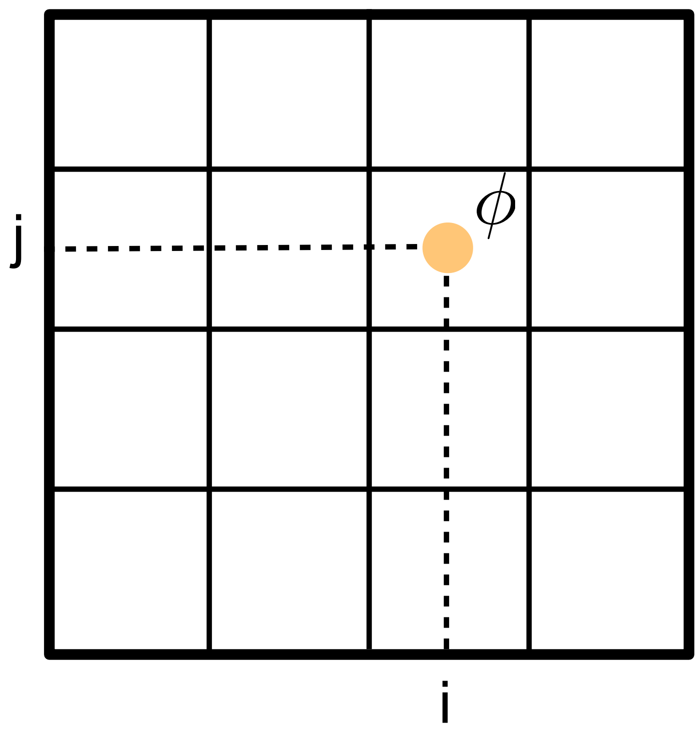
- until now, we have used the following meshes
  - variables are located at the intersection of grid lines

node based mesh




  - but, we could also locate variables @ cell centers!

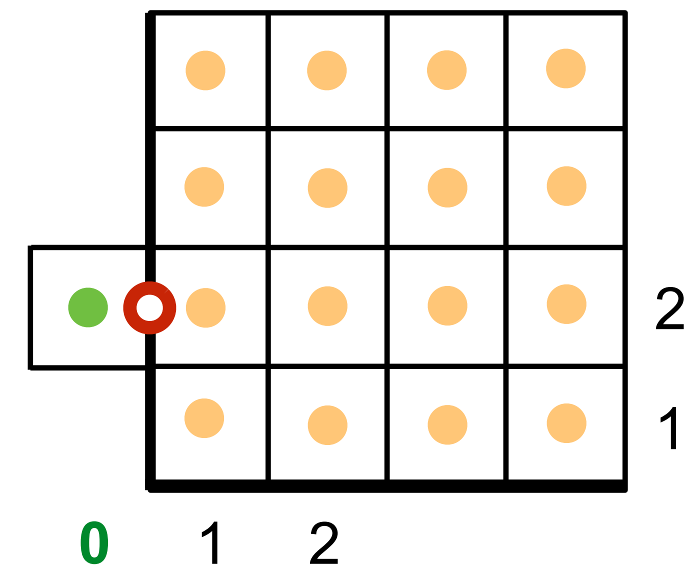
cell centered mesh




  - index i,j refers to cell (element) center

How do cell centered meshes impact boundary conditions?

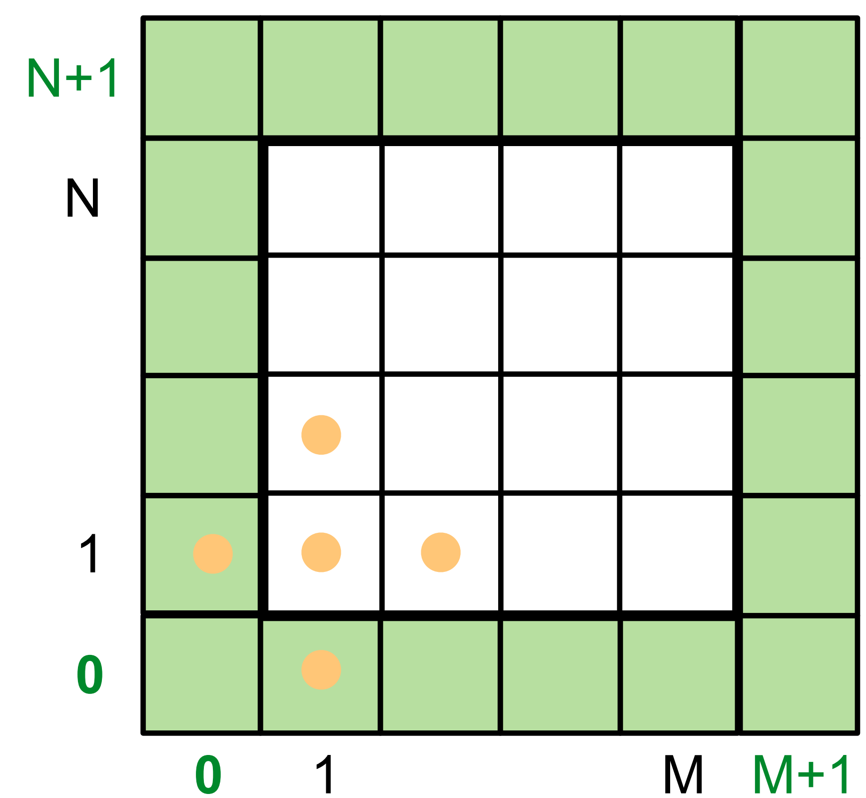
- **Dirichlet boundary:**
  - there's no longer a variable located on the boundary to set to the given Dirichlet value
  - Trick: add a "virtual" **ghost cell** outside the boundary
  - choose the ghost cells' value such that an interpolation to the boundary location with appropriate order is equal to the Dirichlet value
    - Example: 2nd-order




$$\phi_{bc,j} = \frac{\phi_{0,j} + \phi_{1,j}}{2} \quad \Rightarrow \quad \phi_{0,j} = 2\phi_{bc,j} - \phi_{1,j}$$

  - extends the mesh by a layer of ghost cells all around

```
phi(0:M+1,0:N+1)
```




  - Benefit: can use regular stencil even adjacent to boundaries with ghost cell values

```
for j=1:N
   for i=1:M
```

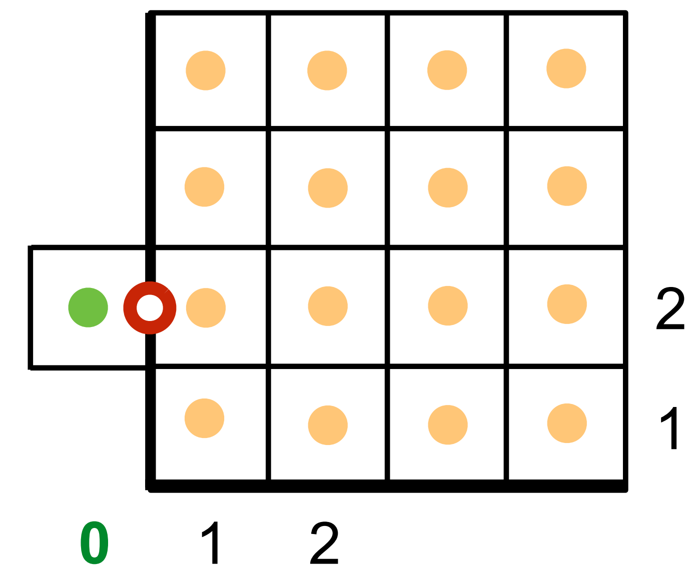
How do cell centered meshes impact boundary conditions?

- **Neumann boundary:**
  - Trick: use ghost cell value to calculate derivative on the boundary
    - Example: 2nd-order

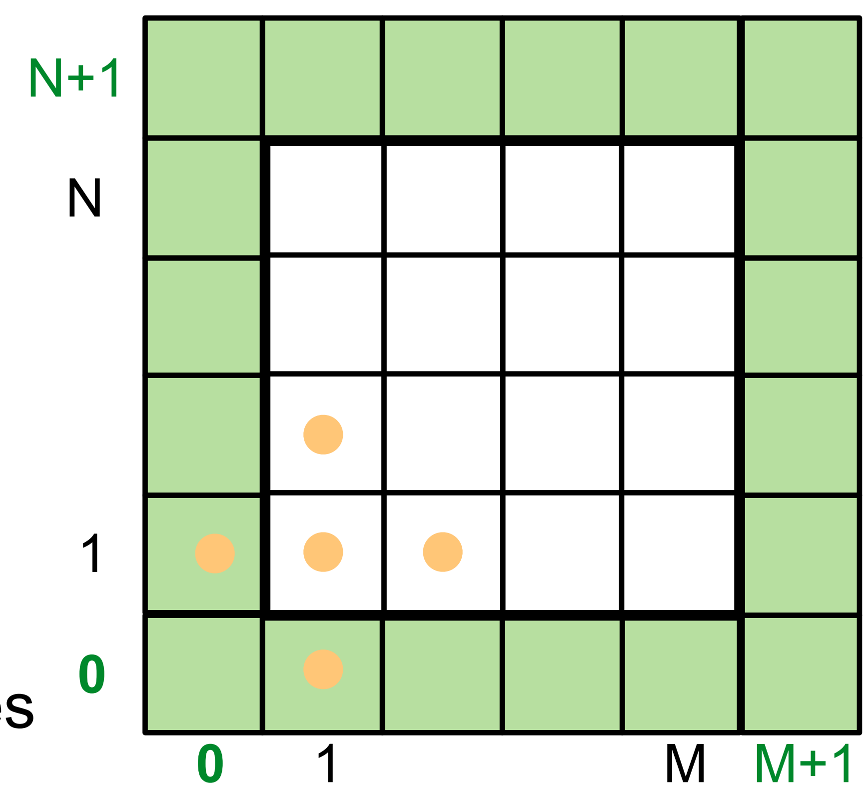
$$g = \left.\frac{\partial \phi}{\partial x}\right|_{bc,j} = \frac{\phi_{1,j} - \phi_{0,j}}{2\,\frac{h}{2}} + O(h^2)$$

$$\Rightarrow \quad \phi_{0,j} = \phi_{1,j} - hg$$

    - this sets the ghost cell value!




  - Again: can use regular stencil even adjacent to boundaries with ghost cell values
  - for higher order, add additional ghost cells

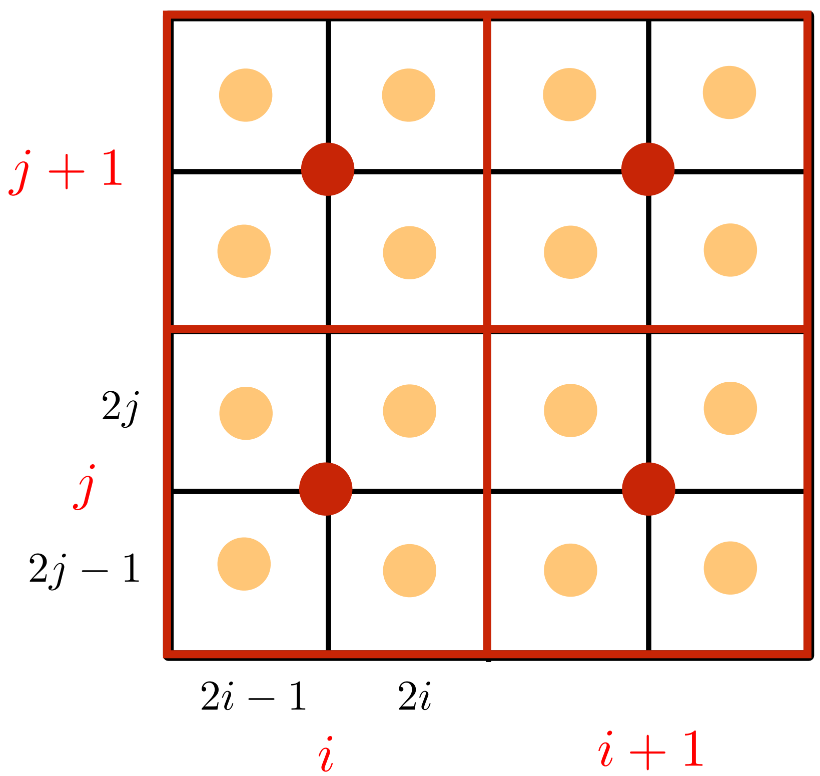



- **Solution procedure for cell centered meshes**
  - update interior cells (j=1:N, i=1:M)
  - calculate ghost cell values with updated interior values

- Small drawback: ghost cells in Gauss-Seidel are not updated and thus may lag one iteration

How do cell centered meshes impact Multigrid methods?

- **Prolongation**




here: i,j are coarse grid indices

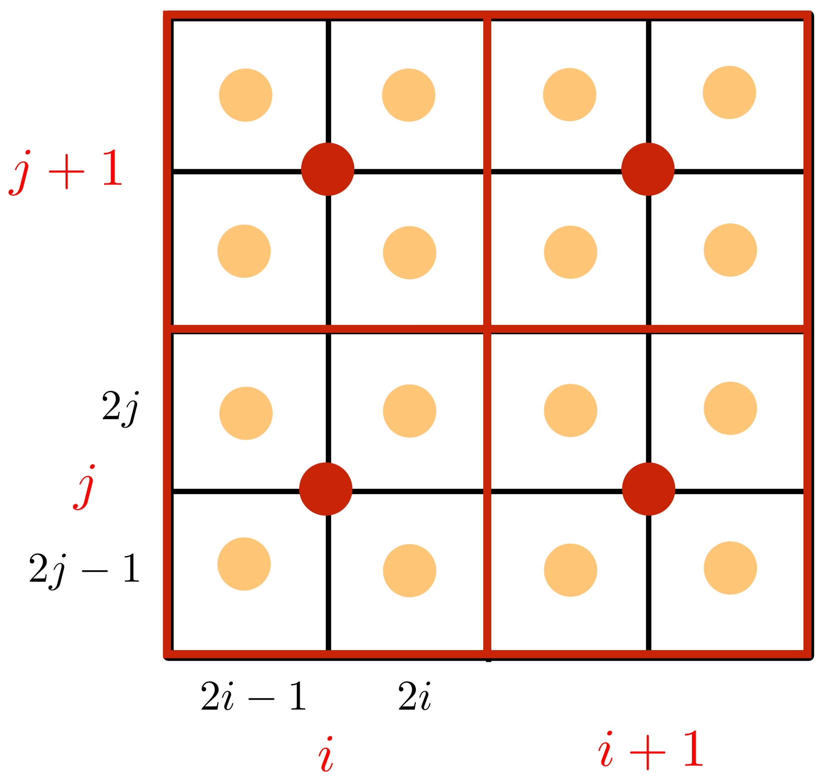
- Option #1: Constant "interpolation"

$$\epsilon^{2h\rightarrow h}_{2i-1:2i,2j-1:2j} = \epsilon^{2h}_{i} \qquad i = 1, 2, \ldots, M^{2h}\,,\; j = 1, 2, \ldots, M^{2h}$$

- Option #2: Bilinear interpolation

How do cell centered meshes impact Multigrid methods?

- **Restriction** (needs to be adjoint of Prolongation)




here: i,j are coarse grid indices

- Option #1: Adjoint to constant "interpolation"

$$r_{i,j}^{h\rightarrow 2h} = \frac{1}{4} \sum_{j'=2j-1}^{2j} \sum_{i'=2i-1}^{2i} r_{i',j'}^{h} \qquad i = 1, 2, \ldots, M^{2h}, \; j = 1, 2, \ldots, M^{2h}$$

- Option #2: Adjoint to bilinear interpolation

- Finally, a comment on Poisson equation with all Neumann boundaries

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x,y) \qquad \left.\frac{\partial \phi}{\partial n}\right|_{bc} = g(x,y)$$

  - if $\phi(x,y)$ is a solution, so is $\phi(x,y) + const$
  - iterative solution may “drift”
  - this is usually not a problem for convergence checks, since these use the residual

$$r(x,y) = f(x,y) - \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

  - but, excessive “drift” may cause finite precision problems, since it can lead to differences of large numbers
  - Fix: subtract the mean of $\phi$ from $\phi$ after convergence or after some number of iterations

$$\phi_{i,j} \rightarrow \phi_{i,j} - \frac{1}{MN} \sum_{j=1}^{N} \sum_{i=1}^{M} \phi_{i,j}$$

# Second Model Problem: Parabolic Equations

- 1D heat equation

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{with} \qquad \varphi = \varphi(x,t)$$

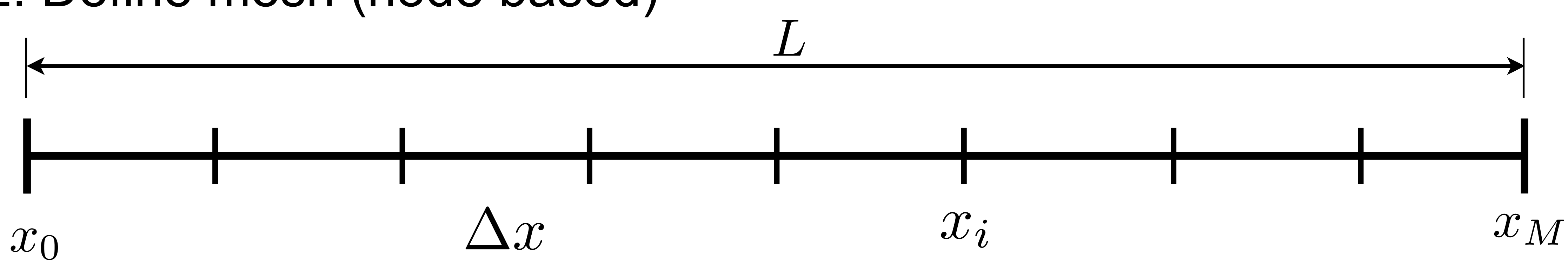
boundary conditions: $\varphi(x=0,t) = \varphi(x=L,t) = 0$

initial condition: $\varphi(x,t=0) = g(x)$

Step 1: Define solution domain

$$0 \leq x \leq L$$

Step 2: Define mesh (node based)




$$\Delta x = h = \frac{L}{M} \qquad\qquad x_i = ih, \quad i = 0 \ldots M$$

# Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 3: Approximate spatial derivatives

for example: 2nd-order central:

$$\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2} + O(\Delta h^2)$$

Step 4: Substitute into PDE

$$\left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right)$$ ⇒ now an ODE!

Step 5: Incorporate boundary conditions

$$\varphi(x=0,t) = \varphi(x=L,t) = 0 \quad \Rightarrow \quad \varphi_0 = \varphi_M = 0$$

# Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 6: Matrix form (only for illustration, never code!)

$$\left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right)$$

$$\frac{d}{dt}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \frac{\alpha}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix}$$

$$\frac{d\vec{\varphi}}{dt} = A\vec{\varphi}$$ ⇒ semi-discrete form ⇒ many ODEs

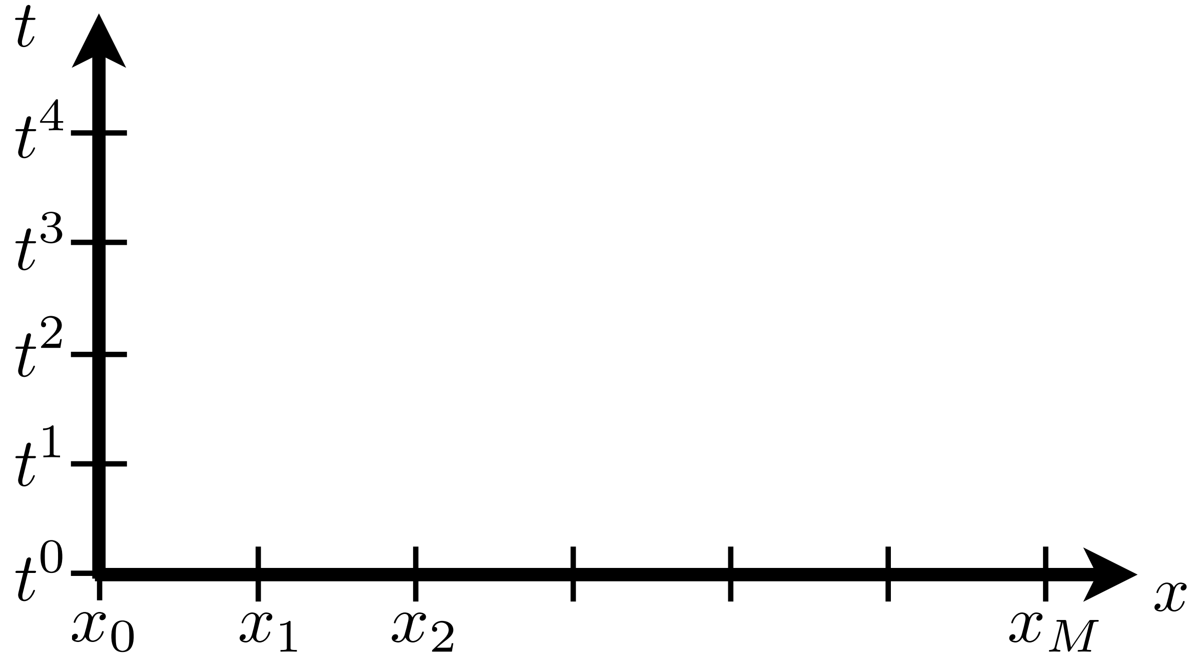
never solve this directly

# Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

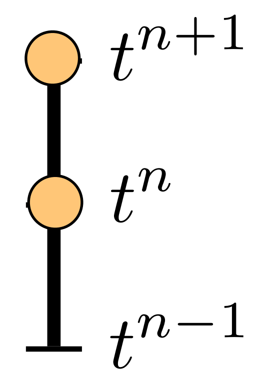
- discretize in time: $t^n = n\Delta t\,, \qquad n = 0, 1, 2, \ldots$




- use finite difference approximation for $\left.\frac{d\varphi}{dt}\right|_i$
  - for example: 1st-order forward

$$\left.\frac{d\varphi}{dt}\right|_i^n = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

stencil:



# Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

- substitute into ODE

$$\left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right) \qquad \left.\frac{d\varphi}{dt}\right|_i^n = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

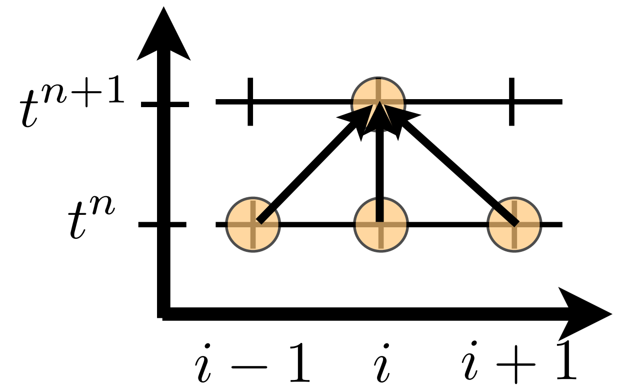
$$\boxed{\varphi_i^{n+1} = \varphi_i^n + \frac{\alpha \Delta t}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)}$$

**FTCS** <u>F</u>orward <u>T</u>ime <u>C</u>entral <u>S</u>pace

- FTCS expresses a single unknown, $\varphi_i^{n+1}$, as a function of only knowns!

⇒ feature of **explicit** methods

solution at $t^{n+1}$ depends only on solution at $t^n$

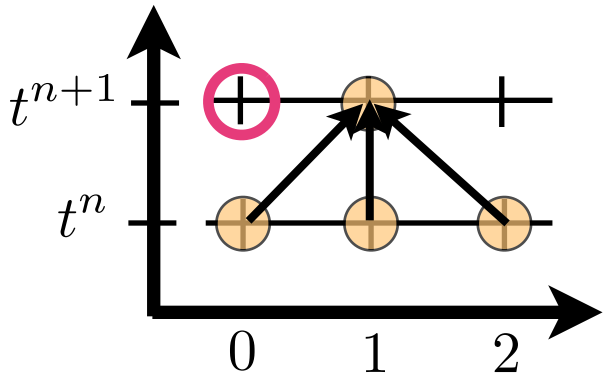


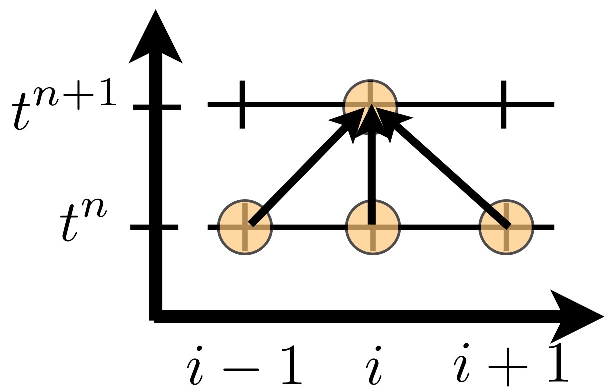
# Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

- BUT: problem at boundary




boundary point (bc) does not influence the solution at same $t$ !

- boundaries lag by one time step
- this violates characteristics of parabolic equations

# Second Model Problem: Parabolic Equations

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Step 7: Solve (but how?)

- Alternative: use backwards time difference: **Laasonen Method (BTCS)**

$$\left.\frac{d\varphi}{dt}\right|_i^{\boxed{n+1}} = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) \qquad \left.\frac{d\varphi}{dt}\right|_i = \frac{\alpha}{h^2}\left(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}\right)$$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{h^2}\left(\varphi_{i+1}^{\boxed{n+1}} - 2\varphi_i^{\boxed{n+1}} + \varphi_{i-1}^{\boxed{n+1}}\right)$$

Problem: no longer explicit, but now implicit

- gather all *n+1* terms on left hand side

$$\boxed{\frac{\alpha \Delta t}{h^2}\varphi_{i-1}^{n+1} - \left(1 + 2\frac{\alpha \Delta t}{h^2}\right)\varphi_i^{n+1} + \frac{\alpha \Delta t}{h^2}\varphi_{i+1}^{n+1} = -\varphi_i^n}$$

$$\Rightarrow \quad a_i^n \varphi_{i-1}^{n+1} + b_i^n \varphi_i^{n+1} + c_i^n \varphi_{i+1}^{n+1} = d_i^n$$

⇒ tri-diagonal system

⇒ solve directly using Gauss (see Class 5)

⇒ much more work than FTCS! So, what's the benefit?

⇒ need to discuss accuracy, stability, and consistency

- Definitions:

1. **Consistency**: numerical approximation approaches PDE as $\Delta x, \Delta y, \Delta t \to 0$

2. **Stability**: numerical solution remains bounded

3. **Convergence**: numerical solution approaches PDE solution as $\Delta x, \Delta y, \Delta t \to 0$

turns out if 1. and 2. are true, then 3. is true for linear, well posed initial value problems

# Accuracy

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

- both FTCS and BTCS use

$$\left.\frac{d\varphi}{dt}\right|_i^n \approx \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) \qquad \left.\frac{d\varphi}{dt}\right|_i^{n+1} \approx \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right)$$

Taylor series:

$$\varphi_i^{n+1} = \varphi_i^n + \Delta t \left.\frac{\partial \varphi_i}{\partial t}\right|^n + O(\Delta t^2) \qquad \varphi_i^n = \varphi_i^{n+1} - \Delta t \left.\frac{\partial \varphi_i}{\partial t}\right|^{n+1} + O(\Delta t^2)$$

$$\left.\frac{d\varphi}{dt}\right|_i^n = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) + O(\Delta t) \qquad \left.\frac{d\varphi}{dt}\right|_i^{n+1} = \frac{1}{\Delta t}\left(\varphi_i^{n+1} - \varphi_i^n\right) + O(\Delta t)$$

both are first order in time

# Consistency

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Consistency ≜ numerical approximation approaches PDE

<u>Example: FTCS</u>

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

- Question: Does this approach the PDE, as $\Delta x, \Delta t \rightarrow 0$ ?

# Consistency

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

Need Taylor series for each term in FTCS

$$\varphi_i^{n+1} = \varphi_i^n + \Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3)$$

$$\varphi_{i+1}^n = \varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5)$$

$$\varphi_{i-1}^n = \varphi_i^n - \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i - \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5)$$

Substitute Taylor series into FTCS

$$\frac{1}{\Delta t}\left(\varphi_i^n + \Delta t \left.\frac{\partial \varphi}{\partial t}\right|^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 \varphi}{\partial t^2}\right|^n + O(\Delta t^3) - \varphi_i^n\right) = \frac{\alpha}{\Delta x^2}\left(\varphi_i^n + \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i \right.$$

$$\left. + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5) - 2\varphi_i^n + \varphi_i^n - \Delta x \left.\frac{\partial \varphi}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 \varphi}{\partial x^2}\right|_i - \frac{\Delta x^3}{6}\left.\frac{\partial^3 \varphi}{\partial x^3}\right|_i + \frac{\Delta x^4}{24}\left.\frac{\partial^4 \varphi}{\partial x^4}\right|_i + O(\Delta x^5)\right)$$

$$\frac{\partial \varphi}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2} + O(\Delta t^2) = \alpha\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\Delta x^2}{12}\frac{\partial^4 \varphi}{\partial x^4} + O(\Delta x)^3\right)$$

# Consistency

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Consistency ≜ numerical approximation approaches PDE

<u>Example: FTCS</u>

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

Modified equation:

$$\frac{\partial \varphi}{\partial t} + \cancel{\frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2}} + \cancel{O(\Delta t^2)} = \alpha\left(\frac{\partial^2 \varphi}{\partial x^2} + \cancel{\frac{\Delta x^2}{12}\frac{\partial^4 \varphi}{\partial x^4}} + \cancel{O(\Delta x)^3}\right)$$

- Question: Does this approach the PDE, as $\Delta x, \Delta t \to 0$ ?

as $\Delta x, \Delta t \to 0$: $\quad \frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$ ⇒ original PDE ⇒ FTCS is consistent

- Similar analysis shows that BTCS is consistent, too