ENO-Schemes

Essentially **N**on-**O**scillatory Schemes

Observations:

- higher order schemes often are dispersive
- Is there a way to modify/choose stencil such that dispersive errors are minimized?

Idea:

- build stencils based on local “smoothness”

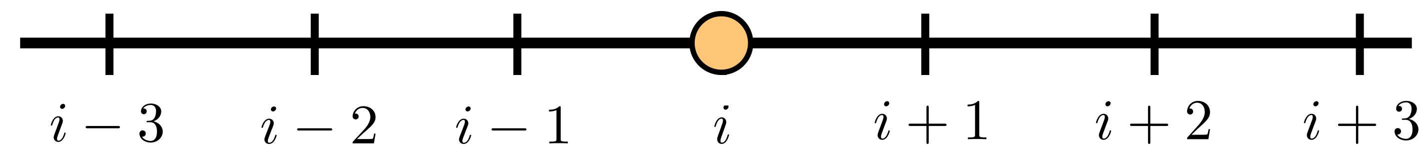
References:

- Harten et al., J. Comput. Phys., 131, pp. 3-47, 1997.
- Shu, NASA/CR-97-206253; ICASE 97-65.

ENO-Schemes

Procedure to build stencils:

- start at i
- add either i+1 or i-1 depending on which has "smoother" data



for equidistant meshes: if $|u_{i+1} - u_i| < |u_i - u_{i-1}|$ $\rightarrow$ take i +*1*

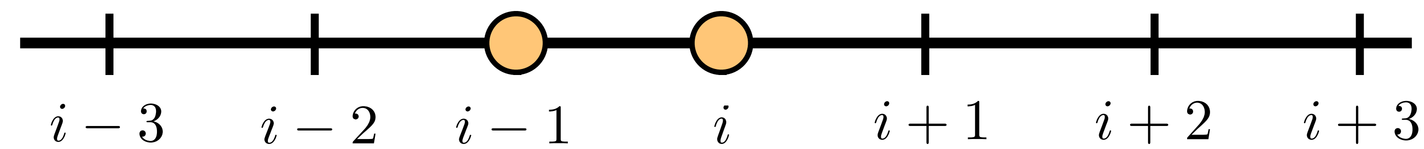
if $|u_{i+1} - u_i| \geq |u_i - u_{i-1}|$ $\rightarrow$ take i - *1*

let's introduce shorthands: $\Delta^+ u_i = u_{i+1} - u_i$ and $\Delta^- u_i = u_i - u_{i-1}$

ENO-Schemes

$$\Delta^+ u_i = u_{i+1} - u_i$$
$$\Delta^- u_i = u_i - u_{i-1}$$

Procedure to build stencils:



- start at i
- add either i+1 or i-1 depending on which has "smoother" data

  for equidistant meshes: if $|\Delta^+ u_i| < |\Delta^- u_i|$ $\rightarrow$ take *i +1*

  if $|\Delta^+ u_i| \geq |\Delta^- u_i|$ $\rightarrow$ take *i - 1*

  - as an example, let's assume $|\Delta^+ u_i| \geq |\Delta^- u_i|$ $\rightarrow$ take *i - 1*

- add either i+1 or i-2 depending on which has "smoother" data

  for equidistant meshes: if $|\Delta^+ \Delta^- u_i| < |\Delta^- \Delta^- u_i|$ $\rightarrow$ take *i +1*

  if $|\Delta^+ \Delta^- u_i| \geq |\Delta^- \Delta^- u_i|$ $\rightarrow$ take *i - 2*

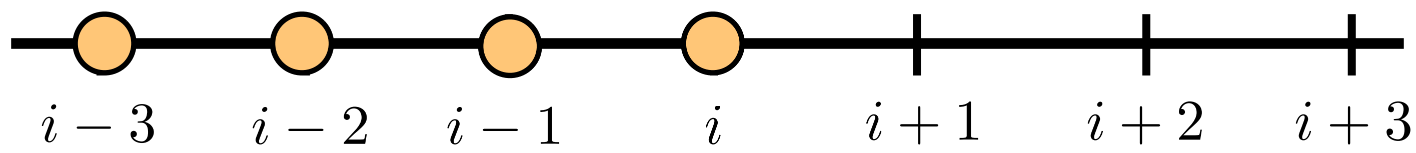
- keep on adding stencil points up to the desired number/order of accuracy
- once stencil points are found, use Taylor table coefficients
- BUT: may need other considerations besides "smoothness": upwind bias

3rd

$$\frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} \qquad a > 0$$

- 3rd-order method ⇒ 4 stencil points
- requires upwind bias
- 3 stencils are possible

S₀:



$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-3} - \frac{7}{6}\Delta^+\varphi_{i-2} + \frac{11}{6}\Delta^+\varphi_{i-1}\right)$$

3^{rd}

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \qquad a > 0$$

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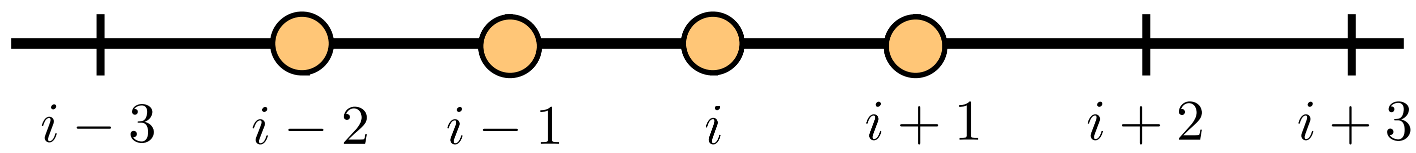
S_1:



$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-3} - \frac{7}{6}\Delta^+\varphi_{i-2} + \frac{11}{6}\Delta^+\varphi_{i-1}\right)$$

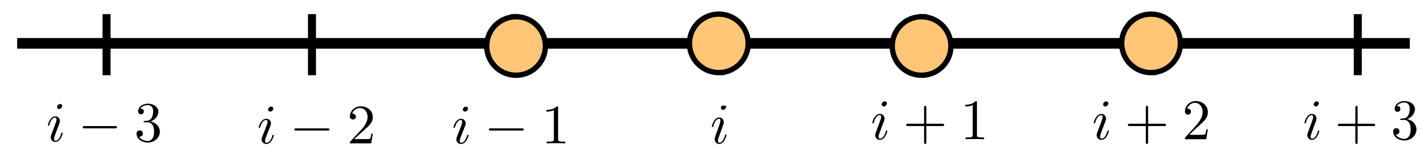
$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} = \frac{1}{\Delta x}\left(-\frac{1}{6}\Delta^+\varphi_{i-2} + \frac{5}{6}\Delta^+\varphi_{i-1} + \frac{1}{3}\Delta^+\varphi_{i}\right)$$

3rd

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \qquad a > 0$$

- 3rd-order method ⇒ 4 stencil points
- requires upwind bias
- 3 stencils are possible

S_2:



$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-3} - \frac{7}{6}\Delta^+\varphi_{i-2} + \frac{11}{6}\Delta^+\varphi_{i-1}\right)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} = \frac{1}{\Delta x}\left(-\frac{1}{6}\Delta^+\varphi_{i-2} + \frac{5}{6}\Delta^+\varphi_{i-1} + \frac{1}{3}\Delta^+\varphi_{i}\right)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,2} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-1} + \frac{5}{6}\Delta^+\varphi_{i} - \frac{1}{6}\Delta^+\varphi_{i+1}\right)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-} = \begin{cases} \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} & \text{if} \quad |\Delta^-\Delta^+\varphi_{i-1}| < |\Delta^-\Delta^+\varphi_{i}| \text{ and } |\Delta^-\Delta^-\Delta^+\varphi_{i-1}| < |\Delta^+\Delta^-\Delta^+\varphi_{i-1}| \\ \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,2} & \text{if} \quad |\Delta^-\Delta^+\varphi_{i-1}| > |\Delta^-\Delta^+\varphi_{i}| \text{ and } |\Delta^-\Delta^-\Delta^+\varphi_{i}| > |\Delta^+\Delta^-\Delta^+\varphi_{i}| \\ \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} & \quad \text{otherwise} \end{cases}$$

- for $a < 0$: use right-biased stencils by simply mirroring all indices and changing + supescripts to - superscripts and vice versa

TVD RK-3 time integration

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \qquad a > 0$$

- ENO methods are often combined with specially optimized RK time integration methods, designed to be TVD
  - start: $\varphi_i^{(0)} = \varphi_i^n$
  - step 1: $\varphi_i^{(1)} = \varphi_i^{(0)} - \alpha_{1,0}\left(a\Delta t \left.\frac{\partial \varphi^{(0)}}{\partial x}\right|_i^-\right)$
  - step 2: $\varphi_i^{(2)} = \varphi_i^{(1)} - \alpha_{2,0}\left(a\Delta t \left.\frac{\partial \varphi^{(0)}}{\partial x}\right|_i^-\right) - \alpha_{2,1}\left(a\Delta t \left.\frac{\partial \varphi^{(1)}}{\partial x}\right|_i^-\right)$
  - step 3: $\varphi_i^{(3)} = \varphi_i^{(2)} - \alpha_{3,0}\left(a\Delta t \left.\frac{\partial \varphi^{(0)}}{\partial x}\right|_i^-\right) - \alpha_{3,1}\left(a\Delta t \left.\frac{\partial \varphi^{(1)}}{\partial x}\right|_i^-\right) - \alpha_{3,2}\left(a\Delta t \left.\frac{\partial \varphi^{(2)}}{\partial x}\right|_i^-\right)$
  - step q: $\varphi_i^{(q)} = \varphi_i^{(q-1)} - \sum_{r=0}^{q-1}\alpha_{q,r}\left(a\Delta t \left.\frac{\partial \varphi^{(r)}}{\partial x}\right|_i^-\right)$
- for example: TVD RK-3

$$\alpha_{1,0} = 1 \qquad \alpha_{2,0} = -\frac{3}{4}\,,\ \alpha_{2,1} = \frac{1}{4} \qquad \alpha_{3,0} = -\frac{1}{12}\,,\ \alpha_{3,1} = -\frac{1}{12}\,,\ \alpha_{3,2} = \frac{2}{3}$$

Code

WENO-Schemes

Weighted **E**ssentially **N**on-**O**scillatory Schemes

Idea:

- if we have all these stencils, why choose only one and let the others go to waste?
  - can use all stencils only in "smooth" regions
  - switch back to ENO in "non-smooth" regions
  - in "smooth" regions, use optimal combination of stencils:

$$\left.\frac{\partial\varphi}{\partial x}\right|_i^- = \omega_0 \left.\frac{\partial\varphi}{\partial x}\right|_i^{-,0} + \omega_1 \left.\frac{\partial\varphi}{\partial x}\right|_i^{-,1} + \omega_2 \left.\frac{\partial\varphi}{\partial x}\right|_i^{-,2} \qquad \text{with} \qquad \omega_0 + \omega_1 + \omega_2 = 1$$

    - if $\omega_0 = 0.1\,,\ \omega_1 = 0.6\,,\ \omega_2 = 0.3$ ⇒ optimal **5th-order** stencil
  - but, calculating ω_0, ω_1, and ω_2 from the smoothness is an art
  - for example: Jiang & Peng, SIAM J. Sci. Comput. 21(6), 2000: WENO-5

WENO-5

Jiang & Peng, SIAM J. Sci. Comput. 21(6), 2000

$$\left.\frac{\partial\varphi}{\partial x}\right|_i^- = \frac{1}{12\Delta x}\left(-\Delta^+\varphi_{i-2} + 7\Delta^+\varphi_{i-1} + 7\Delta^+\varphi_i - \Delta^+\varphi_{i+1}\right)$$
$$-\Psi_{WENO}\left(\frac{\Delta^-\Delta^+\varphi_{i-2}}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_{i-i}}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_i}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_{i+1}}{\Delta x}\right)$$

with $\Psi_{WENO}(a,b,c,d) = \frac{1}{3}\omega_0(a-2b+c) + \frac{1}{6}\left(\omega_2 - \frac{1}{2}\right)(b-2c+d)$

with $\omega_0 = \frac{\alpha_0}{\alpha_0+\alpha_1+\alpha_2}$ and $\omega_2 = \frac{\alpha_2}{\alpha_0+\alpha_1+\alpha_2}$

with $\alpha_0 = \frac{1}{(\epsilon + IS_0)^2}$ and $\alpha_1 = \frac{6}{(\epsilon + IS_1)^2}$ and $\alpha_2 = \frac{3}{(\epsilon + IS_2)^2}$

with $IS_0 = 13(a-b)^2 + 3(a-3b)^2$

$IS_1 = 13(b-c)^2 + 3(b+c)^2$

$IS_2 = 13(c-d)^2 + 3(3c-d)^2$

- for $a < 0$: use right-biased gradient by simply mirroring all indices & terms and exchanging + and - superscripts
- combine with TVD RK-3

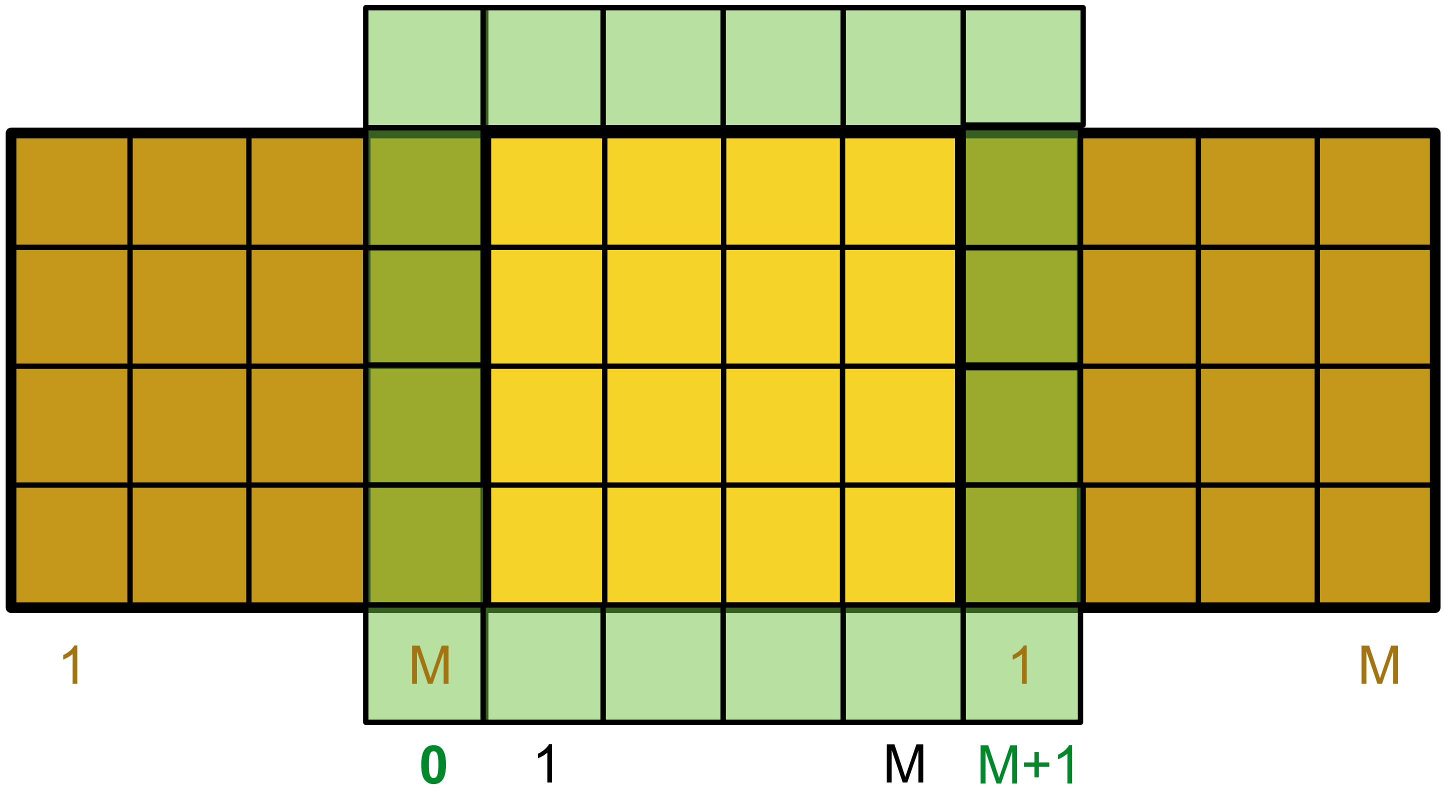
Code

WENO-Schemes

Comments:

- one can construct even higher order schemes
- based on r^{th}-order individual stencils, the resulting scheme is order $2r-1$
- usually WENO is combined with high-order TVD RK, i.e. TVD RK-3 (Shu, SIAM J. Sci. Comput. 9(6), 1988)
- for WENO schemes, the max CFL number to control oscillations is usually smaller than the stability limit
- smaller CFL numbers are usually better
- newer enhancements include:
  - MPWENO (**M**onotonicity **P**reserving WENO) (Suresh & Huynth, J. Comput. Phys. 136, p.83, 1997)
  - MPWENO distinguishes between “smooth” local extrema and O(1) discontinuities

Periodic boundary conditions for cell centered meshes & ghost cells



$$\varphi_{i,N+1} = \varphi_{i,1}$$

$$\varphi_{i,0} = \varphi_{i,N}$$

$$\varphi_{0,j} = \varphi_{M,j} \qquad \varphi_{M+1,j} = \varphi_{1,j}$$

- for larger stencil schemes: add additional ghost cells, e.g.,

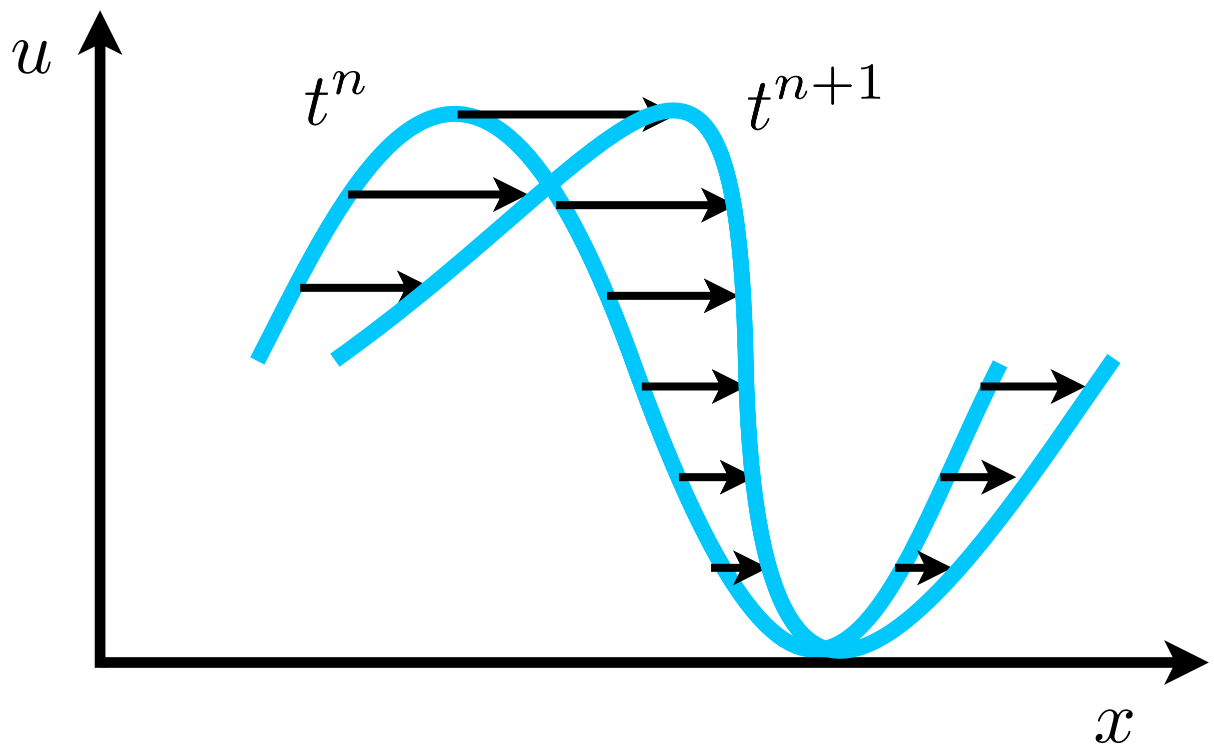
$$\varphi_{-1,j} = \varphi_{M-1,j}$$
$$\varphi_{-2,j} = \varphi_{M-2,j}$$

etc.

Non-linear Hyperbolic Equations

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \quad \text{with} \quad E = \frac{1}{2}u^2$$

- Example:



shocks may be formed!

Lax-Method

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = \overline{u_i^n} - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) \quad \text{with} \quad \overline{u_i^n} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)$$

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)}$$

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{4\Delta x}\left(\left(u_{i+1}^n\right)^2 - \left(u_{i-1}^n\right)^2\right)}$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^2 u}{\partial x^2}$ ⇒ dissipative

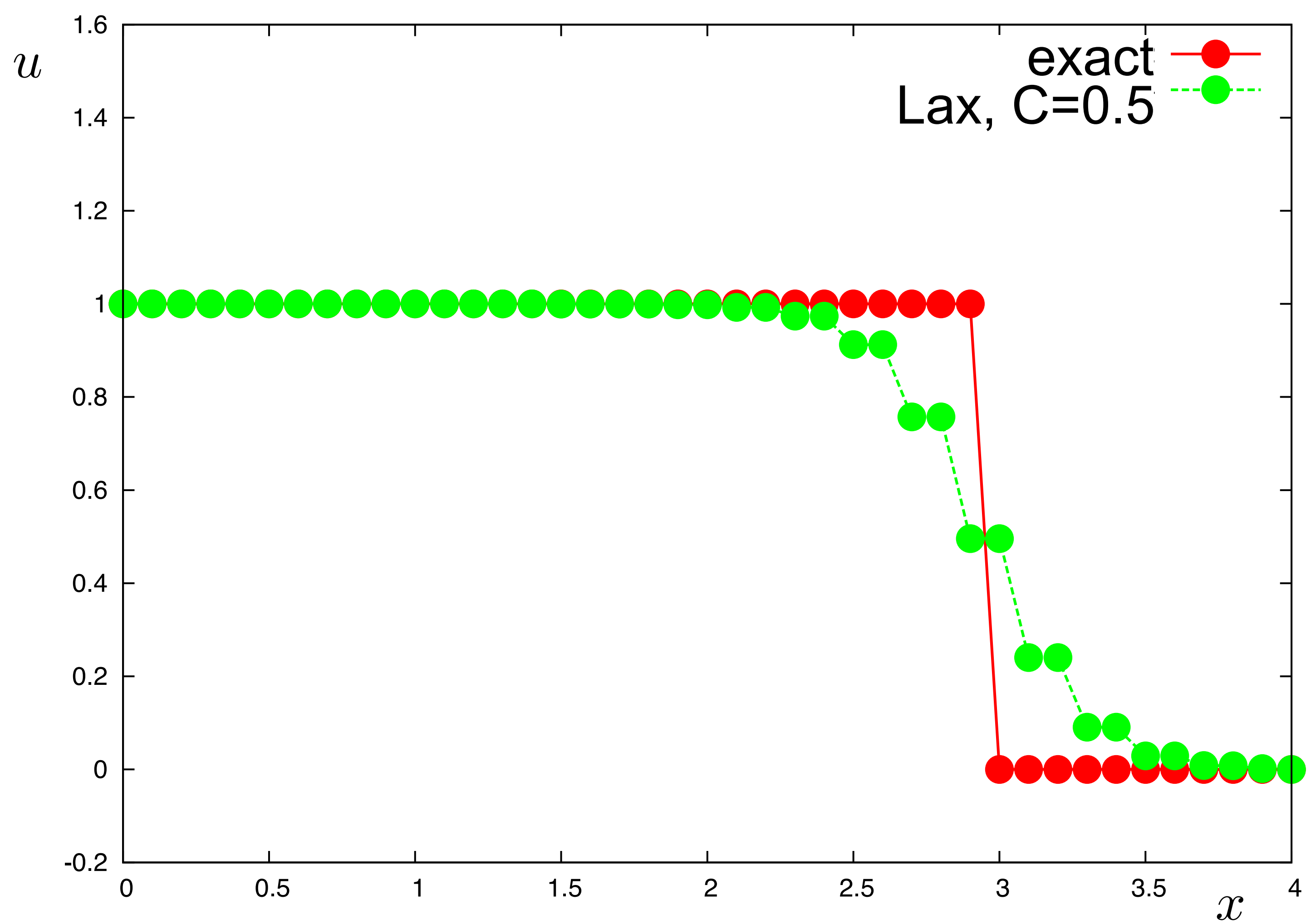
$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

- Example:

$$0 \leq x \leq 4\,, \quad M = 40$$

initial condition: $u(x, t=0) = \begin{cases} 1 & x \leq 2 \\ 0 & x > 2 \end{cases}$

boundary conditions: $u(x=0,t) = 1\,, \quad u(x=4,t) = 0$



Code:
C=0.5,
C=1,
C=0.01

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

Idea: Start from Taylor series at t^n for t^{n+1}

$$u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3)$$

use PDE: $$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\frac{\partial E}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t}\right)$$

but: $$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial E}{\partial u}\left(-\frac{\partial E}{\partial x}\right) = -A\frac{\partial E}{\partial x}$$

define Jacobian as $A = \dfrac{\partial E}{\partial u}$ here: $A = \dfrac{\partial E}{\partial u} = \dfrac{\partial(\frac{1}{2}u^2)}{\partial u} = u$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right)$$

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \Delta t\frac{\partial E}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right) + O(\Delta t^3)$$

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \Delta t \frac{\partial E}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right) + O(\Delta t^3)$$

central   midpoint central

$$u_i^{n+1} = u_i^n - \Delta t \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \frac{\Delta t^2}{2}\frac{\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n - \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n}{\Delta x} + O(\Delta t^3)$$

$$\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n = A_{i+1/2}^n \frac{E_{i+1}^n - E_i^n}{\Delta x} \qquad \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n = A_{i-1/2}^n \frac{E_i^n - E_{i-1}^n}{\Delta x}$$

$$A_{i+1/2}^n = \frac{1}{2}\left(A_i^n + A_{i+1}^n\right) = \frac{1}{2}\left(u_i^n + u_{i+1}^n\right) \qquad A_{i-1/2}^n = \frac{1}{2}\left(A_i^n + A_{i-1}^n\right) = \frac{1}{2}\left(u_i^n + u_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\left[\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\right]$$

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\left(\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\right)$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^3 u}{\partial x^3}$ ⇒ dispersive
- dispersive errors are smallest for $C = \frac{\Delta t}{\Delta x}\max\left(|u|\right) = 1$, they increase for smaller C

Code:
C=0.5,
C=1, C=0.1

MacCormack

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^* = u_i^n + -\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right)$$
$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{\Delta t}{\Delta x}\left(E_i^* - E_{i-1}^*\right)\right]$$

Code:
C=0.5,
C=1, C=0.1

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^3 u}{\partial x^3}$ $\Rightarrow$ dispersive
- quite good in general!
- dispersive errors are smallest for $C = \frac{\Delta t}{\Delta x}\max\left(|u|\right) = 1$, they increase for smaller C
- different from Lax-Wendroff

Code: cmp. C=0.5