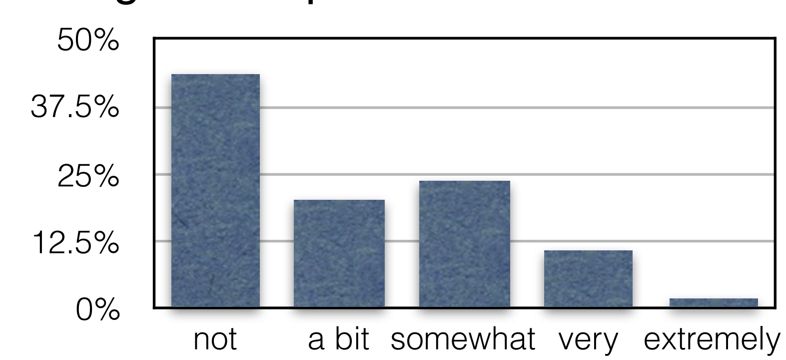
- Muddiest Points from Class 01/11
  - "[…] Are these derivations critical to know and understand for this course or were they shown for more background and understanding?"
  - "[…] can you let us know to what level of detail we would be expected to use the derivations shown today?"
  - "Are we going to use derivations in our homework assignments? Will the final formula suffice when solving problems?"
  - "[…] But are these derivations necessary to know, or should we just be focusing on understanding the assumptions and other key steps that allowed us to arrive at these equations." […]
    - The derivations of the governing equations (PDEs) were shown for completeness sake and as background/reference (they are the topic of Fluid Mechanics classes)
    - Going forward we will start with given PDEs that we will then solve numerically
  - "are there equations with ∇ meaning both "gradient" and "divergence" and if so, how do we tell which is which?"
  - "[…] difference between divergence and a gradient […]"

gradient: just the ∇ followed by a scalar: $\nabla\varphi$ result is a vector: $\nabla\varphi = \begin{bmatrix} \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \\ \frac{\partial\varphi}{\partial z} \end{bmatrix}$

divergence: ∇ followed by a dot and a vector: $\nabla\cdot\vec{v}$ result is a scalar: $\nabla\cdot\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

- "Unrelated to lecture, my piazza link seems to be not working, perhaps others are having similar problems."
  - I've included a link to the Piazza sign-up page on Blackboard
  - Also please read the note posted on Blackboard about Piazza Careers

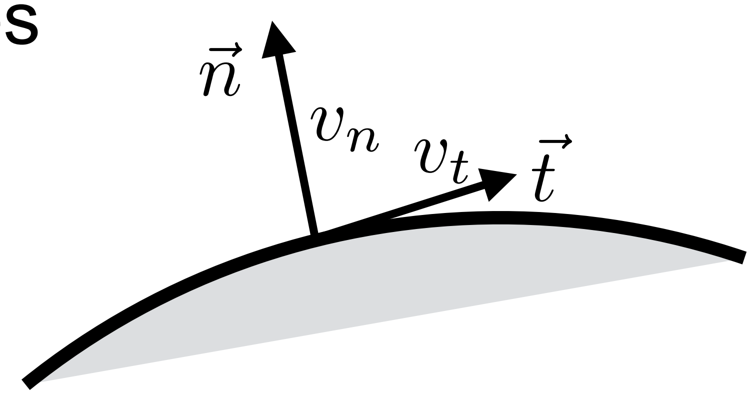


- **Governing Equations**
  - Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
  - Momentum: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$
  - Energy: $$\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho \vec{v} h) = \nabla \cdot (k \nabla T) + \vec{v} \cdot \nabla p + \overline{\overline{\tau}} : \nabla p + \frac{\partial p}{\partial t}$$
  - EOS: $$p = \rho R T$$

• Missing piece: Boundary Conditions (BC)

- Geometry of the problem dictates the type of boundary

I) Solid surfaces



v_n : surface normal velocity

$\Rightarrow v_n = 0$: no flow though surface

v_t : surface tangential velocity

$\Rightarrow v_t = 0$: fluid adheres to surface

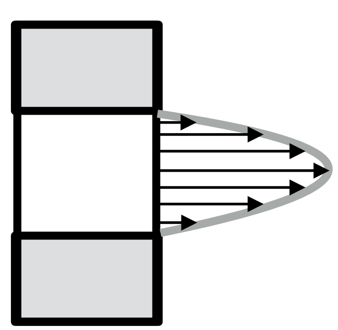
$\Rightarrow$ no slip condition

BUT: boundary condition has to be consistent with the simplifications used in the governing equations!

Example: if Euler equations $\Rightarrow$ inviscid $\Rightarrow$ cannot enforce no-slip

$\Rightarrow$ need slip bc: $v_t \neq 0$

II) Inlets



given (measured) velocity profile: $\vec{v}_{in} = \vec{v}(\vec{x})$

III) Outlets later …

- In general we have 3 types of boundary conditions:
  - Dirichlet BC: specify dependent variable on boundary

    Example: no-slip wall: $\vec{v} = \vec{0}$

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  - Neumann BC: specify normal gradient of dependent variable on boundary

    Example: adiabatic wall: $\dfrac{\partial T}{\partial n} = 0$
  - Robin BC: combination of the above two

- Classification of Differential Equations
  - Why? Numerical solution procedures depend on the type of differential equation

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I) Linear vs non-linear

- Linear: the dependent variable and its derivatives do not appear in products or powers

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Example: - 1D wave equation: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

let u_1 and u_2 be two solutions, then u_1+u_2 is a solution, too!

$\frac{\partial u_1}{\partial t} + a\frac{\partial u_1}{\partial x} = 0$ and $\frac{\partial u_2}{\partial t} + a\frac{\partial u_2}{\partial x} = 0$ $\overset{\text{add}}{\Rightarrow}$ $\frac{\partial (u_1+u_2)}{\partial t} + a\frac{\partial (u_1+u_2)}{\partial x} = 0$

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- Stokes flow

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Example: - Burgers equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

- Navier-Stokes

- Classification of Differential Equations
  - Why? Numerical solution procedures depend on the type of differential equation

I) Linear vs non-linear

- Linear: the dependent variable and its derivatives do not appear in products or powers
- Non-linear: the dependent variable and/or its derivatives appear in products and/or powers
- quasi-linear: PDE with linear highest derivative

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I) Linear vs non-linear

II) Order of highest derivative

  - Navier-Stokes: 2nd-order

- Classification of Differential Equations
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- Navier-Stokes: 2nd-order
- Let's look in more detail at 2nd-order PDEs:
  - 2D model PDE:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0$$

$$\phi = \phi(x, y) \qquad A,\ B,\ \ldots,\ G = f_i(x, y, \phi)$$

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turns out, type is dictated solely by $B^2 - 4AC$

II)Order of highest derivative

- 2D model PDE:

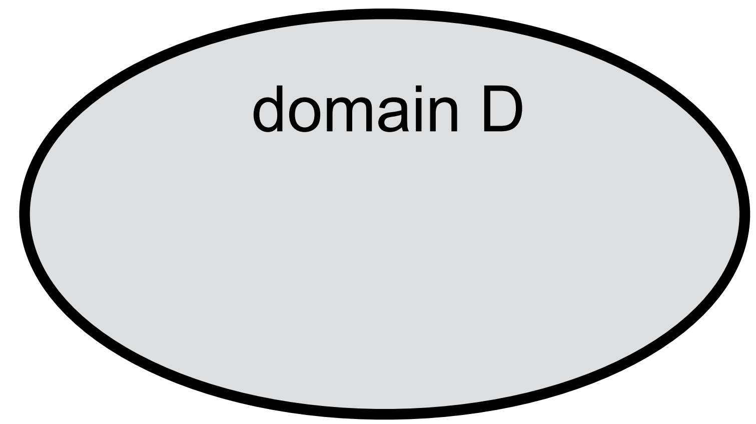
$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}+D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+F\phi+G=0$$

$$\phi=\phi(x,y)\qquad A,\ B,\ \ldots,\ G=f_i(x,y,\phi)$$

- $B^2-4AC<0$: elliptical PDE
- $B^2-4AC=0$: parabolic PDE
- $B^2-4AC>0$: hyperbolic PDE

- WARNING: PDEs can change type, since B^2-4AC is a function of x,y,Φ!

- Elliptic PDEs: $B^2\text{-}4AC < 0$ everywhere
  - no real characteristic curves (curves along which information/disturbances travel)
  - disturbances travel **instantly** in **all** directions
  - domain of solution is a **closed** domain



  - Examples:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(Laplace equation)}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y) \qquad \text{(Poisson equation)}$$

must provide bc on border:

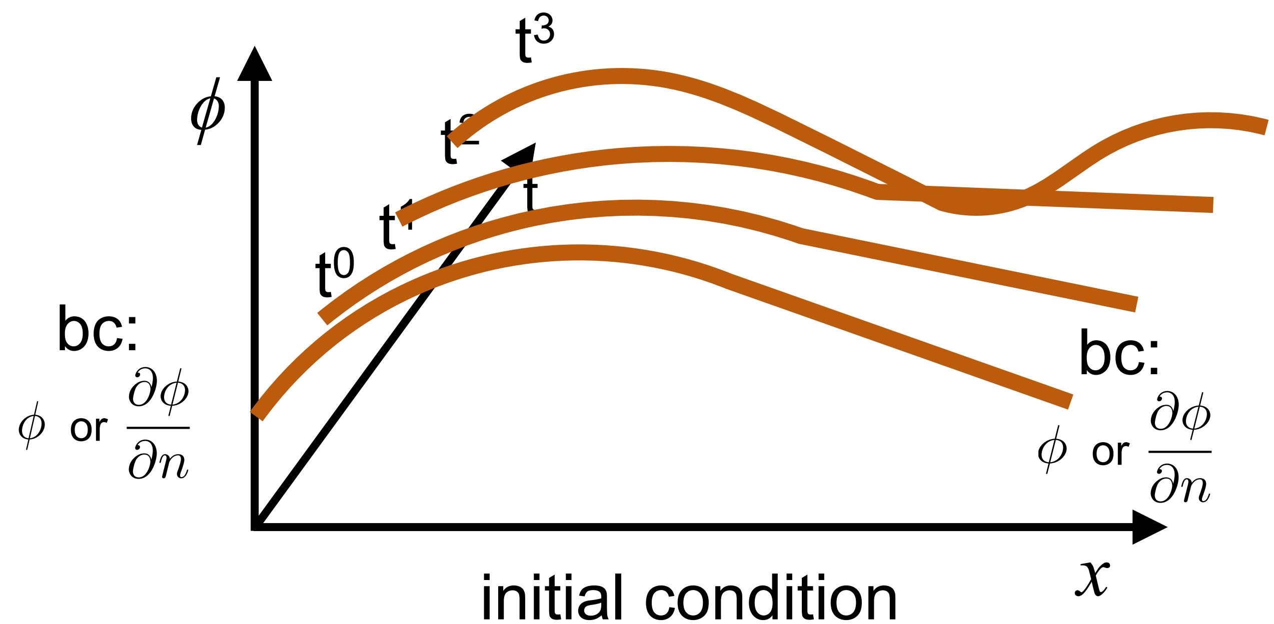
either ϕ or $\dfrac{\partial \phi}{\partial n}$

- Parabolic PDEs: B^2-$4AC = 0$ everywhere
  - domain of solution is an **open** region
  - comparable to initial value ODE ⇒ solution marches forward in time
  - Examples:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$ (heat conduction)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

boundary layer approximations



- Hyperbolic PDEs: B^2-$4AC > 0$ everywhere
  - Example:

$$\frac{\partial^2 \phi}{\partial t^2} = a^2 \frac{\partial^2 \phi}{\partial x^2}$$ (2nd order wave equation)

  - requires 2 initial conditions: $\phi(x, t=0) = f(x)$ and $\dfrac{\partial \phi(x, t=0)}{\partial t} = g(x)$
  - requires 2 boundary conditions
  - hyperbolic PDEs can be solved by the “Method of Characteristics”

    ⇒ reduces PDE to ODE along characteristic lines

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}+D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+F\phi+G=0$$

- Example
  - 2D velocity potential ϕ in incompressible, inviscid flow

$$\left(1-M^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

$M=\frac{u}{a}$: Mach number

$$\Rightarrow \quad A=\left(1-M^2\right) \qquad B=0 \qquad C=1$$

$$B^2-4AC=0^2-4\left(1-M^2\right)\cdot 1=4\left(M^2-1\right)$$

$\Rightarrow \quad M<1: \; B^2-4AC<0$: elliptic

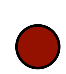
$M=1: \; B^2-4AC=0$: parabolic

$M>1: \; B^2-4AC>0$: hyperbolic

Example: Body moving at speed u, creating a disturbance moving with speed of sound a

Mach number: $M = \dfrac{u}{a}$

M < 1 : elliptic



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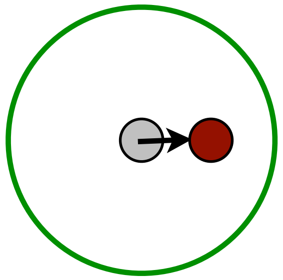

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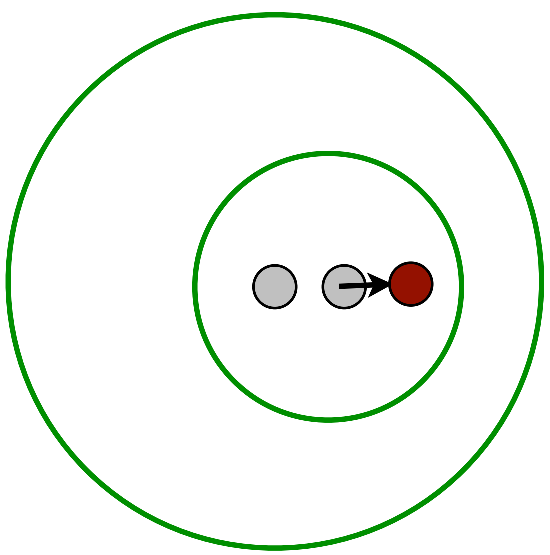

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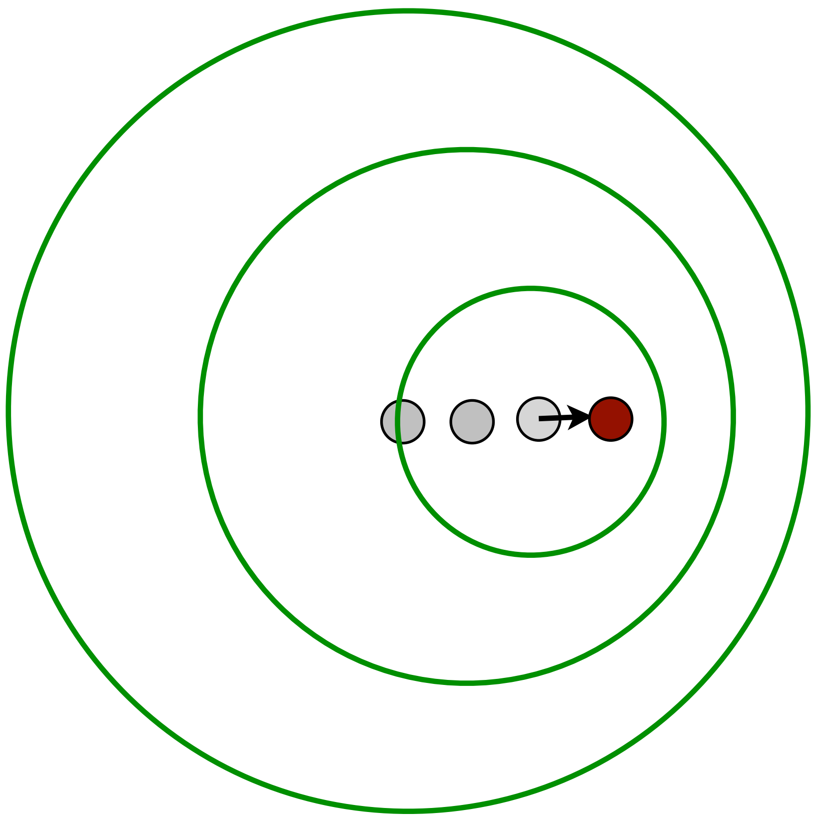

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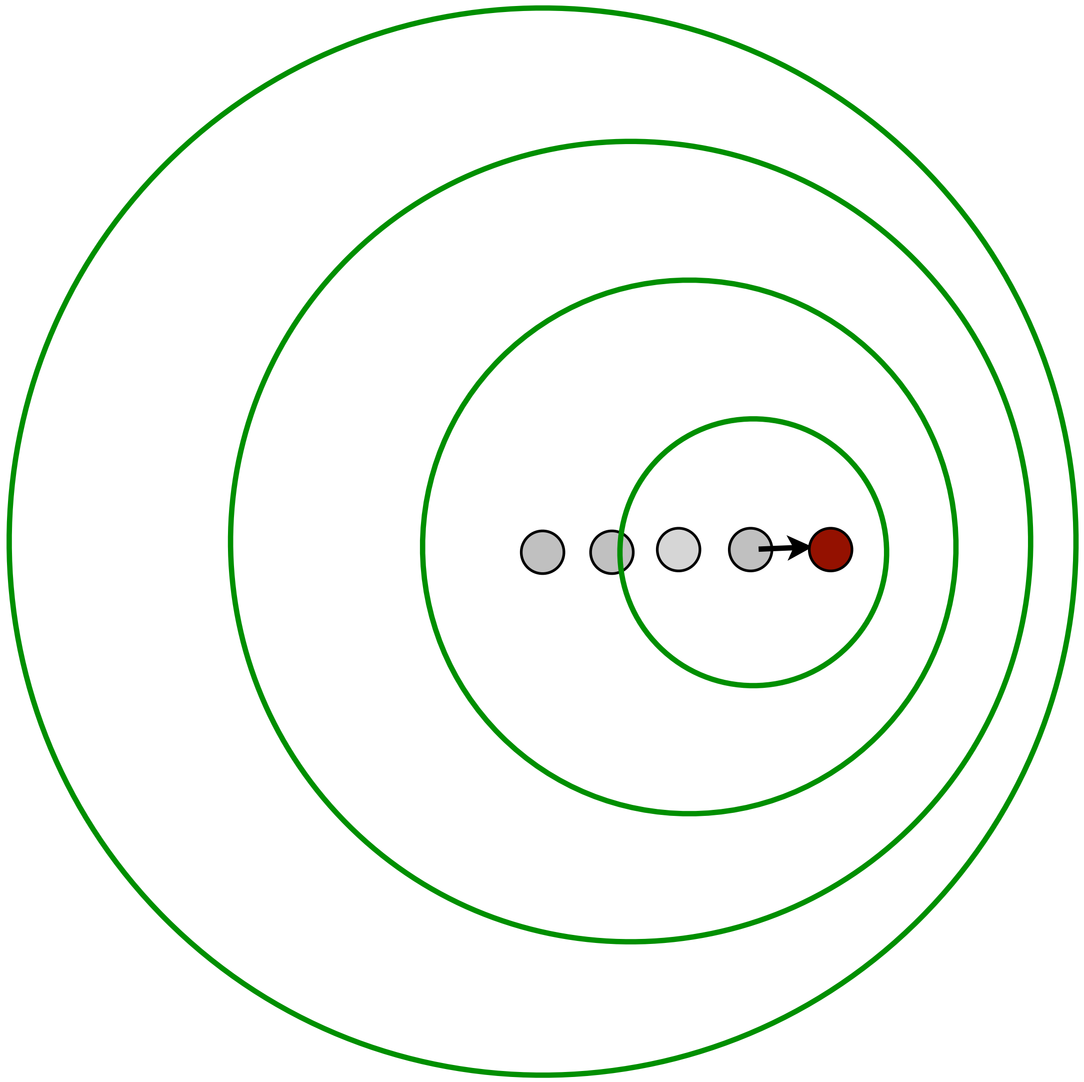

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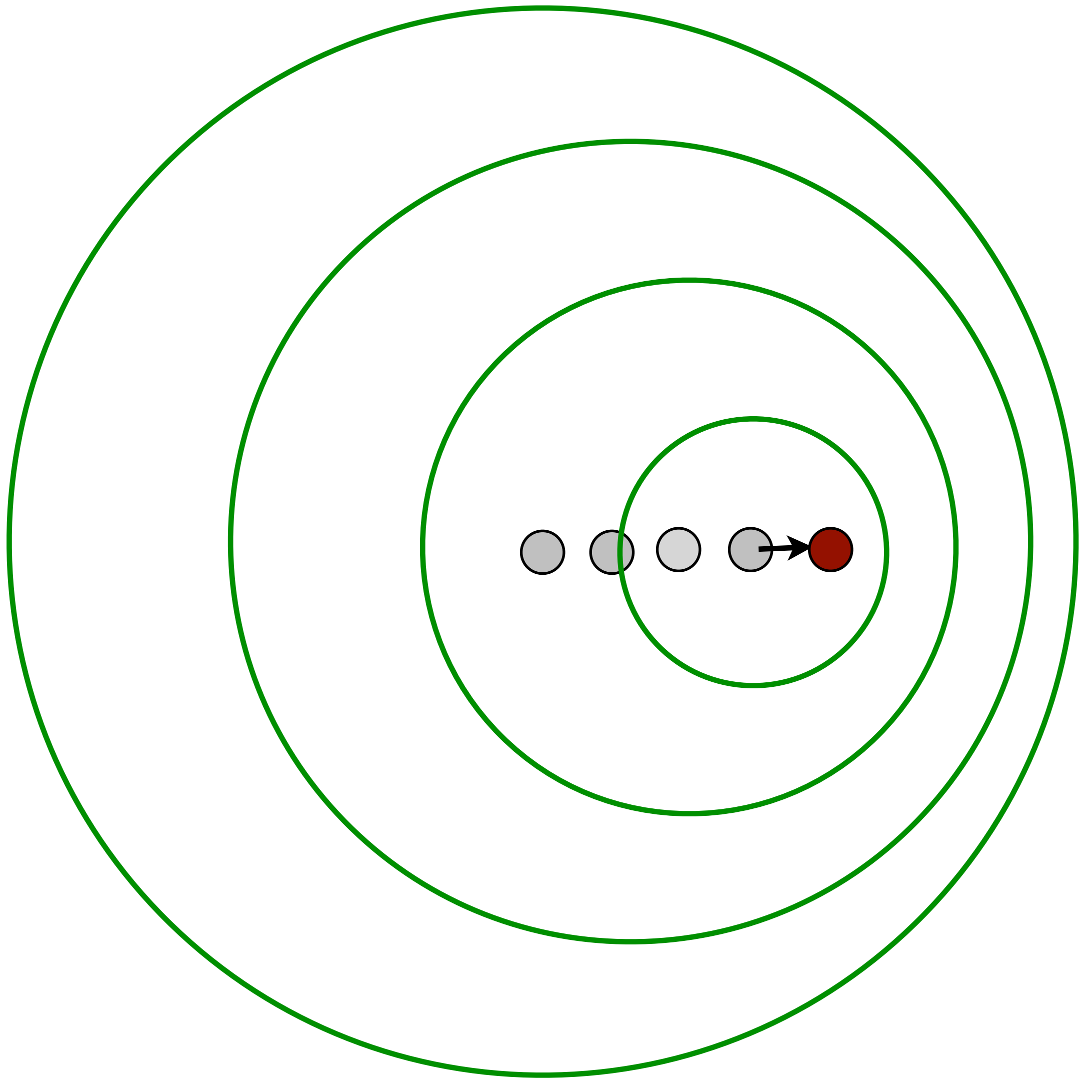

disturbance is felt everywhere

Example: Body moving at speed u, creating a disturbance moving with speed of sound a

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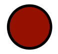
M = 1 : parabolic



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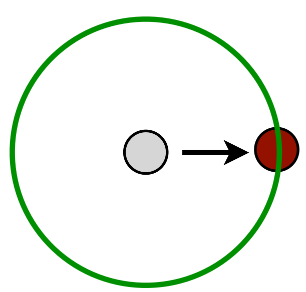

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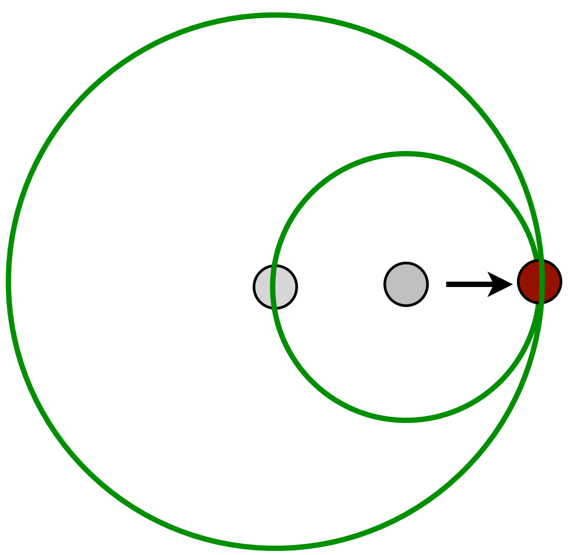

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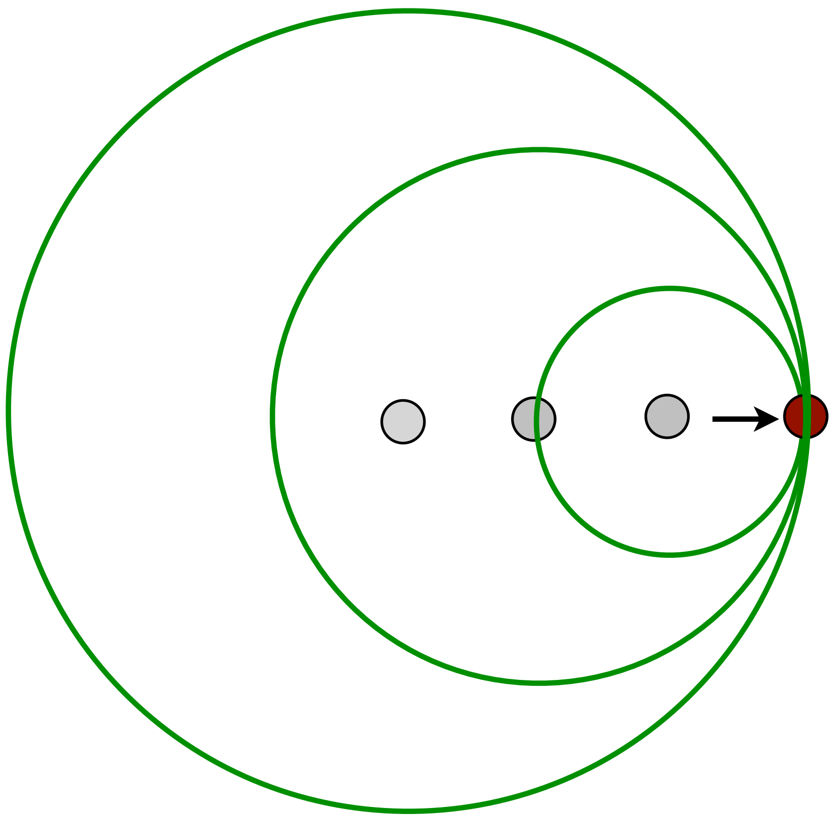

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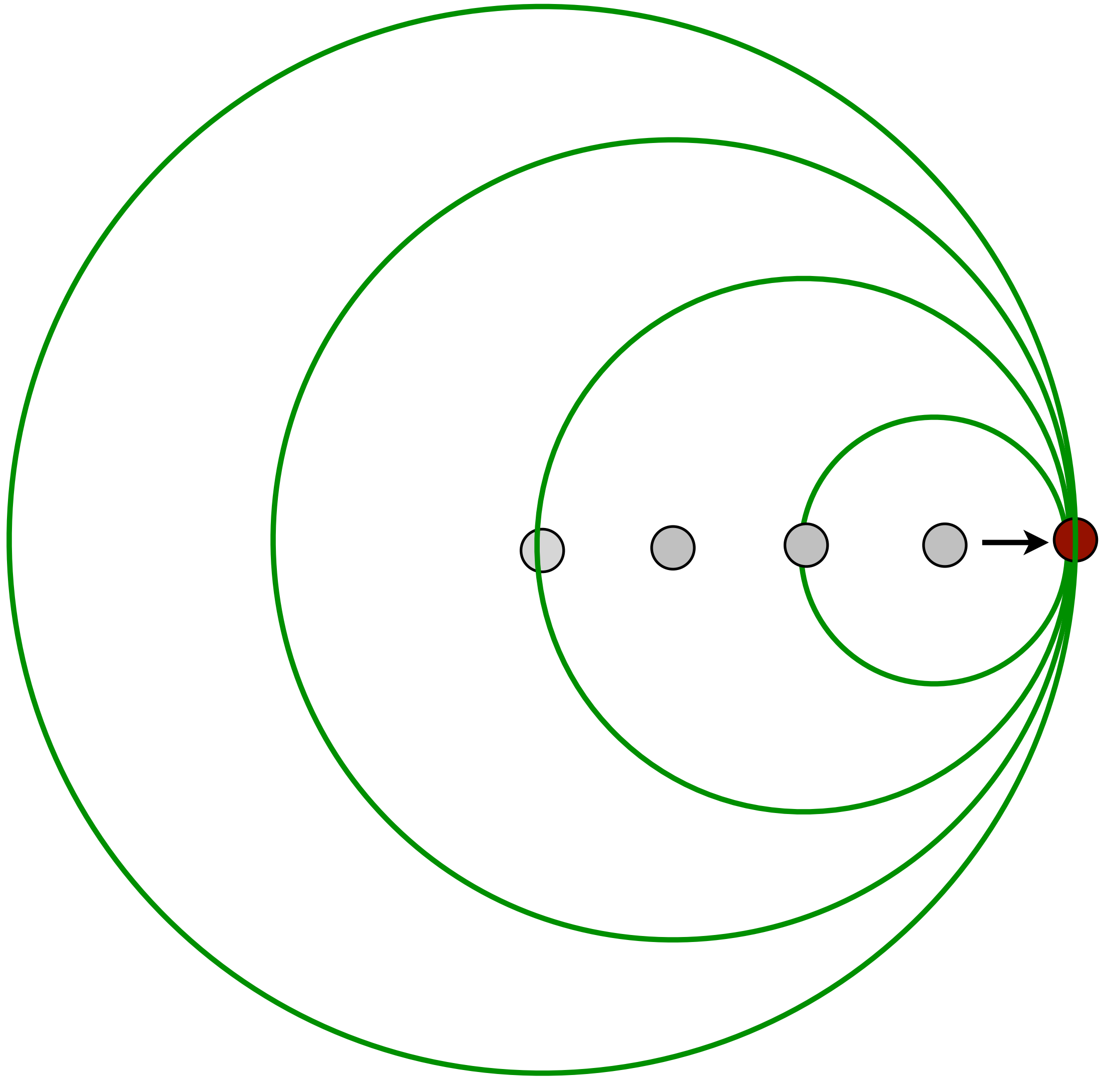

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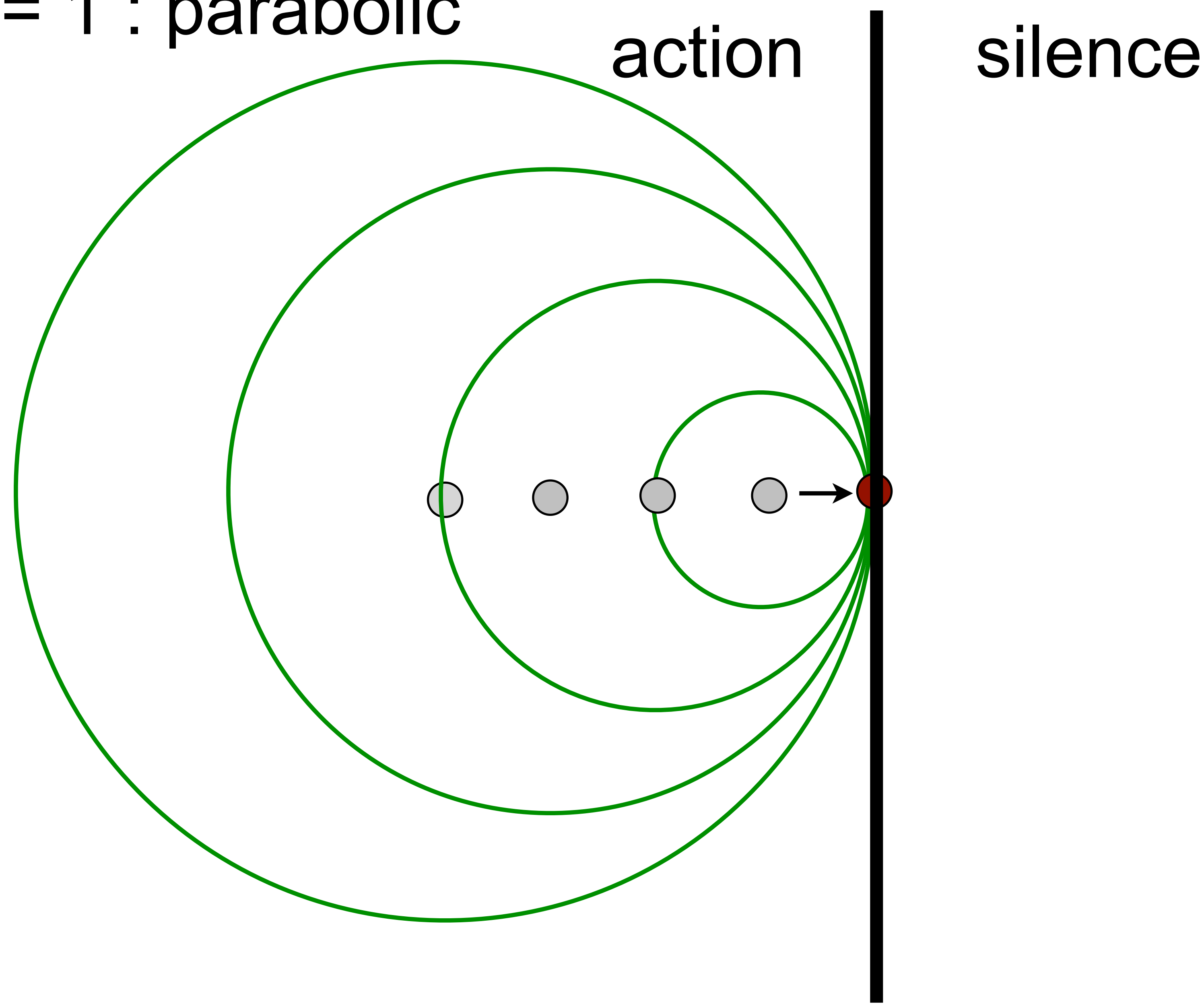

Example: Body moving at speed u, creating a disturbance moving with speed of sound a

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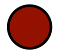
M > 1 : hyperbolic



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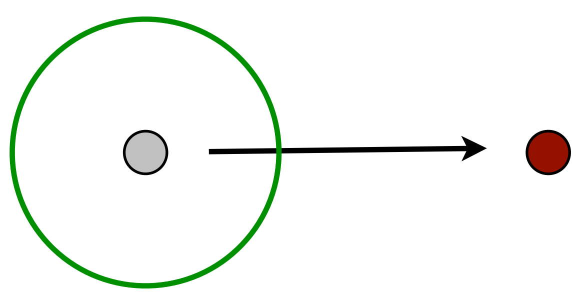

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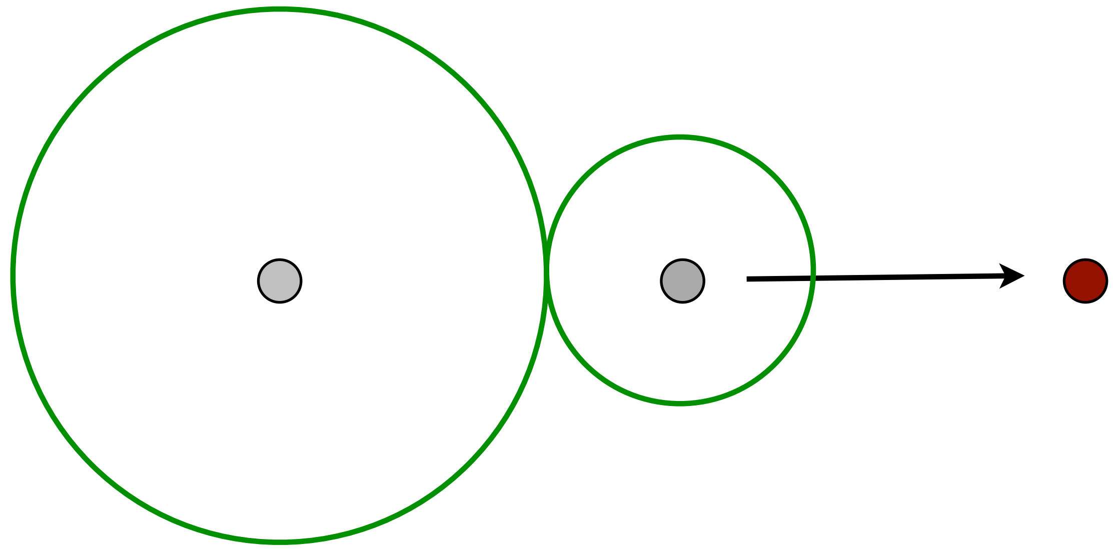

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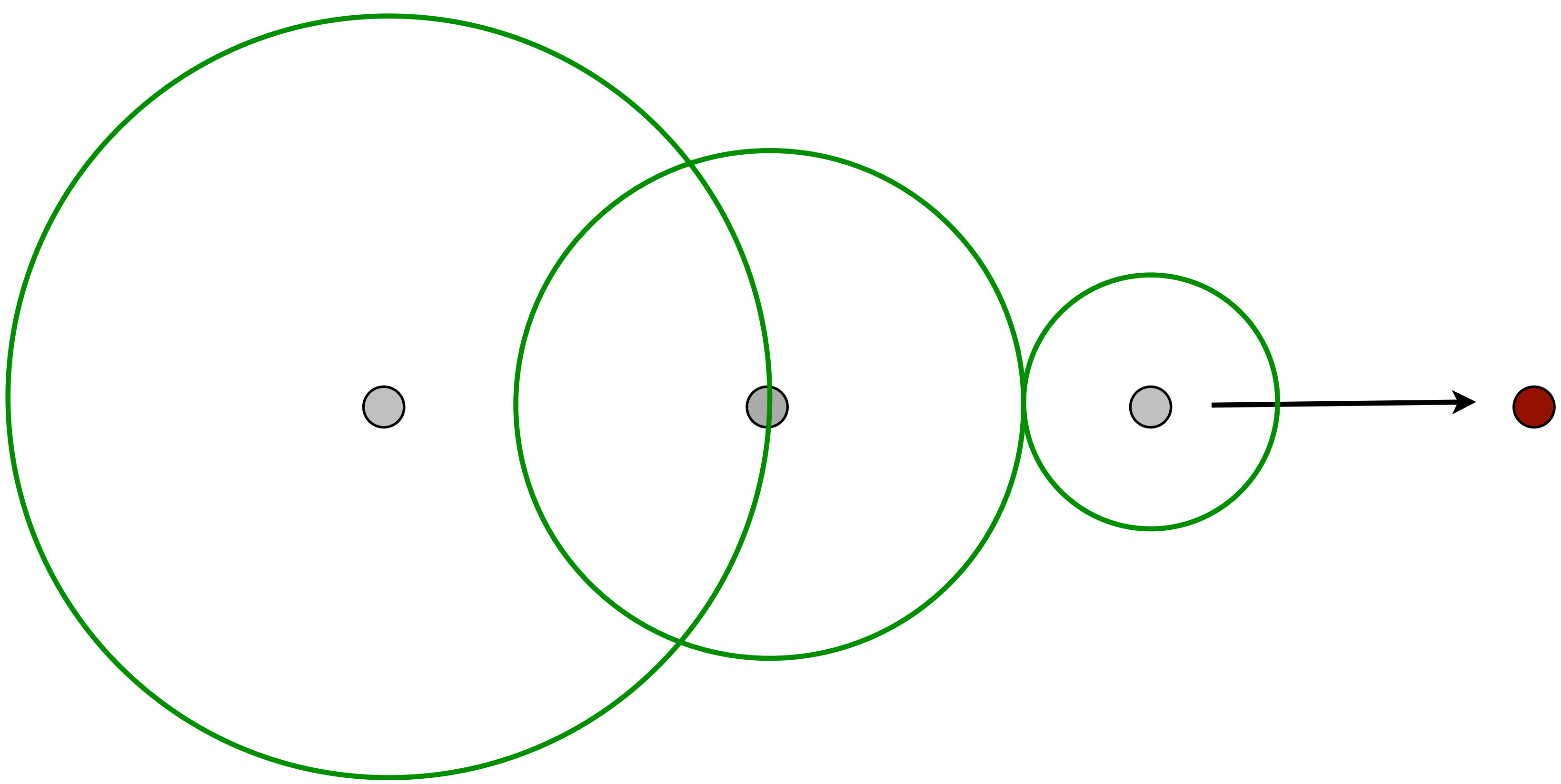

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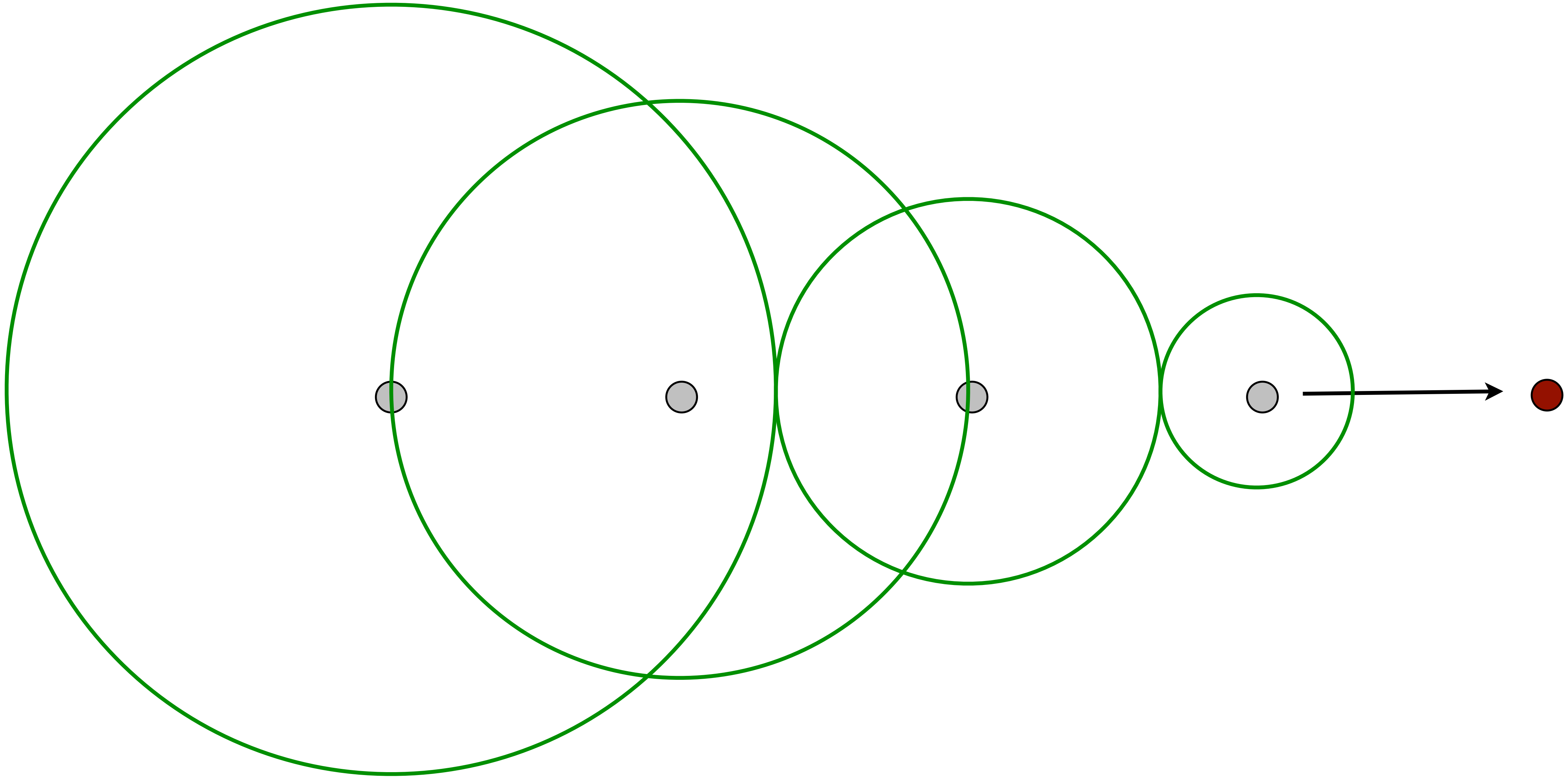

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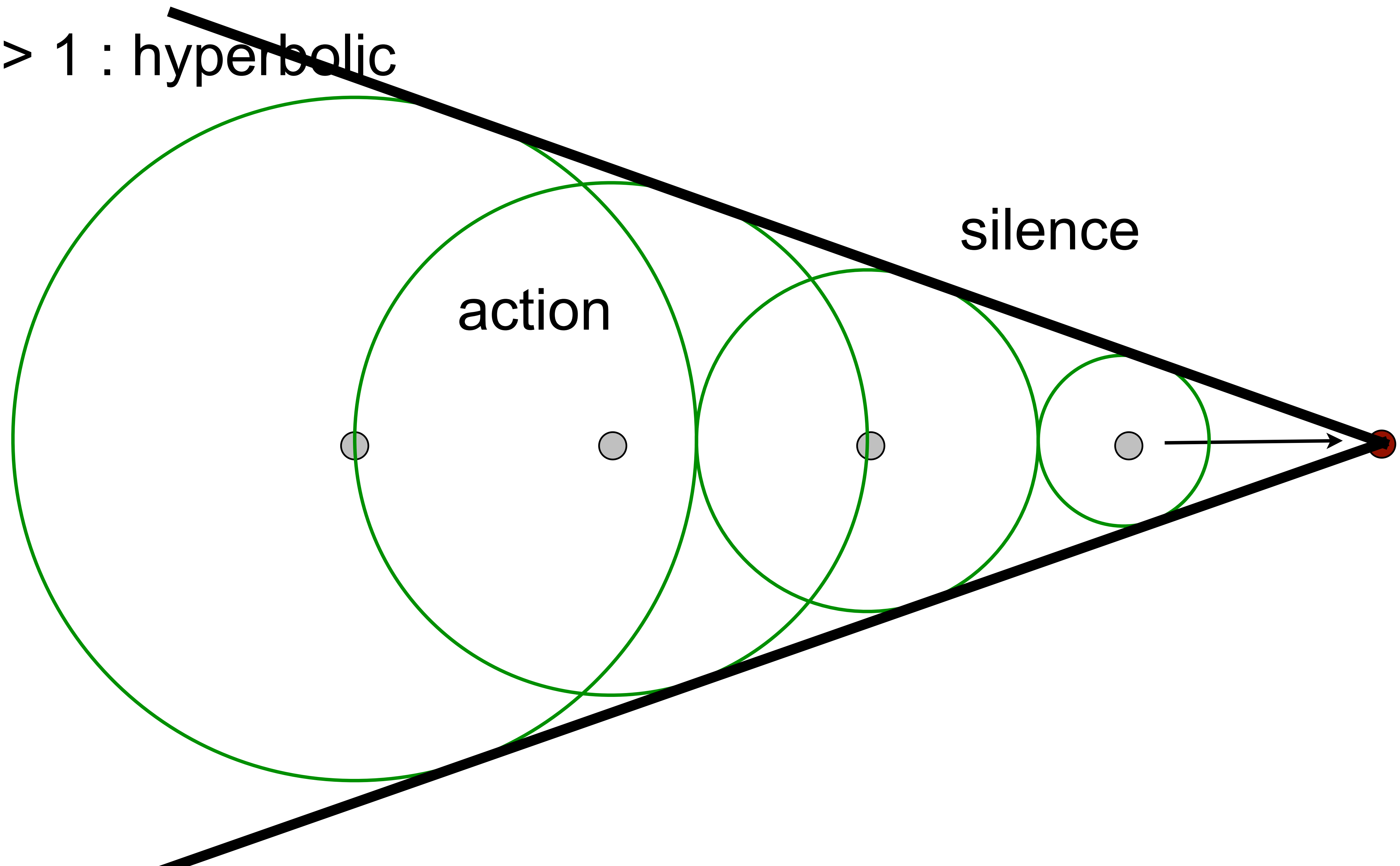


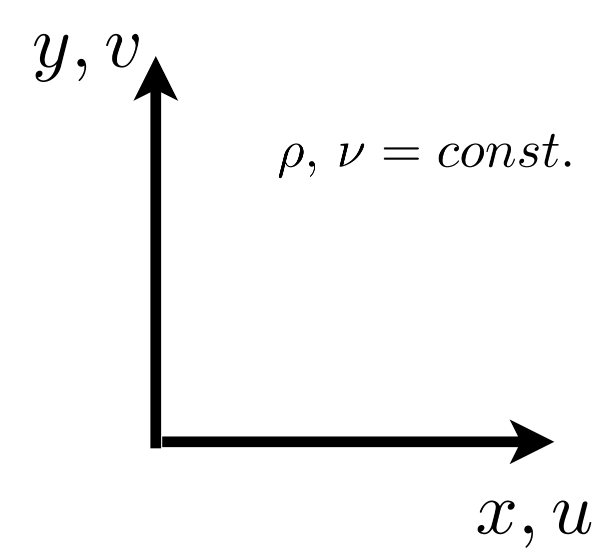
- **What have we done so far?**
  - derived governing equations
  - looked at simplifications
  - looked at classifications of PDEs

- **What's the final goal of this class?**
  - your own code to solve Navier-Stokes in 2D in the incompressible limit

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$



- but, to do this, we need simpler model equations first
- Why?
  to understand and apply numerical methods used in CFD one by one

I) Laplace and Poisson equations

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f(x, y)$$

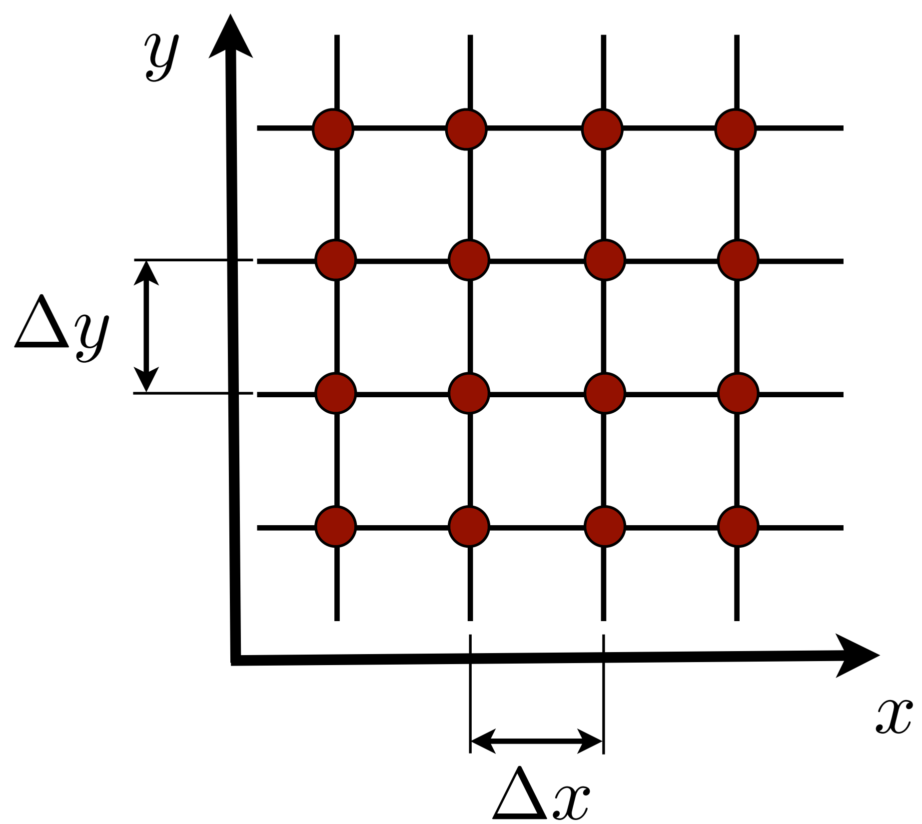
II) Heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{(1D)} \qquad \frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad \text{(2D)}$$

III) Wave and Burgers' equation

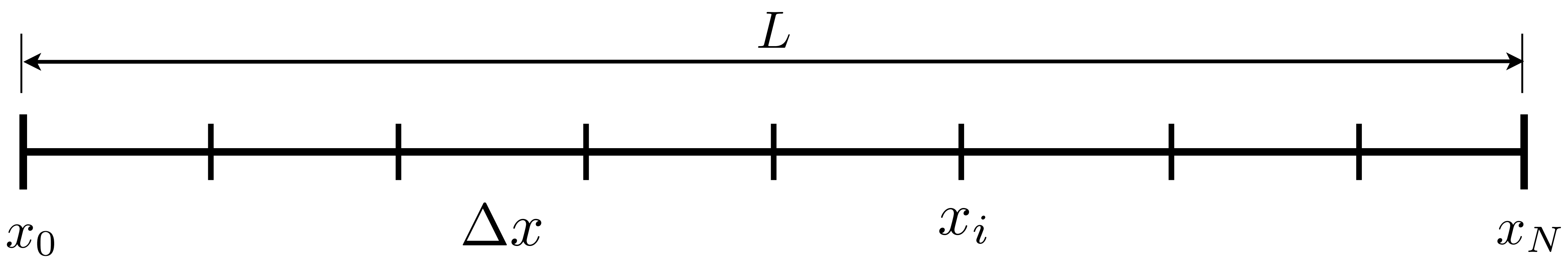
$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

- More definitions and conventions
  - reality is a continuum (at least on the macro/micro scale)
  - but: we'll represent it by solutions on a network of discrete points (grid)



- grid spacing: $\Delta x,\ \Delta y$
  need not be equal or even constant
- grid point: $x_i,\ y_j$

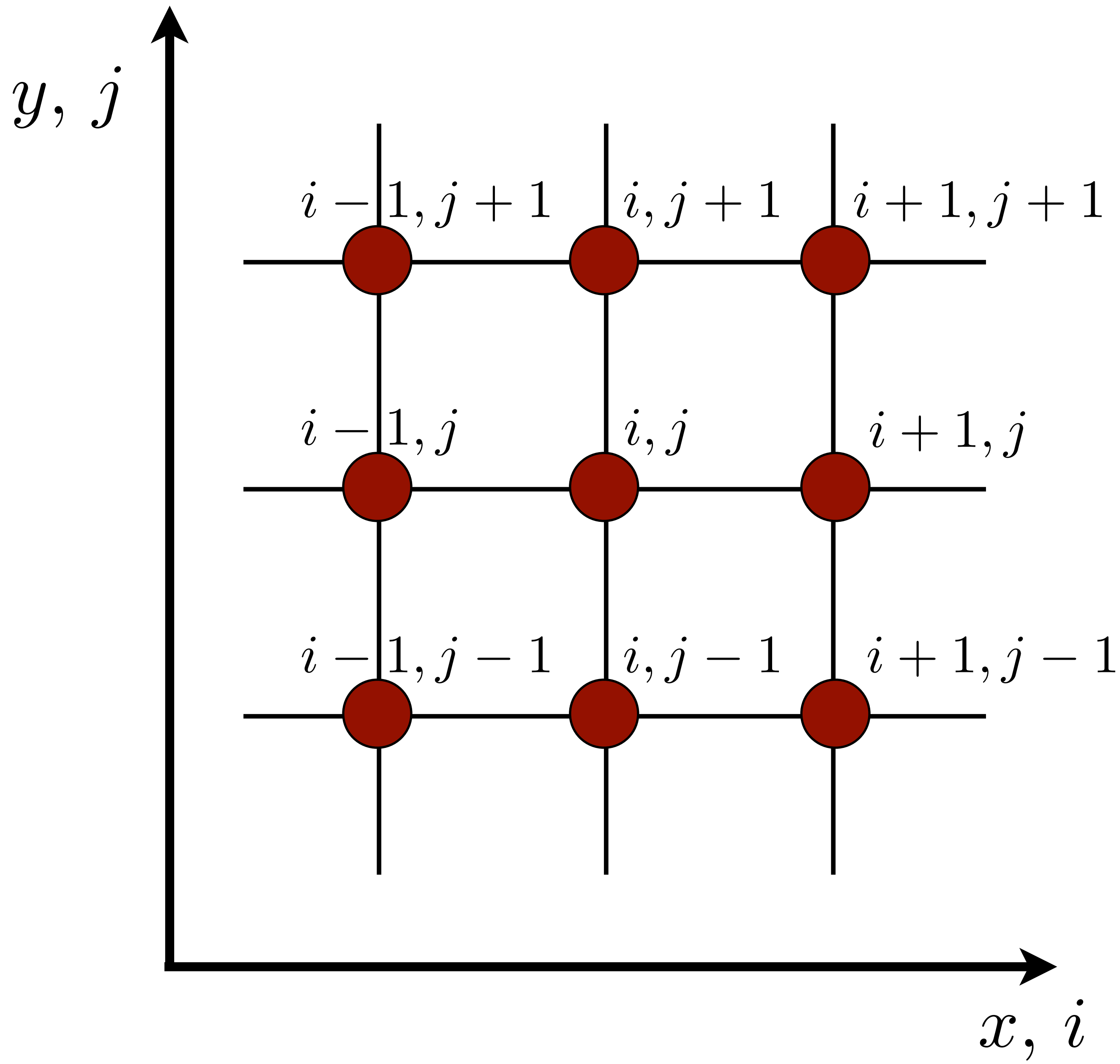
- More definitions and conventions
  - Example:
    - divide domain starting at x_0 and length *L* into *N* equal sized elements



    - grid point spacing: $\Delta x = \dfrac{L}{N}$
    - grid point location: $x_i = x_0 + i\Delta x = x_0 + i\dfrac{L}{N} \qquad i = 0,\ 1, \ldots,\ N$
    - total number of elements: ***N***
    - total number of grid points: ***N + 1***
  - Same can be done for y-direction:

$$y_j = y_0 + j\Delta y \qquad j = 0,\ 1, \ldots,\ M$$

- More definitions and conventions
  - we can identify a specific grid point by its (i,j) coordinate



- We want to enforce the PDEs following application of some numerical scheme at every grid point

- So finally here goes:
  - most equations we want to solve look like this:

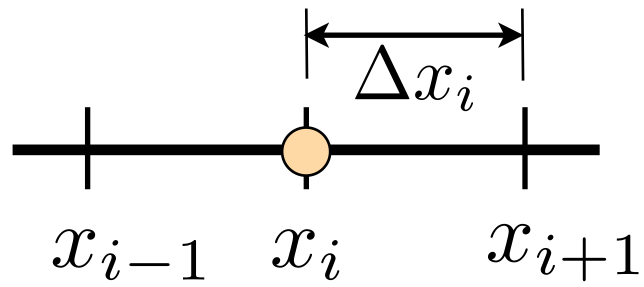
$$\frac{\partial}{\partial t}(\ldots) + \text{spatial derivatives} = 0$$

  - Solution strategy:
    1) approximate spatial derivatives
    2) integrate resulting ODEs using some method

1) Approximate spatial derivates by **finite differences**

- Example #1:

$$\left.\frac{\partial f}{\partial x}\right|_{x_i}$$



$$\Delta x_i = x_{i+1} - x_i$$

- How? **Taylor Series!**

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)\left.\frac{df}{dx}\right|_{x_i} + \frac{1}{2}(x_{i+1} - x_i)^2 \left.\frac{d^2 f}{dx^2}\right|_{x_i} + \ldots$$

or

$$f_{i+1} = f_i + \Delta x_i f_i' + \frac{1}{2}\Delta x_i^2 f_i'' + \ldots$$

let's assume $\Delta x_i = const. = h$

$$f_{i+1} = f_i + h f_i' + \frac{1}{2} h^2 f_i'' + \ldots$$

solve for f_i':

$$f_i' = \frac{f_{i+1} - f_i}{h} - \frac{1}{2} h f_i'' + \ldots \quad \Leftrightarrow \quad \boxed{f_i' = \frac{f_{i+1} - f_i}{h} + O(h)}$$

order

Forward difference

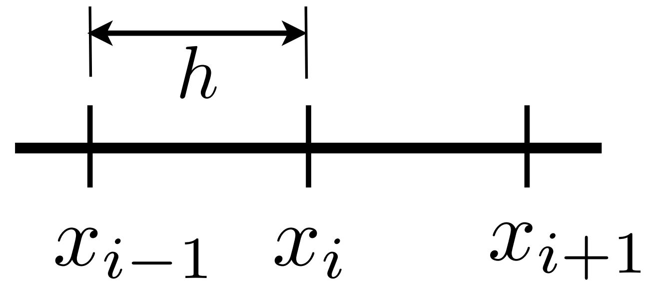
- Forward difference

$$f_i' = \frac{f_{i+1} - f_i}{h} + O(h^1)$$

  - exponent of *h* in *O(h)* is the **order of accuracy** of the method
    - here: order = 1
  - the order indicates how fast the error (the *O(h)* term) decreases with a reduction in *h*
    - here: reduce *h* by a factor 2 ⇒ error reduces by a factor $2^1 = 2$
  - Note: only leading order error term is important!
    Higher order error terms decrease faster = are smaller
    (provided h is sufficiently small)

- Example #2:

$\left.\dfrac{\partial f}{\partial x}\right|_{x_i}$ again, but TS for f_{i-1}



$$h = x_i - x_{i-1}$$

$$f(x_{i-1}) = f(x_i) + (x_{i-1} - x_i) \left.\frac{df}{dx}\right|_{x_i} + \frac{1}{2}(x_{i-1} - x_i)^2 \left.\frac{d^2 f}{dx^2}\right|_{x_i} + \ldots$$

$$f_{i-1} = f_i - h f_i' + \frac{1}{2}(-h)^2 f_i'' + \ldots$$

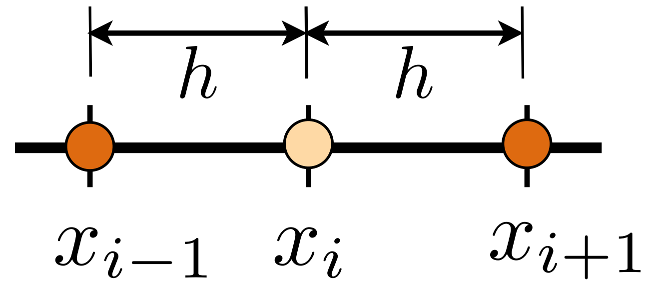
$$\Leftrightarrow \quad f_i' = \frac{f_i - f_{i-1}}{h} + O(h)$$

Backward difference

Question: What's the order? Answer: 1

- Example #3:

$\left.\dfrac{\partial f}{\partial x}\right|_{x_i}$ again, but TS for f_{i+1} & f_{i-1}



$$
\begin{array}{rcl}
f_{i+1} & = & f_i + h f_i' + \frac{1}{2} h^2 f_i'' + \frac{1}{6} h^3 f_i''' + \ldots \\
- \quad f_{i-1} & = & f_i - h f_i' + \frac{1}{2} h^2 f_i'' - \frac{1}{6} h^3 f_i''' + \ldots \\
\hline
f_{i+1} - f_{i-1} & = & 2h f_i' + \frac{1}{3} h^3 f_i''' + \ldots
\end{array}
$$

$$\Leftrightarrow \quad f_i' = \frac{f_{i+1} - f_{i-1}}{2h} - \frac{1}{6} h^2 f_i''' + \ldots$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$$

Central difference

Question: What's the order? Answer: 2

$$\text{as } h \rightarrow \frac{h}{2} : \ \text{error} \rightarrow \frac{\text{error}}{4}$$