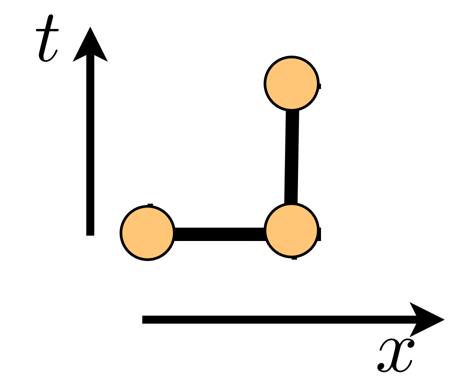
Linear 1D Wave Equation

$$\frac{\varphi_i^{n+1}-\varphi_i^n}{\Delta t} = -a\left(\frac{\varphi_i^n-\varphi_{i-1}^n}{\Delta x}\right) \qquad a > 0$$

stencil:



- Consistency: $\boxed{\text{Board}}$
  - modified equation has diffusion-like leading error term

$$\frac{\partial\varphi}{\partial t} = -a\frac{\partial\varphi}{\partial x} + \frac{a\Delta x}{2}(1-C)\frac{\partial^2\varphi}{\partial x^2} + \ldots$$

  - typical for upwind methods

- Issue: the above upwind FDE is only $O(\Delta x)$
  - Use higher order one-sided (upwind biased) approximations to $\dfrac{\partial\varphi}{\partial x}$?

$$\frac{\varphi_i^{n+1}-\varphi_i^n}{\Delta t} = -a\left(\frac{3\varphi_i^n-4\varphi_{i-1}^n+\varphi_{i-2}^n}{2\Delta x}\right) \qquad a > 0 \qquad O(\Delta x^2)$$

but @ boundaries: need to drop order to make stencil fit (1st-order at bc)

=

Consistency: $\varphi_i^{n+1} = \varphi_i^n - \frac{a\Delta t}{\Delta x}\left(\varphi_i^n - \varphi_{i-1}^n\right)$

T.S.:

$$\cancel{\varphi_i^n} + \Delta t \frac{\partial \varphi}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 \varphi}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 \varphi}{\partial t^3} + O(\Delta t^4) = \cancel{\varphi_i^n} - \frac{a\Delta t}{\Delta x}\left(\cancel{\varphi_i^n} - (\cancel{\varphi_i^n} - \Delta x \frac{\partial \varphi}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 \varphi}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 \varphi}{\partial x^3} + O(\Delta x^4)\right]$$

$$\frac{\partial \varphi}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 \varphi}{\partial t^3} + O(\Delta t^3) = -a\left(\frac{\partial \varphi}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\Delta x^2}{6}\frac{\partial^3 \varphi}{\partial x^3} + O(\Delta x^3)\right)$$

$$\Leftrightarrow \frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial t^2} + a\frac{\Delta x}{2}\frac{\partial^2 \varphi}{\partial x^2} - \frac{\Delta t^2}{6}\frac{\partial^3 \varphi}{\partial t^3} - a\frac{\Delta x^2}{6}\frac{\partial^3 \varphi}{\partial x^3} + O(\Delta x^3) + O(\Delta t^3)$$

Goal: try to express r.h.s in terms of $\frac{\partial}{\partial x^n}$ only.

to maintain accuracy, we thus need:

$$\left.\begin{array}{l} \frac{\partial^2 \varphi}{\partial t^2} \text{ with } O(\Delta t^2), O(\Delta x^2) \\ \frac{\partial^3 \varphi}{\partial t^3} \text{ with } O(\Delta t)\ O(\Delta x) \end{array}\right\}$$ long and tedious. See Appendix C of Hoffman & Chiang.

Shortcut: let's use the PDE (exact formulas!) ↯

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \quad \Big|\frac{\partial}{\partial t} \qquad \Rightarrow \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right) \overset{PDE}{=} \frac{\partial}{\partial t}\left(-a\frac{\partial \varphi}{\partial x}\right) = -\frac{\partial}{\partial x}\left(a\frac{\partial \varphi}{\partial t}\right) = a^2\frac{\partial^2 \varphi}{\partial x^2}$$

$$\Rightarrow \frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} - a^2\frac{\Delta t}{2}\frac{\partial^2 \varphi}{\partial x^2} + a\frac{\Delta x}{2}\frac{\partial^2 \varphi}{\partial x^2} + \ldots$$

$$\Leftrightarrow \quad \frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} + \frac{1}{2}\left(a \Delta x - a^2 \Delta t\right) \frac{\partial^2 \varphi}{\partial x^2} + \ldots\ldots$$

$$\Leftrightarrow \quad \frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} + \frac{a \Delta x}{2}\left(1 - \underbrace{\frac{a^2 \Delta t}{a \Delta x}}_{C}\right) \frac{\partial^2 \varphi}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} + \underbrace{\frac{a \Delta x}{2}(1 - C) \frac{\partial^2 \varphi}{\partial x^2}}_{\text{diffusion like term!}} + \ldots\ldots$$

modified equation!

Differencing gives a leading order term that acts like diffusion! (typical for upwind methods)

- Is there a way to fix FTCS?

$$\varphi_i^{n+1} = \boxed{\varphi_i^n} - \frac{C}{2}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) \qquad a > 0$$

- remember Du Fort-Frankel fix to Richardson?
  - let's try something similar, but in space instead of time:

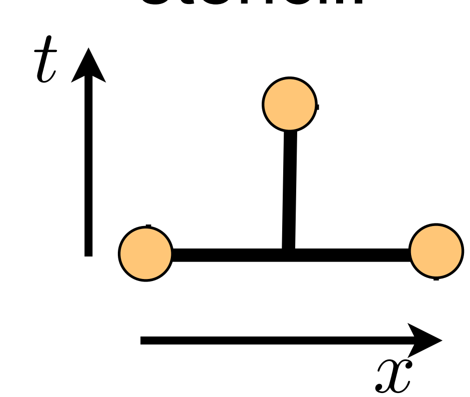
$$\varphi_i^{n+1} = \overline{\varphi_i^n} - \frac{C}{2}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) \qquad \overline{\varphi_i^n} = \frac{1}{2}\left(\varphi_{i+1}^n + \varphi_{i-1}^n\right)$$

$$\varphi_i^{n+1} = \frac{1}{2}\left(\varphi_{i+1}^n + \varphi_{i-1}^n\right) - \frac{C}{2}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right)$$

$$\boxed{\varphi_i^{n+1} = \frac{1}{2}\left(1 - C\right)\varphi_{i+1}^n + \frac{1}{2}\left(1 + C\right)\varphi_{i-1}^n}$$

Lax-Method

stencil:



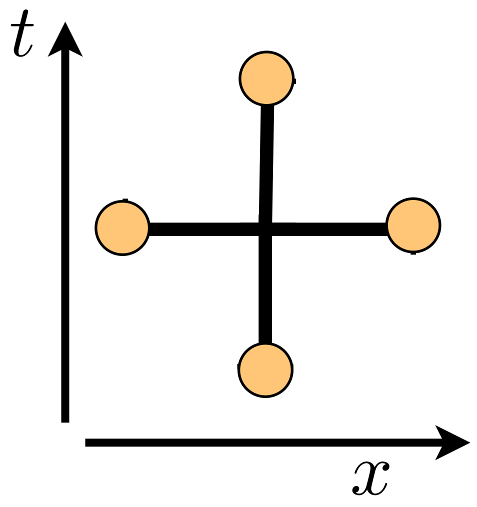
  - $O(\Delta x^2)$ and $O(\Delta t)$
  - stable for $C \leq 1$

- ok, now we have 2nd-order in space, how to get 2nd-order in time?
  - go central in time!

Midpoint Leapfrog

$$\frac{\varphi_i^{n+1} - \varphi_i^{n-1}}{2\Delta t} = -a\left(\frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2\Delta x}\right)$$

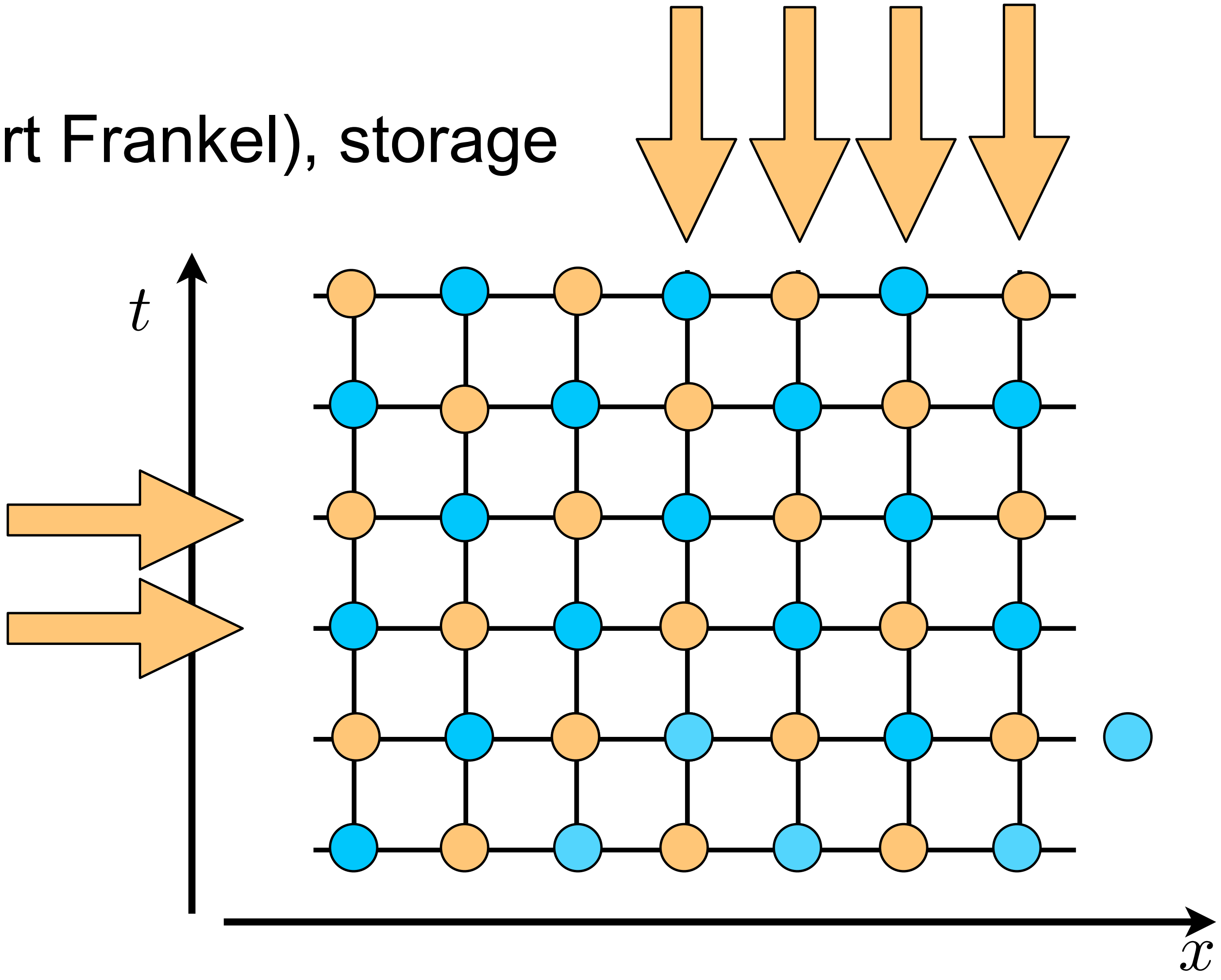
stencil:



- $O(\Delta x^2)$ and $O(\Delta t^2)$
- stable for $C \leq 1$

- But:
  - start-up problem (cp. Du-Fort Frankel), storage
  - de-coupling of solutions!

checker boarding!



Lax-Wendroff

- Idea: Let's revisit Taylor Series **Board**

$$\varphi_i^{n+1} = \varphi_i^n - \frac{C}{2}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) + \frac{C^2}{2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

- $O(\Delta x^2)$ and $O(\Delta t^2)$
- stable for $C \leq 1$

Lax-Wendroff

Let's revisit Taylor Series: $\varphi(x,t+\Delta t) = \varphi(x,t) + \Delta t \frac{\partial \varphi}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 \varphi}{\partial t^2} + O(\Delta t^3)$

$$\Rightarrow \varphi_i^{n+1} = \varphi_i^n + \Delta t \frac{\partial \varphi}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 \varphi}{\partial t^2} + O(\Delta t^3)$$

What's $\frac{\partial^2 \varphi}{\partial t^2}$? Let's use PDE (as before): $\frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} \Rightarrow \frac{\partial^2 \varphi}{\partial t^2} = a^2 \frac{\partial^2 \varphi}{\partial x^2}$

$$\Rightarrow \varphi_i^{n+1} = \varphi_i^n + \Delta t \left(-a \frac{\partial \varphi}{\partial x}\right) + \frac{\Delta t^2}{2} a^2 \frac{\partial^2 \varphi}{\partial x^2} + O(\Delta t^3)$$

FDE:

$$\varphi_i^{n+1} = \varphi_i^n - a \Delta t \frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2 \Delta x} + \frac{1}{2} a^2 \Delta t^2 \frac{\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n}{\Delta x^2}$$

$$\Leftrightarrow \varphi_i^{n+1} = \varphi_i^n - \frac{C}{2}\left(\varphi_{i+1}^n - \varphi_{i-1}^n\right) + \frac{1}{2} C^2 \left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

$\Rightarrow O(\Delta t^2), O(\Delta x^2)$, stable for $C \leq 1$