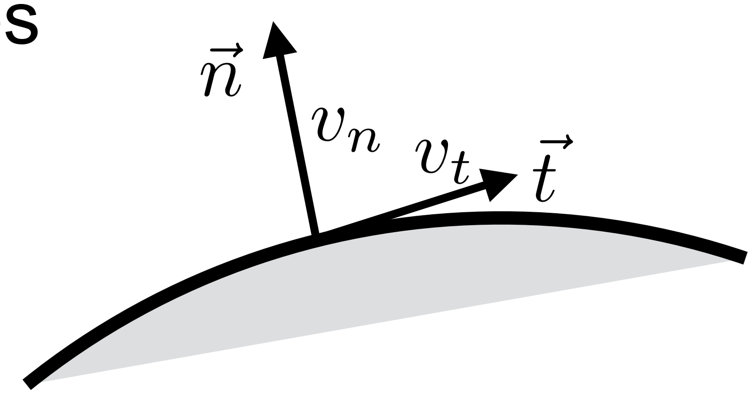
• Missing piece: Boundary Conditions (BC)

- Geometry of the problem dictates the type of boundary

I) Solid surfaces



v_n : surface normal velocity

$\Rightarrow v_n = 0$: no flow though surface

v_t : surface tangential velocity

$\Rightarrow v_t = 0$: fluid adheres to surface

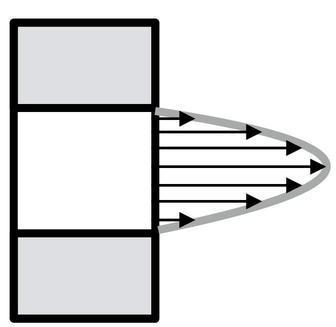
$\Rightarrow$ no slip condition

BUT: boundary condition has to be consistent with the simplifications used in the governing equations!

Example: if Euler equations $\Rightarrow$ inviscid $\Rightarrow$ cannot enforce no-slip

$\Rightarrow$ need slip bc: $v_t \neq 0$

II) Inlets



given (measured) velocity profile: $\vec{v}_{in} = \vec{v}(\vec{x})$

III) Outlets later …

- In general we have 3 types of boundary conditions:
  - Dirichlet BC: specify dependent variable on boundary
    Example: no-slip wall: $\vec{v} = \vec{0}$
  - Neumann BC: specify normal gradient of dependent variable on boundary
    Example: adiabatic wall: $\frac{\partial T}{\partial n} = 0$
  - Robin BC: combination of the above two

- Classification of Differential Equations
  - Why? Numerical solution procedures depend on the type of differential equation

I) Linear vs non-linear

- Linear: the dependent variable and its derivatives do not appear in products or powers

⇒ two solutions can be superposed!

Example: - 1D wave equation: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

let u_1 and u_2 be two solutions, then u_1+u_2 is a solution, too!

$\frac{\partial u_1}{\partial t} + a\frac{\partial u_1}{\partial x} = 0$ and $\frac{\partial u_2}{\partial t} + a\frac{\partial u_2}{\partial x} = 0$ $\overset{\text{add}}{\Rightarrow}$ $\frac{\partial (u_1+u_2)}{\partial t} + a\frac{\partial (u_1+u_2)}{\partial x} = 0$

- Classification of Differential Equations
  - Why? Numerical solution procedures depend on the type of differential equation

I) Linear vs non-linear

- Linear: the dependent variable and its derivatives do not appear in products or powers

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Example: - 1D wave equation: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

- Stokes flow

- Classification of Differential Equations
  - Why? Numerical solution procedures depend on the type of differential equation

I) Linear vs non-linear

- Linear: the dependent variable and its derivatives do not appear in products or powers
- Non-linear: the dependent variable and/or its derivatives appear in products and/or powers

⇒ solutions do not superpose!

Example: - Burgers equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

- Navier-Stokes

- Classification of Differential Equations
  - Why? Numerical solution procedures depend on the type of differential equation

I) Linear vs non-linear

II) Order of highest derivative

- Navier-Stokes: 2nd-order
- Let's look in more detail at 2nd-order PDEs:
  - 2D model PDE:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0$$

$$\phi = \phi(x,y) \qquad A,\ B,\ \ldots,\ G = f_i(x,y,\phi)$$

turns out, type is dictated solely by $B^2 - 4AC$

II)Order of highest derivative

- 2D model PDE:

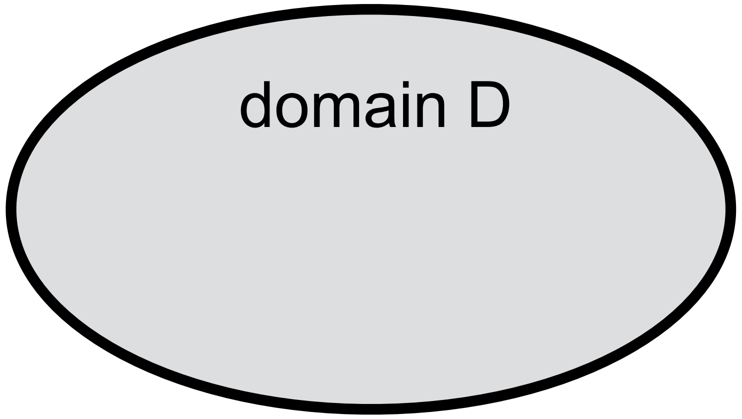
$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0$$

$$\phi = \phi(x, y) \qquad A,\ B,\ \ldots,\ G = f_i(x, y, \phi)$$

- $B^2 - 4AC < 0$: elliptical PDE
- $B^2 - 4AC = 0$: parabolic PDE
- $B^2 - 4AC > 0$: hyperbolic PDE

- WARNING: PDEs can change type, since B^2-4AC is a function of x,y,Φ!

- Elliptic PDEs: $B^2\text{-}4AC < 0$ everywhere
  - no real characteristic curves (curves along which information/disturbances travel)
  - disturbances travel **instantly** in **all** directions
  - domain of solution is a **closed** domain



must provide bc on border:

either ϕ or $\dfrac{\partial \phi}{\partial n}$

  - Examples:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(Laplace equation)}$$

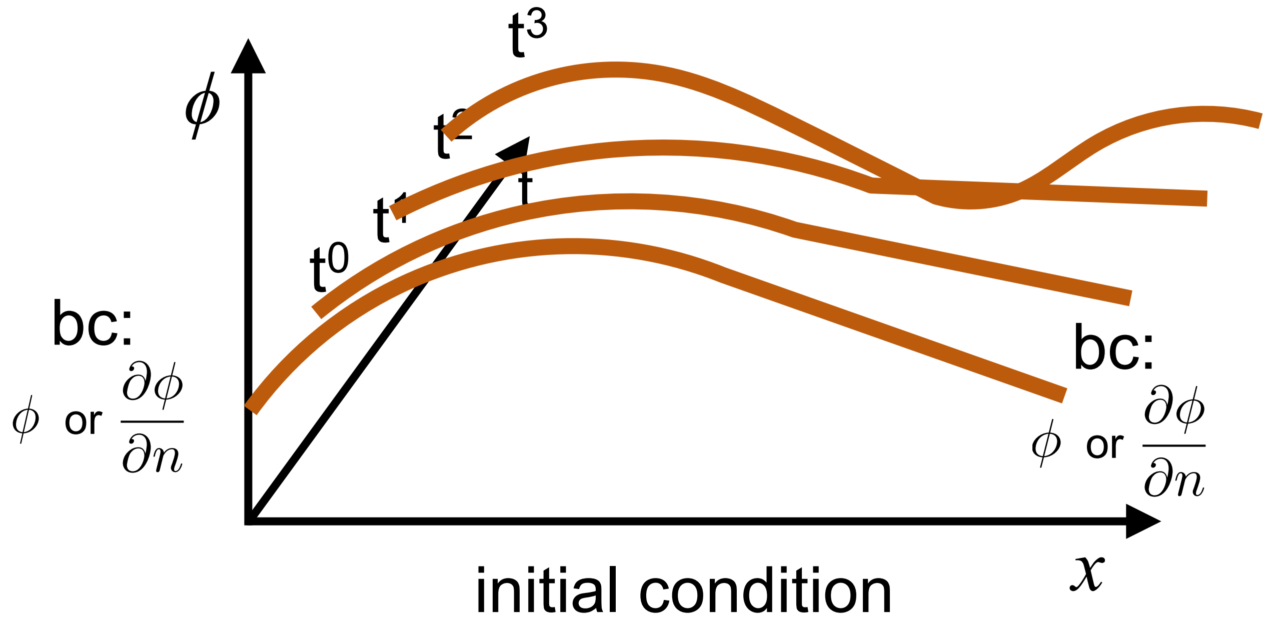
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y) \qquad \text{(Poisson equation)}$$

- Parabolic PDEs: $B^2\text{-}4AC = 0$ everywhere
  - domain of solution is an **open** region
  - comparable to initial value ODE ⇒ solution marches forward in time
  - Examples:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{(heat conduction)}$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

boundary layer approximations



- Hyperbolic PDEs: $B^2\text{-}4AC > 0$ everywhere
  - Example:

$$\frac{\partial^2 \phi}{\partial t^2} = a^2 \frac{\partial^2 \phi}{\partial x^2}$$ (2^{nd} order wave equation)

  - requires 2 initial conditions: $\phi(x, t = 0) = f(x)$ and $\dfrac{\partial \phi(x, t = 0)}{\partial t} = g(x)$
  - requires 2 boundary conditions
  - hyperbolic PDEs can be solved by the “Method of Characteristics”

    ⇒ reduces PDE to ODE along characteristic lines

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} + D\frac{\partial \phi}{\partial x} + E\frac{\partial \phi}{\partial y} + F\phi + G = 0$$

- Example
  - 2D velocity potential ϕ in incompressible, inviscid flow

$$\left(1 - M^2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad M = \frac{u}{a} \text{ : Mach number}$$

$$\Rightarrow \quad A = \left(1 - M^2\right) \qquad B = 0 \qquad C = 1$$

$$B^2 - 4AC = 0^2 - 4\left(1 - M^2\right) \cdot 1 = 4\left(M^2 - 1\right)$$

$$\Rightarrow \quad M < 1 : \; B^2 - 4AC < 0 \text{ : elliptic}$$

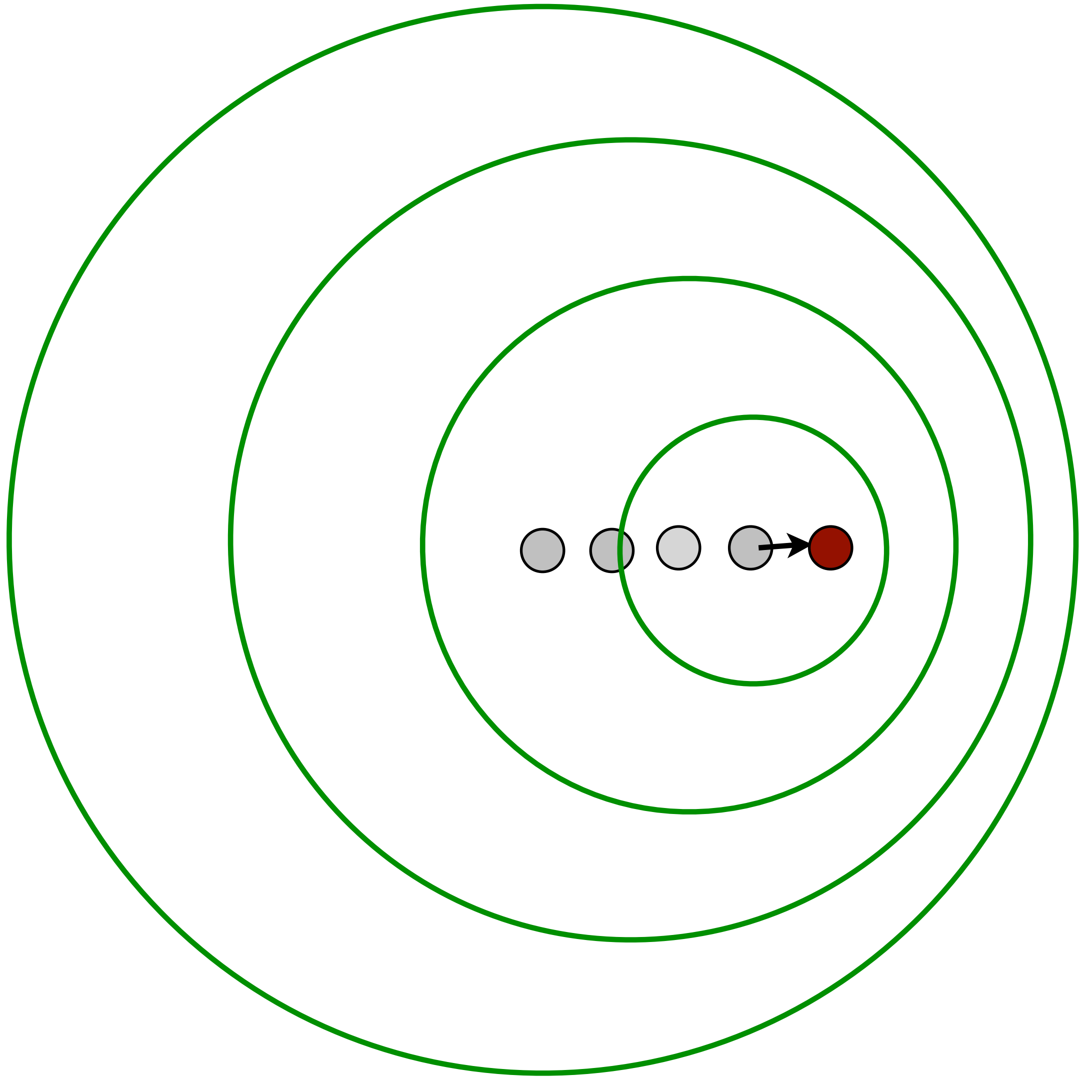
$$M = 1 : \; B^2 - 4AC = 0 \text{ : parabolic}$$

$$M > 1 : \; B^2 - 4AC > 0 \text{ : hyperbolic}$$

Example: Body moving at speed u, creating a disturbance moving with speed of sound a

Mach number: $M = \dfrac{u}{a}$

M < 1 : elliptic

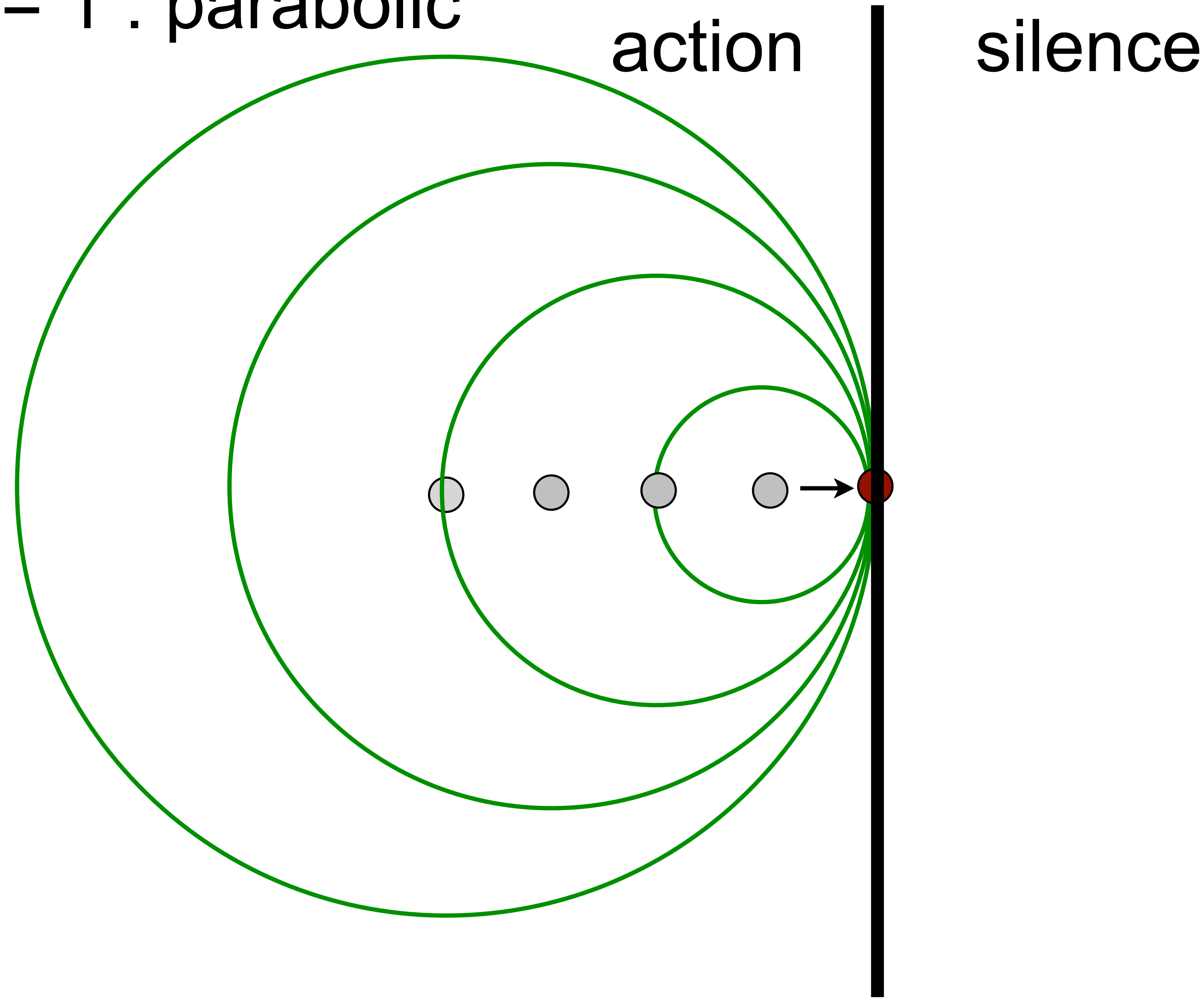


disturbance is felt everywhere

Example: Body moving at speed u, creating a disturbance moving with speed of sound a

Mach number: $M = \dfrac{u}{a}$

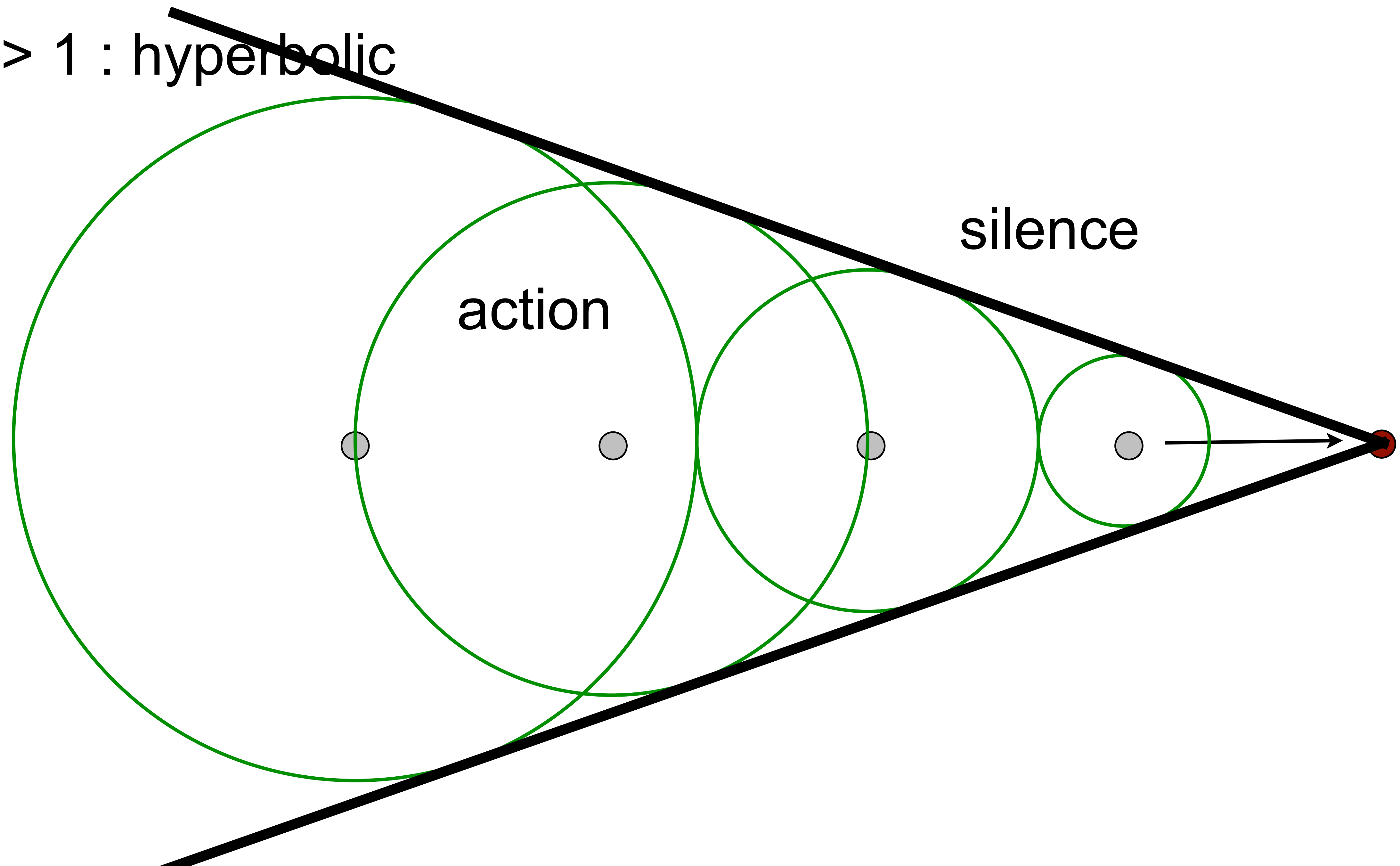
M = 1 : parabolic



Example: Body moving at speed u, creating a disturbance moving with speed of sound a

Mach number: $M = \dfrac{u}{a}$

M > 1 : hyperbolic



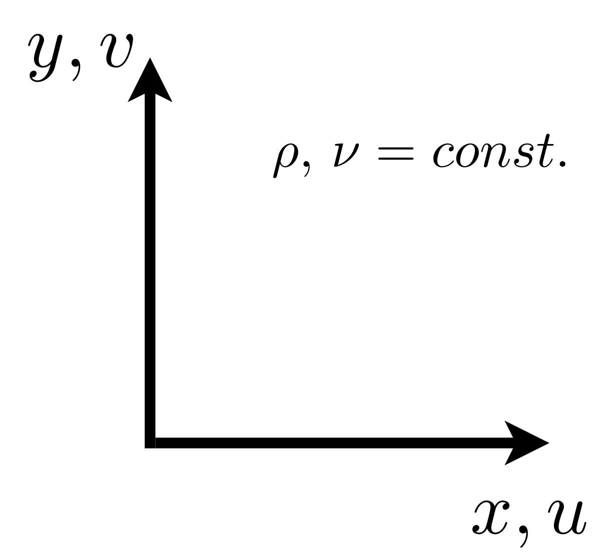
- **What have we done so far?**
  - derived governing equations
  - looked at simplifications
  - looked at classifications of PDEs

- **What's the final goal of this class?**
  - your own code to solve Navier-Stokes in 2D in the incompressible limit

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$



- but, to do this, we need simpler model equations first
- Why?
  to understand and apply numerical methods used in CFD one by one

I) Laplace and Poisson equations

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f(x, y)$$

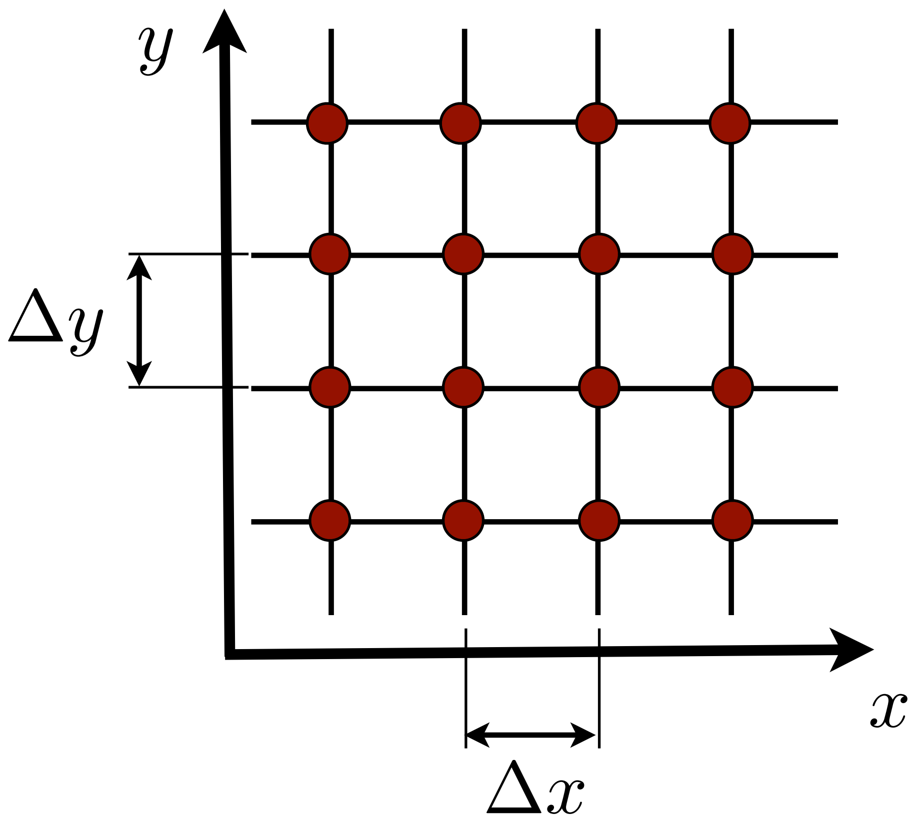
II) Heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{(1D)} \qquad \frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad \text{(2D)}$$

III) Wave and Burgers' equation

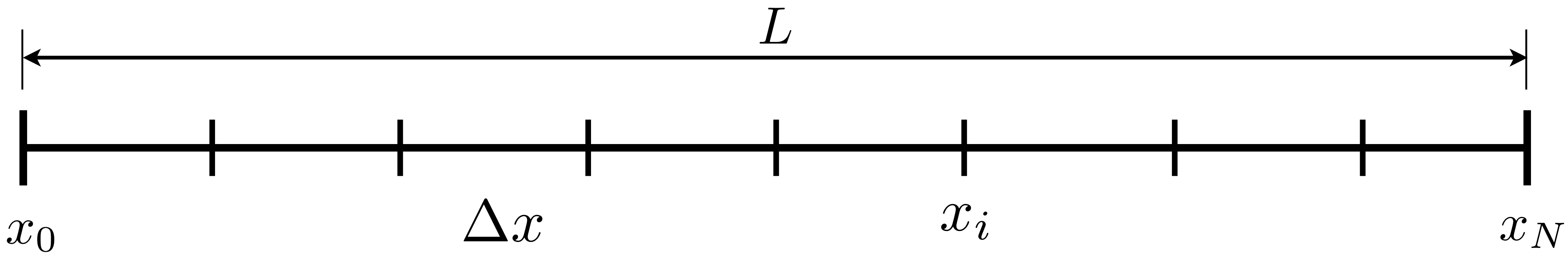
$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

- More definitions and conventions
  - reality is a continuum (at least on the macro/micro scale)
  - but: we'll represent it by solutions on a network of discrete points (grid)



- grid spacing: $\Delta x,\ \Delta y$
  need not be equal or even constant
- grid point: $x_i,\ y_j$

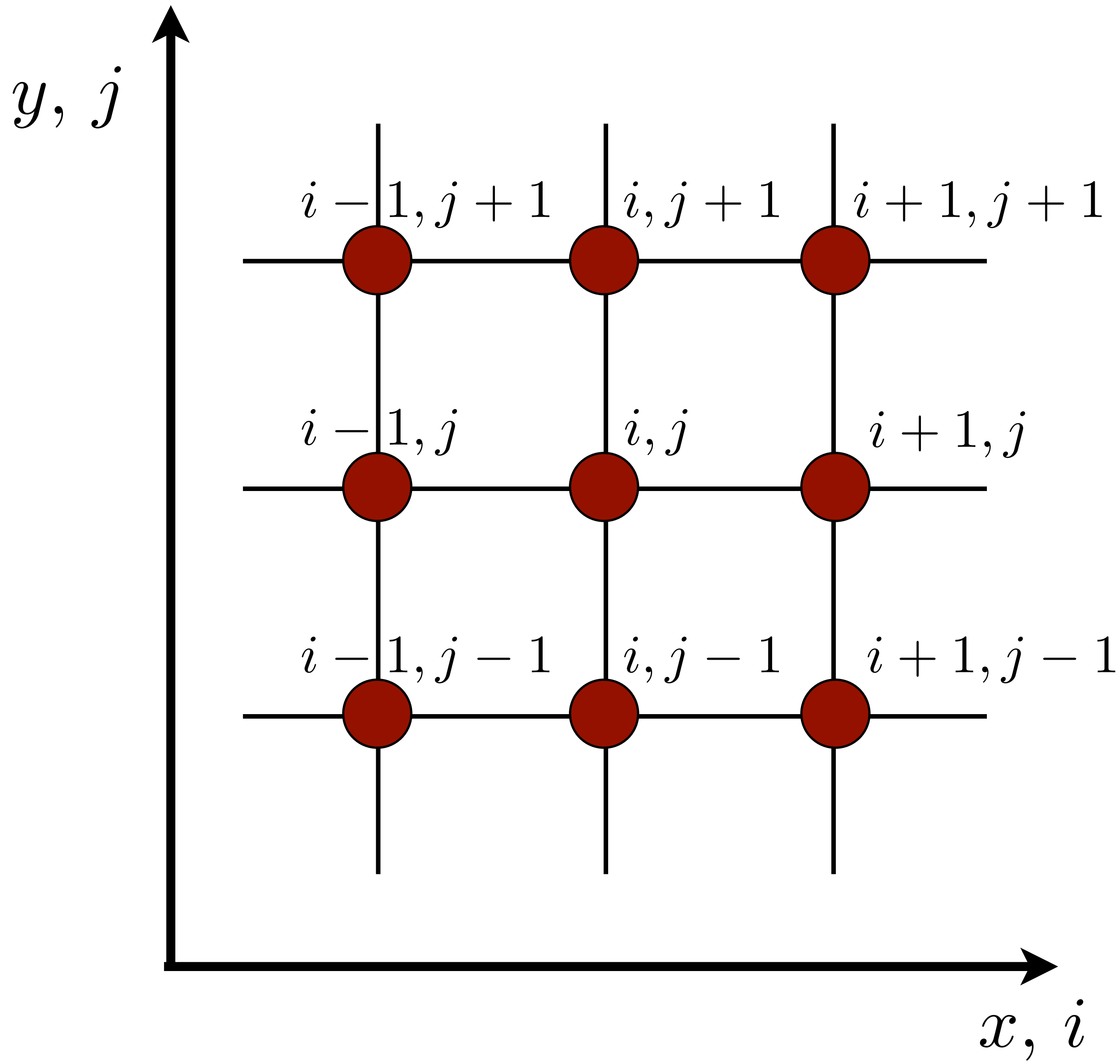
- More definitions and conventions
  - Example:
    - divide domain starting at x_0 and length *L* into *N* equal sized elements



    - grid point spacing: $\Delta x = \dfrac{L}{N}$
    - grid point location: $x_i = x_0 + i\Delta x = x_0 + i\dfrac{L}{N} \qquad i = 0,\ 1, \ldots,\ N$
    - total number of elements: ***N***
    - total number of grid points: ***N + 1***
  - Same can be done for y-direction:

$$y_j = y_0 + j\Delta y \qquad j = 0,\ 1, \ldots,\ M$$

- More definitions and conventions
  - we can identify a specific grid point by its (i,j) coordinate



- We want to enforce the PDEs following application of some numerical scheme at every grid point

- So finally here goes:
  - most equations we want to solve look like this:

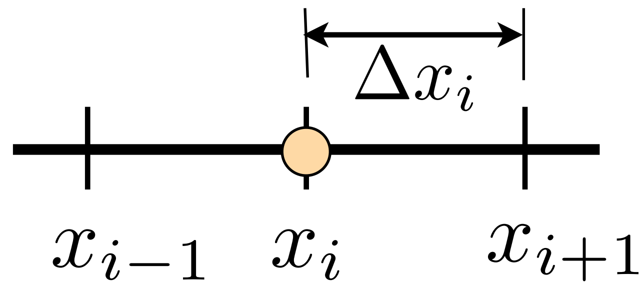
$$\frac{\partial}{\partial t}(\ldots) + \text{spatial derivatives} = 0$$

  - Solution strategy:
    1) approximate spatial derivatives
    2) integrate resulting ODEs using some method

1) Approximate spatial derivates by **finite differences**

- Example #1:

$$\left.\frac{\partial f}{\partial x}\right|_{x_i}$$



$$\Delta x_i = x_{i+1} - x_i$$

- How? **Taylor Series!**

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)\left.\frac{df}{dx}\right|_{x_i} + \frac{1}{2}(x_{i+1} - x_i)^2 \left.\frac{d^2 f}{dx^2}\right|_{x_i} + \dots$$

or

$$f_{i+1} = f_i + \Delta x_i f_i' + \frac{1}{2}\Delta x_i^2 f_i'' + \dots$$

let's assume $\Delta x_i = const. = h$

$$f_{i+1} = f_i + h f_i' + \frac{1}{2} h^2 f_i'' + \dots$$

solve for f_i':

order

$$f_i' = \frac{f_{i+1} - f_i}{h} - \frac{1}{2} h f_i'' + \dots \quad \Leftrightarrow \quad f_i' = \frac{f_{i+1} - f_i}{h} + O(h)$$

Forward difference

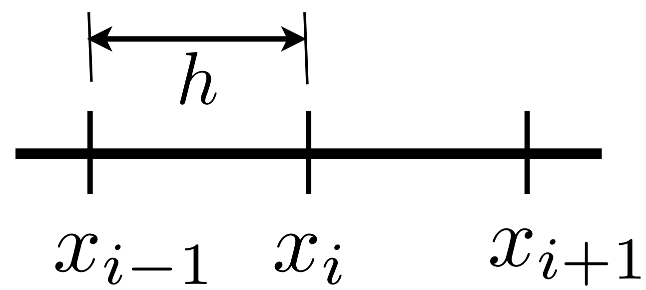
- Forward difference

$$f_i' = \frac{f_{i+1} - f_i}{h} + O(h^1)$$

  - exponent of *h* in *O(h)* is the **order of accuracy** of the method
    - here: order = 1
  - the order indicates how fast the error (the *O(h)* term) decreases with a reduction in *h*
    - here: reduce *h* by a factor 2 ⇒ error reduces by a factor $2^1 = 2$
  - Note: only leading order error term is important!
    Higher order error terms decrease faster = are smaller
    (provided h is sufficiently small)

- Example #2:

$\left.\dfrac{\partial f}{\partial x}\right|_{x_i}$ again, but TS for f_{i-1}



$h = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) + (x_{i-1} - x_i)\left.\frac{df}{dx}\right|_{x_i} + \frac{1}{2}(x_{i-1} - x_i)^2 \left.\frac{d^2 f}{dx^2}\right|_{x_i} + \ldots$$

$$f_{i-1} = f_i - h f_i' + \frac{1}{2}(-h)^2 f_i'' + \ldots$$

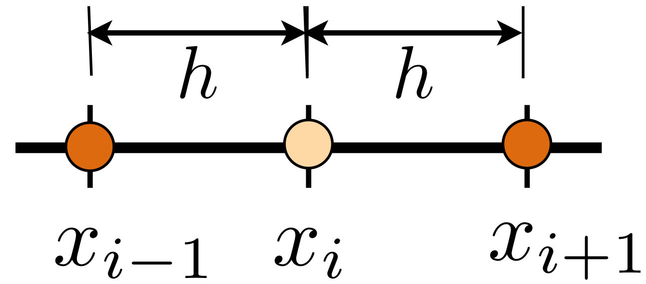
$$\Leftrightarrow \quad \boxed{f_i' = \frac{f_i - f_{i-1}}{h} + O(h)}$$

Backward difference

Question: What’s the order? Answer: 1

- Example #3:

$\left.\dfrac{\partial f}{\partial x}\right|_{x_i}$ again, but TS for f_{i+1} & f_{i-1}



$$
\begin{aligned}
f_{i+1} &= f_i + hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{6}h^3 f_i''' + \ldots \\
- \quad f_{i-1} &= f_i - hf_i' + \frac{1}{2}h^2 f_i'' - \frac{1}{6}h^3 f_i''' + \ldots \\
\hline
f_{i+1} - f_{i-1} &= 2hf_i' + \frac{1}{3}h^3 f_i''' + \ldots
\end{aligned}
$$

$$\Leftrightarrow \quad f_i' = \frac{f_{i+1} - f_{i-1}}{2h} - \frac{1}{6}h^2 f_i''' + \ldots$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$$

Central difference

Question: What's the order? Answer: 2

$$\text{as } h \to \frac{h}{2} : \text{ error} \to \frac{\text{error}}{4}$$