AEE 471 / MAE 561

Homework #3 - Due: Wednesday, February 18th, at the beginning of class

Please submit handwritten and/or printed out answers at the beginning of class. Combine all code you used to solve the problems into a single text file, and upload the text file to Blackboard using the SafeAssign mechanism. No credit will be given, if your code is not uploaded as a text file using SafeAssign. Also ensure that your code contains adequate comments.

Problem 1 (40 points) AEE471: Core Course Outcome #2
Consider the second-order differential equation

$$\frac{d^2\phi}{dx^2} = \sin(2\pi x)\left(-5\pi^2 x^4 + 4\pi^2 x^3 + 15x^2 - 6x\right) + \cos(2\pi x)\left(20\pi x^3 - 12\pi x^2\right)\,, \tag{1}$$

with boundary conditions $\phi(x=0) = \phi(x=1) = 0$. Solve this equation on the interval $0 \leq x \leq 1$ using 257 equally spaced points ($x_0 = 0$, $x_{256} = 1$). As initial guess $\phi^{(0)}(x)$ use

$$\phi^{(0)}(x) = \frac{1}{50}\sin(2\pi x) + \frac{1}{200}\sin(64\pi x)\,. \tag{2}$$

Use all of the following methods: Point Jacobi, Gauss-Seidel, and SOR with **theoretical optimum** over-relaxation factor.

a) For each method, plot the initial guess and the solutions after 100, 200, 300, 400, and 500 iterations into a single plot. There should thus be 3 figures with 6 curves each, clearly annotated.

b) Compare and discuss the solutions from each of the three methods. Of particular interest is to notice how the various wavelengths are damped as the iteration progresses.

c) In one additional graph, plot the infinity norm of the residual vs the iteration number (0-500) for all three methods. Use a log scale for the residual norm and a linear scale for the iteration number. Discuss the results.

Required submission:
- report the ω used for SOR;
- 1 clearly annotated figure containing Point Jacobi solution after 0, 100, 200, 300, 400, and 500 iterations;
- 1 clearly annotated figure containing Gauss-Seidel solution after 0, 100, 200, 300, 400, and 500 iterations;
- 1 clearly annotated figure containing SOR solution after 0, 100, 200, 300, 400, and 500 iterations;
- 1 clearly annotated log-linear plot containing residual norm vs iteration number of Point Jacobi, Gauss-Seidel and SOR including the initial guess residual;
- discussion, incl. comparison of high to low wave number convergence behavior;
- printout of well commented code;
- SafeAssign upload of all used, well commented code.

Problem 2 (60 points) AEE471: Core Course Outcome #2
Consider the 2-dimensional second-order differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 10 - 10\cos^2(x) + 10\sin(y), \qquad \phi(\pm 1, y) = \phi(x, \pm 1) = 0. \tag{3}$$

Solve this equation on the interval $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ using 81×81 equally spaced points. As initial guess use

$$\phi^{(0)}(x, y) = \frac{1}{2}\sin(\pi x)\sin(4\pi x)\,. \tag{4}$$

Use all of the following methods: Point Jacobi, Gauss-Seidel, and SOR with **theoretical optimum** over-relaxation factor.

a) Plot the initial guess and for each method, the solution after 10, 50, 100, and 600 iterations as a surface plot. Use the **same range** for ϕ in all of your plots ($-1 \leq \phi \leq 0.5$ is a good range).

b) Plot the infinity norm of the residual vs the iteration number (0-600) for all three methods into a single plot. Use a log scale for the residual and a linear scale for the iteration number. Discuss the results.

Required submission:
- report the ω used for SOR;
- 1 clearly annotated surface plot of the initial guess;
- 4 clearly annotated surface plots of the Point Jacobi solution after 10, 50, 100, respectively 600 iterations;
- 4 clearly annotated surface plots of the Gauss Seidel solution after 10, 50, 100, respectively 600 iterations;
- 4 clearly annotated surface plots of the SOR solution after 10, 50, 100, respectively 600 iterations;
- 1 clearly annotated log-linear plot containing infinity norm of the residual vs iteration number of Point Jacobi, Gauss-Seidel and SOR;
- printout of well commented code and SafeAssign upload of all used, well commented code.

Problem 3 (required for MAE561, bonus for AEE471)
(MAE561: 20 points, AEE471: Core Course Outcomes #1 & #2: 10 bonus points)
Solve problem 1 using 129, as well as 513 equally spaced points employing the Gauss Seidel method with 500 iterations. Calculate the L_∞, L_1, and L_2 error norms of the solution for the 129, 257, 513 point meshes, if the exact solution is given by

$$\phi_{exact}(x) = \sin(2\pi x)\left(\frac{5}{4}x^4 - x^3\right)\,. \tag{5}$$

In a table present the number of mesh points, together with the 3 error norms and their observed order of convergence for the finer two meshes. Discuss the error norms and the observed order of convergence. Is the observed order close to the formal order of the finite difference approximation? If not, why not, and how can a value close to the formal order be recovered?

Required submission:
- table containing number of mesh points, L_∞, L_1, and L_2 error norms and their observed order of convergence;
- discussion of the obtained results addressing the questions outlined in the problem statement;
- SafeAssign upload of all used, well commented code.

Bonus Problem 4 (10 bonus points) AEE471: Core Course Outcomes #1 & #2
If possible, modify the solution procedure of problem 3 in such a way that the observed order of convergence is close to the formal order of the finite difference discretization. Document the modified solution procedure and redo problem 3 using the new procedure.

Required submission:
- documentation of the modified solution procedure;
- table containing number of mesh points, L_∞, L_1, and L_2 error norms and their observed order of convergence for modified procedure;
- SafeAssign upload of all used, well commented code.