Higher Dimensions

$$C_x = \frac{a\Delta t}{\Delta x} \qquad C_y = \frac{b\Delta t}{\Delta y}$$

next: hyperbolic equation

- let's start with linear hyperbolic equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \qquad a > 0\,,\ b > 0$$

- Stability? for example for 1st-order upwind?

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = -a\frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x} - b\frac{u_{i,j}^n - u_{i,j-1}^n}{\Delta y}$$

- von Neuman stability analysis

$$\rho^{n+1} = \rho^n - C_x\rho^n\left(1 - e^{-ik\Delta x}\right) - C_y\rho^n\left(1 - e^{-ik\Delta y}\right)$$

$$\Rightarrow \quad G = \frac{\rho^{n+1}}{\rho^n} = 1 - C_x\left(1 - e^{-ik\Delta x}\right) - C_y\left(1 - e^{-ik\Delta y}\right)$$

$$= 1 - C_x\left(1 - \cos(k\Delta x) + i\sin(k\Delta x)\right) - C_y\left(1 - \cos(k\Delta y) + i\sin(k\Delta y)\right)$$

worst case:

$$|G|^2 = \left[1 - C_x\left(1 - \cos(k\Delta x)\right) - C_y\left(1 - \cos(k\Delta x)\right)\right]^2 + \left[C_x\sin(k\Delta x) + C_y\sin(k\Delta y)\right]^2$$

[....] $C_x + C_y \le 1 \quad \Rightarrow \quad \boxed{\frac{a\Delta t}{\Delta x} + \frac{b\Delta t}{\Delta y} \le 1}$ (valid for linear equations only) so far, we've done:

- Poisson equation: elliptic equations
- viscous terms: parabolic equations
- convective terms: hyperbolic equations

next:

- combine convective and viscous terms

Viscous Burger's Equation

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (vv)}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

or:

$$\frac{\partial \vec{Q}}{\partial t} + \frac{\partial \vec{E}}{\partial x} + \frac{\partial \vec{F}}{\partial y} = \nu \left(\frac{\partial^2 \vec{Q}}{\partial x^2} + \frac{\partial^2 \vec{Q}}{\partial y^2} \right) \text{ with } \vec{Q} = \begin{bmatrix} u \\ v \end{bmatrix}; \; \vec{E} = \begin{bmatrix} u^2 \\ uv \end{bmatrix}; \; \vec{F} = \begin{bmatrix} uv \\ v^2 \end{bmatrix}$$

Note: This is **NOT** Navier-Stokes!

In the following, we'll look at 1D cases first, but doing 2D is easy by simply using vectors but let's start with an even simpler model problem $C = \frac{a\Delta t}{\Delta x}$ $d = \frac{\alpha \Delta t}{\Delta x^2}$

- linear 1D:

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2}$$

FTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

- Stability? Board

  - cell Reynolds number: $\mathrm{Re}_c = \frac{a\Delta x}{\alpha} = \frac{C}{d}$

$\mathrm{Re}_c \leq 2$ and $d \leq \frac{1}{2}$

$\mathrm{Re}_c \leq \frac{2}{C}$

  - Remember: previously we found that FTCS for hyperbolic equations is **always** unstable!
  - Check: pure hyperbolic $\Rightarrow \alpha = 0$ $\Rightarrow Re_c = \infty$ $\Rightarrow$ always unstable

Let's start with a simpler model problem:

linear 1D: $\boxed{\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2}}$

FTCS: $\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$ Let $c = \frac{a\Delta t}{\Delta x}$; $d = \frac{\alpha\Delta t}{\Delta x^2}$

$\Rightarrow u_i^{n+1} = u_i^n - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$

→ Stability?

$$\xi^{n+1} = \xi^n - \frac{c}{2}\xi^n\underbrace{\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}_{2i\sin(k\Delta x)} + d\,\xi^n\left(\underbrace{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}_{2\cos(k\Delta x)}\right)$$

$\Rightarrow G = \frac{\xi^{n+1}}{\xi^n} = 1 - iC\sin(k\Delta x) - 2d\left(1 - \cos(k\Delta x)\right)$

stable, if $|G|^2 \le 1$: $\left(1 - 2d(1 - \cos(k\Delta x))\right)^2 + C^2\sin^2(k\Delta x)) \le 1$

$$\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$$

$\Rightarrow C \le 2d$ and $d \le \frac{1}{2}$ and $c \le 1$ (but this is included in the first 2 conditions)

$\Rightarrow \frac{a\Delta t}{\Delta x} \le 2\alpha\frac{\Delta t}{\Delta x^2} \Leftrightarrow \frac{a\Delta x}{\alpha} \le 2$

cell Reynoldsnumber! $Re_c = \frac{a\Delta t}{\alpha} = \frac{c}{d}$

$\Rightarrow$ $Re_c \le 2$ and $d \le \frac{1}{2}$ but also, since $c \le 2d$ and $c \le 1 \Rightarrow c^2 \le 2d \Rightarrow c \le \frac{2d}{c}$

$\Rightarrow$ $Re_c \le \frac{2}{c}$

Remember: Previously we found that FTCS for hyperbolic eqs. is <u>always</u> unstable!

Check: hyperbolic: $\alpha = 0 \Rightarrow Re_c = \infty \Rightarrow$ always unstable ✓

FTBCS $a > 0$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

1st-order upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

- large dissipative errors in convective term
- dissipative errors may be larger than physical viscous term!

Remedy:

- could go 2nd-order ➙ dispersive errors
- could go 3rd-order ➙ upwind bias stencil: *i-2, i-1, i, i+1*

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

Du-Fort Frankel

central in time and space with fix in viscous term

$$\boxed{\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \alpha\frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2}}$$

- Order: $O\left((\Delta t)^2, (\Delta x)^2, \left(\frac{\Delta t}{\Delta x}\right)^2\right)$
- startup problem!

MacCormack

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha \Delta t}{\Delta x^2}$$

step 1: $$u_i^* = u_i^n - \frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

step 2: $$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{a\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right)\right]$$

alternate form:

step 1: $$\Delta u_i^n = -\frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$u_i^* = u_i^n + \Delta u_i^n$$

step 2: $$\Delta u_i^* = -\frac{a\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right)$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* + \Delta u_i^*\right]$$

- Stability: $\Delta t \leq \dfrac{1}{\frac{a}{\Delta x} + \frac{2\alpha}{\Delta x^2}}$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

BTCS

Implicit & central in space

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}}$$

$$\Leftrightarrow \quad -\left(\frac{C}{2} + d\right)u_{i-1}^{n+1} + (1 + 2d)\,u_i^{n+1} + \left(\frac{C}{2} - d\right)u_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!

BTBCS $a > 0$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

Implicit & upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}}$$

(1st-order)

$$\Leftrightarrow \quad -(C+d)\,u_{i-1}^{n+1} + (1 + C + 2d)\,u_i^{n+1} - du_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!
- but again: dissipative errors may dominate physical viscous forces!
- go 2nd-order upwind for the convective term:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{3u_i^{n+1} - 4u_{i-1}^{n+1} + u_{i-2}^{n+1}}{2\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

$$\Leftrightarrow \quad \frac{C}{2}u_{i-2}^{n+1} - (2C+d)\,u_{i-1}^{n+1} + \left(1 + \frac{3}{2}C + 2d\right)u_i^{n+1} - du_{i-1}^{n+1} = u_i^n$$

- no-longer tri-diagonal! $\Rightarrow$ time-lag u_{i-2} to recover tri-diagonal structure

$$\Leftrightarrow \quad -(2C+d)\,u_{i-1}^{n+1} + \left(1 + \frac{3}{2}C + 2d\right)u_i^{n+1} - du_{i-1}^{n+1} = u_i^n - \frac{C}{2}u_{i-2}^n$$

- go 3rd-order upwind: again time-lag u_{i-2} to recover tri-diagonal structure

Crank-Nicholson

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

shorthands:

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \qquad \delta_x u_i = u_{i+1} - u_{i-1}$$

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2}\left(\frac{\delta_x u_i^n}{2\Delta x} + \frac{\delta_x u_i^{n+1}}{2\Delta x}\right) + \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)}$$

$$\Leftrightarrow \quad u_i^{n+1} + \frac{a\Delta t}{2}\frac{\delta_x u_i^{n+1}}{2\Delta x} - \frac{\alpha\Delta t}{2}\frac{\delta_x^2 u_i^{n+1}}{\Delta x^2} = u_i^n - \frac{a\Delta t}{2}\frac{\delta_x u_i^n}{2\Delta x} + \frac{\alpha\Delta t}{2}\frac{\delta_x^2 u_i^n}{\Delta x^2}$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha \Delta t}{\Delta x^2}$$

Mixed Treatment

Idea: apply different schemes to the individual terms

Goal: $O(\Delta t^2, \Delta x^2)$

- convective terms
  - central in space: $$\frac{\partial u}{\partial t} = -a\frac{u_{i+1} - u_{i-1}}{2\Delta x} = H_i$$
  - Adams-Bashforth in time: $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1}$$
- diffusive terms
  - central in space: $$\frac{\partial u}{\partial t} = \alpha\frac{\delta_x^2 u_i}{\Delta x^2}$$
  - Crank-Nicholson in time: $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)$$
- combine: $$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1} + \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)}$$
- since viscous terms are implicit ⇒ no viscous time step restriction!