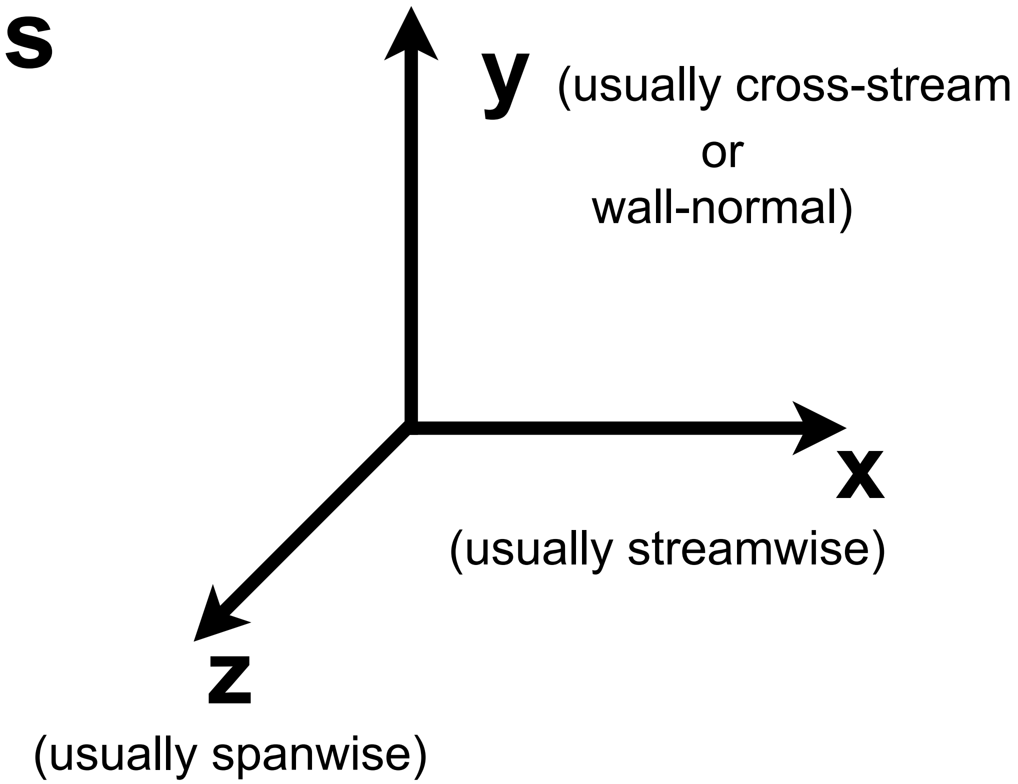
- **Some conventions and definitions**
  - we usually work in Cartesian coordinates:
  - some variable names:
    - velocity: $\vec{v} = \vec{u} = (u, v, w) = u_i$
    - density: ρ
    - pressure: p
    - internal energy: e
    - enthalpy: h
    - temperature: T
    - stress tensor: $\overline{\overline{T}} = T_{ij}$
  - Recall:

$$\nabla \cdot \vec{v} = \frac{\partial u_i}{\partial x_i} = u_{i,i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

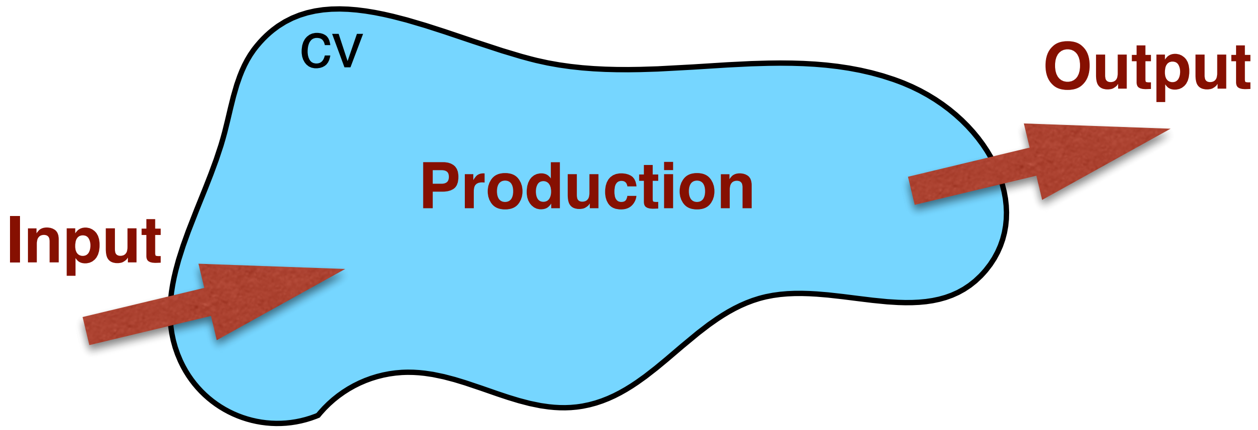


• Equations of Motion

- A conservation law can be defined as

Change in storage = Input - Output + Production

- Let's consider a closed volume = control volume (cv)

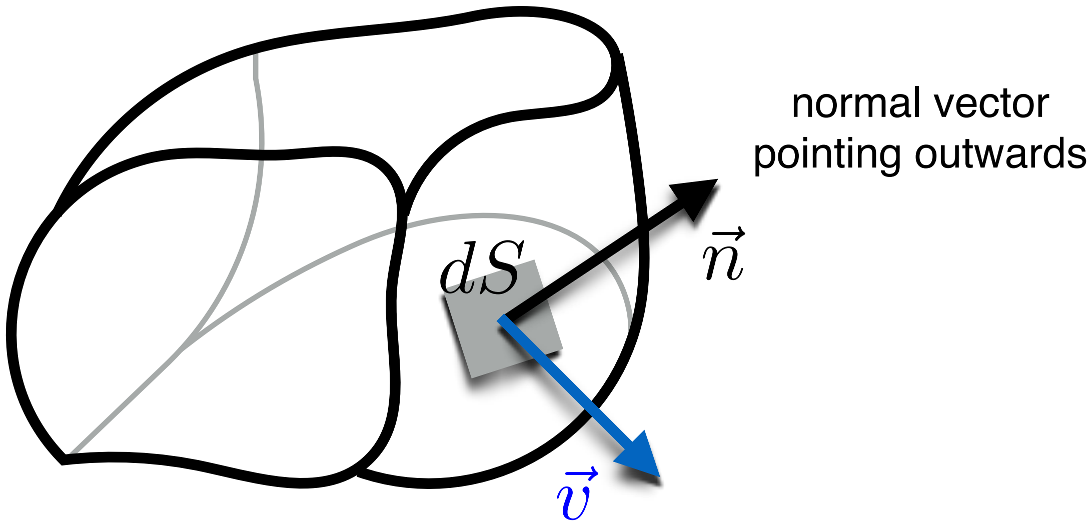


drawing is 2D,
but cv is really 3D

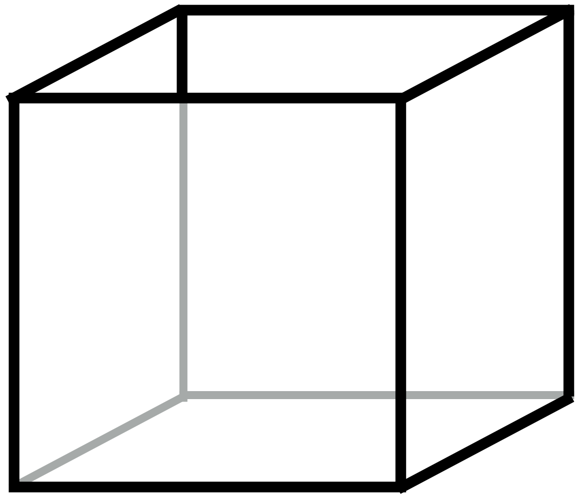
- Examples for production:
  - mass: 0 ⇒ mass is conserved
  - momentum: $\sum \vec{F}$ ⇒ forces: gravity, viscous forces, etc.
  - energy: 0 ⇒ energy is conserved (it may change form though)

Conservation of Mass (Continuity Equation)

- consider a fixed control volume (cv):



or



- mass inside cv: $m = \int_V \rho dV$

- change in mass = Inflow - Outflow of mass

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho dV = -\int_S \rho\vec{v}\cdot\vec{n}dS$$

negative sign due to normal pointing outwards

$$\int_V \frac{d\rho}{dt}dV + \int_S \rho\vec{v}\cdot\vec{n}dS = 0$$

Integral form ⇒ Finite Volume Methods

Conservation of Mass (Continuity Equation)

- use Gauss Theorem

$$\int_V \frac{d\rho}{dt} dV + \int_S \rho\vec{v}\cdot\vec{n}dS = 0$$

$$\int_S \rho\vec{v}\cdot\vec{n}dS = \int_V \nabla\cdot(\rho\vec{v})\,dV$$

divergence (div)

$$\Rightarrow \quad \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right] dV = 0$$

⬅ must be valid for any control volume V

$$\Rightarrow \quad \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

Differential form ⇒ Finite Difference Methods

other ways to write this:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + 0$$

$$\rho_{,t} + (\rho u_i)_{,i} = 0$$

⬅ Einstein:

- sum over equal indices (here i)
- comma: partial derivative

- Reynold's transport theorem:
  - let φ be a conserved intensity

then $\phi = \int\limits_{\Omega_{CM}} \rho\varphi d\Omega$ with Ω_{CM} : volume of a control mass
for example: small particle,
fluid parcel, etc.

  - transport theorem for fixed control volumes

$$\frac{d\phi}{dt} = \frac{d}{dt}\int\limits_{\Omega_{CM}} \rho\varphi d\Omega = \frac{d}{dt}\int\limits_{V} \rho\varphi dV + \int\limits_{S} \rho\varphi\vec{v}\cdot\vec{n} dS$$

$$\frac{d}{dt}\int_{\Omega_{CM}} \rho\varphi d\Omega = \frac{d}{dt}\int_V \rho\varphi dV + \int_S \rho\varphi\vec{v}\cdot\vec{n}dS$$

- Momentum
  - $\varphi = \vec{v}$
  - How can we produce momentum? $\Rightarrow$ Newton's 2nd law

$$\frac{d(m\vec{v})}{dt} = \sum \vec{F}$$

$\vec{F}$: forces acting on control mass

$$\frac{d(m\vec{v})}{dt} = \frac{\partial}{\partial t}\int_V \rho\vec{v}dV + \int_S \rho\vec{v}\,\vec{v}\cdot\vec{n}dS = \sum \vec{F}$$

  - What are potential forces?
    - pressure, normal & shear stress : surface forces
    - gravity, electro/magnetic : body forces

Surface Forces

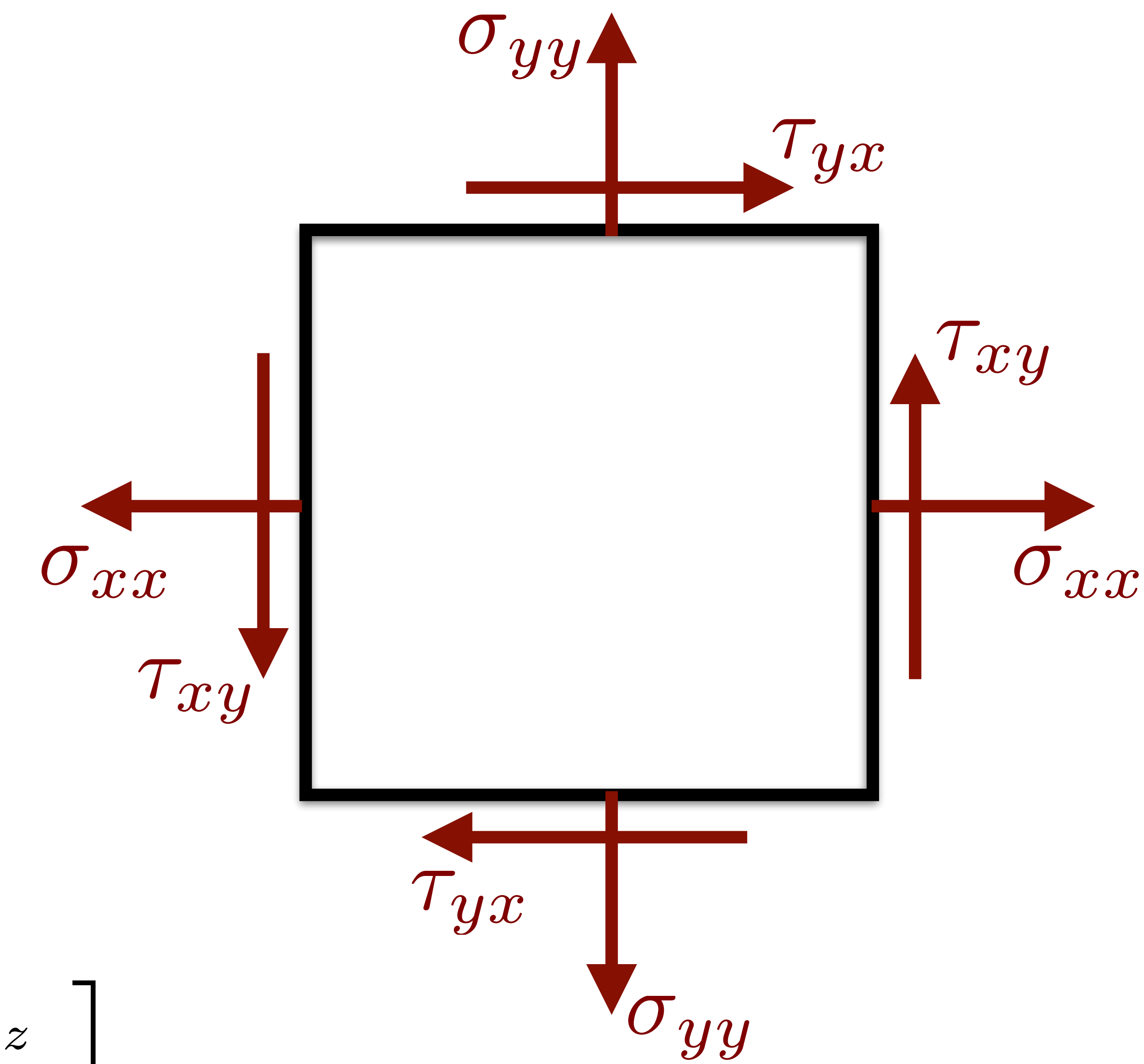
- consider a control volume enclosed by control surfaces

  σ: normal stress

  τ: shear stress

  1st subindex: direction of face normal

  2nd subindex: direction of stress



- stress tensor: $\overline{\overline{T}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$

- surface force: $\vec{F}_{surf} = \int_S \overline{\overline{T}} \cdot \vec{n} dS$

- Volume Forces
  - for example gravity: $\vec{F}_{vol} = \int\limits_V \rho\vec{g}dV$
- Momentum equation

$$\int\limits_V \frac{\partial \rho\vec{v}}{\partial t}dV + \int\limits_S \rho\vec{v}\,\vec{v}\cdot\vec{n}dS = \int\limits_S \overline{\overline{T}}\cdot\vec{n}dS + \int\limits_V \rho\vec{g}dV$$

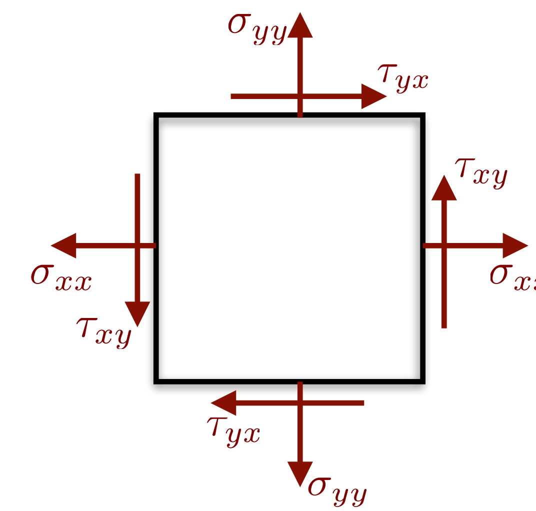
  - or using Gauss theorem

$$\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\,\vec{v}) = \nabla\cdot\overline{\overline{T}} + \rho\vec{g}$$

  - but what's $\overline{\overline{T}}$?

- Stress tensor

$$\overline{\overline{T}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



  - normal stress:

$$\sigma_{xx} = -p + \tau_{xx}$$
$$\sigma_{yy} = -p + \tau_{yy}$$
$$\sigma_{zz} = -p + \tau_{zz}$$

  - for fluids: stresses are function of rate of strain
    - if function is linear ⇒ Newtonian fluid (most fluids are)

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\vec{v} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\vec{v} \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\vec{v} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

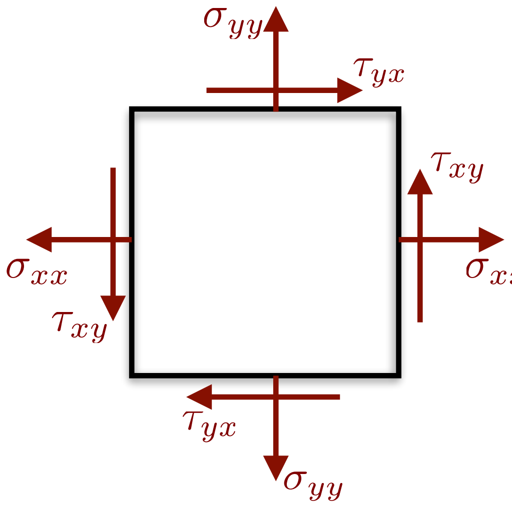
μ: dynamic viscosity
λ: 2nd viscosity

Stoke's hypothesis: $\lambda = -\frac{2}{3}\mu$

- Stress tensor

$$\sigma_{xx} = -p + \tau_{xx} \qquad \tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\vec{v} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\sigma_{yy} = -p + \tau_{yy} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\vec{v} \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$
$$\sigma_{zz} = -p + \tau_{zz} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\vec{v} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\overline{\overline{T}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



  - put it all together:

$$\overline{\overline{T}} = T_{ij} = -\left(p + \frac{2}{3}\mu\frac{\partial u_j}{\partial x_j}\right)\delta_{ij} + 2\mu D_{ij}$$

    - Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

    - deformation tensor: $D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

  - alternative way to write this:

$$\overline{\overline{T}} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{v}\right)\overline{\overline{I}} + 2\mu\overline{\overline{D}} \qquad \text{with} \qquad \overline{\overline{D}} = \frac{1}{2}\left(\nabla\vec{v} + (\nabla\vec{v})^T\right)$$

  - usually one splits this into pressure and viscous terms:

$$\overline{\overline{T}} = -p\overline{\overline{I}} + \overline{\overline{\tau}} \quad \text{with} \quad \overline{\overline{\tau}} = 2\mu\overline{\overline{D}} - \frac{2}{3}\mu\nabla\cdot\vec{v}\overline{\overline{I}}$$

$$\overline{\overline{T}} = -p\overline{\overline{I}} + \overline{\overline{\tau}}$$

- Momentum equation
  - put it all together:

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\,\vec{v}) = \nabla \cdot \overline{\overline{T}} + \rho \vec{g}$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\,\vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$

$\leftarrow$ Navier-Stokes equation in **conservative** form

  - let's look at the left-hand-side for component i: u_i

$$\frac{\partial \rho u_i}{\partial t} + \nabla \cdot (\rho \vec{v}\, u_i) \overset{\text{product rule}}{=} \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + \rho \vec{v} \cdot \nabla u_i + u_i \nabla \cdot (\rho \vec{v})$$

$$\overset{\text{re-arrange}}{=} \rho \left(\frac{\partial u_i}{\partial t} + \vec{v} \cdot \nabla u_i \right) + u_i \left(\cancel{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v})} \right)$$

= 0: continuity!

  - put it all together:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \overline{\overline{\tau}} + \vec{g}}$$

$\leftarrow$ Navier-Stokes equation in **non-conservative** form

$$\frac{d}{dt}\int_{\Omega_{CM}} \rho\varphi d\Omega = \frac{d}{dt}\int_V \rho\varphi dV + \int_S \rho\varphi\vec{v}\cdot\vec{n}dS$$

- Energy equation
  - there are many equivalent forms
  - here, let's use enthalpy: $h = h_{ref} + c_p T$ with c_p: const. specific heat @ p=const.
  - $\varphi = h$

$$\frac{\partial}{\partial t}\int_V \rho h dV + \int_S \rho h\vec{v}\cdot\vec{n}dS = \int_S k\nabla T\cdot\vec{n}dS + \int_V \left(\vec{v}\cdot\nabla p + \overline{\overline{\tau}} : \nabla\vec{v}\right) dV + \frac{\partial}{\partial t}\int_V p dV$$

with k: thermal conductivity

work done by pressure and viscous forces

  - use Gauss:

$$\boxed{\frac{\partial \rho h}{\partial t} + \nabla\cdot(\rho h\vec{v}) = \nabla\cdot(k\nabla T) + \vec{v}\cdot\nabla p + \overline{\overline{\tau}} : \nabla\vec{v} + \frac{\partial p}{\partial t}}$$

- So what do we have so far?
  - Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
  - Momentum: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$
  - Energy: $$\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho \vec{v} h) = \nabla \cdot (k \nabla T) + \vec{v} \cdot \nabla p + \overline{\overline{\tau}} : \nabla p + \frac{\partial p}{\partial t}$$
  - 5 equations (continuity + 3 momentum + energy)
  - but 6 unknowns (ρ, u, v, w, p, T)

⇒ need one more equation! ⇒ equation of state (EOS)

for most gases: $p = \rho R T$ (ideal gas law) R: gas constant

• Governing Equations

- Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
- Momentum: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$
- Energy: $$\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho \vec{v} h) = \nabla \cdot (k \nabla T) + \vec{v} \cdot \nabla p + \overline{\overline{\tau}} : \nabla p + \frac{\partial p}{\partial t}$$
- EOS: $$p = \rho R T$$

• Governing Equations

- Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
- Momentum: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$
- Energy: $$\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho \vec{v} h) = \nabla \cdot (k \nabla T) + \cancel{\vec{v} \cdot \nabla p} + \cancel{\overline{\overline{\tau}} \cdot \nabla p} + \cancel{\frac{\partial p}{\partial t}}$$
- EOS: $$p = \rho R T$$

• Simplifications

I) neglect work done by pressure and viscous forces

- use $h = c_p T$

Governing Equations

- Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
- Momentum: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$
- Energy: $$\frac{\partial \rho T}{\partial t} + \nabla \cdot (\rho \vec{v} T) = \nabla \cdot \left(\frac{k}{c_p} \nabla T \right)$$
- EOS: $$p = \rho R T$$

Simplifications

I) neglect work done by pressure and viscous forces
   - use $h = c_p T$

Governing Equations

- Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
- Momentum: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$$
- Energy: $$\cancel{\frac{\partial \rho T}{\partial t} + \nabla \cdot (\rho \vec{v} T) = \nabla \cdot \left(\frac{k}{c_p} \nabla T \right)}$$
- EOS: $$p = \rho R T$$

Simplifications

I) neglect work done by pressure and viscous forces
- use $h = c_p T$

II) assume
- iso-thermal: T = const.; μ = const.

- **Governing Equations**
  - Continuity: $\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\rho \vec{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \vec{v} = 0$
  - Momentum: $\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g}$
  - Energy: $\cancel{\frac{\partial \rho T}{\partial t} + \nabla \cdot (\rho \vec{v} T) = \nabla \cdot \left(\frac{k}{c_p} \nabla T \right)}$
  - EOS: $\cancel{p = \rho R T}$
- **Simplifications**

I) neglect work done by pressure and viscous forces

- use $h = c_p T$

II) assume

- iso-thermal: T = const.; μ = const.
- incompressible: ρ = const.

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \nabla \cdot \overline{\overline{\tau}} + \rho \vec{g} \quad \text{or} \quad \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \overline{\overline{\tau}} + \vec{g}$$

x-direction: $$\left(\nabla \cdot \overline{\overline{\tau}}\right)_{x-dir} = \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}$$

$$= \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} + \lambda \cancel{\nabla \cdot \vec{v}}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right)$$

$$= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)} = \mu\nabla^2 u$$

$$\Rightarrow \quad \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \vec{v} + \vec{g}$$

with kinematic viscosity: $\nu = \frac{\mu}{\rho}$

- **Simplifications**

  I) neglect work done by pressure and viscous forces
  - use $h = c_p T$

  II) assume
  - iso-thermal: T = const.; μ = const.
  - incompressible: ρ = const. $\quad \nabla \cdot \vec{v} = 0$

• Governing Equations

- Continuity: $$\nabla \cdot \vec{v} = 0$$
- Momentum: $$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} + \vec{g}$$

valid for incompressible, iso-thermal flow

• Simplifications

I) neglect work done by pressure and viscous forces
   - use $h = c_p T$

II) assume
   - iso-thermal: T = const.; μ = const.
   - incompressible: ρ = const.

- **Governing Equations**
  - Continuity: $\nabla \cdot \vec{v} = 0$
  - Momentum: $\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \cancel{\nu \nabla^2 \vec{v}} + \vec{g}$

Euler equations

- **Simplifications**

I) neglect work done by pressure and viscous forces
- use $h = c_p T$

II) assume
- iso-thermal: T = const.; μ = const.
- incompressible: ρ = const.

III) assume inviscid flow: μ = 0; ν = 0

$$\nabla \cdot \vec{v} = 0$$
$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \vec{g}$$

• Simplifications

IV) Creeping flow ⇔ Stokes flow

- Let's start by making equations dimensionless
- consider only incompressible, iso-thermal flow
- need reference scales (use subindex 0) ⇒ dimensionless (superindex *)

$$t^* = \frac{t}{t_0} \qquad x^* = \frac{x}{L_0} \qquad \vec{v}^* = \frac{\vec{v}}{u_0} \qquad p^* = \frac{p}{\rho u_0^2}$$

with $\rho = const.$

- substitute into equations:

$$\nabla^* \cdot \vec{v}^* = 0$$
$$St \frac{\partial \vec{v}^*}{\partial t^*} + \cancel{\vec{v}^* \cdot \nabla^* \vec{v}^*} = -\nabla^* p^* + \frac{1}{Re} {\nabla^*}^2 \vec{v}^* + \frac{1}{Fr^2}$$

with $St = \frac{L_0}{u_0 t_0}$ Strouhal number; $Re = \frac{\rho u_0 L_0}{\mu}$ Reynolds number; $Fr = \frac{u_0}{\sqrt{g L_0}}$ Froude number

- creeping flows: Re << 1: $\vec{v}^* \cdot \nabla^* \vec{v}^* << \frac{1}{Re} {\nabla^*}^2 \vec{v}^*$

• Simplifications

$$\nabla \cdot \vec{v} = 0$$
$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \vec{g}$$

IV) Creeping flow ⇔ Stokes flow

- Let's start by making equations dimensionless
- consider only incompressible, iso-thermal flow
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$$t^* = \frac{t}{t_0} \qquad x^* = \frac{x}{L_0} \qquad \vec{v}^* = \frac{\vec{v}}{u_0} \qquad p^* = \frac{p}{\rho u_0^2}$$ with $\rho = const.$

- substitute into equations:

$$\boxed{\begin{aligned} &\nabla \cdot \vec{v} = 0 \\ &St \frac{\partial \vec{v}}{\partial t} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{v} + \frac{1}{Fr^2} \end{aligned}}$$

dropped *

with $St = \frac{L_0}{u_0 t_0}$ Strouhal number, $Re = \frac{\rho u_0 L_0}{\mu}$ Reynolds number, $Fr = \frac{u_0}{\sqrt{g L_0}}$ Froude number

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