$$\frac{d^2\varphi}{dx^2} = \sin(k\pi x) \qquad 0 \le x \le 1 \qquad \varphi(0) = \varphi(1) = 0$$

Multigrid Methods

- Let $k = k_m$ with $1 \le \frac{k_m}{2\pi} \le \frac{M}{4}$

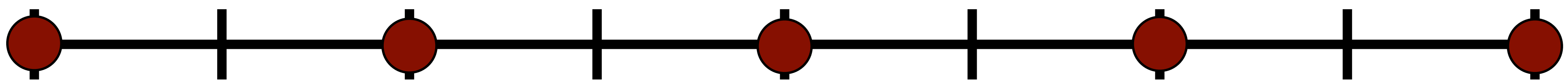
  - need 2 points/wavelength minimum

  $$\Rightarrow \frac{k_{\max}}{2\pi} = \frac{M}{2}$$

  - lower half of allowed wavenumbers

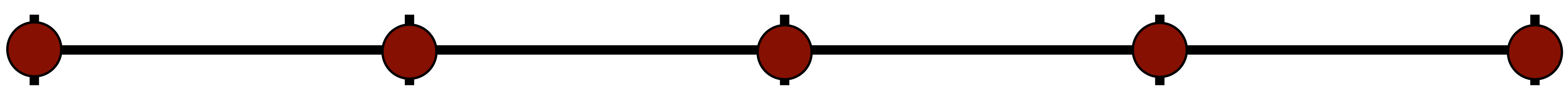
  ⇒ slowly varying

- Let's evaluate $\sin(k_m x_i)$ at even index points only: $x_i = 2i\frac{L}{M} = \frac{2i}{M}$



$$\Rightarrow \quad \sin(k_m x_i) = \sin\left(\frac{k_m 2i}{M}\right) = \sin\left(\frac{k_m i}{\frac{M}{2}}\right)$$

  - same as function evaluated at **every** point of mesh with spacing $\frac{L}{\frac{M}{2}}$



  - on mesh with $M/2+1$ mesh points, function goes up to the maximum allowed wavenumber **⇒ rapidly varying**

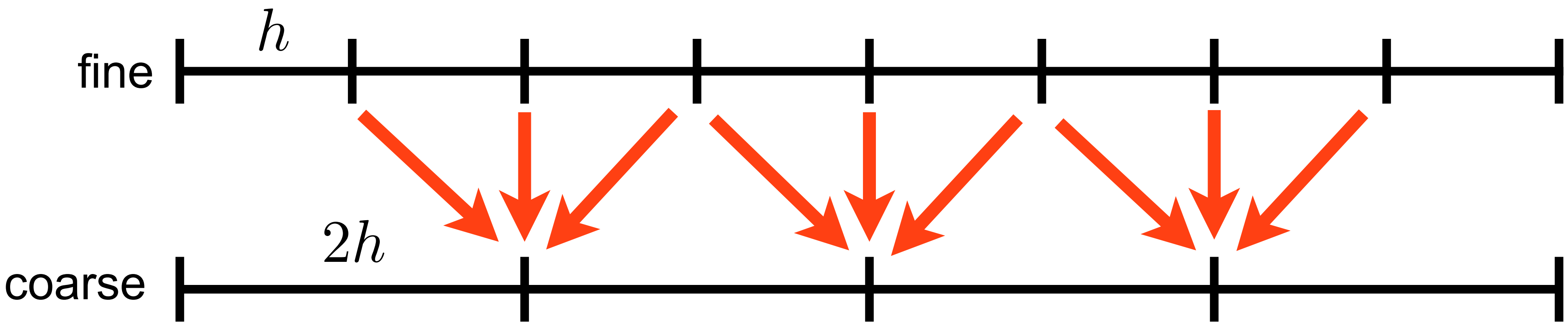
- Can turn slowly varying function into rapidly varying function by coarsening mesh ('skipping' mesh points)

Multigrid Methods

- Ideas behind Multigrid Methods
  - reduce slowly varying parts of residual on coarser grids, since they are rapidly varying there
  - transfer improved solution from coarse grid back to fine grid

- This will require transfer operations between grids
  - fine ➙ coarse: **restriction**
  - coarse ➙ fine: **prolongation**

Restriction



$M^h + 1$ points

$M^{2h} + 1$ points

$$M^{2h} = \frac{M^h}{2}$$

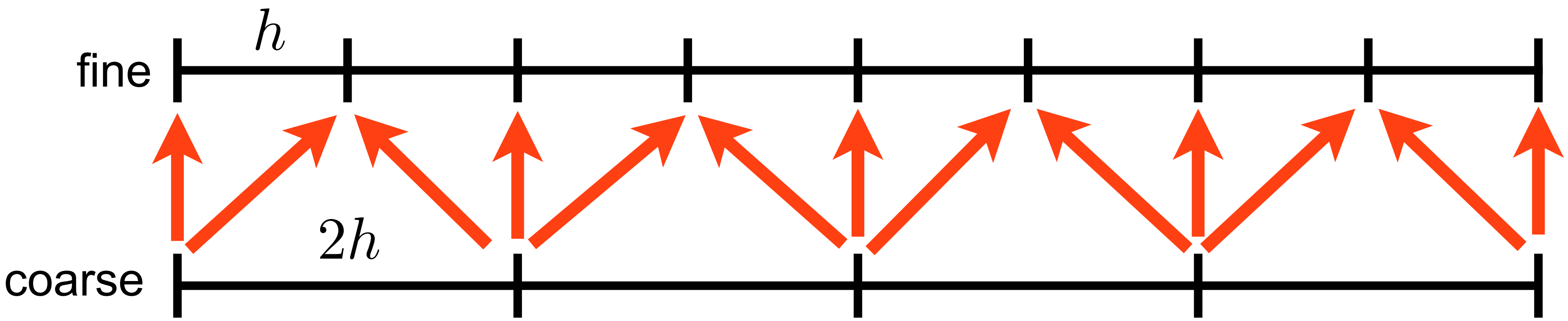
- Option #1:
  - take every 2nd point

$$r_i^{h \to 2h} = r_{2i}^h \qquad i = 1, 2, \ldots, M^{2h} - 1 \qquad \text{(all interior points)}$$

- Option #2:
  - average (usually better)

$$r_i^{h \to 2h} = \frac{1}{4}\left(r_{2i-1}^h + 2r_{2i}^h + r_{2i+1}^h\right) \qquad i = 1, 2, \ldots, M^{2h} - 1$$

Prolongation



$M^h + 1$ points

$M^{2h} + 1$ points

$$M^{2h} = \frac{M^h}{2}$$

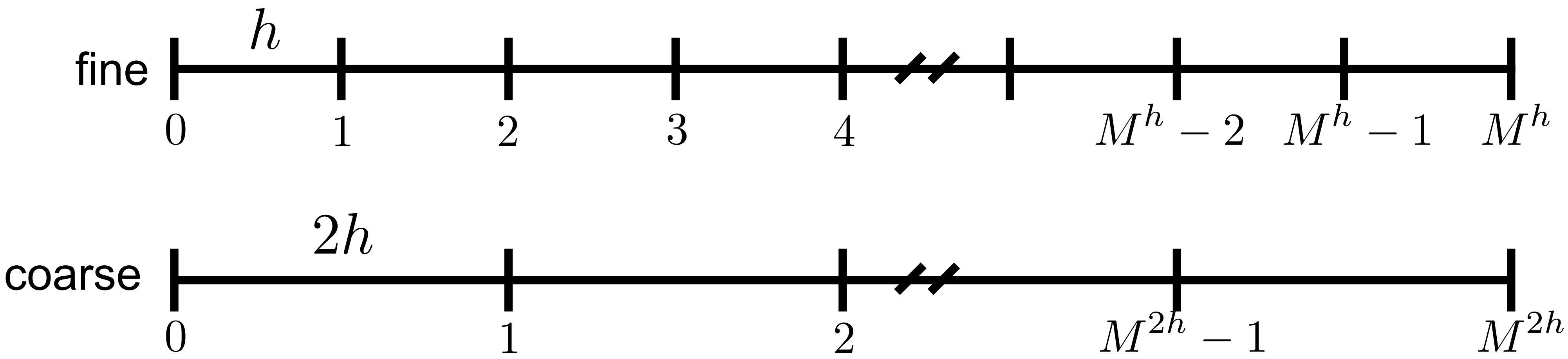
- simple copy of aligned mesh points

$$\epsilon_{2i}^{2h \to h} = \epsilon_i^{2h} \qquad i = 0, 1, \ldots, M^{2h}$$

- average non-aligned mesh points

$$\epsilon_{2i+1}^{2h \to h} = \frac{1}{2}\left(\epsilon_i^{2h} + \epsilon_{i+1}^{2h}\right) \qquad i = 0, 1, \ldots, M^{2h} - 1$$

Dual Grid Multigrid



$$M^{2h}=\frac{M^h}{2}$$

a) on fine gird, perform a few iterations for $A\vec{\varphi}=\vec{f}$ with ${\vec{\varphi}^h}^{(0)}$ as initial guess

b) calculate residual: $\vec{r}^h=\vec{f}-A{\vec{\varphi}^h}^{(k)}$

c) restrict residual to coarse grid: $\vec{r}^{h\rightarrow 2h}$

d) on coarse grid, perform a few iterations for $A\vec{\epsilon}^{2h}=\vec{r}^{h\rightarrow 2h}$ with ${\vec{\epsilon}^{2h}}^{(0)}=0$

e) prolong error to fine grid: $\vec{\epsilon}^{2h\rightarrow h}$

f) apply correction to fine grid solution: ${\vec{\varphi}^h}^{(0)}={\vec{\varphi}^h}^{(k)}+\vec{\epsilon}^{2h\rightarrow h}$

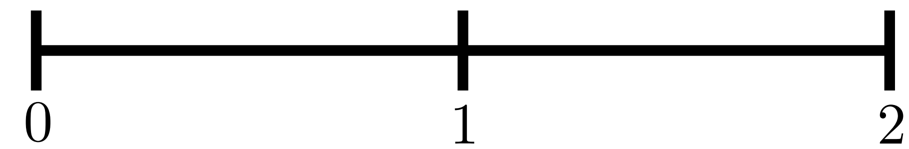
g) goto step a) until norm of residual drops below acceptable threshold

reminder: never code matrices, use loops with index form instead

Multigrid

- Why stop at 2 grids?
  - go all the way to coarsest possible mesh
  - possible if $M^h = 2^p$

$$M^{2^p h} = 2$$



Multigrid

a) on fine gird, perform a few iterations for $A\vec{\varphi} = \vec{f}$ with $\vec{\varphi}^{h^{(0)}}$ as initial guess

b) calculate residual: $\vec{r}^{h} = \vec{f} - A\vec{\varphi}^{h^{(k)}}$

c) restrict residual to coarse grid: $\vec{r}^{h\rightarrow 2h}$

d) on coarse grid, perform a few iterations for $A\vec{\epsilon}^{2h} = \vec{r}^{h\rightarrow 2h}$ with $\vec{\epsilon}^{2h^{(0)}} = 0$

 d.b) calculate residual to error equation: $\vec{r}^{2h} = \vec{r}^{h\rightarrow 2h} - A\vec{\epsilon}^{2h^{(k)}}$

 d.c) restrict residual to next coarser grid: $\vec{r}^{2h\rightarrow 4h}$

 d.d) on next coarser grid, perform a few iterations for $A\vec{\epsilon}^{4h} = \vec{r}^{2h\rightarrow 4h}$ with $\vec{\epsilon}^{4h^{(0)}} = 0$

  d.d.b-d) continue with ...b) - ...d) on next coarser mesh $4h$ all the way to ph mesh

  d.d.e-g) do steps ...e) - ...g) from coarsest mesh ph all the way to $4h$ mesh

 d.e) prolong error to next finer grid: $\vec{\epsilon}^{4h\rightarrow 2h}$

 d.f) apply correction to next finer grid solution (error): $\vec{\epsilon}^{2h^{(0)}} = \vec{\epsilon}^{2h^{(k)}} + \vec{\epsilon}^{4h\rightarrow 2h}$

 d.g) on next finer grid, perform a few iterations for $A\vec{\epsilon}^{2h} = \vec{r}^{h\rightarrow 2h}$

e) prolong error to fine grid: $\vec{\epsilon}^{2h\rightarrow h}$

f) apply correction to coarse grid solution: $\vec{\varphi}^{h^{(0)}} = \vec{\varphi}^{h^{(k)}} + \vec{\epsilon}^{2h\rightarrow h}$

g) goto step a) until norm of residual drops below acceptable threshold

Recursive algorithm (but we know the number of levels beforehand)

Multigrid

- How to code this?
  - write iteration, prolongation, restriction as subroutines/functions
  - could use recursive calls with dynamic memory allocations

  OR

  - pre-compute and store grid level support data for all grid levels in vectors `M(1:p), N(1:p), h(1:p)`
  - do not make new arrays/variables for each grid level, instead use

`rhs (0:M(1),0:N(1),1:p):` $\vec{f}$, $\vec{r}^{h\to 2h}$, $\vec{r}^{2h\to 4h}$, ...

`r   (0:M(1),0:N(1),1:p):` $\vec{r}^{h}$, $\vec{r}^{2h}$, ...

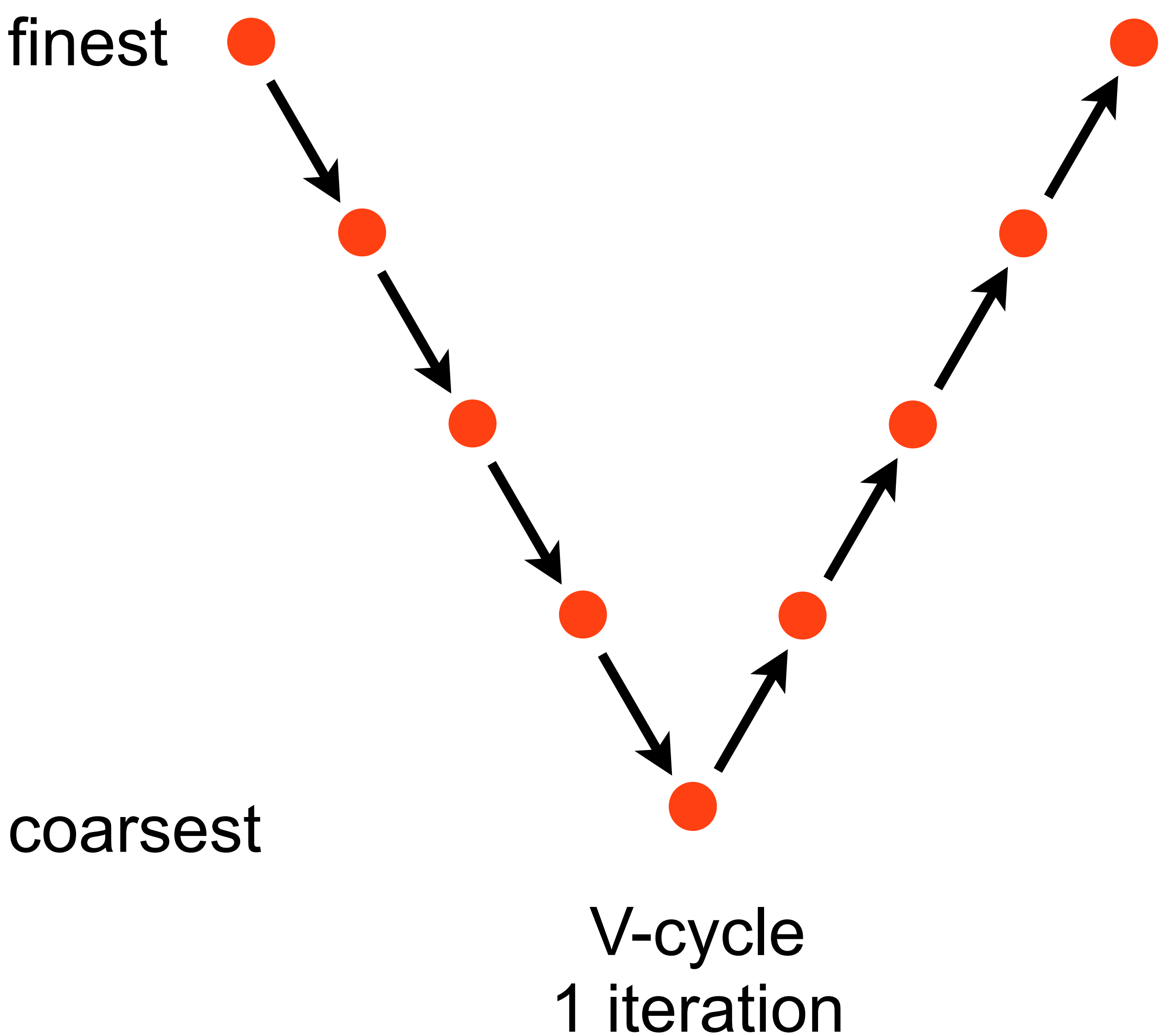
`eps (0:M(1),0:N(1),1:p):` — , $\vec{\epsilon}^{2h}$, $\vec{\epsilon}^{4h}$, ...

`epsc(0:M(1),0:N(1),1:p):` $\vec{\epsilon}^{2h\to h}$, $\vec{\epsilon}^{4h\to 2h}$, ...

  - this wastes some memory but makes coding easier

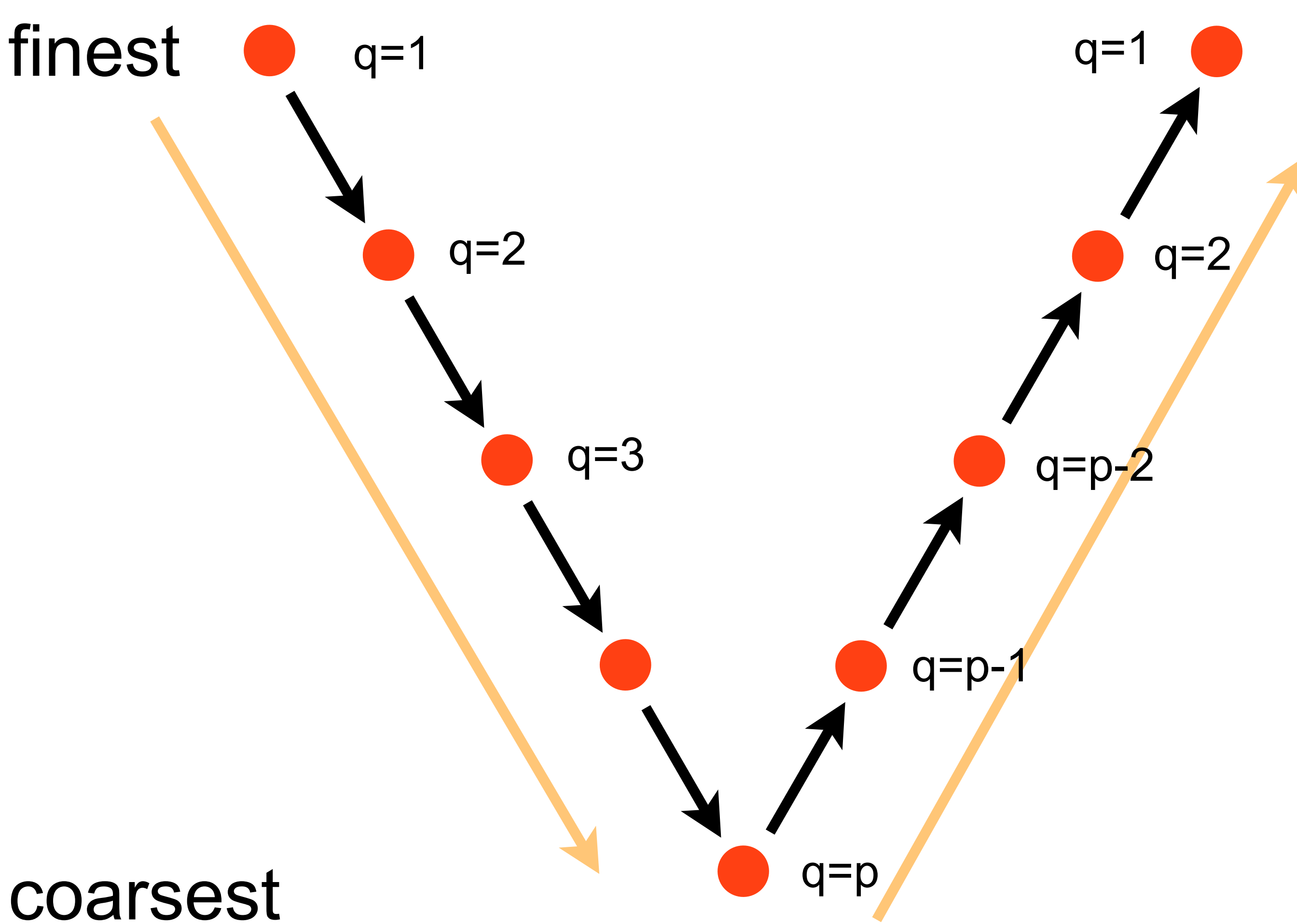
Multigrid: V-cycle

- How to traverse the different grid levels?
  - Many options!



Multigrid

- How to code single V-cycle iteration?



V-cycle
1 iteration

```
phi    = GaussSeidel (phi,rhs(:,1),h(1),M(1))
r(:,1) = calcResidual(phi,rhs(:,1),h(1),M(1))
```

```
loop q from 2 to p
   rhs(:,q) = restrict     (r(:,q-1)
   eps(:,q) = GaussSeidel  (eps(:,q),rhs(:,q),
                            h(q),M(q)
   r  (:,q) = calcResidual(eps(:,q),rhs(:,q),

end loop q
```

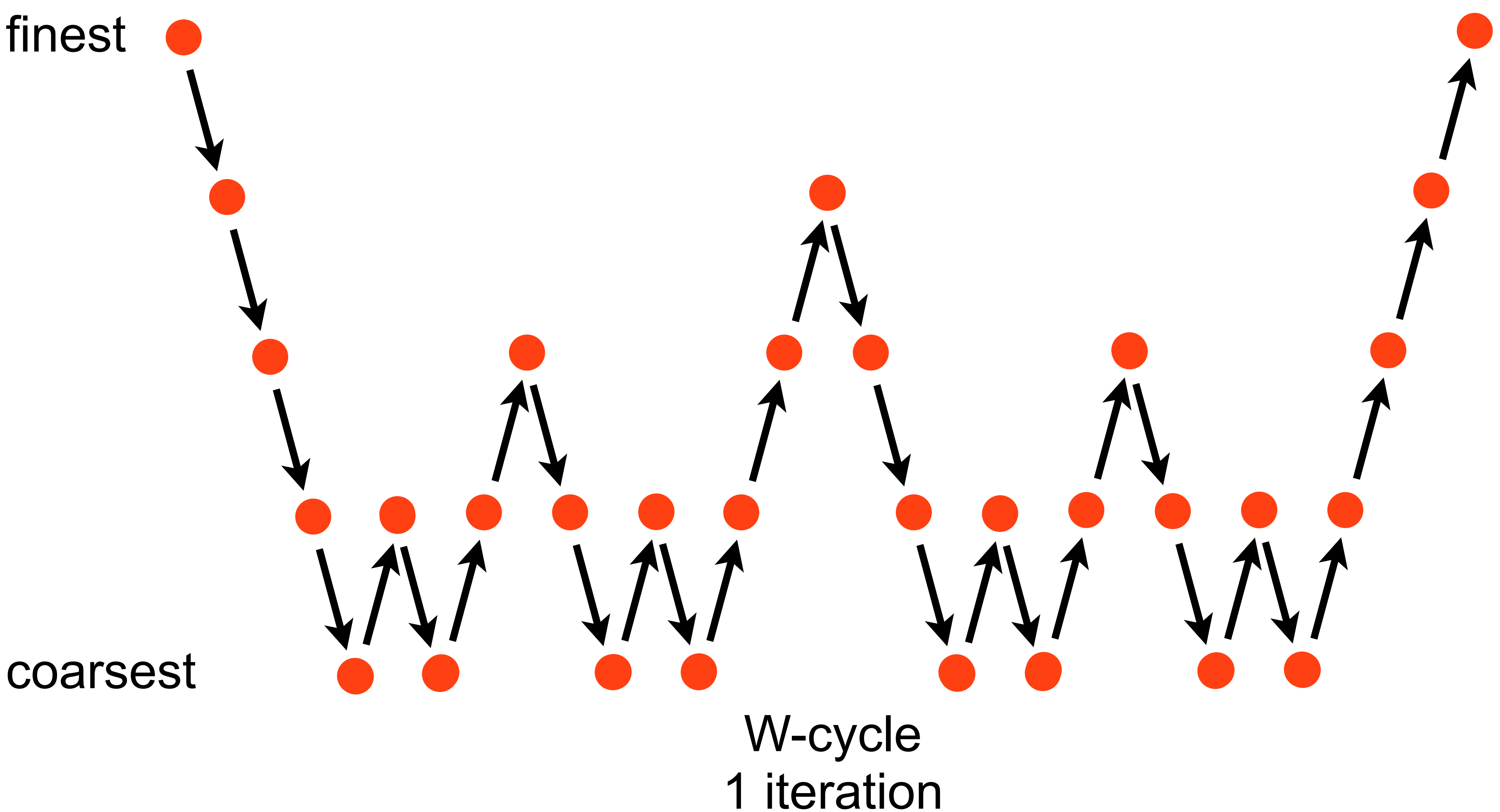
```
loop q from p-1 to 2
   epsc(:,q) = prolong(eps(:,q+1),M(q))
   eps (:,q) = correct(eps(:,q),epsc(:,q)
   eps (:,q) = GaussSeidel(eps(:,q),rhs(:,q),
                            h(q),M(q)
end loop q
```

```
epsc(:,1) = prolong(eps(:,2),M(1))
phi       = correct(phi,epsc(:,1),M(1
```

comment: `rhs(:,1)` must contain PDE right hand side

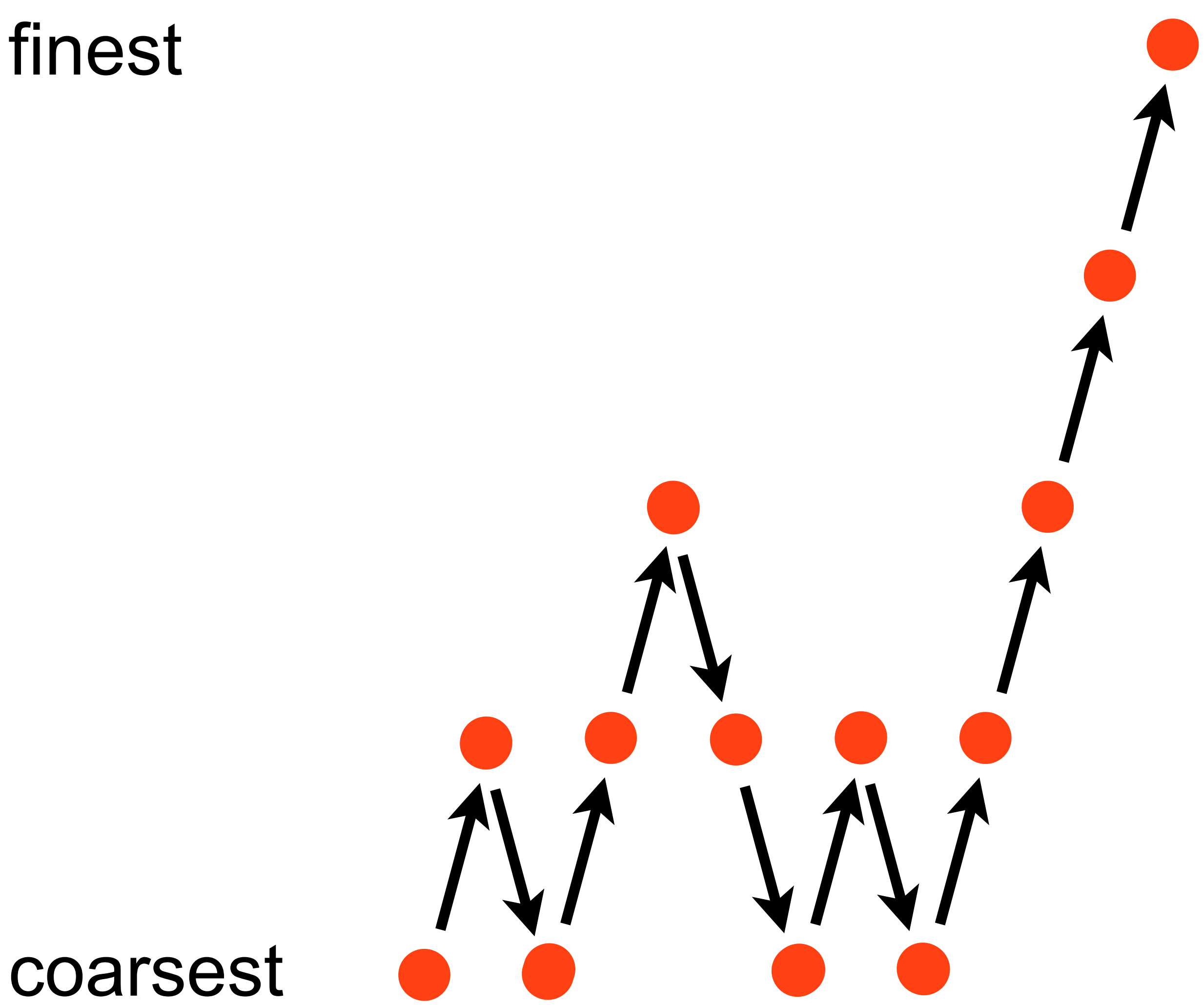
Multigrid: W-Cycle

- How to traverse the different grid levels?



Multigrid: FMC

- Full Multi Grid cycle:
    - start at coarsest grid level



FMC-cycle
1 iteration