• Muddiest Points from Class 04/12

- *"I am confused as to the size of the mesh we need for a second order scheme for the full navier stokes. Using C/fortran notation do we have P(0:N+1,0:M+1),U(1:N,-1:M+2),and V(-1:N+2,1:M)? Or is that Marker and Cell method?"*
  - MAC and fractional step use the same staggered mesh. The sizes are phi(0:M+1,0:N+1), u(1:M+1,0:N+1), v(0:M+1,1:N+1)

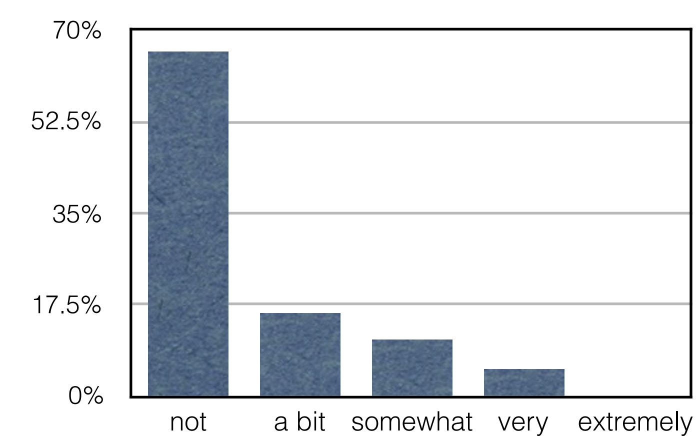
- *"Is phi (sic) identical to pressure in the cases we will analyze?"*
  - if your volume flux into the domain is constant in time, then yes, otherwise, no.

- *"So if I understand correctly, after hw #10, we're basically done with the final project? All we have to do is add the Lagrangian step?"*
  - You only need to add the Poisson equation solve (HW5) and the projection/correction step (a Lagrangian step is something else entirely, you perhaps mean Lagrange multiplier)

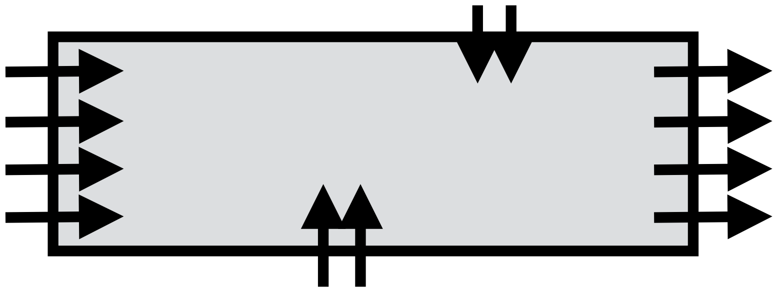
- *"You mentioned that there was going to be an extra credit homework 11. Is that going to be overlapping with the project?"*
  - Yes

- *"Everything was clear, however I think the normal lecture slides were step-by-step."*
  - Fixed

- *"I'm quite confused about getting v* into a particular subspace and don't even know what questions to ask at the moment. Hopefully next lecture will clear things up for me."*
- *"In the derivation of the pressure Poisson equation, is that fact that the div(v^n+1) is enforced to be zero is what causes phi to become the Lagrange Multiplier that project v* onto the subset of divergence free functions, or does this quality originate from elsewhere?"*
- *"I still have some doubts about Fractional step method discussed in class today. Can u please recap it in the next class?"*
  - see today's class



Outlet Correction for Fractional Step Method



Overall mass conservation (continuity equation) requires:

- *volume flow into domain = volume flow out of domain*

$$\dot{q}_{in} = \int_\Gamma \max(0, \vec{v}\cdot\vec{n})d\Gamma = \sum_{j=1}^{N}\Big(\max(u_{\frac{1}{2},j},0)+\max(-u_{M+\frac{1}{2},j},0)\Big)\Delta y + \sum_{i=1}^{M}\Big(\max(v_{i,\frac{1}{2}},0)+\max(-v_{i,N+\frac{1}{2}},0)\Big)\Delta x$$

$$\dot{q}_{out} = \int_\Gamma \max(0, -\vec{v}\cdot\vec{n})d\Gamma = \sum_{j=1}^{N}\Big(\max(-u_{\frac{1}{2},j},0)+\max(u_{M+\frac{1}{2},j},0)\Big)\Delta y + \sum_{i=1}^{M}\Big(\max(-v_{i,\frac{1}{2}},0)+\max(v_{i,N+\frac{1}{2}},0)\Big)\Delta x$$

- to simplify the discussion, let's assume there's only 1 outlet at the right boundary with all positive velocities (outwards pointing)

$$\dot{q}_{out} = \sum_{j=1}^{N} u_{M+\frac{1}{2},j}\Delta y$$

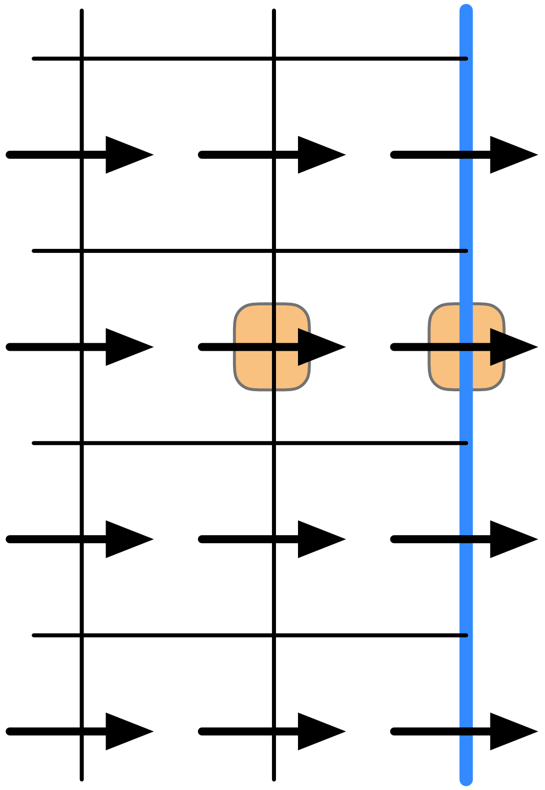
- since q_{in} is fixed by Dirichlet conditions, but the outlet velocities are determined by extrapolation there's no guarantee that $q_{in} = q_{out}$

- need to correct outlet velocities to enforce $q_{in} = q_{out}$:

$$\dot{q}_{corr} = \dot{q}_{in} - \dot{q}_{out} \quad\Rightarrow\quad u_{corr} = \frac{\dot{q}_{corr}}{L_y} = \frac{\dot{q}_{in}-\dot{q}_{out}}{L_y} \quad\Rightarrow\quad u_{M+\frac{1}{2},j} = u_{M+\frac{1}{2},j} + u_{corr}$$

- this correction needs to be done for outlets before the right hand side of the Poisson equation is calculated (it ensures that the rhs is in the column space of the Poisson matrix)

Advanced Outlet Boundary Conditions: Convective BC



- Recall that for hyperbolic equations, information travels along characteristic curves with a certain speed

- Let's consider only transport in the x-direction near the outlet

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \ldots \qquad \frac{\partial v}{\partial t} = u\frac{\partial v}{\partial x} + \ldots \qquad \frac{\partial Y}{\partial t} = u\frac{\partial Y}{\partial x} + \ldots$$

- Information from inside the domain should travel to the outlet with speed u

- If we use const. extrapolation, what's the speed information travels from $u_{M-\frac{1}{2},j}$ to $u_{M+\frac{1}{2},j}$?

$$a = \infty$$

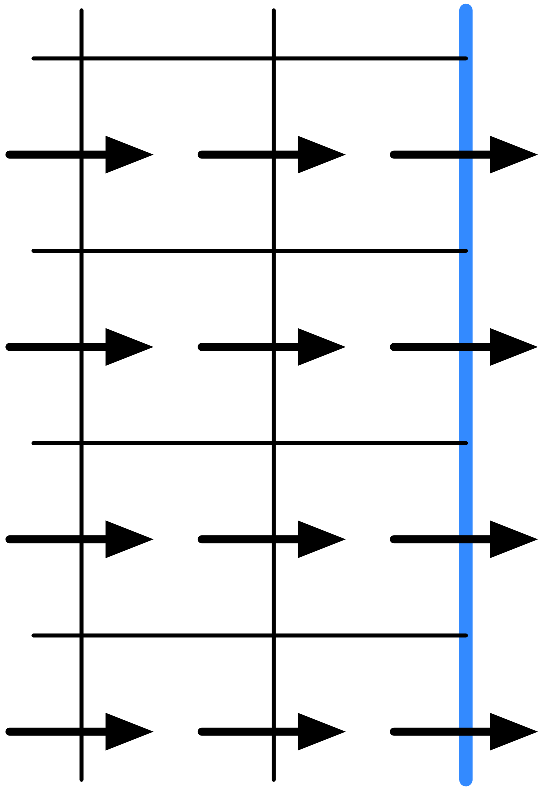
- Why not use a more realistic speed a and solve the transport in the x-direction for outlet bc values?

$$\left.\frac{\partial u}{\partial t}\right|_{M+\frac{1}{2},j} = a\left.\frac{\partial u}{\partial x}\right|_{M+\frac{1}{2},j}$$

use upwind w/ Crank-Nicholson: $$\frac{u^{n+1}_{M+\frac{1}{2},j} - u^{n}_{M+\frac{1}{2},j}}{\Delta t} = \frac{a}{2}\left(\frac{u^{n+1}_{M+\frac{1}{2},j} - u^{n+1}_{M-\frac{1}{2},j}}{\Delta x} + \frac{u^{n}_{M+\frac{1}{2},j} - u^{n}_{M-\frac{1}{2},j}}{\Delta x}\right)$$

Advanced Outlet Boundary Conditions: Convective BC

$$\sigma = \frac{a\Delta t}{2\Delta x}$$



$$\frac{u^{n+1}_{M+\frac{1}{2},j} - u^{n}_{M+\frac{1}{2},j}}{\Delta t} = \frac{a}{2}\left(\frac{u^{n+1}_{M+\frac{1}{2},j} - u^{n+1}_{M-\frac{1}{2},j}}{\Delta x} + \frac{u^{n}_{M+\frac{1}{2},j} - u^{n}_{M-\frac{1}{2},j}}{\Delta x}\right)$$

- This doesn't look too promising. But what do we know and not know?

  we apply boundary conditions after having solved for the interior!

$$u^{n+1}_{M+\frac{1}{2},j} = \frac{u^{n}_{M+\frac{1}{2},j}(1+\sigma) - \sigma\left(u^{n+1}_{M-\frac{1}{2},j} + u^{n}_{M-\frac{1}{2},j}\right)}{1-\sigma}$$

$\Rightarrow$ this can be directly solved for the boundary condition velocity at time n+1!

- But what's the speed a?
  - it has to be positive, but there are many choices. For example:
    - maximum of all u velocities at M-1/2,j
- This convective boundary condition works for all other quantities as well:

$$v^{n+1}_{M+1,j+\frac{1}{2}} = \frac{v^{n}_{M+1,j+\frac{1}{2}}(1+\sigma) - \sigma\left(v^{n+1}_{M,j+\frac{1}{2}} + v^{n}_{M,j+\frac{1}{2}}\right)}{1-\sigma}$$

$$Y^{n+1}_{M+1,j+\frac{1}{2}} = \frac{Y^{n}_{M+1,j+\frac{1}{2}}(1+\sigma) - \sigma\left(Y^{n+1}_{M,j+\frac{1}{2}} + Y^{n}_{M,j+\frac{1}{2}}\right)}{1-\sigma}$$

- Finally, to initialize the boundary condition, use constant extrapolation at t=0

Fractional Step Method Algorithm

1. Initialize variables and apply boundary conditions (see HW 10)
2. calculate stable time step (see HW10)
3. solve any additional scalar equations (see HW10)
4. solve viscous Burger's equations in interior: $u^*_{i+\frac{1}{2},j},\ v^*_{i,j+\frac{1}{2}}$ (see HW10)
5. apply velocity boundary conditions (see HW10)
6. correct outlet velocities to ensure volume flux conservation (see this class)
7. calculate right hand side of Poisson equation

$$f_{i,j} = \frac{1}{\Delta t}\left(\frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{\Delta y}\right)$$

(can check that $\sum_{j=1}^{N}\sum_{i=1}^{M} f_{i,j} = 0$)

8. solve Poisson equation for Lagrange multiplier φ using Neumann boundary conditions (see HW5)
9. project/correct velocities using Lagrange multiplier

$$u^{n+1}_{i+\frac{1}{2},j} = u^*_{i+\frac{1}{2},j} - \Delta t\frac{\varphi_{i+1,j} - \varphi_{i,j}}{\Delta x} \qquad v^{n+1}_{i,j+\frac{1}{2}} = v^*_{i,j+\frac{1}{2}} - \Delta t\frac{\varphi_{i,j+1} - \varphi_{i,j}}{\Delta y}$$

10. apply velocity boundary conditions (see HW10)
11. go to 2 until done

- Comment on GCI analysis for steady state solutions
  - Common way to reach a steady state solution for time dependent PDEs is to time advance the solution until in discrete form

$$\frac{\partial \phi}{\partial t} < \epsilon$$

  - To perform GCI analysis for spatial discretization errors, must make sure that "non-steady-state" error ϵ is much smaller than spatial errors

- Please fill out the course evaluation forms online!
- Questions about the Final Project?