• 4th-order PADE

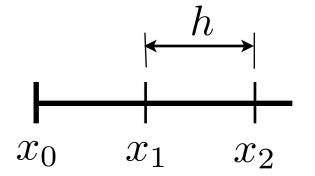
$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h} (f_{i+1} - f_{i-1}) + O(h^4)$$

- only 2 slight problems:
 - to get f'_{i} , we need f'_{i-1} and $f'_{i+1} \Rightarrow$ coupled system \Rightarrow implicit
 - to get f'_{0} , we need f'_{-1} , and to get f'_{N} we need f'_{N+1}
 - ⇒ N-1 equations for N+1 unknowns!
- Solution: apply difference formula of lower order at boundaries using only "inner" points
- Why lower order?

Example 6: left boundary

$$f_0' + a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_1' = O(?)$$

stencil:



Following procedure results in

$$f_0' + 2f_1' = \frac{1}{h} \left(-\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2 \right) + O(h^3)$$

similar for right boundary

$$2f'_{N-1} + f'_N = \frac{1}{h} \left(\frac{5}{2} f_N - 2f_{N-1} - \frac{1}{2} f_{N-2} \right) + O(h^3)$$

recap: in the interior we had

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h} (f_{i+1} - f_{i-1}) + O(h^4)$$

So what do we have now?

Spring 2015

AEE471/MAE561 Computational Fluid Dynamics

AEE471/MAE561 Computational Fluid Dynamics
$$\begin{bmatrix} 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 4 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2 \\ 3(f_2 - f_0) \\ 3(f_3 - f_1) \\ \vdots \\ 3(f_{N-1} - f_{N-3}) \\ 3(f_N - f_{N-2}) \\ \frac{5}{2}f_N - 2f_{N-1} - \frac{1}{2}f_{N-2} \end{bmatrix}$$

$$f'_{0} \qquad f'_{1} \qquad f'_{2} \qquad f'_{N-2} \qquad f'_{N-1} \qquad f'_{N}$$

$$x_{0} \qquad x_{1} \qquad x_{2} \qquad x_{N-2} \qquad x_{N-1} \qquad x_{N}$$

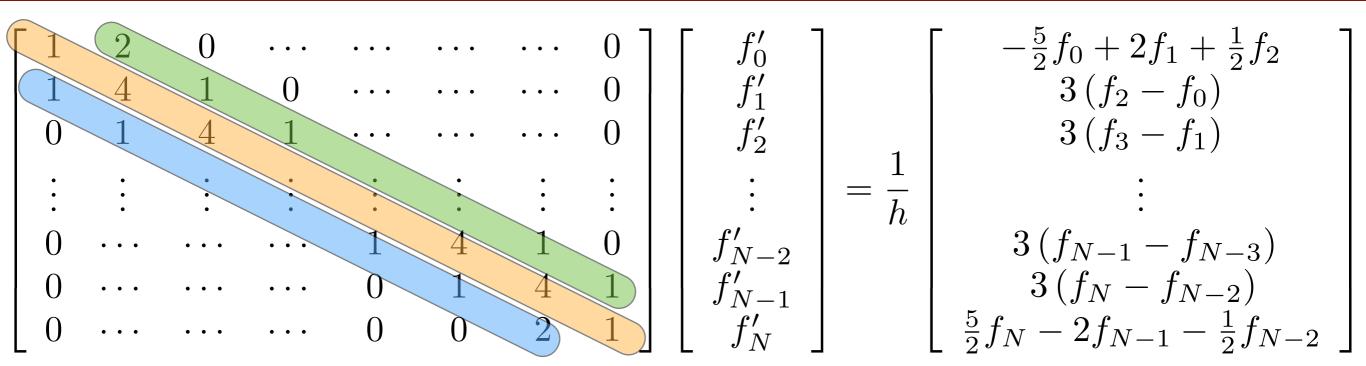
$$f'_{0} + 2f'_{1} = \frac{1}{h} \left(-\frac{5}{2} f_{0} + 2f_{1} + \frac{1}{2} f_{2} \right) \qquad f'_{N-3} + 4f'_{N-2} + f'_{N-1} = \frac{3}{h} \left(f_{N-1} - f_{N-3} \right)$$

$$f'_{0} + 4f'_{1} + f'_{2} = \frac{3}{h} \left(f_{2} - f_{0} \right) \qquad \qquad f'_{N-2} + 4f'_{N-1} + f'_{N} = \frac{3}{h} \left(f_{N} - f_{N-2} \right)$$

$$f'_{1} + 4f'_{2} + f'_{3} = \frac{3}{h} \left(f_{3} - f_{1} \right) \qquad \qquad 2f'_{N-1} + f'_{N} = \frac{1}{h} \left(\frac{5}{2} f_{N} - 2f_{N-1} - \frac{1}{2} f_{N-2} \right)$$

Spring 2015

AEE471/MAE561 Computational Fluid Dynamics



- So what do we have now? A tri-diagonal system!
- Was this inevitable? No, it depends on the choice of boundary scheme!
- Recipe:
 - Choose a boundary scheme such that
 - a) matrix form is preserved (here tri-diagonal)
 - b) boundary stencil ≤ 1st interior stencil (reasons later)
- Does the lower order @ boundary not pollute the higher order in the interior?
 - Depends on the case. In most cases, the additional error remains @ or near the boundary
- How do we solve the resulting tri-diagonal system?

How to solve a tri-diagonal system?

▶ Gaussian elimination (direct solve)

▶ instead: store the 3 diagonals in separate vectors (1D arrays)

$egin{array}{c} b_0 \ a_1 \ 0 \end{array}$	$c_0\\b_1\\a_2$	$egin{array}{c} 0 \ c_1 \ b_2 \end{array}$	$0 \\ 0 \\ c_2$	• • •	0 0 0	0 0 0	0 0 0	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} f_0' \\ f_1' \\ f_2' \end{bmatrix}$		$egin{pmatrix} d_0 \ d_1 \ d_2 \ \end{pmatrix}$
•	•	•	•	٠.	•	•	•		:		•
•	•	•	•	٠.	•	•	•		•	=	•
0	0	0	0		a_{N-2}	b_{N-2}	c_{N-2}	0	$ f'_{N-2} $		d_{N-2}
0	0	0	0	• • •	0	a_{N-1}	b_{N-1}	c_{N-1}	$ f'_{N-1} $		$ d_{N-1} $
0	0	0	0		0	0	a_N	b_N $ floor$	$\lfloor f_N' \rfloor$		$\lfloor d_N \rfloor$

• store right hand side in vector d_i with $i = 0 \dots N$

▶ instead: store the 3 diagonals in separate vectors (1D arrays)

$\begin{bmatrix} b_0 \\ a_1 \\ 0 \end{bmatrix}$	c_0 b_1 a_2	0 c_1 b_2	$0\\0\\c_2$	• • •	0 0 0	0 0 0	0 0 0	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\left[\begin{array}{c}f_0'\\f_1'\\f_2'\end{array}\right]$		$\left[\begin{array}{c}d_0\\d_1\\d_2\end{array}\right]$
	•		:	· · ·		• •	•	:			
	•	•	:	• • •		:	•	•		=	
0	0	0	0	• • •	a_{N-2}	b_{N-2}	c_{N-2}	0	$\left f_{N-2}'\right $		d_{N-2}
0	0	0	0	• • •	0	a_{N-1}	b_{N-1}	c_{N-1}	$\left f_{N-1}'\right $		$ d_{N-1} $
	0	0	0	• • •	0	0	a_N	b_N	$\lfloor f'_N \rfloor$		$\lfloor d_N \rfloor$

- store right hand side in vector d_i with $i = 0 \dots N$
- store main diagonal in vector b_i with $i = 0 \dots N$

▶ instead: store the 3 diagonals in separate vectors (1D arrays)

1	b_0	c_0	0	0		0	0	0	0	$\begin{bmatrix} f'_0 \end{bmatrix}$		$\lceil d_0 \rceil$
	a_1	b_1	c_1	0		0	0	0	0	f_1'		d_1
	0	a_2	b_2	c_2	• • •	0	0	0	0	f_2'		d_2
	•			•	٠.	•	•	•				
	•	•		· · ·	· · .	:	•	•		•	=	
	0	0	0	0		a_{N-2}	b_{N-2}	c_{N-2}	0	$\left f_{N-2}' \right $		$\left d_{N-2} \right $
	0	0	0	0		0	a_{N-1}	b_{N-1}	c_{N-1}	$ f'_{N-1} $		$ d_{N-1} $
	0	0	0	0		0	0	(a_N)	b_N \rfloor	$\lfloor f'_N \rfloor$		$\lfloor d_N \rfloor$

- store right hand side in vector d_i with $i = 0 \dots N$
- store main diagonal in vector b_i with $i = 0 \dots N$
- store lower diagonal in vector a_i with $i = 1 \dots N$?

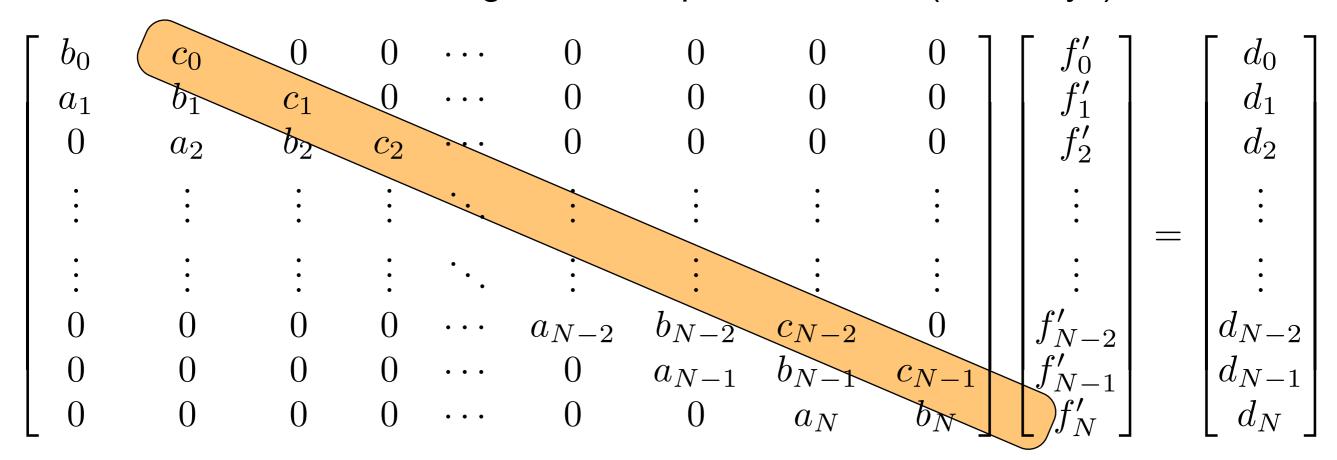
a bit impractical! extend vector to $i=0\dots N$ and skip i=0

▶ instead: store the 3 diagonals in separate vectors (1D arrays)

b_0 a_1	c_0 b_1	0 c_1	$0 \\ 0$	• • •	$0 \\ 0$	0	0	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$ \mid \int f_0' f_1' $		$\left[\begin{array}{c} d_0 \\ d_1 \end{array}\right]$	
0	a_2	b_2	c_2	• • •	0	0	0	0	f_2'		d_2	
•			•	•••	•	•	•	•		_	•	
•	•		·:	··.		•	•	•			•	
0	0	0	0		a_{N-2}	b_{N-2}	c_{N-2}	0	$ f'_{N-} $	-2	d_{N-2}	
0	0	0	0	• • •	0	a_{N-1}	b_{N-1}	c_{N-1}	$ f'_{N-} $	$\cdot 1$	d_{N-1}	
0	0	0	0		0	0	(a_N)	b_N $_$	$\mid \; \mid \; f_N'$.]	$\lfloor d_N \rfloor$	

- store right hand side in vector d_i with $i = 0 \dots N$
- store main diagonal in vector b_i with $i = 0 \dots N$
- store lower diagonal in vector a_i with $i = 0 \dots N$

▶ instead: store the 3 diagonals in separate vectors (1D arrays)



- store right hand side in vector d_i with $i = 0 \dots N$
- store main diagonal in vector b_i with $i = 0 \dots N$
- store lower diagonal in vector a_i with $i = 0 \dots N$
- store upper diagonal in vector c_i with $i = 0 \dots N 1$?

again extend vector

to $i = 0 \dots N$ and skip i = N

▶ instead: store the 3 diagonals in separate vectors (1D arrays)

$\lceil b_0 \rceil$	c_0	0	0		0	0	0	0	$\mid f'_0 \mid$		$\lceil d_0 \rceil$	
a_1	b_1	c_1	0		0	0	0	0	$\mid f_1' \mid$		d_1	
0	a_2	b_2	c_2	• • •	0	0	0	0	f_2'		d_2	
•	•	•	•	•	•	•	•	•			•	
•	•	•	•	•	•	•	•	•		=	•	
•	•	•	•	•	•	•	•	•			•	
0	0	0	0	• • •	a_{N-2}	b_{N-2}	c_{N-2}	0	$ f'_{N-2} $		d_{N-2}	
0	0	0	0	• • •	0	a_{N-1}	b_{N-1}	c_{N-1}	l et		$ d_{N-1} $	
0	0	0	0		0	0	a_N	b_N	$\mid f_N' \mid$		d_N	

- store right hand side in vector d_i with $i = 0 \dots N$
- store main diagonal in vector b_i with $i = 0 \dots N$
- store lower diagonal in vector a_i with $i = 0 \dots N$
- store upper diagonal in vector c_i with $i = 0 \dots N$

2) Elimination

▶ 1st pivot b₀: subtraction multiplier: a₁/b₀

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$
 $d_1 \to d_1 - d_0 \frac{a_1}{b_0}$ $(c_1 \to c_1 \quad a_1 \to 0)$

$$d_1 \to d_1 - d_0 \frac{a_1}{b_0}$$

$$(c_1 \rightarrow c_1)$$

$$a_1 \to 0$$

2) Elimination

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{N-2} & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ 1st pivot b₀: subtraction multiplier: a₁/b₀

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$
 $d_1 \to d_1 - d_0 \frac{a_1}{b_0}$ $(c_1 \to c_1 \quad a_1 \to 0)$

$$(c_1 \to c_1)$$

$$a_1 \to 0$$

▶ 2nd pivot b₁: subtraction multiplier: a₂/b₁

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1}$$

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1}$$
 $d_2 \to d_2 - d_1 \frac{a_2}{b_1}$ $(c_2 \to c_2 \quad a_2 \to 0)$

$$(c_2 \rightarrow c_2)$$

$$a_2 \to 0$$

2) Elimination

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{N-2} & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ 1st pivot b₀: subtraction multiplier: a₁/b₀

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$
 $d_1 \to d_1 - d_0 \frac{a_1}{b_0}$ $(c_1 \to c_1 \quad a_1 \to 0)$

$$(c_1 \to c_1)$$

$$a_1 \to 0$$

▶ 2nd pivot b₁: subtraction multiplier: a₂/b₁

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1}$$

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1}$$
 $d_2 \to d_2 - d_1 \frac{a_2}{b_1}$ $(c_2 \to c_2 \quad a_2 \to 0)$

$$(c_2 \rightarrow c_2)$$

$$a_2 \to 0$$

ith pivot b_{i-1}: subtraction multiplier: a_i/b_{i-1}

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}}$$

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}}$$
 $d_i \to d_i - d_{i-1} \frac{a_i}{b_{i-1}}$ $(c_i \to c_i \quad a_i \to 0)$

$$(c_i \rightarrow c_i)$$

$$a_i \to 0$$

2) Elimination

▶ 1st pivot b₀: subtraction multiplier: a₁/b₀

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$

$$b_1 \to b_1 - c_0 \frac{a_1}{b_0}$$
 $d_1 \to d_1 - d_0 \frac{a_1}{b_0}$ $(c_1 \to c_1 \quad a_1 \to 0)$

$$(c_1 \rightarrow c_1$$

$$a_1 \to 0$$

▶ 2nd pivot b₁: subtraction multiplier: a₂/b₁

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1}$$

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1}$$
 $d_2 \to d_2 - d_1 \frac{a_2}{b_1}$ $(c_2 \to c_2 \quad a_2 \to 0)$

$$(c_2 \rightarrow c_2)$$

$$a_2 \to 0$$

ith pivot b_{i-1}: subtraction multiplier: a_i/b_{i-1}

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}}$$

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}} \qquad d_i \to d_i - d_{i-1} \frac{a_i}{b_{i-1}} \qquad (c_i \to c_i \qquad a_i \to 0)$$

$$(c_i \to c_i)$$

$$a_i \to 0$$

Nth pivot b_{N-1}: subtraction multiplier: a_N/b_{N-1}

$$b_N \to b_N - c_{N-1} \frac{a_N}{b_{N-1}} \qquad d_N \to d_N - d_{N-1} \frac{a_N}{b_{N-1}} \quad (c_N \to c_N \quad a_N \to 0)$$

$$d_N \rightarrow d_N - d_{N-1} \frac{a_N}{b_N}$$

$$(c_N \to c_N$$

2) Elimination

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{N} \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ ith pivot b_{i-1}: subtraction multiplier: a_i/b_{i-1}

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}} \qquad d_i \to d_i - d_{i-1} \frac{a_i}{b_{i-1}} \qquad (c_i \to c_i \qquad a_i \to 0)$$

```
loop from i = 1 to N  # loop over rows to eliminate
  b(i) = b(i) - c(i-1)*a(i)/b(i-1)
  d(i) = d(i) - d(i-1)*a(i)/b(i-1)
end loop i
```

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{N-1} & c_{N-1} \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

▶ last row:

$$d_N o rac{d_N}{b_N} \quad (b_N o 1)$$

3) Back-Substitution

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{N-1} & c_{N-1} \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

▶ last row:

$$d_N \to \frac{d_N}{b_N} \quad (b_N \to 1)$$

3) Back-Substitution

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 & f'_{N-2} & d_{N-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & f'_N & d_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-2} \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ last row:

$$d_N \to \frac{d_N}{b_N} \quad (b_N \to 1)$$

▶ row N-1:

3) Back-Substitution

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 & f'_{N-2} & d_{N-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & f'_N & d_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ last row:

$$d_N o rac{d_N}{b_N} \quad (b_N o 1)$$

▶ row N-1:

$$d_{N-1} \to (d_{N-1} - c_{N-1} d_N) \qquad (c_{N-1} \to 0)$$

Class 05 20

3) Back-Substitution

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 & f'_{N-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & f'_{N-1} & d_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & f'_{N} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-2} \\ d_{N-2} \\ d_{N-2} \\ d_{N-1} \\ d_{$$

▶ last row:

$$d_N o rac{d_N}{b_N} \quad (b_N o 1)$$

▶ row N-1:

$$d_{N-1} \to (d_{N-1} - c_{N-1}d_N) \frac{1}{b_{N-1}} \qquad (c_{N-1} \to 0 \qquad b_{N-1} \to 1)$$

$$\begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & c_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ last row:

$$d_N o rac{d_N}{b_N} \quad (b_N o 1)$$

▶ row N-1:

$$d_{N-1} \to (d_{N-1} - c_{N-1}d_N) \frac{1}{b_{N-1}} \qquad (c_{N-1} \to 0 \qquad b_{N-1} \to 1)$$

▶ ith row:

$$d_i \to (d_i - c_i d_{i+1}) \frac{1}{b_i}$$
 $(c_i \to 0 \quad b_i \to 1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ last row:

$$d_N o rac{d_N}{b_N} \quad (b_N o 1)$$

▶ row N-1:

$$d_{N-1} \to (d_{N-1} - c_{N-1} d_N) \frac{1}{b_{N-1}} \qquad (c_{N-1} \to 0 \qquad b_{N-1} \to 1)$$

▶ ith row:

$$d_i \to (d_i - c_i d_{i+1}) \frac{1}{b_i}$$
 $(c_i \to 0 \quad b_i \to 1)$

▶ 1st row:

$$d_0 \to (d_0 - c_0 d_1) \frac{1}{b_0}$$
 $(c_0 \to 0 \quad b_0 \to 1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}$$

▶ last row:

$$d_N \to \frac{d_N}{b_N} \quad (b_N \to 1)$$

▶ ith row:

$$d_N \to \frac{d_N}{b_N} \quad (b_N \to 1) \qquad \qquad d_i \to (d_i - c_i d_{i+1}) \frac{1}{b_i} \qquad (c_i \to 0 \qquad b_i \to 1)$$

```
d(N) = d(N)/b(N)
loop from i = N-1 to 0 backwards # loop over rows
   d(i) = (d(i) - c(i)*d(i+1))/b(i)
end loop i
```

How to solve a tri-diagonal system?

- Gaussian elimination (direct solve)
- 1) Store diagonals in vectors
- 2) Elimination
- 3) Back-Substitution

```
loop from i = 1 to N  # loop over rows to eliminate
     b(i) = b(i) - c(i-1)*a(i)/b(i-1)
     d(i) = d(i) - d(i-1)*a(i)/b(i-1)
end loop i
```

```
d(N) = d(N)/b(N)
loop from i = N-1 to 0 backwards # loop over rows
d(i) = (d(i) - c(i)*d(i+1))/b(i)
end loop i
```

Some notes for this algorithm:

- destroys (overwrites) a, b, and d
- solution is in vector d
- implement as function/subroutine
- alternatives can be found in 'Numerical Recipes' online.

Class 05 25