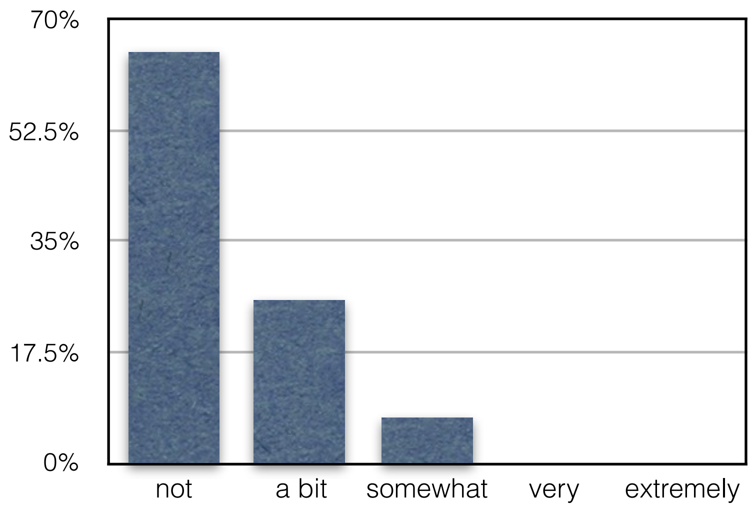
Muddiest Points from Class 03/20

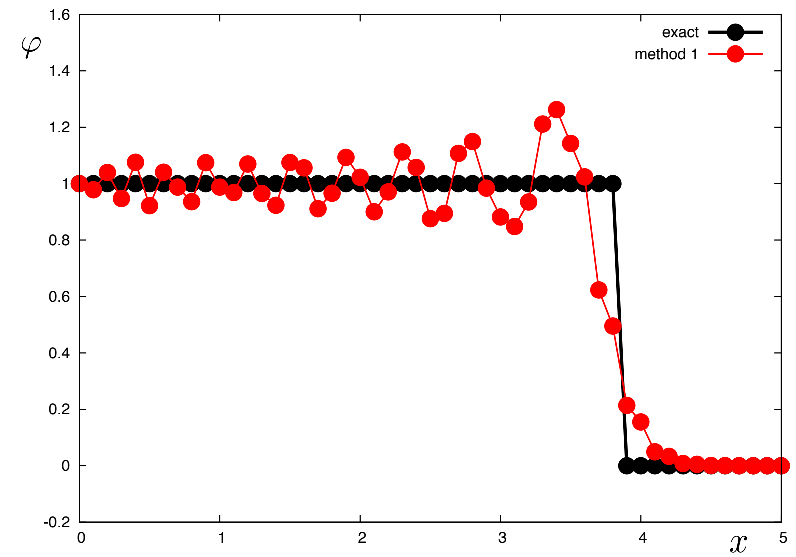
- *“Based on the discussion regarding turbulence and viscous effects, will there be situations where the solution method used is dictated by the method's ability to accurately model the dissipation (or lack thereof) of the physical problem? “*
  - Yes. There’s a dispute in the research community what’s better: higher order & not discretely conserving (kinetic energy) vs. lower order & discretely conserving (kinetic energy)
- *“How do we account for half a time step when coding the formulas in Matlab?”*
  - just do a time step with 1/2 dt.
  - if this asks about how to store this with integer indices: DON’T STORE ALL TIME STEPS
  - for RK: use a different variable names
- *“Is there any benefit to not using a crank-nicholson implicit method? It seems as though most problems can be formulated as such and it is very powerful and stable.”*
  - CN as discussed w/ ADI is only second order! Many situations require higher order: WENO-TVD-RK (today)
- *“Can more sub steps be added to the runge kutta method like steps of 1/3 or 1/4 to time to increase the order?”*
  - Yes, there’s a huge number of papers written on “optimal” RK methods
- *“Which of the methods discussed is most used in practice?”*
  - Everyone has their favorite method, but a quite popular method is discussed today: WENO-TVD-RK



ENO-Schemes

Essentially **N**on-**O**scillatory Schemes

$$\frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x}$$



Observations:

- higher order schemes often are dispersive
- Is there a way to modify/choose stencil such that dispersive errors are minimized?

Idea:

- build stencils based on local "smoothness"

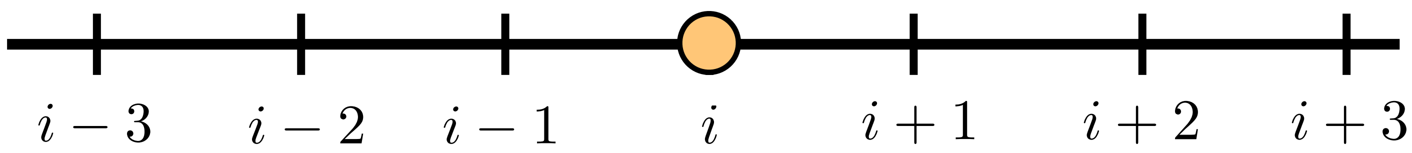
References:

- Harten et al., J. Comput. Phys., 131, pp. 3-47, 1997.
- Shu, NASA/CR-97-206253; ICASE 97-65.

ENO-Schemes

Procedure to build stencils:

- start at i
- add either i+1 or i-1 depending on which has “smoother” data



for equidistant meshes: if $|u_{i+1} - u_i| < |u_i - u_{i-1}|$ → take *i +1*

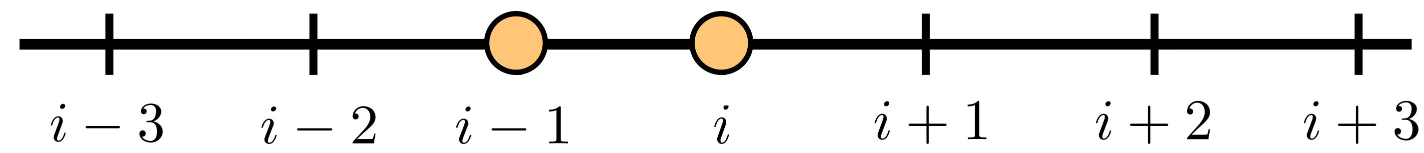
if $|u_{i+1} - u_i| \geq |u_i - u_{i-1}|$ → take *i - 1*

let’s introduce shorthands: $\Delta^+ u_i = u_{i+1} - u_i$ and $\Delta^- u_i = u_i - u_{i-1}$

ENO-Schemes

$$\Delta^+ u_i = u_{i+1} - u_i$$
$$\Delta^- u_i = u_i - u_{i-1}$$

Procedure to build stencils:



- start at i
- add either i+1 or i-1 depending on which has “smoother” data

for equidistant meshes: if $|\Delta^+ u_i| < |\Delta^- u_i|$ $\rightarrow$ take *i +1*

if $|\Delta^+ u_i| \geq |\Delta^- u_i|$ $\rightarrow$ take *i - 1*

- as an example, let’s assume $|\Delta^+ u_i| \geq |\Delta^- u_i|$ $\rightarrow$ take *i - 1*

- add either i+1 or i-2 depending on which has “smoother” data

for equidistant meshes: if $|\Delta^+ \Delta^- u_i| < |\Delta^- \Delta^- u_i|$ $\rightarrow$ take *i +1*

if $|\Delta^+ \Delta^- u_i| \geq |\Delta^- \Delta^- u_i|$ $\rightarrow$ take *i - 2*

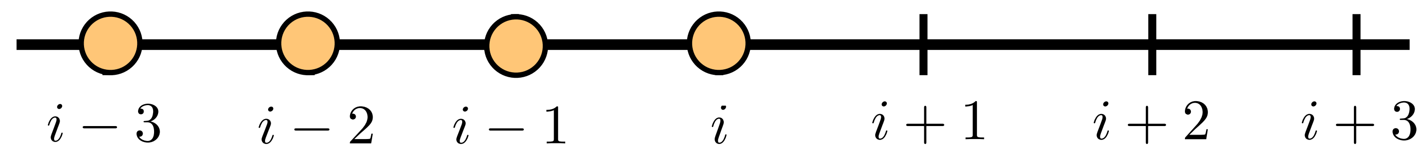
- keep on adding stencil points up to the desired number/order of accuracy
- once stencil points are found, use Taylor table coefficients
- BUT: may need other considerations besides “smoothness”: upwind bias

3rd-order ENO for linear equations

$$\frac{\partial \varphi}{\partial t} = -a \frac{\partial \varphi}{\partial x} \qquad a > 0$$

- 3rd-order method ⇒ 4 stencil points
- requires upwind bias
- 3 stencils are possible

S_0:



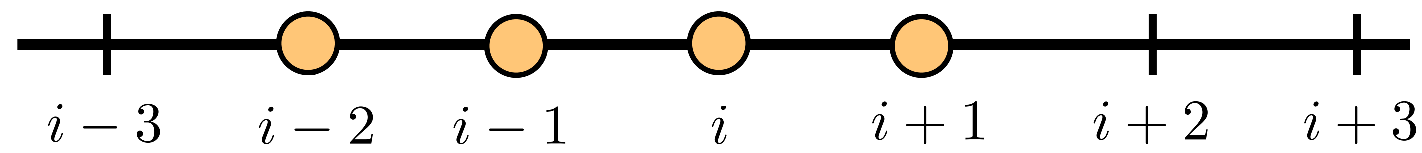
$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-3} - \frac{7}{6}\Delta^+\varphi_{i-2} + \frac{11}{6}\Delta^+\varphi_{i-1}\right)$$

3rd-order ENO for linear equations

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \qquad a > 0$$

- 3rd-order method ⇒ 4 stencil points
- requires upwind bias
- 3 stencils are possible

S[1]:



$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-3} - \frac{7}{6}\Delta^+\varphi_{i-2} + \frac{11}{6}\Delta^+\varphi_{i-1}\right)$$

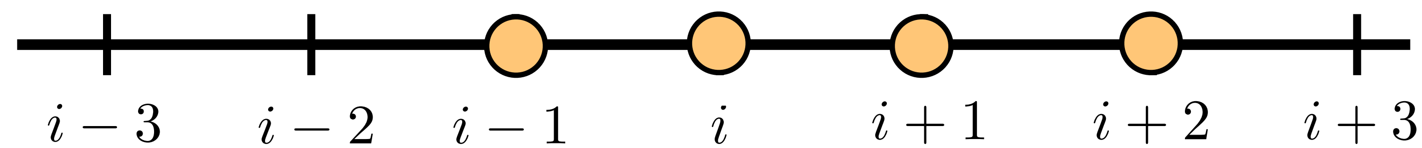
$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} = \frac{1}{\Delta x}\left(-\frac{1}{6}\Delta^+\varphi_{i-2} + \frac{5}{6}\Delta^+\varphi_{i-1} + \frac{1}{3}\Delta^+\varphi_{i}\right)$$

3rd-order ENO for linear equations

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \qquad a > 0$$

- 3rd-order method ⇒ 4 stencil points
- requires upwind bias
- 3 stencils are possible

S_2:



$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-3} - \frac{7}{6}\Delta^+\varphi_{i-2} + \frac{11}{6}\Delta^+\varphi_{i-1}\right)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} = \frac{1}{\Delta x}\left(-\frac{1}{6}\Delta^+\varphi_{i-2} + \frac{5}{6}\Delta^+\varphi_{i-1} + \frac{1}{3}\Delta^+\varphi_{i}\right)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-,2} = \frac{1}{\Delta x}\left(\frac{1}{3}\Delta^+\varphi_{i-1} + \frac{5}{6}\Delta^+\varphi_{i} - \frac{1}{6}\Delta^+\varphi_{i+1}\right)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^{-} = \begin{cases} \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} & \text{if} \quad |\Delta^-\Delta^+\varphi_{i-1}| < |\Delta^-\Delta^+\varphi_i| \text{ and } |\Delta^-\Delta^-\Delta^+\varphi_{i-1}| < |\Delta^+\Delta^-\Delta^+\varphi_{i-1}| \\ \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,2} & \text{if} \quad |\Delta^-\Delta^+\varphi_{i-1}| > |\Delta^-\Delta^+\varphi_i| \text{ and } |\Delta^-\Delta^-\Delta^+\varphi_{i}| > |\Delta^+\Delta^-\Delta^+\varphi_{i}| \\ \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} & \quad \text{otherwise} \end{cases}$$

- for $a < 0$: use right-biased stencils by simply mirroring all indices and changing + superscripts to - superscripts and vice versa

TVD RK-3 time integration

$$\frac{\partial \varphi}{\partial t} = -a\frac{\partial \varphi}{\partial x} \qquad a > 0$$

- ENO methods are often combined with specially optimized RK time integration methods, designed to be TVD
  - start: $\varphi_i^{(0)} = \varphi_i^n$
  - step 1: $\varphi_i^{(1)} = \varphi_i^{(0)} - \alpha_{1,0}\left(a\Delta t\left.\frac{\partial \varphi^{(0)}}{\partial x}\right|_i^-\right)$
  - step 2: $\varphi_i^{(2)} = \varphi_i^{(1)} - \alpha_{2,0}\left(a\Delta t\left.\frac{\partial \varphi^{(0)}}{\partial x}\right|_i^-\right) - \alpha_{2,1}\left(a\Delta t\left.\frac{\partial \varphi^{(1)}}{\partial x}\right|_i^-\right)$
  - step 3: $\varphi_i^{(3)} = \varphi_i^{(2)} - \alpha_{3,0}\left(a\Delta t\left.\frac{\partial \varphi^{(0)}}{\partial x}\right|_i^-\right) - \alpha_{3,1}\left(a\Delta t\left.\frac{\partial \varphi^{(1)}}{\partial x}\right|_i^-\right) - \alpha_{3,2}\left(a\Delta t\left.\frac{\partial \varphi^{(2)}}{\partial x}\right|_i^-\right)$
  - step q: $\varphi_i^{(q)} = \varphi_i^{(q-1)} - \sum_{r=0}^{q-1}\alpha_{q,r}\left(a\Delta t\left.\frac{\partial \varphi^{(r)}}{\partial x}\right|_i^-\right)$
- for example: TVD RK-3

$$\alpha_{1,0} = 1 \qquad \alpha_{2,0} = -\frac{3}{4},\ \alpha_{2,1} = \frac{1}{4} \qquad \alpha_{3,0} = -\frac{1}{12},\ \alpha_{3,1} = -\frac{1}{12},\ \alpha_{3,2} = \frac{2}{3}$$

Code

WENO-Schemes

Weighted **E**ssentially **N**on-**O**scillatory Schemes

Idea:

- if we have all these stencils, why choose only one and let the others go to waste?
  - can use all stencils only in "smooth" regions
  - switch back to ENO in "non-smooth" regions
  - in "smooth" regions, use optimal combination of stencils:

$$\left.\frac{\partial \varphi}{\partial x}\right|_i^- = \omega_0 \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,0} + \omega_1 \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,1} + \omega_2 \left.\frac{\partial \varphi}{\partial x}\right|_i^{-,2} \qquad \text{with} \qquad \omega_0 + \omega_1 + \omega_2 = 1$$

    - if $\omega_0 = 0.1\,,\ \omega_1 = 0.6\,,\ \omega_2 = 0.3$ ⇒ optimal **5th-order** stencil

  - but, calculating ω_0, ω_1, and ω_2 from the smoothness is an art
  - for example: Jiang & Peng, SIAM J. Sci. Comput. 21(6), 2000: WENO-5

WENO-5

Jiang & Peng, SIAM J. Sci. Comput. 21(6), 2000

$$\left.\frac{\partial\varphi}{\partial x}\right|_i^- = \frac{1}{12\Delta x}\left(-\Delta^+\varphi_{i-2} + 7\Delta^+\varphi_{i-1} + 7\Delta^+\varphi_i - \Delta^+\varphi_{i+1}\right)$$
$$-\Psi_{WENO}\left(\frac{\Delta^-\Delta^+\varphi_{i-2}}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_{i-1}}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_i}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_{i+1}}{\Delta x}\right)$$

with $\Psi_{WENO}(a,b,c,d) = \frac{1}{3}\omega_0(a-2b+c) + \frac{1}{6}\left(\omega_2 - \frac{1}{2}\right)(b-2c+d)$

with $\omega_0 = \frac{\alpha_0}{\alpha_0+\alpha_1+\alpha_2}$ and $\omega_2 = \frac{\alpha_2}{\alpha_0+\alpha_1+\alpha_2}$

with $\alpha_0 = \frac{1}{(\epsilon + IS_0)^2}$ and $\alpha_1 = \frac{6}{(\epsilon + IS_1)^2}$ and $\alpha_2 = \frac{3}{(\epsilon + IS_2)^2}$

with $IS_0 = 13(a-b)^2 + 3(a-3b)^2$ $\quad IS_1 = 13(b-c)^2 + 3(b+c)^2$ $\quad IS_2 = 13(c-d)^2 + 3(3c-d)^2$

- for $a < 0$: $\left.\frac{\partial\varphi}{\partial x}\right|_i^+ = \frac{1}{12\Delta x}\left(-\Delta^+\varphi_{i-2} + 7\Delta^+\varphi_{i-1} + 7\Delta^+\varphi_i - \Delta^+\varphi_{i+1}\right)$

$$+\Psi_{WENO}\left(\frac{\Delta^-\Delta^+\varphi_{i+2}}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_{i+1}}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_i}{\Delta x}, \frac{\Delta^-\Delta^+\varphi_{i-1}}{\Delta x}\right)$$

- combine with TVD RK-3

Code

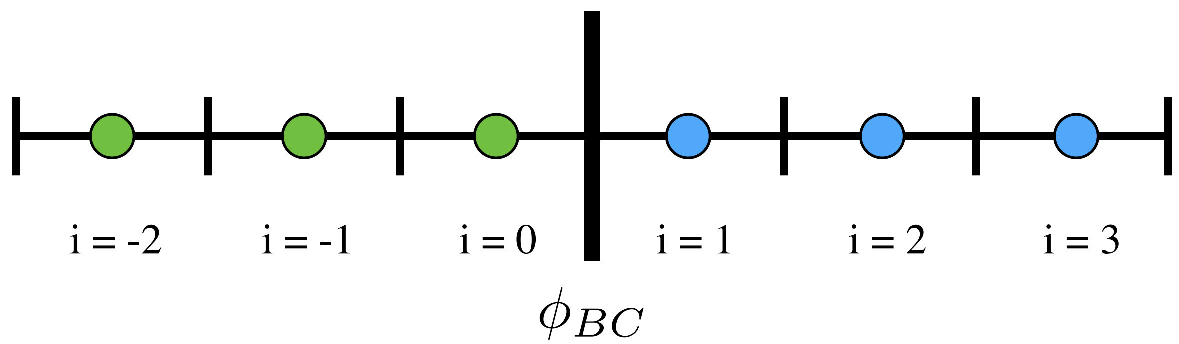
WENO-Schemes

Comments:

- one can construct even higher order schemes
- based on r^{th}-order individual stencils, the resulting scheme is order $2r-1$
- usually WENO is combined with high-order TVD RK, i.e. TVD RK-3 (Shu, SIAM J. Sci. Comput. 9(6), 1988)
- for WENO schemes, the max CFL number to control oscillations is usually smaller than the stability limit
- smaller CFL numbers are usually better
- newer enhancements include:
  - MPWENO (**M**onotonicity **P**reserving WENO) (Suresh & Huynth, J. Comput. Phys. 136, p.83, 1997)
  - MPWENO distinguishes between "smooth" local extrema and O(1) discontinuities

One issue: ENO & WENO use stencils that are much larger than we used so far

- for larger stencil schemes: add additional ghost cells, e.g.,



left boundary

Dirichlet:

$$\phi_0 = 2\phi_{BC} - \phi_1$$

$$\phi_{-1} = 2\phi_{BC} - \phi_2$$

$$\phi_{-2} = 2\phi_{BC} - \phi_3$$

zero-Neumann:

$$\phi_0 = \phi_1$$

$$\phi_{-1} = \phi_2$$

$$\phi_{-2} = \phi_3$$

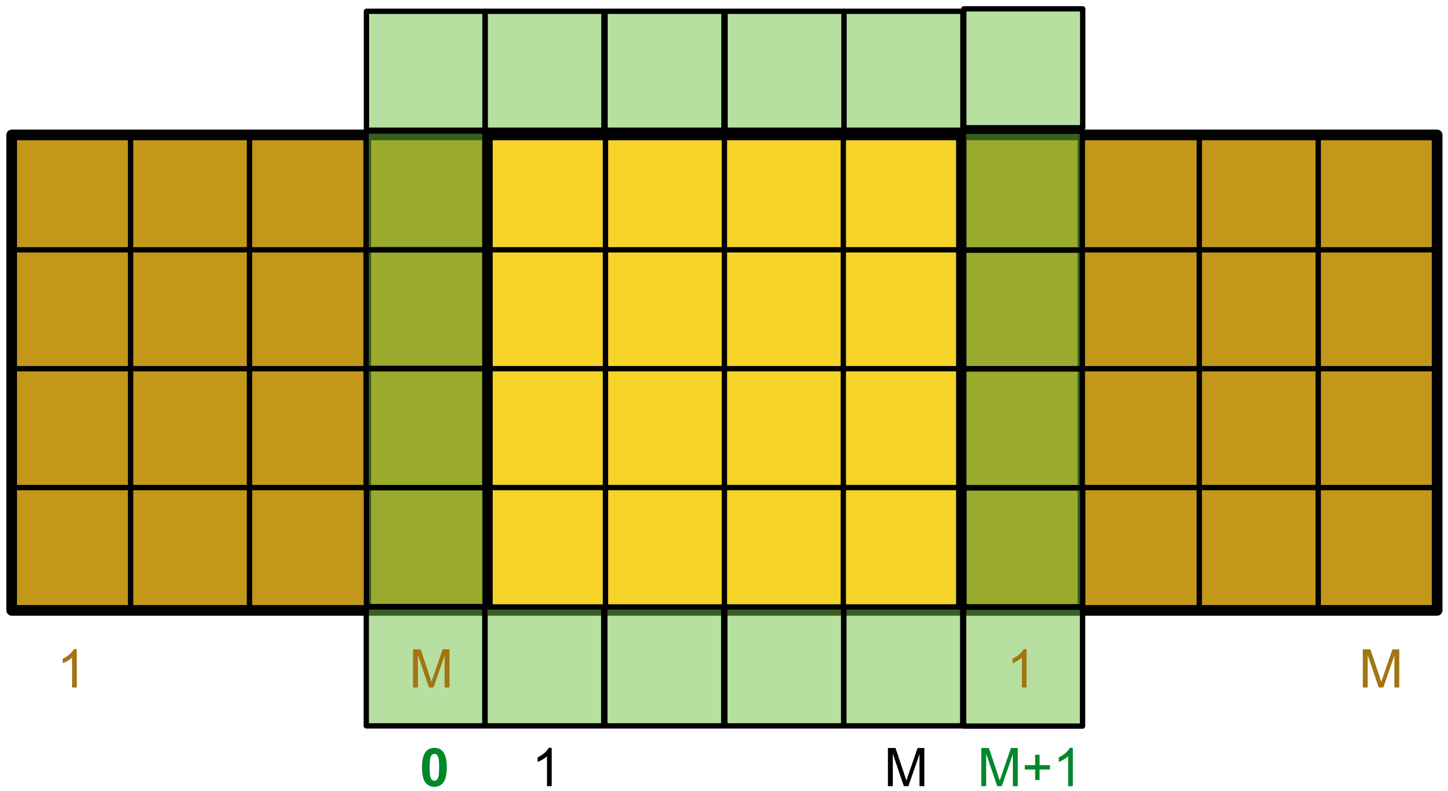
- similar for right boundaries and 2D problems

- Q: What's the order of these ghost cell values?

  2nd-order only!

Is there a simple “exact” boundary condition one can use for code verification?

Yes: periodic boundaries



$$\varphi_{i,N+1} = \varphi_{i,1}$$

$$\varphi_{i,0} = \varphi_{i,N}$$

$$\varphi_{0,j} = \varphi_{M,j} \qquad \varphi_{M+1,j} = \varphi_{1,j}$$

- for larger stencil schemes: add additional ghost cells, e.g.,

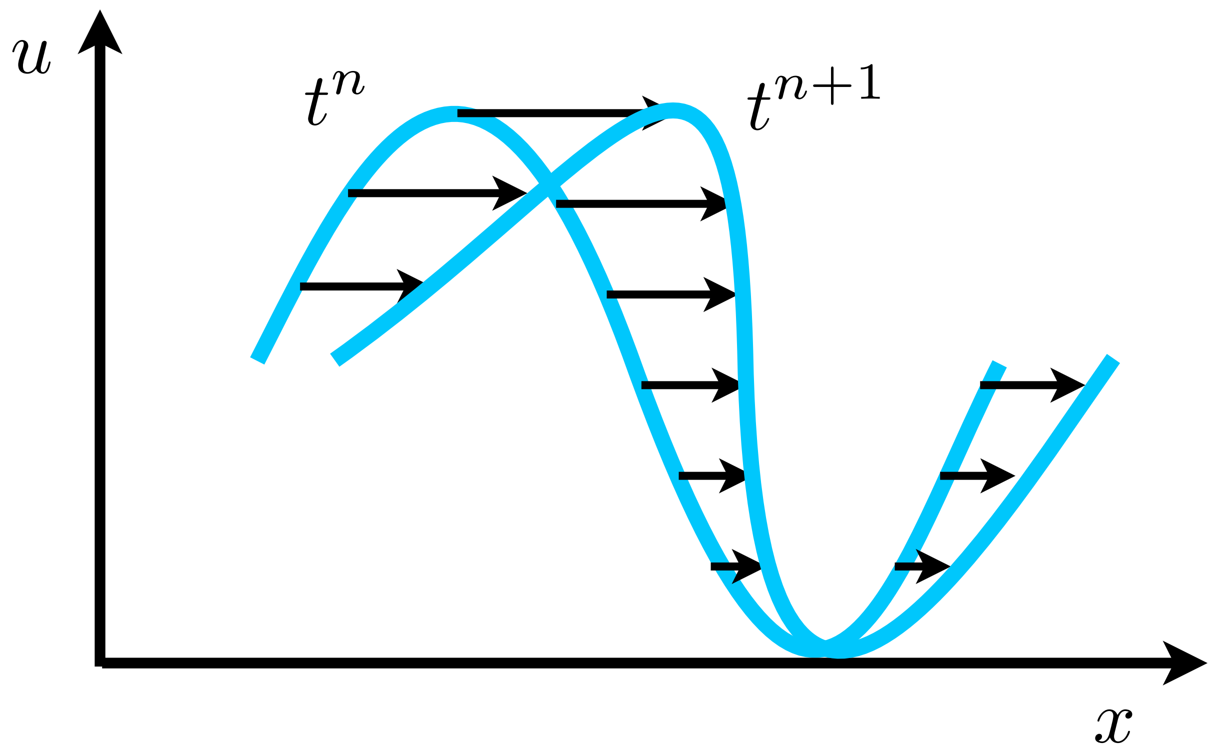
$$\varphi_{-1,j} = \varphi_{M-1,j}$$
$$\varphi_{-2,j} = \varphi_{M-2,j}$$

etc.

Non-linear Hyperbolic Equations

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \quad \text{with} \quad E = \frac{1}{2}u^2$$

- Example:



shocks may be formed!

Lax-Method

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = \overline{u_i^n} - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) \quad \text{with} \quad \overline{u_i^n} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)$$

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)}$$

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{4\Delta x}\left(\left(u_{i+1}^n\right)^2 - \left(u_{i-1}^n\right)^2\right)}$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^2 u}{\partial x^2}$ ⇒ dissipative

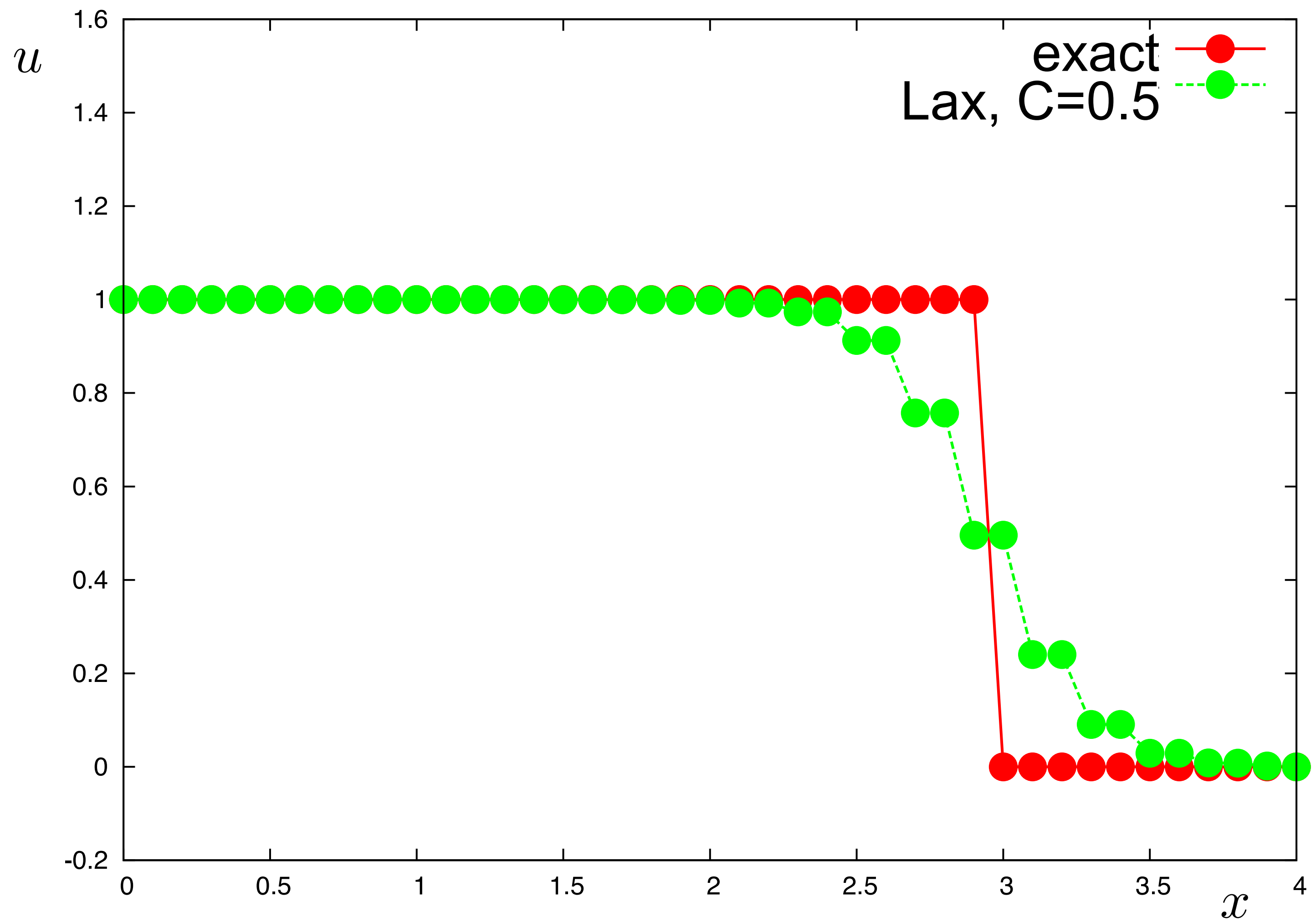
- Example:

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$0 \leq x \leq 4\,, \quad M = 40$$

initial condition: $u(x, t=0) = \begin{cases} 1 & x \leq 2 \\ 0 & x > 2 \end{cases}$

boundary conditions: $u(x=0, t) = 1\,, \quad u(x=4, t) = 0$



Code:
C=0.5,
C=1,
C=0.1

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

Idea: Start from Taylor series at t^n for t^{n+1}

$$u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3)$$

use PDE: $\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x}$ $\qquad$ $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\frac{\partial E}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t}\right)$

but: $\frac{\partial E}{\partial t} = \frac{\partial E}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial E}{\partial u}\left(-\frac{\partial E}{\partial x}\right) = -A\frac{\partial E}{\partial x}$

define Jacobian as $A = \frac{\partial E}{\partial u}$ $\qquad$ here: $A = \frac{\partial E}{\partial u} = \frac{\partial(\frac{1}{2}u^2)}{\partial u} = u$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right)$$

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \Delta t\frac{\partial E}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right) + O(\Delta t^3)$$

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \Delta t \frac{\partial E}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right) + O(\Delta t^3)$$

central — midpoint central

$$u_i^{n+1} = u_i^n - \Delta t \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \frac{\Delta t^2}{2}\frac{\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n - \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n}{\Delta x} + O(\Delta t^3)$$

$$\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n = A_{i+1/2}^n \frac{E_{i+1}^n - E_i^n}{\Delta x} \qquad \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n = A_{i-1/2}^n \frac{E_i^n - E_{i-1}^n}{\Delta x}$$

$$A_{i+1/2}^n = \frac{1}{2}\left(A_i^n + A_{i+1}^n\right) = \frac{1}{2}\left(u_i^n + u_{i+1}^n\right) \qquad A_{i-1/2}^n = \frac{1}{2}\left(A_i^n + A_{i-1}^n\right) = \frac{1}{2}\left(u_i^n + u_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\left[\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\right]$$

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\left(\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\right)$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^3 u}{\partial x^3}$ $\Rightarrow$ dispersive
- dispersive errors are smallest for $C = \frac{\Delta t}{\Delta x}\max\left(|u|\right) = 1$, they increase for smaller C

Code:
C=0.5,
C=1, C=0.1

MacCormack

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^* = u_i^n + -\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right)$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{\Delta t}{\Delta x}\left(E_i^* - E_{i-1}^*\right)\right]$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^3 u}{\partial x^3}$ $\Rightarrow$ dispersive
- quite good in general!
- dispersive errors are smallest for $C = \frac{\Delta t}{\Delta x}\max\left(|u|\right) = 1$, they increase for smaller C
- different from Lax-Wendroff

Code:
C=0.5,
C=1, C=0.1

Code: cmp. C=0.5

Beam-Warming Implicit

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

Idea: Start again from Taylor series, but in $\pm\Delta t$ direction

$$u_i^{n+1} = u_i^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + O(\Delta t^3)$$

-

$$u_i^n = u_i^{n+1} - \Delta t \left.\frac{\partial u}{\partial t}\right|_i^{n+1} + \frac{\Delta t^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1} + O(\Delta t^3)$$

subtract:

$$2u_i^{n+1} = 2u_i^n + \Delta t\left(\left.\frac{\partial u}{\partial t}\right|_i^n + \left.\frac{\partial u}{\partial t}\right|_i^{n+1}\right) + \frac{\Delta t^2}{2}\left(\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n - \left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1}\right) + O(\Delta t^3)$$

Determine $\left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1}$ by Taylor series: $\left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1} = \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \Delta t \frac{\partial}{\partial t}\left.\left(\frac{\partial^2 u}{\partial t^2}\right)\right|_i^n + O(\Delta t^2)$

Substitute in

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left(\left.\frac{\partial u}{\partial t}\right|_i^n + \left.\frac{\partial u}{\partial t}\right|_i^{n+1}\right) + \frac{\Delta t^2}{4}\left(\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n - \left(\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \Delta t\frac{\partial}{\partial t}\left.\left(\frac{\partial^2 u}{\partial t^2}\right)\right|_i^n + O(\Delta t^2)\right)\right) + O(\Delta t^3)$$

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left(\left.\frac{\partial u}{\partial t}\right|_i^n + \left.\frac{\partial u}{\partial t}\right|_i^{n+1}\right) + O(\Delta t^3)$$

Beam-Warming Implicit

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$A = \frac{\partial E}{\partial u}$$

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left(\left.\frac{\partial u}{\partial t}\right|_i^n + \left.\frac{\partial u}{\partial t}\right|_i^{n+1}\right) + O(\Delta t^3)$$

now use the PDE:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left(-\left.\frac{\partial E}{\partial x}\right|_i^n - \left.\frac{\partial E}{\partial x}\right|_i^{n+1}\right) + O(\Delta t^3)$$

we need E @ n+1:

Taylor Series: $$E^{n+1} = E^n + \Delta t \left.\frac{\partial E}{\partial t}\right|^n + O(\Delta t^2) = E^n + \Delta t \left.\frac{\partial E}{\partial u}\frac{\partial u}{\partial t}\right|^n + O(\Delta t^2)$$

replace $\frac{\partial u}{\partial t}$ with finite difference approximation: $$\left.\frac{\partial u}{\partial t}\right|^n = \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t)$$

$$E^{n+1} = E^n + \Delta t A^n \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t^2)$$

take $\frac{\partial}{\partial x}$: $$\left.\frac{\partial E}{\partial x}\right|_i^{n+1} = \left.\frac{\partial E}{\partial x}\right|_i^n + \frac{\partial}{\partial x}\left(A^n\left(u^{n+1} - u^n\right)\right) + O(\Delta t^2)$$

substitute back in: $$u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left(2\left.\frac{\partial E}{\partial x}\right|_i^n + \frac{\partial}{\partial x}\left[A^n\left(u^{n+1} - u^n\right)\right]\right) + O(\Delta t^3)$$

Beam-Warming Implicit

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left(2\left.\frac{\partial E}{\partial x}\right|_i^n + \frac{\partial}{\partial x}\left[A^n\left(u^{n+1} - u^n\right)\right]\right) + O(\Delta t^3) \qquad A = \frac{\partial E}{\partial u}$$

use 2nd order central

$$\frac{\partial}{\partial x}\left[A^n\left(u^{n+1} - u^n\right)\right] \approx \frac{A_{i+1}^n u_{i+1}^{n+1} - A_{i-1}^n u_{i-1}^{n+1}}{2\Delta x} - \frac{A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n}{2\Delta x}$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left(2\frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \frac{A_{i+1}^n u_{i+1}^{n+1} - A_{i-1}^n u_{i-1}^{n+1}}{2\Delta x} - \frac{A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n}{2\Delta x}\right)$$

rearrange:

$$-\frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^{n+1} + u_i^{n+1} + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t}{4\Delta x}\left(A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n\right)$$

tri-diagonal!

Beam-Warming Implicit

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$-\frac{\Delta t}{4\Delta x} A_{i-1}^n u_{i-1}^{n+1} + u_i^{n+1} + \frac{\Delta t}{4\Delta x} A_{i+1}^n u_{i+1}^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t}{4\Delta x}\left(A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n\right)$$

- order: $O(\Delta x^2)$, $O(\Delta t^2)$
- stability: unconditionally stable!
- Drawback: large dispersive errors!
- Idea: Why not add a dissipative term to the scheme?
  - add 4th-order damping: $D = -\epsilon_l \left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$
  - this adds a stability constraint: $0 \leq -\epsilon_l \leq \frac{1}{8}$

Code: Δx = 0.1, C = 2.5

Code: Δx = 0.1, C = 2.5, ε = 0.1

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order Explicit Upwind

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(E_i^n - E_{i-1}^n\right) \quad \text{for} \quad u_i^n > 0$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right) \quad \text{for} \quad u_i^n < 0$$

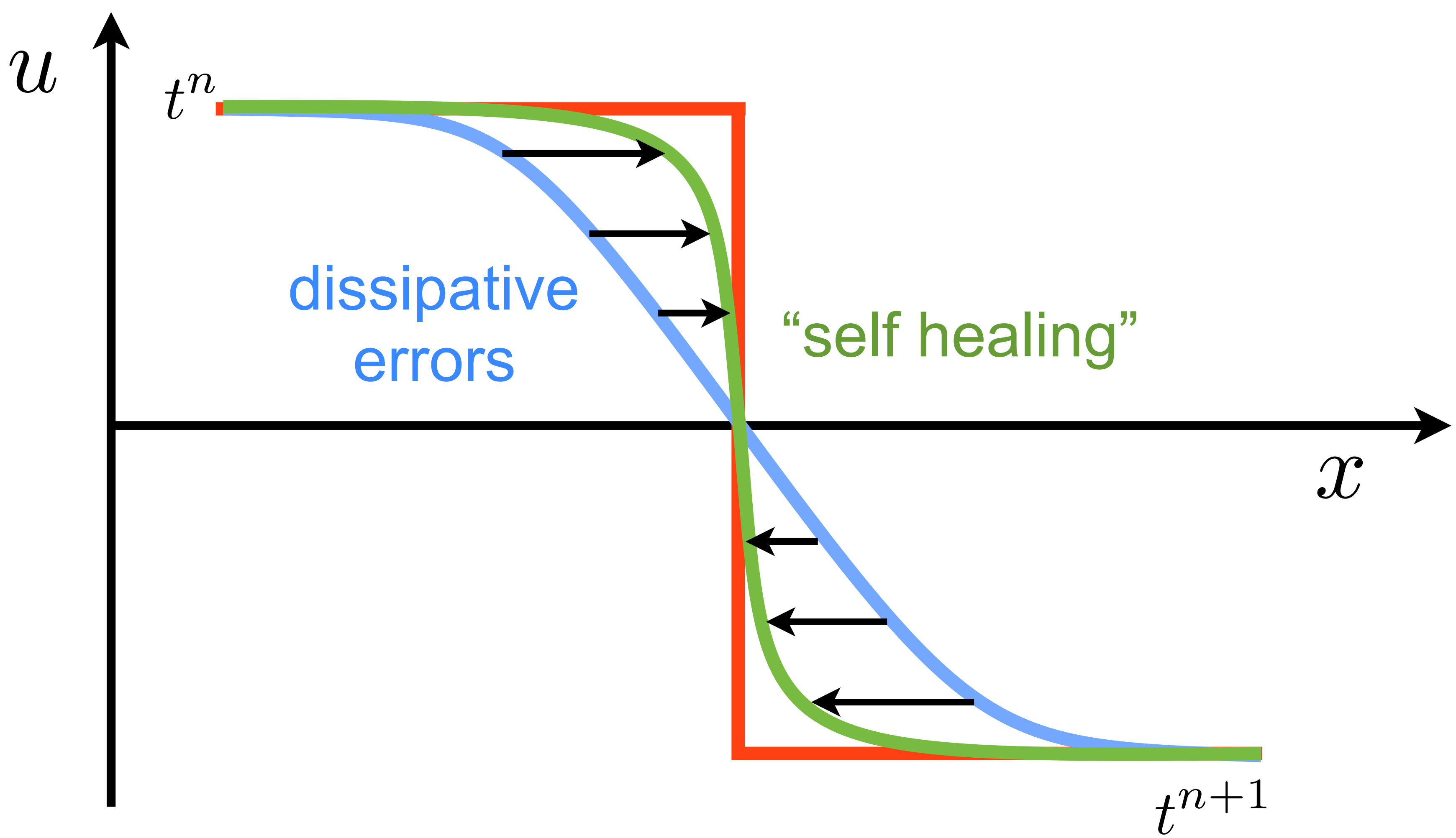
Code:
C=0.5,
C=1, C=0.1

- order: $O(\Delta t)$, $O(\Delta x)$
- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: dissipative
- much better here than for wave equation!
- Why?

1st-order Explicit Upwind

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

- much better here than for wave equation!
- Why?



1st-order Implicit Upwind

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

if $u_i^n > 0$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{E_i^{n+1} - E_{i-1}^{n+1}}{\Delta x} = -\frac{\frac{(u_i^{n+1})^2}{2} - \frac{(u_{i-1}^{n+1})^2}{2}}{\Delta x}$$

- Problem: non-linear system! $\Rightarrow$ linearize the non-linear terms!

$$(u_i^{n+1})^2 \approx u_i^n u_i^{n+1} \qquad (u_{i-1}^{n+1})^2 \approx u_{i-1}^n u_{i-1}^{n+1}$$

$$\Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left(u_i^n u_i^{n+1} - u_{i-1}^n u_{i-1}^{n+1}\right)$$

- rearrange: $\boxed{\left(\frac{\Delta t}{2\Delta x}u_{i-1}^n\right)u_{i-1}^{n+1} - \left(1 - \frac{\Delta t}{2\Delta x}u_i^n\right)u_i^{n+1} = -u_i^n}$
  - order: $O(\Delta t)$, $O(\Delta x)$
  - stability: unconditionally stable
  - leading order error term: dissipative
  - @ $C=1$: explicit is better than implicit

Code:
C=0.5,
C=1, C=1.5