Verification & Validation

(Part II)

or

How can I trust my CFD results?

or

Can I quantify the error/uncertainty of a CFD simulation?

Definition of Terms:

- **Error**

  *"A **recognizable** deficiency in any phase or activity of modeling and simulation that is **not** due to lack of knowledge." (AIAA G-077-1998)*

➡ Acknowledged errors are errors that can be estimated, bounded, or ordered

- ✓ Finite precision arithmetic in a digital computer
- ✓ Insufficient spatial discretization
- ✓ Insufficient temporal discretization
- ✓ Insufficient iterative convergence

➡ Unacknowledged errors are mistakes or blunders

- ✓ Computer programming errors (source code or compiler)
- ✓ Use of incorrect input files (geometry, material properties)

The following slides rely/copy heavily on the following tutorial/talk:

Verification and Validation in Computational Simulation

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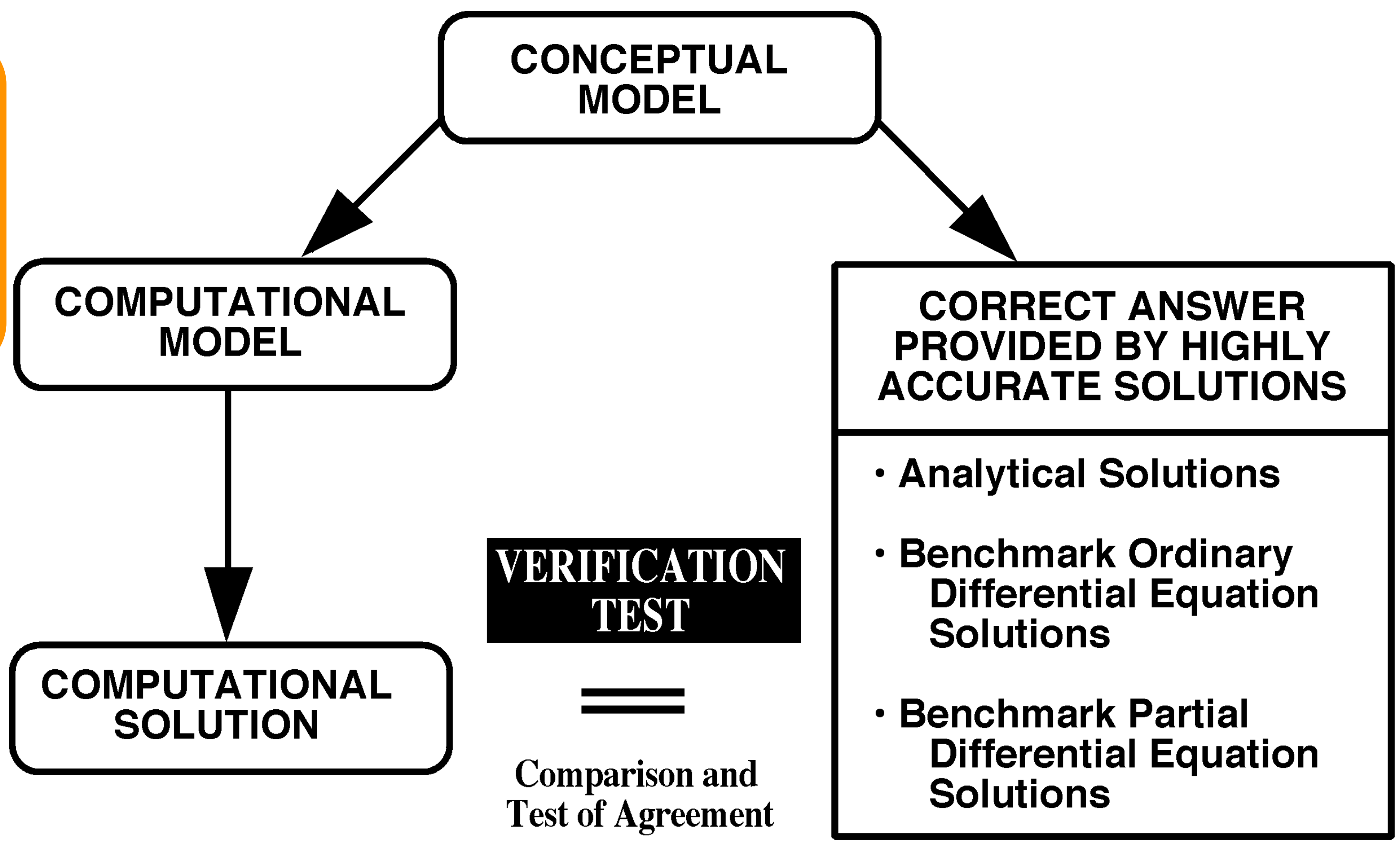
Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

Terminology: Verification

American Institute of Aeronautics and Astronautics, Committee on Standards in Computational Fluid Dynamics definition (1998):

Verification: **The process of determining that a model implementation accurately represents the developer's conceptual description of the model and the solution to the model**

! **Verification deals with mathematics**



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Two Types of Verification

- Verification is now commonly divided into two types:
- Code Verification: Verification activities directed toward:
  - Finding and removing mistakes in the source code
  - Finding and removing errors in numerical algorithms
  - Improving software using software quality assurance practices
- Solution Verification: Verification activities directed toward:
  - Assuring the accuracy of input data for the problem of interest
  - Estimating the numerical solution error
  - Assuring the accuracy of output data for the problem of interest

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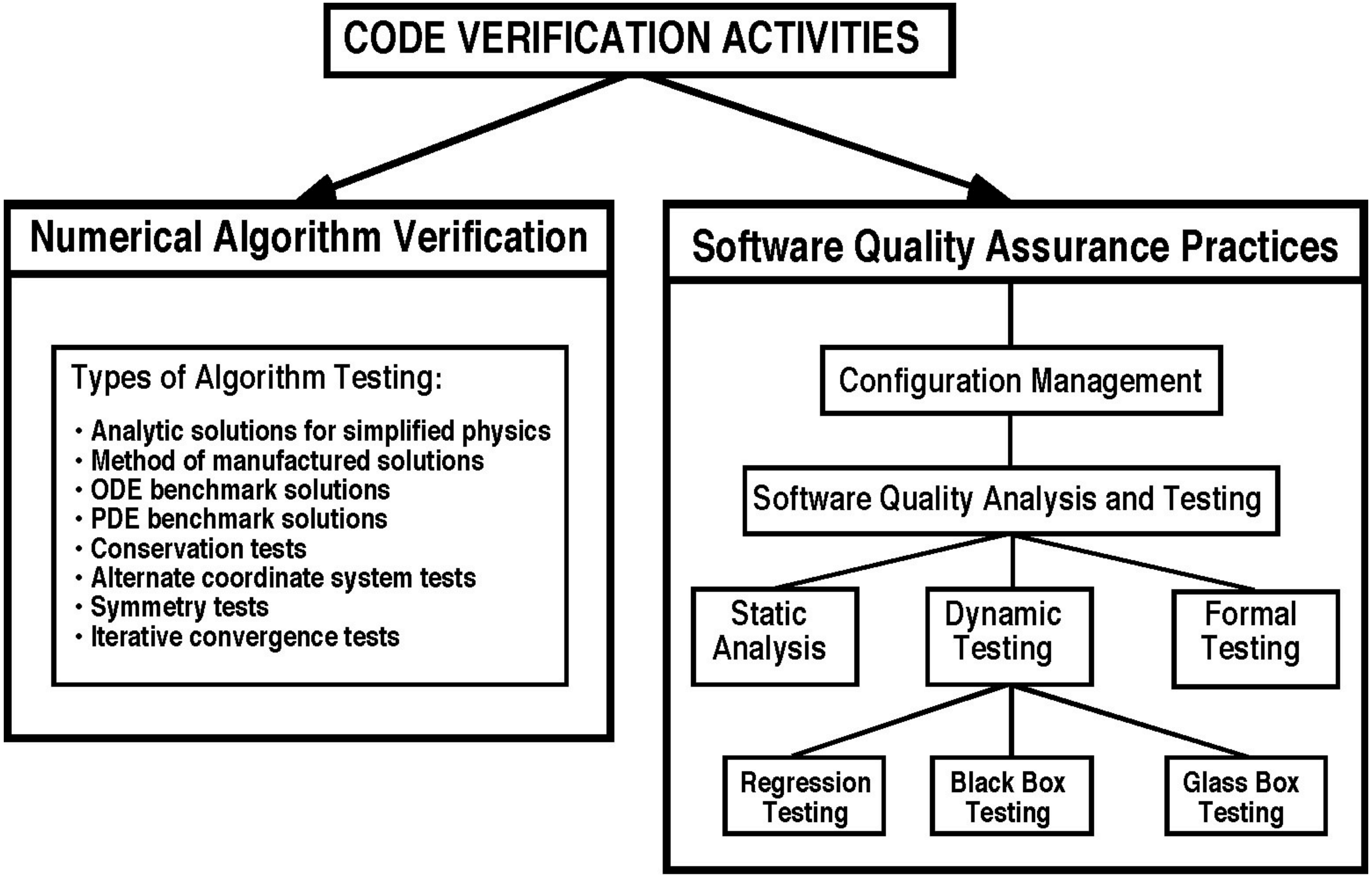
Code Verification



Numerical Algorithm Verification

- Formal order of accuracy of a numerical method is determined by:
  - Taylor series analysis for finite-difference and finite volume methods
  - Interpolation theory for finite-element methods
- Consider the 1-D unsteady heat conduction equation:

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$$

- Using a forward difference in time and a centered difference in space, the Taylor series analysis results in:

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = \left[-\frac{1}{2} \frac{\partial^2 T}{\partial t^2} \right] \Delta t + \left[\frac{\alpha}{12} \frac{\partial^4 T}{\partial x^4} \right] (\Delta x)^2 + O\left(\Delta t^2\right) + O\left(\Delta x^4\right)$$

Observed Order of Accuracy

- Computed solutions do not typically reproduce the formal order of accuracy
- Factors that can degrade the formal order of accuracy include:
  - Mistakes in the computer code, i.e., programming errors
  - Δx, Δy, Δz, Δt are not sufficiently small for the solution to be in the asymptotic convergence region, i.e., truncation errors
  - Singularities or discontinuities in the solution domain and on the boundaries
  - Insufficient iterative convergence for solving nonlinear equations
  - Round-off error due to finite word length in the computer
- We use the term "observed" order of accuracy for the actual accuracy determined from computed solutions

What's needed?

- We need the exact solution to compare our numerical solution to
- Ideally we would have an analytical exact solution!

Methods for Determining the Observed Order of Accuracy

- **Method of Exact Solutions (MES):**
  - **MES involves the comparison of a numerical solution to the exact solution to the governing PDEs**
  - **MES is the traditional method for code verification testing**
  - **Number and variety of exact solutions is extremely small**
- Method of Manufactured Solutions (MMS):
  - MMS is a more general and more powerful approach for code verification
  - Rather than trying to find an exact solution to a PDE, we "manufacture" an exact solution *a priori*
  - It is not required that the manufactured solution be physically real
  - Use the PDE operator to analytically generate source terms in a new PDE
  - The manufactured solution is the exact solution to a new (modified) equation: original PDE + source terms
  - MMS involves solving the backward problem: given an original PDE and a chosen solution, find a modified PDE which that chosen solution will satisfy
  - Initial & boundary conditions are determined from the solution, after the fact

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  - **MMS involves solving the backward problem: given an original PDE and a chosen solution, find a modified PDE which that chosen solution will satisfy**
  - **Initial & boundary conditions are determined from the solution, after the fact**

Method of Manufactured Solutions (MMS) Example:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \mathcal{L}(u) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \quad \Rightarrow \quad \mathcal{L}(u) = 0$$

Idea: Modify PDE to: $\mathcal{L}(u) = Q(x,t)$

1. Select/define/**make up** an **<u>exact</u>** solution to the modified PDE

   for example: $U(x,t) = A + \sin(x + Ct)$

2. Find $Q(x,t)$ such that $U(x,t)$ is the exact solution!

   How? substitute $U(x,t)$ into the modified PDE: $Q(x,t) = \mathcal{L}(U) = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} - \nu\frac{\partial^2 U}{\partial x^2}$

   $$Q(x,t) = C\cos(x+Ct) + [A + \sin(x+Ct)]\cos(x+Ct) + \nu\sin(x+Ct)$$

3. Add $Q(x,t)$ as a source term to the CFD code

   $$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

4. Use exact solution as boundary conditions
   - Dirichlet: $u(0,t) = U(0,t) = A + \sin(Ct)$ $\qquad u(1,t) = U(1,t) = A + \sin(1+Ct)$
   - Neuman: $\frac{\partial u}{\partial x}(1,t) = \cos(1+Ct)$ etc.

Important:
- chosen solution need not be physical! example: $U(x,t) = \sin(t)e^x$
- chosen solution must involve all terms of the PDE! (sin,cos,tanh,exp ...)

Method of Manufactured Solutions

the following are examples from P. Brady & M. Herrmann, J. Comput. Phys. (2012).

- Finite volume temperature equation for multiphase flows

$$L(T) = \frac{1}{V_{cv}} \left[\int_{cv} \frac{\partial T}{\partial t} dV + \mathbf{u} \cdot \int_{cv} \nabla T dV - \frac{1}{(\rho c_p)_{cv}} \int_{cv} \nabla \cdot (k \nabla T) dV \right] = 0$$

with

$$(\rho c_p)_{cv} = \psi_{cv} (\rho c_p)_l + (1 - \psi_{cv}) (\rho c_p)_g \qquad k = k_g + (k_l - k_g) H(G)$$

- Source term for MMS is thus

$$L(T) = \frac{1}{V_{cv}} \left[\int_{cv} \frac{\partial T}{\partial t} dV + \mathbf{u} \cdot \int_{cv} \nabla T dV - \frac{1}{(\rho c_p)_{cv}} \left(\int_{cv, G=0} (k_l - k_g) \nabla G \cdot \nabla T dS \right. \right.$$
$$\left. \left. + \int_{cv, G>0} k_l \nabla^2 T dV + \int_{cv, G<0} k_g \nabla^2 T dV \right) \right]$$

- solve source term analytically, or with AMR marching tets

Example 1: 3D Unsteady Single Phase

- Material properties:

$$k_+ = k_- = 0.5, \quad c_{p_+} = c_{p_-} = 10.0, \quad \rho_+ = \rho_- = 5.0 + 1.5T$$

- Velocity field

$$(U, V, W) = (\cos(\pi x), \cos(\pi y), \cos(\pi z))/75$$

- Manufactured solution:

$$T_2(\boldsymbol{x}) = 10^{-4}\exp(2x)\cos(y)z^3\sin(t) + 10\cos^2(t)$$

- Analytical source term:

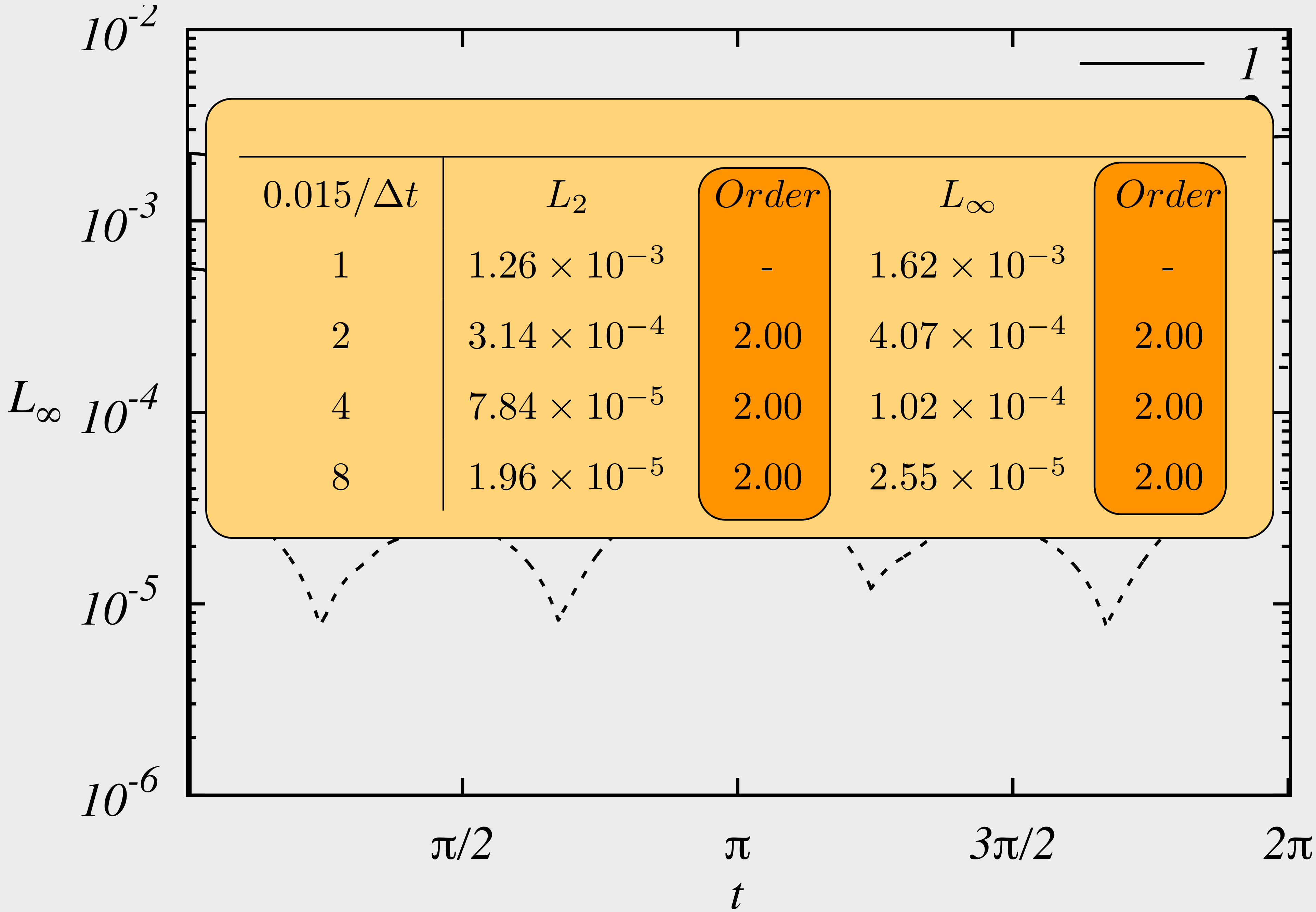
$$L_t = \int_{cv} \frac{\partial T_2}{\partial t} dV = 10^{-4}\frac{\exp(2x_2) - \exp(2x_1)}{2}(\sin(y_2) - \sin(y_1))\frac{z_2^4 - z_1^4}{4}\cos(t) - 20\cos(t)\sin(t)V_{cv}$$

$$L_c = \boldsymbol{u} \cdot \int_{cv} \nabla T_2 dV = 10^{-4}\frac{\exp(2x_2) - \exp(2x_1)}{2}\sin(t)\Big[U(\sin(y_2) - \sin(y_1))\frac{z_2^4 - z_1^4}{2} + V(\cos(y_2) - \cos(y_1))\frac{z_2^4 - z_1^4}{4} + W(\sin(y_2) - \sin(y_1))(z_2^3 - z_1^3)\Big]$$

$$L_{p/m} = \int_{cv} k_{+/-}\nabla^2 T_2 dV = 0.75 \times 10^{-4} k_+ \sin(t)\left[\exp(2x_2) - \exp(2x_1)\right]\left[\sin(y_2) - \sin(y_1)\right] \times [z_2^2 - z_1^2]\left[\frac{z_2^2 + z_1^2}{2} + 2\right]$$

Example 1: 3D Unsteady Single Phase

- Infinity norm of error under Δt refinement and $N_{x,y,z}$ = 25 for 2nd-order Adams-Bashforth



$0.015/\Delta t$	L_2	*Order*	L_∞	*Order*
1	1.26×10^{-3}	-	1.62×10^{-3}	-
2	3.14×10^{-4}	2.00	4.07×10^{-4}	2.00
4	7.84×10^{-5}	2.00	1.02×10^{-4}	2.00
8	1.96×10^{-5}	2.00	2.55×10^{-5}	2.00

Example 2: 3D Steady Single Phase

- Material properties:

$$k_+ = k_- = 0.5, \quad c_{p_+} = c_{p_-} = 10.0, \quad \rho_+ = \rho_- = 5.0 + 1.5T$$

- Manufactured solution:

$$T_3(\boldsymbol{x}) = \exp(2x)\cos(y)z^3$$

- Leading order error of modified equation (convective: 3rd order QUICK, diffusive: 2nd order central)

$$LE \propto \Delta x^2 \left(1 - \frac{U}{\alpha}\Delta x\right)$$

➡ formal order of accuracy is a function of U/α

- Velocity field 1 with $U/\alpha \gg 1 \Rightarrow LE \propto \Delta x^3$

$$(U, V, W) = 10(\cos(\pi x), \cos(\pi y), \cos(\pi z))$$

- Velocity field 2 with $U/\alpha \ll 1 \Rightarrow LE \propto \Delta x^2$

$$(U, V, W) = (\cos(\pi x), \cos(\pi y), \cos(\pi z))/75$$

Example 2: 3D Steady Single Phase

Table 6: Test 3: $(U,V,W) = 10(\cos(\pi x), \cos(\pi y), \cos(\pi z))$

N_x	L_2	$Order$	L_∞	$Order$
25	$1.45 \cdot 10^{-5}$	-	$9.03 \cdot 10^{-5}$	-
50	$1.70 \cdot 10^{-6}$	3.09	$1.13 \cdot 10^{-5}$	3.00
100	$2.00 \cdot 10^{-7}$	3.09	$1.36 \cdot 10^{-6}$	3.05
200	$2.28 \cdot 10^{-8}$	3.13	$1.59 \cdot 10^{-7}$	3.10

✓

Table 7: Test 4: $(U,V,W) = (\cos(\pi x), \cos(\pi y), \cos(\pi z))/75$

N_x	L_2	$Order$	L_∞	$Order$
25	3.13×10^{-5}	-	9.18×10^{-5}	-
50	7.44×10^{-6}	2.07	2.24×10^{-5}	2.03
100	1.81×10^{-6}	2.04	5.53×10^{-6}	2.02
200	4.45×10^{-7}	2.02	1.37×10^{-6}	2.01

✓

Example 3: Bug Finding

- Bug in diffusive term:

$$\nabla \cdot (k \nabla T)_i \approx \frac{\frac{k_{i+1}+k_i}{2}\frac{T_{i+1}-T_i}{\Delta x} - \frac{k_i+k_{i+1}}{2}\frac{T_i-T_{i-1}}{\Delta x}}{\Delta x}$$

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- Manufactured solution:

$$T_4 = \exp(2x)\cos(y)$$

N_x	L_2	$Order$	L_∞	$Order$
25	8.98×10^{-2}	-	3.90×10^{-1}	-
50	9.06×10^{-2}	-0.01	3.98×10^{-1}	-0.0319
100	8.79×10^{-2}	0.04	3.80×10^{-1}	0.0697

Example 3: Bug Finding

- Bug in diffusive term:

$$\nabla \cdot (k\nabla T)_i \approx \frac{\frac{k_{i+1}+k_i}{2}\frac{T_{i+1}-T_i}{\Delta x} - \frac{k_i+k_{i+1}}{2}\frac{T_i-T_{i-1}}{\Delta x}}{\Delta x}$$

- Manufactured solution + all velocities = 0 (turn of convective terms):

$$T_4 = \exp(2x)\cos(y)$$

N_x	L_2	$Order$	L_∞	$Order$
25	2.29×10^{-1}	-	7.64×10^{-1}	-
50	2.33×10^{-1}	-0.02	7.93×10^{-1}	-0.06
100	2.24×10^{-1}	0.06	7.59×10^{-1}	0.06

Example 3: Bug Finding

- Bug in diffusive term:

$$\nabla \cdot (k \nabla T)_i \approx \frac{\frac{k_{i+1} + k_i}{2} \frac{T_{i+1} - T_i}{\Delta x} - \frac{k_i + k_{i+1}}{2} \frac{T_i - T_{i-1}}{\Delta x}}{\Delta x}$$

- Simpler manufactured solution + all velocities = 0 (turn of convective & x-diffusion terms):

$$T = \cos(y)$$

N_x	L_2	$Order$	L_∞	$Order$
25	4.92×10^{-4}	-	4.03×10^{-3}	-
50	1.63×10^{-4}	1.59	2.35×10^{-3}	0.78
100	8.30×10^{-5}	0.97	1.16×10^{-3}	1.02

✓

$\Rightarrow$ Error must be in x-diffusion term

Navier-Stokes

incompressible limit

Recap:

$$\nabla \cdot \vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v}\vec{v}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

in 2D:

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ &\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}}$$

non-conservative:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Navier-Stokes

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in 2D:

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$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

non-dimensionalize:

$$\overline{x} = \frac{x}{L}, \quad \overline{y} = \frac{y}{L}, \quad \overline{u} = \frac{u}{u_{ref}}, \quad \overline{v} = \frac{v}{u_{ref}}, \quad \overline{t} = \frac{t}{L/u_{ref}}, \quad \overline{p} = \frac{p}{\rho u_{ref}^2}$$

drop bars:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = \boxed{-\frac{\partial p}{\partial x}} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = \boxed{-\frac{\partial p}{\partial y}} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\mathrm{Re} = \frac{u_{ref} L}{\nu}$$

so what's new? pressure term & continuity

However: continuity and momentum equations appear decoupled! (?)
for compressible flows, coupling is via density and equation of state!

Vorticity-Streamfunction Formulation

Idea: directly couple/enforce continuity and eliminate pressure

vorticity: $\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{v}$ $\vec{\omega}$: angular velocity

in 2D: $\Omega_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

streamfunction ψ: $u = \dfrac{\partial \psi}{\partial y}$ $v = -\dfrac{\partial \psi}{\partial x}$

Goal: derive a PDE for vorticity from momentum equations, eliminating pressure

Board

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)$$

→ vorticity equation: 1 equation, no pressure! But how do we get the velocities?

→ streamfunction! But how do we get the streamfunction?

But: Continuity and momentum eqs. appear decoupled?

for compressible flows coupling is via ρ and E.O.S.

Solution Strategies:

Option 1: Vorticity - Streamfunction formulation

vorticity: $\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{v}$ in 2D: $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$\vec{\omega}$: angular velocity

streamfunction ψ: $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

to derive a PDE for $\vec{\Omega}$, use momentum eqs (non-conservative) to eliminate p: take $\frac{\partial}{\partial y}$ of u-eq. & $\frac{\partial}{\partial x}$ of v-eq.

$$\frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{Re}\left(\frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial y^3}\right)$$

$$\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{Re}\left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x \partial y^2}\right)$$

subtract:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \cancelto{0}{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) =$$

$$\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right]$$

$$\Rightarrow \frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{Re}\left[\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right] \quad \text{with } \Omega = \Omega_z$$

Vorticity-Streamfunction Formulation

But how do we get the streamfunction?

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

let's just use the definitions and substitute in!

$$\Omega = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \qquad \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega}$$

But what about the continuity equation?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

let's again use the definitions and substitute in!

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

$\Rightarrow$ continuity is automatically satisfied!

Vorticity-Streamfunction Formulation

non-conservative form:

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega$$

conservative form:

$$\frac{\partial \Omega}{\partial t} + \frac{\partial u\Omega}{\partial x} + \frac{\partial v\Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega$$

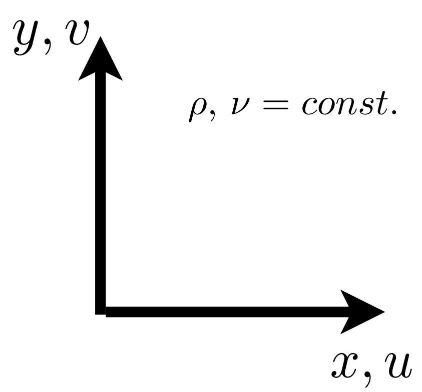
Vorticity-Streamfunction Formulation

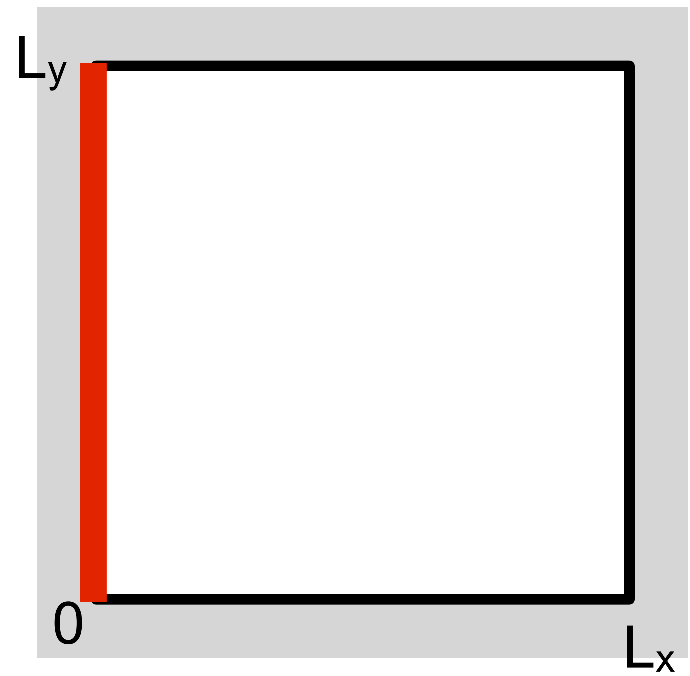
$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

One challenge remains: boundary conditions are usually given in terms of u & v

⇒ need to express as Ω and ψ

Example: Cavity





- all boundaries are impermeable walls
- normal velocity = 0

⇒ walls are streamlines! ⇒ $\boldsymbol{\psi = const}$

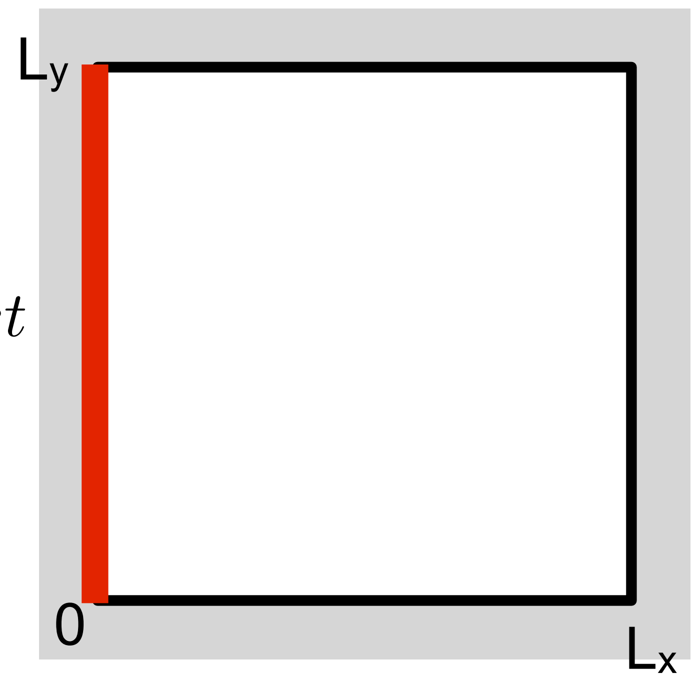
for example: left boundary: $x = 0 \quad \Rightarrow u = 0$ if wall is stationary: $v = 0$

since $u = \dfrac{\partial \psi}{\partial y} \quad \Rightarrow \left.\dfrac{\partial \psi}{\partial y}\right|_{x=0} = 0 \quad \Rightarrow \psi|_{x=0} = const$

But what about Ω?

Vorticity-Streamfunction Formulation

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$



$$\psi|_{x=0} = const$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{x=0} = 0$$

using node based mesh

But what about Ω?

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega$$

Taylor Series for 1st interior point:

$$\psi_{1,j} = \psi_{0,j} + \Delta x \left.\frac{\partial \psi}{\partial x}\right|_{0,j} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{0,j} + \dots$$

if wall is stationary: $v = 0$

$$\left.\frac{\partial \psi}{\partial x}\right|_{0,j} = -v_{0,j} = 0 \qquad \Rightarrow \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{0,j} = \frac{\psi_{1,j} - \psi_{0,j}}{\frac{\Delta x^2}{2}}$$

$$\Rightarrow \quad \Omega_{0,j} = \frac{2(\psi_{0,j} - \psi_{1,j})}{\Delta x^2}$$

similar procedure for all other boundaries

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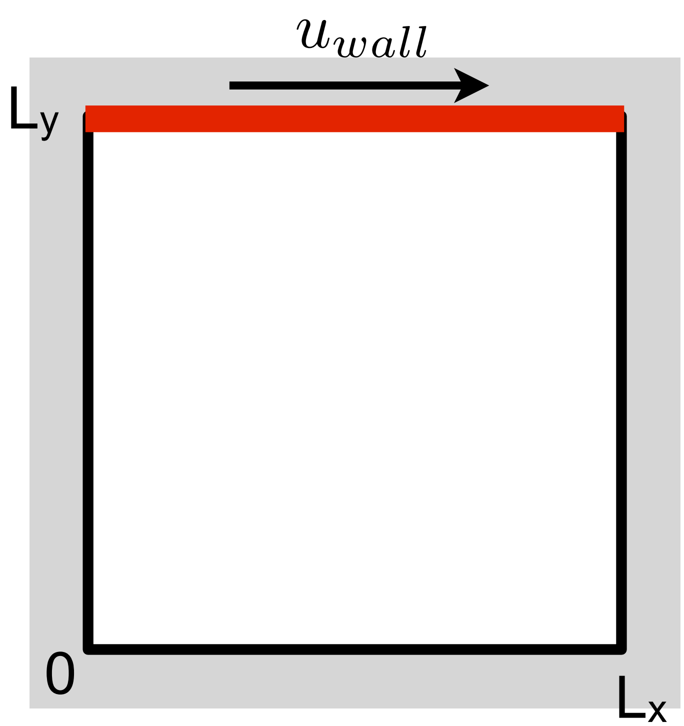

using node based mesh

What if wall moves tangentially? for example: upper wall

$$u(x, y = L_y) = u_{wall}$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{i,N} = u_{wall} \quad \text{and} \quad \left.\frac{\partial \psi}{\partial x}\right|_{i,N} = 0 \quad \text{(impermeable)}$$

Taylor Series for 1st interior point:

$$\psi_{i,N-1} = \psi_{i,N} - \Delta y \left.\frac{\partial \psi}{\partial y}\right|_{i,N} + \frac{\Delta y^2}{2} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,N} + \ldots$$

$$\Rightarrow \quad \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,N} = \frac{2(\psi_{i,N-1} - \psi_{i,N})}{\Delta y^2} + \frac{2}{\Delta y} u_{wall}$$

with

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega \qquad \Rightarrow \qquad \Omega_{i,N} = \frac{2}{\Delta y^2}\left(\psi_{i,N} - \psi_{i,N-1}\right) - \frac{2}{\Delta y} u_{wall}$$

caveat: this is only 1st-order in space! could go 2nd-order:

$$\Omega_{i,N} = \frac{-\psi_{i,N} + 8\psi_{i,N-1} - 7\psi_{i,N-2}}{2\Delta y^2} - \frac{3}{\Delta y} u_{wall}$$

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Solution Procedure

- initialize Ω
- solve for ψ
- solve boundary conditions for Ω
- advance Ω

this works ok in 2D. But what about 3D?

- need 3D vector potential instead of streamfunction
- need to solve 3 elliptic equations

$\Rightarrow$ very cumbersome and expensive!

$\Rightarrow$ vorticity/streamfunction approach usually used just in 2D