AEE 471/ MAE 561

Homework #8 - Due: April 1st, at the beginning of class

Please submit result graphs together with either handwritten or printed out descriptions, equations, and answers at the beginning of class. Combine all code you used to solve the problems into a single text file, and upload the text file to Blackboard using the SafeAssign mechanism. No credit will be given, if your code is not uploaded as a text file using SafeAssign. Also ensure that your code contains adequate comments. Add a printout of all code as an **appendix** to your submission.

Problem 1 (100 points, AEE 471: Core Course Outcomes #1 & #2)
Consider the following one-dimensional non-dimensional PDE

$$\frac{\partial \phi}{\partial t} + a(x,t)\frac{\partial \phi}{\partial x} = 0\,, \tag{1}$$

defined on a periodic domain of size $0 \le x \le 1$ with initial condition

$$\phi(x, t=0) = \begin{cases} 0 & : \quad x < 3/16 \\ 1 & : \quad 3/16 \le x < 5/16 \\ 0 & : \quad 5/16 \le x < 5/8 \\ \left[1 - \cos\left(\pi(8x-5)\right)\right]/2 & : \quad 5/8 \le x < 7/8 \\ 0 & : \quad x \ge 7/8 \end{cases} \tag{2}$$

and $a(x,t) = \frac{2}{5}\cos(\frac{\pi}{4}t)\left(1 + \frac{1}{2}\cos(4\pi x)\right)$.

1. Write in index form for a cell centered mesh of M elements the equation to solve the PDE using the first-order upwind method for cell center i and time level $n+1$, i.e. $\phi_i^{n+1} = \ldots$, the index equations to calculate ghost cell values, and the equation to determine the maximum stable time step size. No points will be given if the formulas merely appear in code.

2. Using a **cell centered** mesh with $M = 256$ and the first-order upwind method, calculate and plot the solution $\phi(x,t)$ at $t = 1$, 2, 3, and 4 time units into separate figures, adding the initial condition at $t = 0$ to the $t = 4$ figure. Use a time step size Δt that is 80% of the maximum stable time step size, but no larger than 2.5e-2 time units. In an additional figure, plot the Δt used vs. t for $0 \le t \le 4$. To help you debug your code, on Blackboard you will find the file `1up.txt` that contains the solution variables for the first 4 time steps using $M = 32$ with a maximum allowable time step size of 2.5e-1.

3. Explain the solution behavior to 2. using the modified equation of the first-order upwind method.

4. Write in index form for a cell centered mesh of M elements the equation to solve the PDE using the Lax-Wendroff method for cell center i and time level $n+1$, i.e. $\phi_i^{n+1} = \ldots$. No points will be given if the formulas merely appear in code.

5. Using a **cell centered** mesh with $M = 256$ and the Lax-Wendroff method, calculate and plot the solution $\phi(x,t)$ at $t = 1$, 2, 3, and 4 time units into the corresponding figures of task 2. Use a time step size Δt that is 80% of the maximum stable time step size of a first order upwind method, but no larger than 2.5e-2 time units. To help you debug your code, on Blackboard you will find the file `lw.txt` that contains the solution variables for the first 4 time steps using $M = 32$ with a maximum allowable time step size of 2.5e-1.

6. Explain the solution behavior to 4. using the modified equation of the Lax-Wendroff method.

7. **Required for MAE561, Bonus for AEE471**: Write in index form for a cell centered mesh of M elements the equation to solve the PDE using the 3rd order ENO method with 3rd order Runge-Kutta TVD time advancement method (ENO3RK3) for cell center i and time level $n+1$, i.e. $\phi_i^{n+1} = \ldots$, including all RK step equations. Also write the index equations to calculate all required ghost cell values. No points will be given if the formulas merely appear in code.

8. **Required for MAE561, Bonus for AEE471**: Using a **cell centered** mesh with $M = 256$ and the ENO3RK3 method, calculate and plot the solution $\phi(x,t)$ at $t = 1$, 2, 3, and 4 time units into the corresponding figures of task 2 & 5. Use a time step size Δt that is 80% of the maximum stable time step size of a first order upwind method, but no larger than 2.5e-2 time units. To help you debug your code, on Blackboard you will find the file `eno3rk3.txt` that contains the solution variables for the first 4 time steps using $M = 32$ with a maximum allowable time step size of 2.5e-1.

Required submission:
- Handwritten or typeset first-order upwind method in index form, ghost cell boundary conditions in index form, maximum time step formula;
- Four clearly annotated plots of $\phi(x)$ at t = 1, 2, 3, respective 4, for $M = 256$ using the first-order upwind method with the $t = 4$ figure containing the solution at $t = 0$ as well;
- Clearly annotated plot of time step size Δt vs t for $0 \leq t \leq 4$ and $M = 256$ using the first-order upwind method;
- Handwritten or typeset derivation of the modified equation for the first-order upwind method;
- Explanation of the observed solution behavior for the first-order upwind method;

- Handwritten or typeset Lax-Wendroff method in index form;
- Four clearly annotated plots of $\phi(x)$ at $t = 1$, 2, 3, respective 4, for $M = 256$ using the first order upwind and Lax-Wendroff methods with the $t = 4$ figure containing the solution at $t = 0$ as well;
- Handwritten or typeset derivation of the modified equation for the Lax-Wendroff method;
- Explanation of the observed solution behavior for the Lax-Wendroff method;

Required for MAE561:
- Handwritten or typeset ENO3RK3 method in index form;
- Four clearly annotated plots of $\phi(x)$ at $t = 1$, 2, 3, respective 4, for $M = 256$ using the first order upwind, Lax-Wendroff, and ENO3RK3 methods, with the $t = 4$ figure containing the solution at $t = 0$ as well.