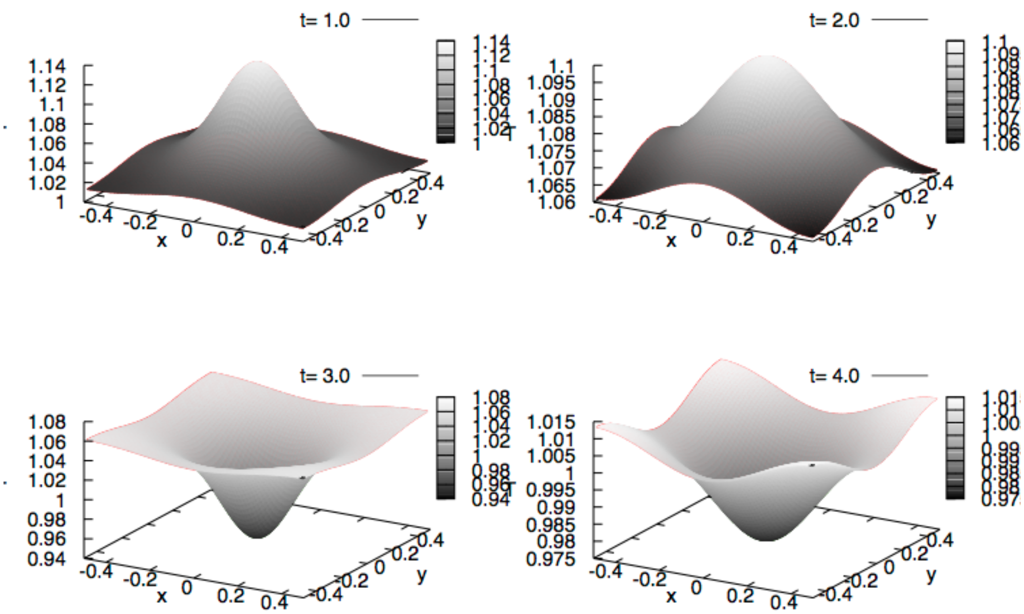
Homework 7 Solution

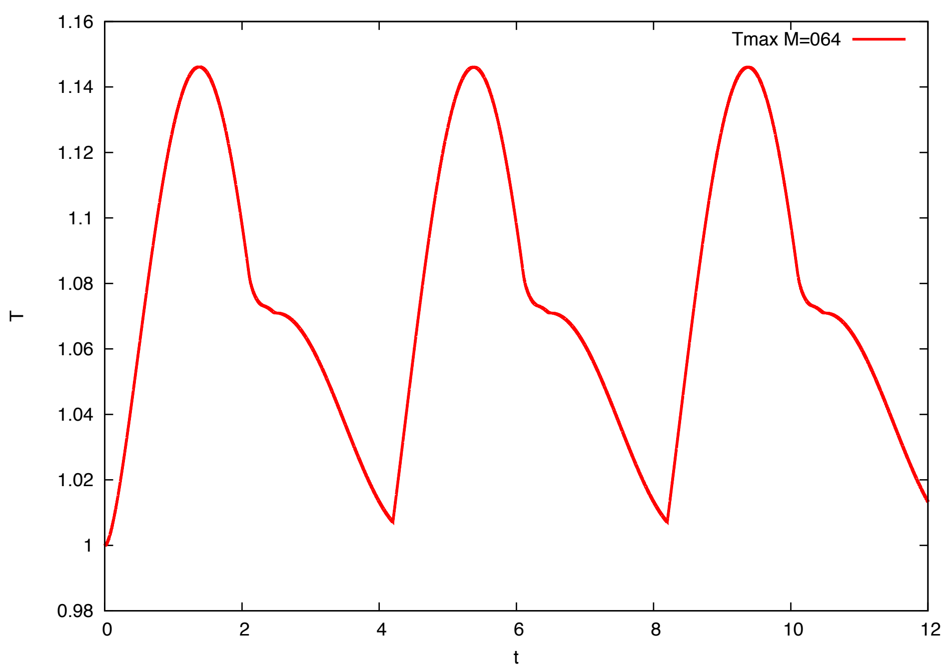
Problem 1(100 points total)

Task 1 (20 points / 10 points)   scan

Task 2 (50 points / 25 points)



20 pts / 10 pts



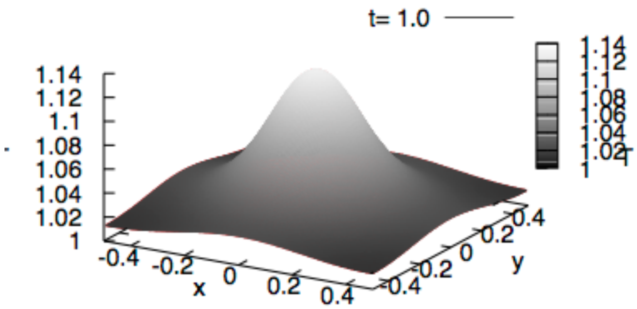
10 pts / 5 pts

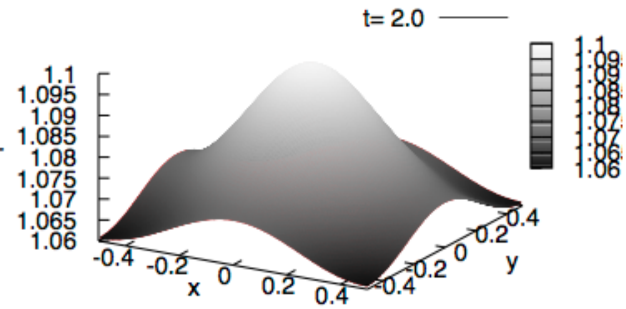
Code: 20 pts / 10 pts

Homework 7 Solution

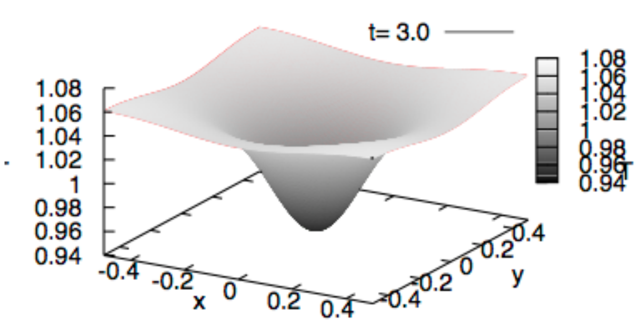
Task 3 (30 points / 24 points) see scan

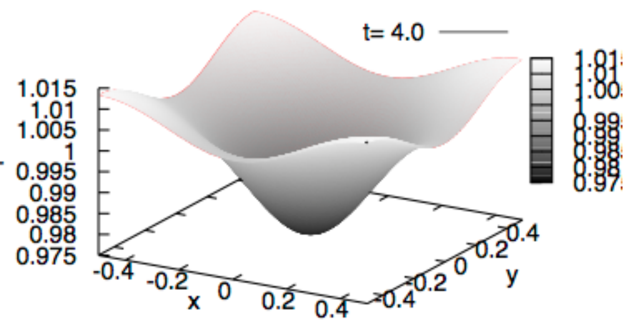
Task 4 (10 bonus points / 41 points)

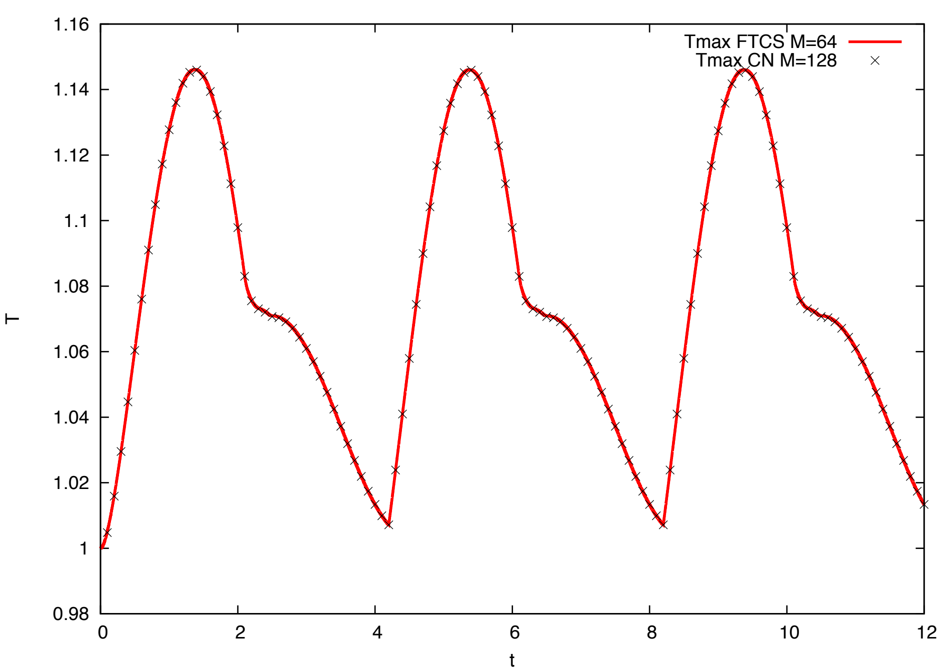




4 bonus pts / 16 points







2 bonus pts / 8 points

Code: 4 bonus pts / 17 pts

Task 5 (10 bonus points / 10 bonus points)

Homework 7

1)

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \alpha \left(\frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{h^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{h^2} \right) + q_i^n$$

$$\Rightarrow \quad T_{i,j}^{n+1} = T_{i,j}^{n} + \frac{\alpha \Delta t}{h^2}\left(T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} - 4T_{i,j}^{n} \right) + q_i^n \Delta t$$

10/4

left boundary: $T_{0,j} = T_{1,j}$ (0 … n+1) 2/1 bottom boundary: $T_{i,0} = T_{i,1}$ 2/1 (0 … n+1)

right boundary: $T_{n+1,j} = T_{n,j}$, $j = 1 \dots n$ 2/1 top boundary: $T_{i,n+1}^{n} = T_{i,n}^{n}$ 2/1 $i = 1 \dots n$

$$\Delta t_{max} = \frac{1}{4}\frac{h^2}{\alpha}$$ 2/2

3) Crank-Nicholson:

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{1}{2}\Bigg[\alpha \left(\frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{h^2} + \frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{h^2} \right) + q_i^{n+1}$$

10/8

$$+ \alpha \left(\frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{h^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{h^2} \right) + q_i^n \Bigg]$$

gather n+1 terms that are unknown.

(can be also multiplied by $\frac{1}{\Delta t}$)

$$-\frac{1}{2}\frac{\alpha \Delta t}{h^2}\left(T_{i+1,j}^{n+1} + T_{i-1,j}^{n+1} + T_{i,j+1}^{n+1} + T_{i,j-1}^{n+1} \right) + \left(1 + \frac{2\alpha \Delta t}{h^2} \right) T_{i,j}^{n+1} =$$

$$\frac{1}{2}\frac{\alpha \Delta t}{h^2}\left(T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} \right) + \left(1 - \frac{2\alpha \Delta t}{h^2} \right) T_{i,j}^{n} + \frac{\Delta t}{2}\left(q_{i,j}^{n+1} + q_{i,j}^{n} \right)$$

$\Rightarrow$ Gauss-Seidel:

$$T_{i,j}^{n+1(k+1)} = \frac{1}{1 + 2\frac{\alpha \Delta t}{h^2}} \Bigg\{ \frac{1}{2}\frac{\alpha \Delta t}{h^2}\left[T_{i-1,j}^{n+1(k+1)} + T_{i,j-1}^{n+1(k+1)} + T_{i+1,j}^{n+1(k)} + T_{i,j+1}^{n+1(k)} \right] + \frac{1}{2}\frac{\alpha \Delta t}{h^2}\left[T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} \right]$$

$$+ \left(1 - \frac{2\alpha \Delta t}{h^2} \right) T_{i,j}^{n} + \frac{\Delta t}{2}\left(q_{i,j}^{n+1} + q_{i,j}^{n} \right) \Bigg\}$$

10/8

(can also be multiplied by $\frac{1}{\Delta t}$)

Residual:

10/8

$$
T_{i,j}^{(k+1)} = \frac{1}{2}\frac{\alpha\Delta t}{h^2}\left(T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n}\right) + \left(1 - \frac{2\alpha\Delta t}{h^2}\right)T_{i,j}^{n} + \frac{\Delta t}{2}\left(q_{i,j}^{n+1} + q_{i,j}^{n}\right)
$$
$$
- \left[\left(1 + \frac{2\alpha\Delta t}{h^2}\right)T_{i,j}^{n+1\,(k+1)} - \frac{\alpha\Delta t}{2h^2}\left(T_{i+1,j}^{n+1\,(k+1)} + T_{i-1,j}^{n+1\,(k+1)} + T_{i,j+1}^{n+1\,(k+1)} + T_{i,j-1}^{n+1\,(k+1)}\right)\right]
$$