Fractional Step Method

(Chorin 1965)

Idea: Split momentum equations into separate parts

⇒ since we have no pressure evolution equation, split pressure from the rest:

$$\frac{\partial \vec{v}}{\partial t} = N(\vec{v}) + \frac{1}{\mathrm{Re}} L(\vec{v}) \qquad : \quad \vec{v}^n \rightarrow \vec{v}^*$$

$$\frac{\partial \vec{v}}{\partial t} = -\nabla p \qquad : \quad \vec{v}^* \rightarrow \vec{v}^{n+1}$$

What are the implications of this split?

- Excursion into Linear Algebra:
  - let's introduce the space of all vector functions
  - for incompressible, unsteady flow, the velocity must evolve in the **subspace** of solenoidal functions ($\nabla \cdot \vec{v} = 0$)
  - when we update $\vec{v}^n \rightarrow \vec{v}^*$, then $\vec{v}^*$ is not necessarily in that subspace anymore, even if $\vec{v}^n$ was!
  - How do we get it back into the subspace? **projection**
  - that's exactly what $\vec{v}^* \rightarrow \vec{v}^{n+1}$ does!

Fractional Step Method

Some very important consequences of this thinking:

- p is not determined by a convection/diffusion equation!
- p is solely used to project $\vec{v}^*$ into the subspace of solenoidal functions!

⇒ thus p is not really a pressure, but rather a Lagrange multiplier

⇒ to make this distinction clear, let's call it φ

$$\boxed{\begin{aligned} \frac{\partial \vec{v}}{\partial t} &= N(\vec{v}) + \frac{1}{\mathrm{Re}} L(\vec{v}) \qquad &: \quad \vec{v}^n \to \vec{v}^* \\ \frac{\partial \vec{v}}{\partial t} &= -\nabla \varphi \qquad &: \quad \vec{v}^* \to \vec{v}^{n+1} \end{aligned}}$$

Fractional Step Method by Kim & Moin (1985)

- step 1: use Adams Bashforth for nonlinear terms and Crank-Nicholson for linear terms

$$\frac{\vec{v}^{*}_{i,j} - \vec{v}^{n}_{i,j}}{\Delta t} = \frac{3}{2}\vec{H}^{n}_{i,j} - \frac{1}{2}\vec{H}^{n}_{i,j} + \frac{1}{2}\frac{1}{\mathrm{Re}}\left(\frac{\partial_x^2}{\Delta x^2} + \frac{\partial_y^2}{\Delta y^2}\right)\left(\vec{v}^{*}_{i,j} + \vec{v}^{n}_{i,j}\right)$$

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$\Rightarrow$ two tri-diagonal solves:

1st: $$\left(1 - \frac{\Delta t}{2}\frac{1}{\mathrm{Re}}\frac{\delta_x^2}{\Delta x^2}\right)\Delta\vec{v}^{**}_{i,j} = \frac{\Delta t}{2}\left(3\vec{H}^{n}_{i,j} - \vec{H}^{n-1}_{i,j}\right) + \frac{\Delta t}{\mathrm{Re}}\left(\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2}\right)\vec{v}^{n}_{i,j}$$

2nd: $$\left(1 - \frac{\Delta t}{2}\frac{1}{\mathrm{Re}}\frac{\delta_y^2}{\Delta y^2}\right)\Delta\vec{v}^{*}_{i,j} = \Delta\vec{v}^{**}_{i,j} \qquad \rightarrow \qquad \vec{v}^{*}_{i,j} = \vec{v}^{n}_{i,j} + \Delta\vec{v}^{*}_{i,j}$$

- step 2: pressure Poisson equation

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$$\mathrm{div}\left(\mathrm{grad}\ \varphi^{n+1}\right) = \frac{1}{\Delta t}\mathrm{div}\left(\vec{v}^{*}\right)$$

- step 3: project into solenoidal subspace

$$\vec{v}^{n+1} = \vec{v}^{*} - \Delta t\ \mathrm{grad}\ \varphi^{n+1}$$

CRUCIAL : You **MUST** use the same discrete grad and div operators in step 2 & 3!

$\Rightarrow$ ∇^2 must be build as div(grad), for example

$$\delta_x^2 = (\varphi_{i+1,j} - \varphi_{i,j}) - (\varphi_{i,j} - \varphi_{i-1,j})$$

Fractional Step Method by Kim & Moin (1985):

- Step 1:

$$\frac{\vec{v}_{ij}^{\,*} - \vec{v}_{ij}^{\,n}}{\Delta t} = \underbrace{\frac{3}{2}\vec{H}_i^{\,n} - \frac{1}{2}\vec{H}_i^{\,n-1}}_{\text{Adams Bashforth}} + \underbrace{\frac{1}{2}\frac{1}{Re}\left(\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2}\right)\left(\vec{v}_{ij}^{\,*} + v_{ij}^{n}\right)}_{\text{Crank-Nicholson}}$$

rewrite:

$$\left(1 - \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_x^2}{\Delta x^2} - \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_y^2}{\Delta y^2}\right)\left(\vec{v}_{ij}^{\,*} - \vec{v}_{ij}^{\,n}\right) = \frac{3\Delta t}{2}\vec{H}_i^{\,n} - \frac{\Delta t}{2}\vec{H}_i^{\,n-1} + 2\left(\frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_x^2}{\Delta x^2} + \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_y^2}{\Delta y^2}\right)\left(\vec{v}_{ij}^{\,n}\right.$$

approximate:

$$\left(1 - \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_x^2}{\Delta x^2}\right)\left(1 - \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_y^2}{\Delta y^2}\right)\left(\vec{v}_{ij}^{\,*} - \vec{v}_{ij}^{\,n}\right) = \frac{\Delta t}{2}\left(3\vec{H}_i^{\,n} - \vec{H}_i^{\,n-1}\right) + \frac{\Delta t}{Re}\left(\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2}\right)\vec{v}_{ij}^{\,n}$$

$\Rightarrow$ 2 tridiagonal solves:

1st: $$\left(1 - \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_x^2}{\Delta x^2}\right)\Delta\vec{v}_{ij}^{\,**} = \frac{\Delta t}{2}\left(3\vec{H}_i^{\,n} - \vec{H}_i^{\,n-1}\right) + \frac{\Delta t}{Re}\left(\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2}\right)\vec{v}_{ij}^{\,n}$$

2nd: $$\left(1 - \frac{\Delta t}{2}\frac{1}{Re}\frac{\delta_y^2}{\Delta y^2}\right)\Delta\vec{v}_{ij}^{\,*} = \Delta\vec{v}_{ij}^{\,**}$$

$$\rightarrow \vec{v}_{ij}^{\,*} = \vec{v}_{ij}^{\,n} + \Delta\vec{v}_{ij}^{\,*}$$

Step 2 Pressure Poisson eq.:

$$\frac{\partial \vec{v}}{\partial t} = -\nabla \varphi^{n+1} \quad \text{or better:} \quad \frac{\partial \vec{v}}{\partial t} = -\operatorname{grad} \varphi^{n+1}$$

$$\rightarrow \frac{\vec{v}^{\,n+1} - \vec{v}^{\,*}}{\Delta t} = -\operatorname{grad} \varphi^{n+1} \quad \Big| \ \operatorname{div}(\ldots.)$$

0 : solenoidal!

$$\frac{\cancel{\operatorname{div}(\vec{v}^{\,n+1})} - \operatorname{div}(\vec{v}^{\,*})}{\Delta t} = -\operatorname{div}(\operatorname{grad} \varphi^{n+1})$$

$$\Rightarrow \boxed{\operatorname{div}(\operatorname{grad} \varphi^{n+1}) = \frac{1}{\Delta t} \operatorname{div}(\vec{v}^{\,*})}$$

Step 3 $\vec{v}^{\,n+1} = \vec{v}^{\,*} - \Delta t \operatorname{grad} \varphi^{n+1}$

Crucial!!: You **MUST** use the same discrete grad and div operators in steps 2 and 3!

$\Rightarrow \nabla^2$ must be build as div(grad) → for example: $\delta_x^2 \varphi_i = (\varphi_{i+1} - \varphi_i) - (\varphi_i - \varphi_{i-1})$

Fractional Step Method by Kim & Moin (1985)

But still need to define boundary conditions!

- for step 1: use velocity boundary conditions with ghost cells as in MAC
- for step 2: let's look at lower wall as an example:

$$\operatorname{div}\left(\operatorname{grad}\,\varphi^{n+1}\right) = \frac{1}{\Delta t}\operatorname{div}\left(\vec{v}^{*}\right)$$

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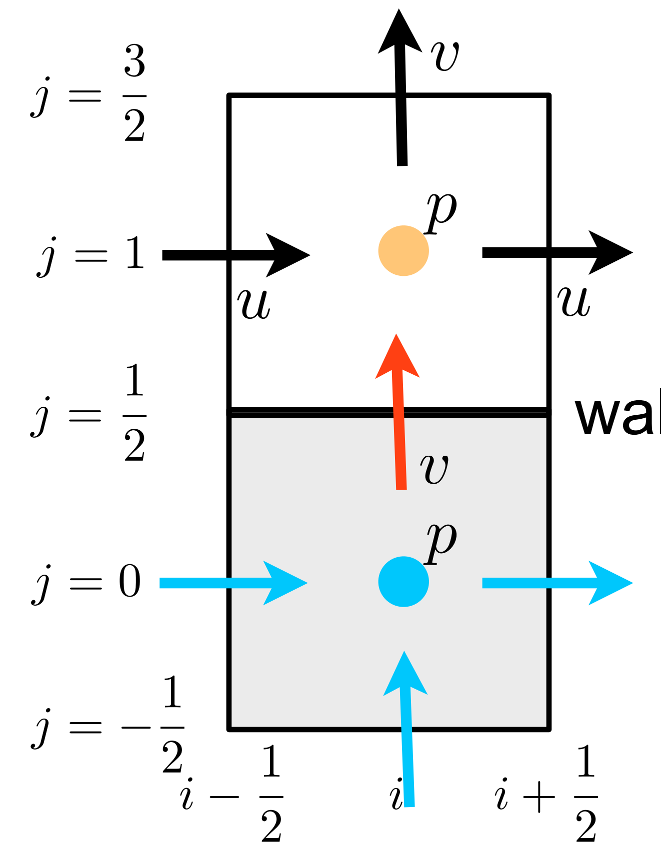

Neumann for φ: $\varphi^{n+1}_{i,0} = \varphi^{n+1}_{i,1}$

velocity: $v^{*}_{i,\frac{1}{2}} = v^{n+1}_{i,\frac{1}{2}}$

but, there's something of an inconsistency in the velocity bc!

➡ imposes a velocity from inside the solenoidal subspace onto a velocity field that is outside!

➡ to avoid this inconsistency (Kim & Moin 1985)

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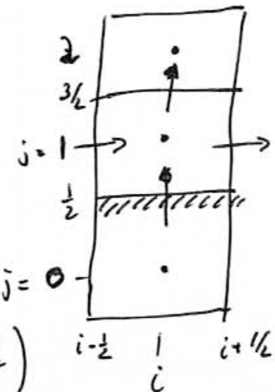
➡ Is it necessary to avoid the inconsistency in the final project? No

Still need boundary conditions!

for step 1: use velocity b.c. for $\vec{v}^n$ with ghost cells as in MAC.

for step 2: Let's look at lower wall as an example:



$$\frac{\delta_x^2}{\Delta x^2}\varphi_{ij}^{n+1} + \frac{\delta_y^2}{\Delta y^2}\varphi_{ij}^{n+1} = \frac{1}{\Delta t}\left(\frac{u^*_{i+\frac12,j}-u^*_{i-\frac12,j}}{\Delta x} + \frac{v^*_{i,j+\frac12}-v^*_{i,j-\frac12}}{\Delta y}\right)$$

at $j=1$:

$$\frac{\varphi_{i+1,1}^{n+1}-2\varphi_{i,1}^{n+1}+\varphi_{i-1,1}^{n+1}}{\Delta x^2} + \frac{\varphi_{i,2}^{n+1}-2\varphi_{i,1}^{n+1}+\varphi_{i,0}^{n+1}}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u^*_{i+\frac12,1}-u^*_{i-\frac12,1}}{\Delta x}+\frac{v^*_{i,\frac32}-v^*_{i,\frac12}}{\Delta y}\right)$$

we could do: Neuman for φ : $\frac{\partial \varphi^{n+1}}{\partial n} = 0 \Rightarrow \varphi_{i,0}^{n+1} = \varphi_{i,1}^{n+1}$

and $v_{i,\frac{1}{2}}^{*} = v_{i,\frac{1}{2}}^{n+1}$, but this last one imposes a velocity b.c. from <u>inside</u> the solenoidal subspace onto a field ($\vec{v}^*$) that is outside!

→ better (Kim & Moin):

for φ: Let's look at step 3 for $v_{i,\frac{1}{2}}$: $v_{i,\frac{1}{2}}^{n+1} = v_{i,\frac{1}{2}}^{*} - \Delta t/_{\Delta y} \left(\varphi_{i,1}^{n+1} - \varphi_{i,0}^{n+1} \right)$

$$\Rightarrow \frac{\varphi_{i,1}^{n+1} - \varphi_{i,0}^{n+1}}{\Delta y} = - \frac{v_{i,\frac{1}{2}}^{n+1} - v_{i,\frac{1}{2}}^{*}}{\Delta t}$$

Substitute into Poisson eq. of step 2:

$$\frac{\varphi_{i+1,1}^{n+1} - 2\varphi_{i,1}^{n+1} + \varphi_{i-1,1}^{n+1}}{\Delta x^2} + \frac{1}{\Delta y}\left(\frac{\varphi_{i,2}^{n+1} - \varphi_{i,1}^{n+1}}{\Delta y} + \frac{v_{i,\frac{1}{2}}^{n+1} - v_{i,\frac{1}{2}}^{*}}{\Delta t \Delta y} \right) = \underbrace{\frac{1}{\Delta t}\left(\frac{u_{i+\frac{1}{2},1}^{*} - u_{i-\frac{1}{2},1}^{*}}{\Delta x} + \frac{v_{i,\frac{3}{2}}^{*} - v_{i,\frac{1}{2}}^{*}}{\Delta y} \right)}_{\frac{1}{\Delta t} D \vec{v}_{i,1}^{*}}$$

$$\Rightarrow \frac{\varphi_{i+1,1}^{n+1} - 2\varphi_{i,1}^{n+1} + \varphi_{i-1,1}^{n+1}}{\Delta x^2} + \frac{\varphi_{i,2}^{n+1} - \varphi_{i,1}^{n+1}}{\Delta y^2} = \frac{1}{\Delta t} D \vec{v}_{i,1}^{*} - \frac{1}{\Delta t} \frac{v_{i,\frac{1}{2}}^{n+1} - v_{i,\frac{1}{2}}^{*}}{\Delta y}$$

→ modifies matrix @ boundary!

for $\vec{v}^*$: same idea as above:

$\vec{v}_{i,j}^{*} = \vec{v}_{i,j}^{n+1} + \Delta t \text{ grad } \varphi^n$ ← use n instead of $n+1$

$$\Rightarrow v_{i,\frac{1}{2}}^{*} = v_{i,\frac{1}{2}}^{n+1} + \Delta t \left(\frac{\varphi_{i,1}^{n} - \varphi_{i,0}^{n}}{\Delta y} \right)$$