Homework 1: Total Points: 70

2) $\boxed{2 = 20}$

$$f_j'' + a_0 f_j + a_1 f_{j+1} + a_2 f_{j+2} + a_3 f_{j+3} = O(h^2) \qquad (1)$$

$$f_j = f_j$$

$$f_{j+1} = f_j + h f_j' + \frac{h^2}{2} f_j'' + \frac{1}{6} h^3 f_j''' + \frac{1}{24} h^4 f_j^{iv} + \ldots$$

$$f_{j+2} = f_j + 2h f_j' + 2h^2 f_j'' + \frac{4}{3} h^3 f_j''' + \frac{2}{3} h^4 f_j^{iv} + \ldots$$

$$f_{j+3} = f_j + 3h f_j' + \frac{9}{2} h^2 f_j'' + \frac{9}{2} h^3 f_j''' + \frac{27}{8} h^4 f_j^{iv} + \ldots$$

Taylor Table

	f_j	f_j'	f_j''	f_j'''	f_j^{iv}	
f_j''	0	0	1	0		(1)
$a_0\ f_j$	a_0	0	0	0	0	(1)
$a_1\ f_{j+1}$	a_1	$+a_1 h$	$+\frac{1}{2} a_1 h^2$	$+\frac{1}{6} a_1 h^3$	$\frac{1}{24} a_1 h^4$	(2)
$a_2\ f_{j+2}$	a_2	$2a_2 h$	$2a_2 h^2$	$\frac{4}{3} a_2 h^3$	$\frac{2}{3} a_2 h^4$	(2)
$a_3\ f_{j+3}$	a_3	$3a_3 h$	$\frac{9}{2} a_3 h^2$	$\frac{9}{2} a_3 h^3$	$\frac{27}{8} a_3 h^4$	(2)
	$=0$	$=0$	$=0$	$=0$		

$$\left.\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 0 \\ a_1 h + 2a_2 h + 3a_3 h &= 0 \\ 1 + \frac{a_1}{2} h^2 + 2a_2 h^2 + \frac{9}{2} a_3 h^2 &= 0 \\ \frac{1}{6} a_1 h^3 + \frac{4}{3} a_2 h^3 + \frac{9}{2} a_3 h^3 &= 0 \end{aligned}\right\} (2) \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1/2 & 2 & 9/2 \\ 0 & 1/6 & 4/3 & 9/2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/h^2 \\ 0 \end{bmatrix} \quad \text{Solve}$$

(1) (1) (1) (1)

$$\Rightarrow a_0 = -2/h^2 \; ; \; a_1 = 5/h^2 \; ; \; a_2 = -4/h^2 \; ; \; a_3 = 1/h^2$$

$$\Rightarrow \text{error} \quad \frac{1}{24}\frac{5}{h^2} h^4 - \frac{2}{3}\frac{4}{h^2} h^4 + \frac{27}{8}\frac{1}{h^2} h^4 = \left(\frac{5}{24} - \frac{8}{3} + \frac{27}{8}\right) h^2 = \underline{\underline{\frac{11}{12} h^2}} \;(2) = O(h^2) \quad (1)$$

$$\Rightarrow \boxed{f_j'' = \frac{2f_j - 5f_{j+1} + 4f_{j+2} - f_{j+3}}{h^2} + O(h^2)} \quad (2)$$

(total: 10 pts)

Problem 2:

l.h.s $\frac{\delta(u_n v_n)}{\delta x_n} = \frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h}$

r.hs : $\bar{u}_n \frac{\delta v_n}{\delta x} + \bar{v}_n \frac{\delta u_n}{\delta x} = \frac{1}{2}(u_{n+1} + u_{n-1}) \frac{v_{n+1} - v_{n-1}}{2h} + \frac{1}{2}(v_{n+1} + v_{n-1}) \frac{u_{n+1} - u_{n-1}}{2h}$

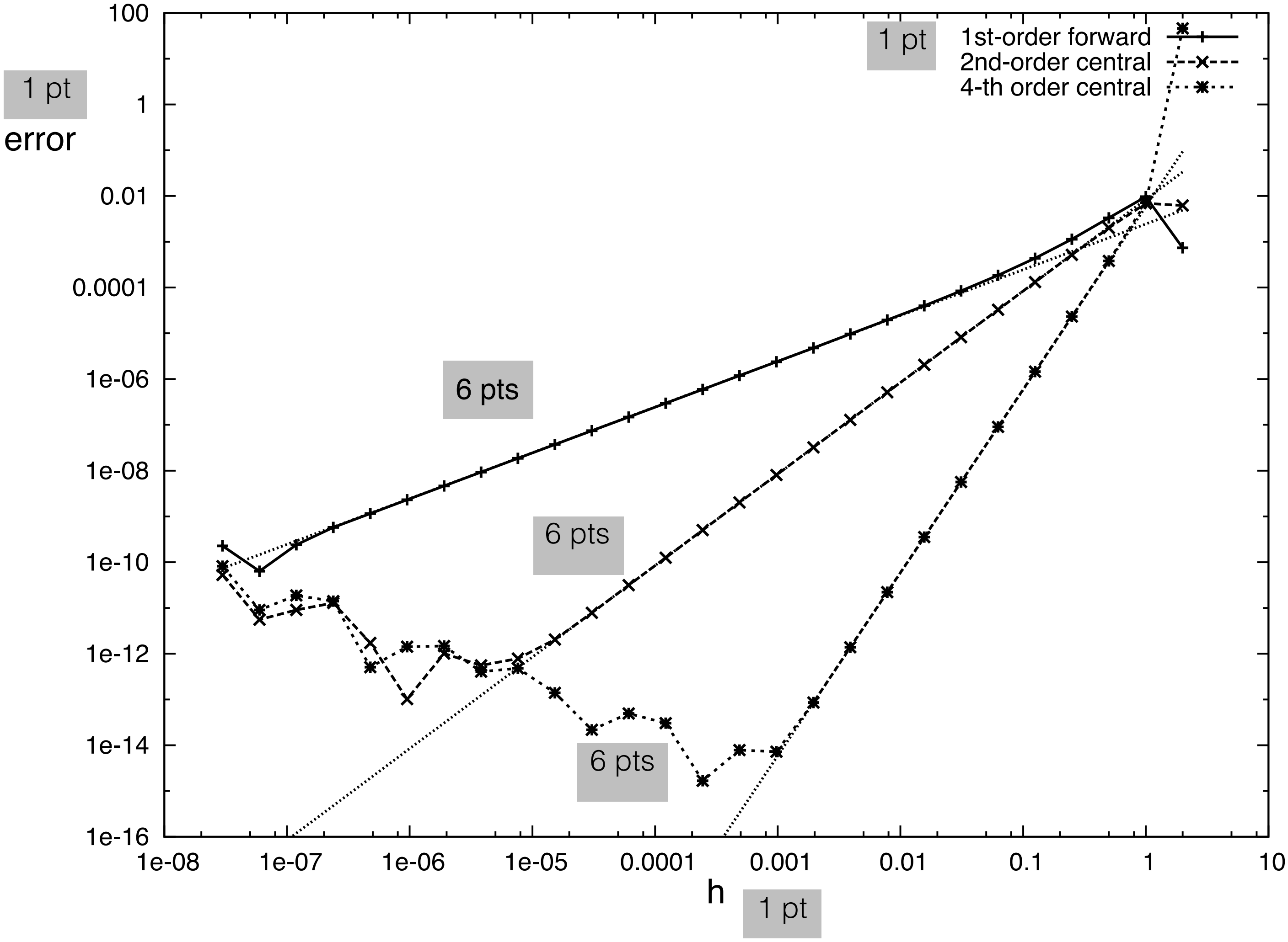
$$= \frac{u_{n+1}v_{n+1} - \cancel{u_{n+1}v_{n-1}} + \cancel{u_{n-1}v_{n+1}} - u_{n-1}v_{n-1}}{4h}$$

$$+ \frac{v_{n+1}u_{n+1} - \cancel{v_{n+1}u_{n-1}} + \cancel{v_{n-1}u_{n+1}} + v_{n-1}u_{n-1}}{4h}$$

$$= \frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h}$$

$\Rightarrow$ l.hs = r.h.s q.ed.

Problem 3: (total: 40 pts)



In a log/log plot of error versus grid spacing, a method of order n, should generate a line with slope n. As shown in the plot, this is the case for the 1st order method with slope 1 (**3 pts**), 2nd order method with slope 12 (**3 pts**) and 4th order method with slope 4 (**3 pts**), at least for certain ranges of h. There are two areas of deviation from this behavior for each method. First, if h is too large, higher order Taylor terms beyond the first error term are no longer negligible, resulting in either error cancellation or addition with higher or lower than expected error (**5 pts**). Second, if h is too small, then finite precision errors in the differences in the finite difference formula can divided by ever smaller h resulting in a linear increase in the error (**5 pts**).

no code: -20 pts

not log/log: -10 pts

log(e) vs log (h): -10pts