Beam-Warming Implicit

Idea: Start again from Taylor-Series in $\pm\Delta t$ direction:

$$u_i^{n+1} = u_i^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + O(\Delta t^3) \qquad (i)$$

$$u_i^n = u_i^{n+1} - \Delta t \left.\frac{\partial u}{\partial t}\right|_i^{n+1} + \frac{\Delta t^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1} + O(\Delta t^3) \qquad (ii)$$

$$(i)-(ii): \quad 2u_i^{n+1} = 2u_i^n + \Delta t\left(\left.\frac{\partial u}{\partial t}\right|_i^n + \left.\frac{\partial u}{\partial t}\right|_i^{n+1}\right) + \frac{\Delta t^2}{2}\left(\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n - \left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1}\right) + O(\Delta t^3)$$

new Taylor-Series for $\left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1}$: $\quad \left(\frac{\partial^2 u}{\partial t^2}\right)_i^{n+1} = \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n + \Delta t \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n + O(\Delta t^2)$

Substitute in:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left(\left.\frac{\partial u}{\partial t}\right|_i^n + \left.\frac{\partial u}{\partial t}\right|_i^{n+1}\right) + \frac{\Delta t^2}{2}\underbrace{\left(\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n - \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n\right)}_{0} + O(\Delta t^3)$$

use PDE: $\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x}$

$$\Rightarrow u_i^{n+1} = u_i^n + \frac{\Delta t}{2}\left(-\left.\frac{\partial E}{\partial x}\right|_i^n - \left.\frac{\partial E}{\partial x}\right|_i^{n+1}\right) + O(\Delta t^3) \qquad (*)$$

Need E @ t^{n+1} $\Rightarrow$ Taylor-Series: $\quad E^{n+1} = E^n + \Delta t \left.\frac{\partial E}{\partial t}\right|^n + O(\Delta t^2)$

$$= E^n + \Delta t \underbrace{\frac{\partial E}{\partial u}}_{A}\frac{\partial u}{\partial t} + O(\Delta t^2)$$

replace $\frac{\partial u}{\partial t}$ with FDE: $\frac{\partial u}{\partial t} = \frac{u^{n+1}-u^n}{\Delta t} + O(\Delta t)$

$$\Rightarrow E^{n+1} = E^n + \Delta t\, A\, \frac{u^{n+1}-u^n}{\Delta t} + O(\Delta t^2)$$

take $\frac{\partial}{\partial x}$: $\quad \left.\frac{\partial E}{\partial x}\right|_i^{n+1} = \left.\frac{\partial E}{\partial x}\right|_i^n + \frac{\partial}{\partial x}\left(A(u_i^{n+1}-u_i^n)\right) + O(\Delta t^2)$

Substitute back in (*): $\quad u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left(2\left.\frac{\partial E}{\partial x}\right|_i^n + \underbrace{\frac{\partial}{\partial x}\left(A(u_i^{n+1}-u_i^n)\right)}_{\text{Use 2nd-order central}}\right) + O(\Delta t^3)$

$$\frac{\partial}{\partial x}\left(A(u_i^{n+1}-u_i^n)\right) \approx \frac{A_{i+1}u_{i+1}^{n+1} - A_{i-1}u_{i-1}^{n+1}}{2\Delta x} - \frac{A_{i+1}u_{i+1}^{n} - A_{i-1}u_{i-1}^{n}}{2\Delta x}$$

(a) what time level is A? Use t^n : $A_{i\pm 1}^n$ etc.

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{2}\left(2\,\frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \frac{A_{i+1}^n u_{i+1}^{n+1} - A_{i-1}^n u_{i-1}^{n+1}}{2\Delta x} - \frac{A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n}{2\Delta x}\right) + O(\Delta t^3)$$

rearrange:

$$-\frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^{n+1} + u_i^{n+1} + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{4}\frac{\Delta t}{\Delta x}\left(A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n\right)$$

Beam-Warming Implicit

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

Idea: Start again from Taylor series, but in $\pm\Delta t$ direction **Board**

$$-\frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^{n+1} + u_i^{n+1} + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t}{4\Delta x}\left(A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n\right)$$

- order: $O(\Delta x^2)$, $O(\Delta t^2)$
- stability: unconditionally stable! **Code: Δx = 0.1, C = 2.5**
- Drawback: large dispersive errors!
- Idea: Why not add a dissipative term to the scheme?
  - add 4th-order damping: $D = -\epsilon_l\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right)$
  - this adds a stability constraint: $0 \le -\epsilon_l \le \frac{1}{8}$

Code: Δx = 0.1, C = 2.5, ε = 0.1

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order Explicit Upwind

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(E_i^n - E_{i-1}^n\right) \quad \text{for} \quad u_i^n > 0$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right) \quad \text{for} \quad u_i^n < 0$$

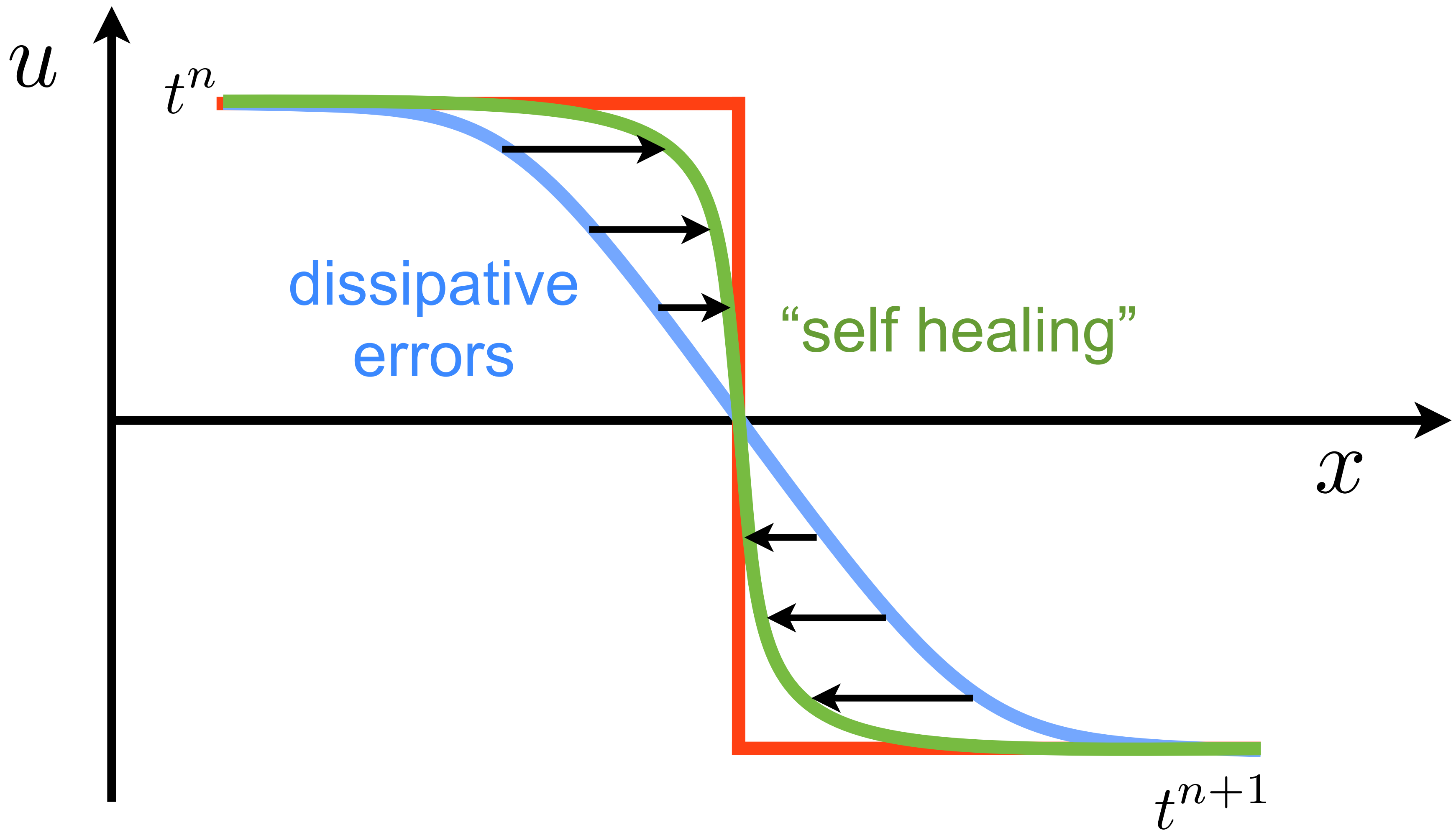
- order: *O(Δt), O(Δx)*
- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: dissipative
- much better here than for wave equation!
- Why?

Code:
C=0.5,
C=1, C=0.1

1st-order Explicit Upwind

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

- much better here than for wave equation!
- Why?



1st-order Implicit Upwind

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

if $u_i^n > 0$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{E_i^{n+1} - E_{i-1}^{n+1}}{\Delta x} = -\frac{\frac{(u_i^{n+1})^2}{2} - \frac{(u_{i-1}^{n+1})^2}{2}}{\Delta x}$$

- Problem: non-linear system! ⇒ linearize the non-linear terms!

$$(u_i^{n+1})^2 \approx u_i^n u_i^{n+1} \qquad (u_{i-1}^{n+1})^2 \approx u_{i-1}^n u_{i-1}^{n+1}$$

$$\Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left(u_i^n u_i^{n+1} - u_{i-1}^n u_{i-1}^{n+1}\right)$$

- rearrange: $$\boxed{\left(\frac{\Delta t}{2\Delta x}u_{i-1}^n\right)u_{i-1}^{n+1} - \left(1 - \frac{\Delta t}{2\Delta x}u_i^n\right)u_i^{n+1} = -u_i^n}$$
  - order: *O(Δt), O(Δx)*
  - stability: unconditionally stable
  - leading order error term: dissipative
  - @ *C=1*: explicit is better than implicit

Code:
C=0.5,
C=1, C=1.5