Observation:

Many of the higher order schemes have large dispersive errors!

How can we deal with this?

- add linear damping
  - 2^{nd} or 4^{th} order with constant coefficients
  - Example:

$$D_l = -\epsilon_l \Delta x^4 \frac{\partial^4 u}{\partial x^4} \quad \text{(1D)} \qquad D_l = -\epsilon_l \left(\Delta x^4 \frac{\partial^4 u}{\partial x^4} + \Delta y^4 \frac{\partial^4 u}{\partial y^4} \right) \quad \text{(2D)}$$

  - Issues:
    - need to choose constant coefficient ε_l a-priori
    - damping is active everywhere
    - can introduce additional stability constraint (Beam-Warming: $\varepsilon_l \leq 1/8$)
    - changes the PDE that is solved ➞ might be unacceptable
  - Solutions:
    - could limit damping to regions, for example:
      inviscid regions ⇒ do damping / boundary layer ⇒ no damping
    - Better: We need a sensor to turn damping on or off automatically

⇒ $\boxed{\text{TVD schemes}}$

TVD schemes

Some definitions:

Monotone schemes

- scheme that does not generate **new** local extrema
  - ⇒ a local minimum does not decrease
  - ⇒ a local maximum does not increase
- Consequences
  - oscillation free
  - dissipative
  - **only first order!**

Example: 1st-order upwind

TVD schemes

Some definitions:

Total Variation (TV)

$$TV(u) = \frac{1}{V_\Omega} \int_\Omega \left| \frac{\partial u}{\partial \vec{x}} \right| d\vec{x}$$

for example using 1-sided, 1st-order differences in 1D:

$$TV(u^n) = \frac{1}{N-1} \sum_{i=1}^{N-1} \left| u_{i+1}^n - u_i^n \right|$$

Total Variation Diminishing (TVD)

$$TV(u^{n+1}) \leq TV(u^n)$$

$\Rightarrow$ monotone schemes are TVD!

TVD schemes

Example: 1D wave equation

- a general explicit finite difference method can be written as

$$u_i^{n+1} = u_i + A^n_{i+1/2}\Delta u^n_{i+1/2} - B^n_{i-1/2}\Delta u^n_{i-1/2}$$

where $\Delta u^n_{i+1/2} = u^n_{i+1} - u^n_i$ and $\Delta u^n_{i-1/2} = u^n_i - u^n_{i-1}$

A and B depend on the chosen scheme

A method is TVD if

$A^n_{i+1/2} \geq 0$ and

$B^n_{i-1/2} \geq 0$ and

$A^n_{i+1/2} + B^n_{i+1/2} \leq 1$

1st-order TVD schemes

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$\Delta u^n_{i+1/2} = u^n_{i+1} - u^n_i$$

let's revisit explicit 1st-order upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\begin{cases} E^n_{i+1} - E^n_i & \text{if} \quad \alpha^n_{i+1/2} < 0 \\ E^n_i - E^n_{i-1} & \text{if} \quad \alpha^n_{i-1/2} > 0 \end{cases}$$

with $$\alpha^n_{i+1/2} = \begin{cases} \frac{E^n_{i+1} - E^n_i}{u^n_{i+1} - u^n_i} & \text{if} \quad |\Delta u^n_{i+1/2}| \geq \epsilon' \\ (u^n_i + u^n_{i+1})/2 & \text{if} \quad |\Delta u^n_{i+1/2}| < \epsilon' \end{cases}$$

with for example $\epsilon' = 10^{-12}$ for u=O(1)

implementation: try to avoid if-statements in loops (vectorization problems)

use $$sign(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha > 0 \\ -1 & \text{if} \quad \alpha < 0 \end{cases}$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - sign(\alpha^n_{i+1/2})\right)\left(E^n_{i+1} - E^n_i\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + sign(\alpha^n_{i-1/2})\right)\left(E^n_i - E^n_{i-1}\right)$$

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order TVD schemes

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - sign(\alpha_{i+1/2}^n)\right)\left(E_{i+1}^n - E_i^n\right)$$
$$- \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + sign(\alpha_{i-1/2}^n)\right)\left(E_i^n - E_{i-1}^n\right)$$

but also: $sign(\alpha) = \dfrac{|\alpha|}{\alpha}$

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - \frac{|\alpha_{i+1/2}^n|}{\alpha_{i+1/2}^n}\right)\left(E_{i+1}^n - E_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + \frac{|\alpha_{i-1/2}^n|}{\alpha_{i-1/2}^n}\right)\left(E_i^n - E_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - \frac{|\alpha_{i+1/2}^n|}{\frac{E_{i+1}^n - E_i^n}{\Delta u_{i+1/2}^n}}\right)\left(E_{i+1}^n - E_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + \frac{|\alpha_{i-1/2}^n|}{\frac{E_i^n - E_{i-1}^n}{\Delta u_{i-1/2}^n}}\right)\left(E_i^n - E_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - \left(E_i^n + E_{i-1}^n\right) + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right) \quad \text{with} \quad h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

numerical flux function

1st-order TVD schemes

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - \left(E_i^n + E_{i-1}^n\right) + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

since $\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases} \quad \Rightarrow \quad E_{i+1}^n - E_i^n = \alpha_{i+1/2}^n \Delta u_{i+1/2}^n$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\alpha_{i+1/2}^n\Delta u_{i+1/2}^n - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n + \alpha_{i-1/2}^n\Delta u_{i-1/2}^n + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

is this TVD? compare to $u_i^{n+1} = u_i + A_{i+1/2}^n \Delta u_{i+1/2}^n - B_{i-1/2}^n \Delta u_{i-1/2}^n$

$$A_{i+1/2}^n = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n| - \alpha_{i+1/2}^n\right) \qquad B_{i-1/2}^n = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i-1/2}^n| + \alpha_{i-1/2}^n\right)$$

$$\Rightarrow A_{i+1/2}^n \geq 0 \qquad \Rightarrow B_{i-1/2}^n \geq 0$$

$0 \leq A_{i+1/2}^n + B_{i+1/2}^n \leq 1$? $\qquad 0 \leq \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n| \cancel{- \alpha_{i+1/2}^n} + |\alpha_{i+1/2}^n| \cancel{+ \alpha_{i+1/2}^n}\right) \leq 1$

$$\boxed{0 \leq \frac{\Delta t}{\Delta x}|\alpha_{i+1/2}^n| \leq 1}$$

$\Rightarrow$ Courant number requirement!

1st-order TVD schemes

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

or: $$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right)$$

central

dissipation

can implement this as multi-step

Step 1: $u_i^* = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$

Step 2: $u_i^{n+1} = u_i^* - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\Phi_{i+1/2}^n - \Phi_{i-1/2}^n\right)$

with $\Phi_{i+1/2}^n = -|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n$: flux limiter function

Code: C=0.5, C=0.1, C=1.0

One other issue for non-linear equations:

- weak solutions of the conservation laws may not be unique!
- How to pick the correct one?
  - make use of physics: 2^{nd} law of thermodynamics
    $\Rightarrow$ entropy may not decrease
  - impose the entropy condition to get the physically correct solution

➡ **need dissipative mechanism**

- but that's exactly what α does in the numerical flux function!

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right)$$

- however, we may have *α* = *0*. What then?

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

define $\psi = \begin{cases} |\alpha| & \text{if} \quad |\alpha| \geq \varepsilon \\ \frac{\alpha^2+\varepsilon^2}{2\varepsilon} & \text{if} \quad |\alpha| < \varepsilon \end{cases}$ with $0 \leq \varepsilon \leq \frac{1}{8}$

- use ψ instead of α in the numerical flux function h

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\psi_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

entropy fix

We now have a 1st-order TVD method, but would prefer at least a 2nd-order TVD method

- idea by Harten: let's modify the flux E by replacing it with $\overline{E} = E + G$
- let's start with a formulation we had earlier (slide 6):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

introduce flux limiter function: $\Phi_{i+1/2}^n = -|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n$

We now have a 1st-order TVD method, but would prefer at least a 2nd-order TVD method

- idea by Harten: let's modify the flux E by replacing it with $\overline{E} = E + G$
- let's start with a formulation we had earlier (slide 6):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) + \Phi_{i+1/2}^n\right]$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$
$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) + \Phi_{i+1/2}^n\right]$$

- Harten-Yee-limiter:

$$\Phi_{i+1/2}^n = (G_{i+1}^n + G_i^n) - \psi(\alpha_{i+1/2}^n + \beta_{i+1/2}^n)\Delta u_{i+1/2}^n$$

with $\psi(y) = \begin{cases} |y| & \text{if} \quad |y| \geq \varepsilon \\ \frac{y^2+\varepsilon^2}{2\varepsilon} & \text{if} \quad |y| < \varepsilon \end{cases}$ $\quad 0 \leq \varepsilon \leq \frac{1}{8}$ (entropy fix function)

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

(same as before)

$$\beta_{i+1/2}^n = \begin{cases} \frac{G_{i+1}^n - G_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ 0 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

but what's G? many many choices!

for example $G_i^n = S \cdot \max(0, \min(\sigma_{i+1/2}|\Delta u_{i+1/2}^n|, S \cdot \sigma_{i-1/2}\Delta u_{i-1/2}^n))$

where $S = \mathrm{sign}(\Delta u_{i+1/2}^n)$ and $\sigma_{i+1/2} = \frac{1}{2}\left[\psi(\alpha_{i+1/2}^n) - \frac{\Delta t}{\Delta x}(\alpha_{i+1/2}^n)^2\right]$

other choices: see book

Code: C=0.5, C=0.1, C=1.0