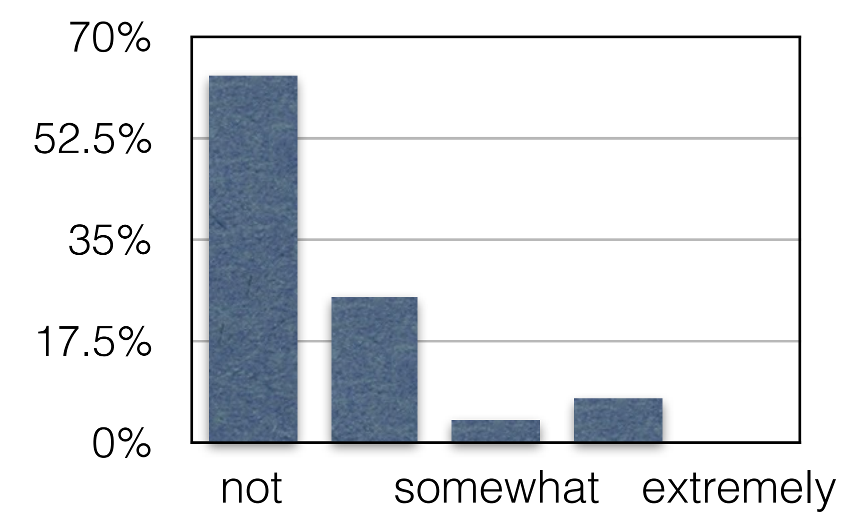
Muddiest Points from Class 01/23

- *"Difference between Centred Difference and developing the finite difference equations using the grid points."*
  - if you determine the most accurate formula for the stencil i-1, i, and i+1 (using a Taylor table) you will find the central difference formula

- *"I wasn't clear on how you know which points to write Taylor series for with regards to the stencil. Is the stencil something provided on each question or am I misinterpreting what the stencil is entirely?*
  - the stencil is always given/pre-defined

- *"Can you go over the error again? For example is the actual answer the one you get from the code and the theoretical is the one you get by doing from hand/calculator? Or vice versa? Could you possibly work through another example?"*
  - to calculate the error, you must be able to calculate the exact solution (analytical form + it's numerical evaluation with at least double precision)
  - the Taylor series form of the error (containing infinitely many terms) is used solely to understand the behavior of the error when changing mesh spacing, not to calculate an actual numerical value for the error

- *"You said that we should use double precision for our numerical answers. To confirm, that means that you want 16 significant figures in the final answer?"*
- *"what's the exact meaning of the "double precision" at the last page of the PPT?"*
  - double precision uses 64bit to store floating point numbers
  - IEE-754: 52 significant mantissa bits (roughly 16 decimal digits)
  - Matlab uses double precision by default
  - Fortran use `double precision`, C/C++ use `double`



- Can we improve on the general technique somehow?
  - Idea: use not only *f* @ stencil points, but also *f'* (PADE)
  - Example 5:

find $\left.\frac{\partial f}{\partial x}\right|_{x_i}$ using only grid points $x_{i-1},\, x_i,\, x_{i+1}$

or $f_i' + a_0 f_i + a_1 f_{i+1} + a_2 f_{i-1} + a_3 f_{i+1}' + a_4 f_{i-1}' = O(?)$

- Step 1: Write Taylor series for each stencil point around point where derivative is requested
- Step 2: Put into Taylor table (can combine with step 1)
- Step 3: Set as many of the lower order terms on the right hand side to zero as possible
- Step 4: Substitute solution back in

- Step 1: Write Taylor series for each stencil point around point where derivative is requested
- Step 2: Put into Taylor table (can combine with step 1)

$$f_i' + a_0 f_i + a_1 f_{i+1} + a_2 f_{i-1} + a_3 f_{i+1}' + a_4 f_{i-1}' = O(?)$$

	f_i	f_i'	f_i''	f_i'''	$f_i^{(IV)}$	$f_i^{(V)}$
f_i'	0	1	0	0	0	0
$a_0 f_i$	a_0	0	0	0	0	0
$a_1 f_{i+1}$	a_1	$a_1 h$	$\frac{1}{2}a_1 h^2$	$\frac{1}{6}a_1 h^3$	$\frac{1}{24}a_1 h^4$	$\frac{1}{120}a_1 h^5$
$a_2 f_{i-1}$	a_2	$-a_2 h$	$\frac{1}{2}a_2 h^2$	$-\frac{1}{6}a_2 h^3$	$\frac{1}{24}a_2 h^4$	$-\frac{1}{120}a_2 h^5$
$a_3 f_{i+1}'$	0	a_3	$a_3 h$	$\frac{1}{2}a_3 h^2$	$\frac{1}{6}a_3 h^3$	$\frac{1}{24}a_3 h^4$
$a_4 f_{i-1}'$						

- Step 1: Write Taylor series for each stencil point around point where derivative is requested
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$$f_i' + a_0 f_i + a_1 f_{i+1} + a_2 f_{i-1} + a_3 f_{i+1}' + a_4 f_{i-1}' = O(?)$$

	f_i	f_i'	f_i''	f_i'''	$f_i^{(IV)}$	$f_i^{(V)}$
f_i'	0	1	0	0	0	0
$a_0 f_i$	a_0	0	0	0	0	0
$a_1 f_{i+1}$	a_1	$a_1 h$	$\frac{1}{2} a_1 h^2$	$\frac{1}{6} a_1 h^3$	$\frac{1}{24} a_1 h^4$	$\frac{1}{120} a_1 h^5$
$a_2 f_{i-1}$	a_2	$-a_2 h$	$\frac{1}{2} a_2 h^2$	$-\frac{1}{6} a_2 h^3$	$\frac{1}{24} a_2 h^4$	$-\frac{1}{120} a_2 h^5$
$a_3 f_{i+1}'$	0	a_3	$a_3 h$	$\frac{1}{2} a_3 h^2$	$\frac{1}{6} a_3 h^3$	$\frac{1}{24} a_3 h^4$
$a_4 f_{i-1}'$	0	a_4	$-a_4 h$	$\frac{1}{2} a_4 h^2$	$-\frac{1}{6} a_4 h^3$	$\frac{1}{24} a_4 h^4$

	f_i	f_i'	f_i''	f_i'''	$f_i^{(IV)}$	$f_i^{(V)}$
f_i'	0	1	0	0	0	0
$a_0 f_i$	a_0	0	0	0	0	0
$a_1 f_{i+1}$	a_1	$a_1 h$	$\frac{1}{2} a_1 h^2$	$\frac{1}{6} a_1 h^3$	$\frac{1}{24} a_1 h^4$	$\frac{1}{120} a_1 h^5$
$a_2 f_{i-1}$	a_2	$-a_2 h$	$\frac{1}{2} a_2 h^2$	$-\frac{1}{6} a_2 h^3$	$\frac{1}{24} a_2 h^4$	$-\frac{1}{120} a_2 h^5$
$a_3 f_{i+1}'$	0	a_3	$a_3 h$	$\frac{1}{2} a_3 h^2$	$\frac{1}{6} a_3 h^3$	$\frac{1}{24} a_3 h^4$
$a_4 f_{i-1}'$	0	a_4	$-a_4 h$	$\frac{1}{2} a_4 h^2$	$-\frac{1}{6} a_4 h^3$	$\frac{1}{24} a_4 h^4$
	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	

- Step 3: Set as many of the lower order terms on the right hand side to zero as possible

$$a_0 + a_1 + a_2 = 0$$

$$1 + a_1 h - a_2 h + a_3 + a_4 = 0$$

$$\frac{1}{2} a_1 h^2 + \frac{1}{2} a_2 h^2 + a_3 h - a_4 h = 0$$

$$\frac{1}{6} a_1 h^3 - \frac{1}{6} a_2 h^3 + \frac{1}{2} a_3 h^2 + \frac{1}{2} a_4 h^2 = 0$$

$$\frac{1}{24} a_1 h^4 + \frac{1}{24} a_2 h^4 + \frac{1}{6} a_3 h^3 - \frac{1}{6} a_4 h^3 = 0$$

$$a_0 + a_1 + a_2 = 0$$

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$$\frac{1}{24} a_1 h^4 + \frac{1}{24} a_2 h^4 + \frac{1}{6} a_3 h^3 - \frac{1}{6} a_4 h^3 = 0$$

Solve this 5x5 system: $a_0 = 0,\ a_1 = -\frac{3}{4h},\ a_2 = \frac{3}{4h},\ a_3 = \frac{1}{4},\ a_4 = \frac{1}{4}$

- Step 4: Substitute solution back in

$$f_i' - \frac{3}{4h} f_{i+1} + \frac{3}{4h} f_{i-1} + \frac{1}{4} f_{i+1}' + \frac{1}{4} f_{i-1}' = \alpha h^4 f^{(V)} + \ldots$$

multiply by 4 and resort:

$$\boxed{f_{i-1}' + 4 f_i' + f_{i+1}' = \frac{3}{h} \left(f_{i+1} - f_{i-1} \right) + O(h^4)}$$

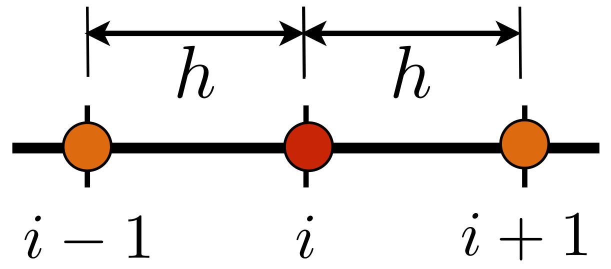
Order? 4th order!

- 4^{th}-order PADE

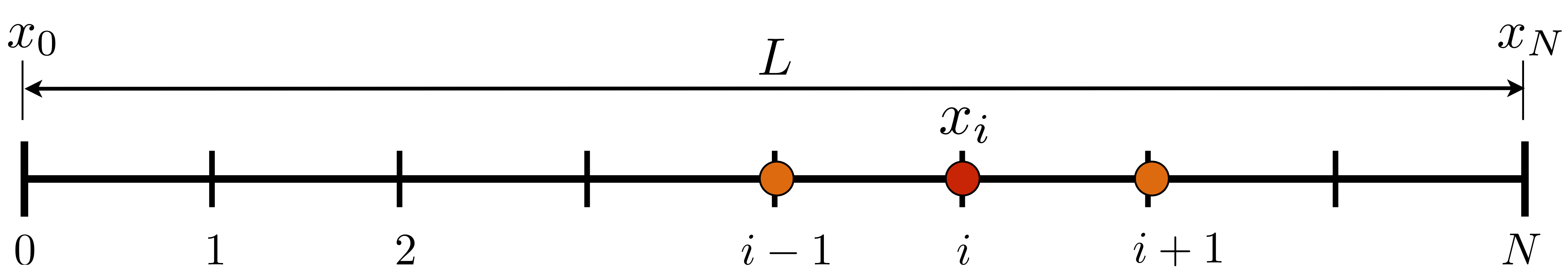
$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}\left(f_{i+1} - f_{i-1}\right) + O(h^4)$$

  - only 2 slight problems:
    - to get f'_i, we need f'_{i-1} and f'_{i+1} ⇒ coupled system ⇒ implicit

Stencil:



Solution domain:



- 4^{th}-order PADE

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}\left(f_{i+1} - f_{i-1}\right) + O(h^4)$$

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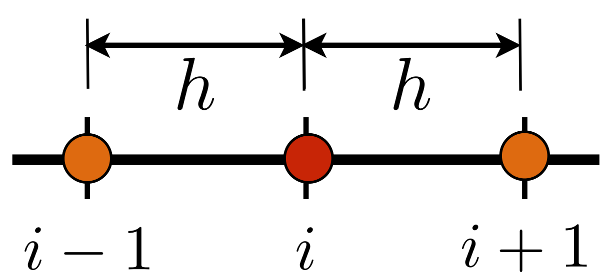
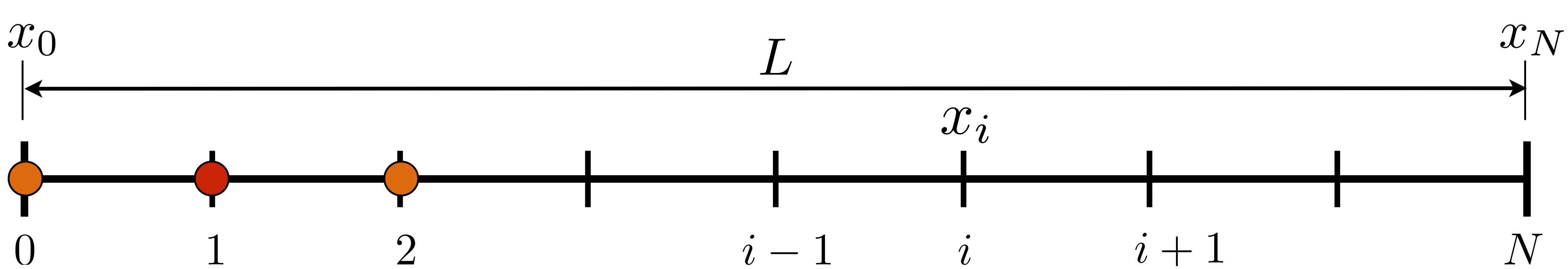


Solution domain:

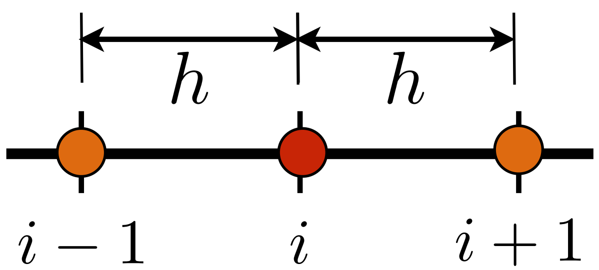


- 4^{th}-order PADE

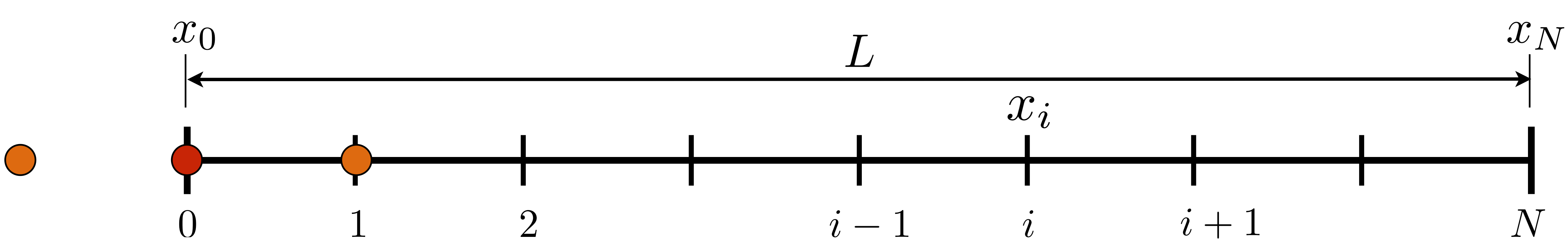
$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}(f_{i+1} - f_{i-1}) + O(h^4)$$

  - only 2 slight problems:
    - to get f'_i, we need f'_{i-1} and f'_{i+1} ⇒ coupled system ⇒ implicit
  - Solution: apply difference formula of lower order at boundaries using only "inner" points

Stencil:



Solution domain:

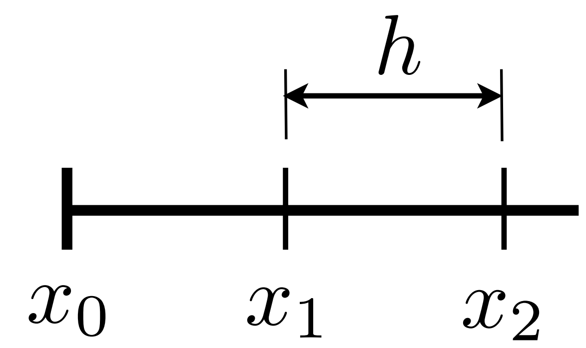


Need to modify the formula @ boundary

- Example 6: left boundary

$$f'_0 + a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f'_1 = O(?)$$

stencil:



- Following the standard procedure results in

$$f'_0 + 2f'_1 = \frac{1}{h}\left(-\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2\right) + O(h^3)$$

- similar for right boundary

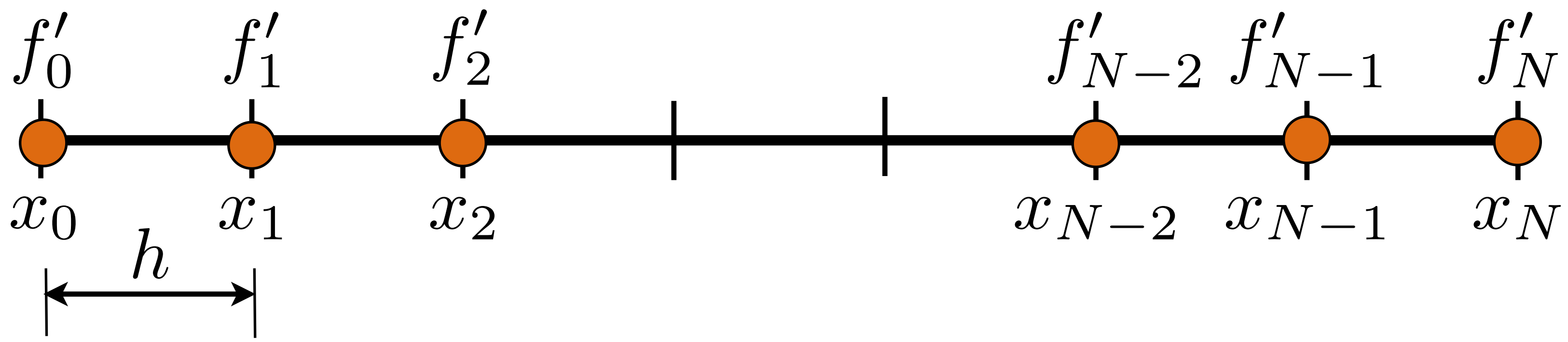
$$2f'_{N-1} + f'_N = \frac{1}{h}\left(\frac{5}{2}f_N - 2f_{N-1} - \frac{1}{2}f_{N-2}\right) + O(h^3)$$

- recap: in the interior we had

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}\left(f_{i+1} - f_{i-1}\right) + O(h^4)$$

- So what do we have now?

$$\begin{bmatrix} 1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 4 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 4 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} f_0' \\ f_1' \\ f_2' \\ \vdots \\ f_{N-2}' \\ f_{N-1}' \\ f_N' \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -\frac{5}{2} f_0 + 2 f_1 + \frac{1}{2} f_2 \\ 3\left(f_2 - f_0\right) \\ 3\left(f_3 - f_1\right) \\ \vdots \\ 3\left(f_{N-1} - f_{N-3}\right) \\ 3\left(f_N - f_{N-2}\right) \\ \frac{5}{2} f_N - 2 f_{N-1} - \frac{1}{2} f_{N-2} \end{bmatrix}$$



$$f_0' + 2 f_1' = \frac{1}{h}\left(-\frac{5}{2} f_0 + 2 f_1 + \frac{1}{2} f_2\right)$$

$$f_0' + 4 f_1' + f_2' = \frac{3}{h}\left(f_2 - f_0\right)$$

$$f_1' + 4 f_2' + f_3' = \frac{3}{h}\left(f_3 - f_1\right)$$

$$f_{N-3}' + 4 f_{N-2}' + f_{N-1}' = \frac{3}{h}\left(f_{N-1} - f_{N-3}\right)$$

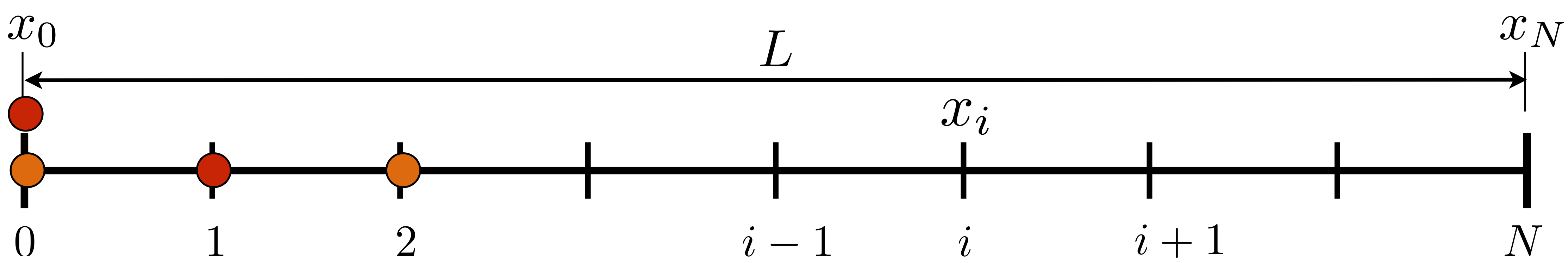
$$f_{N-2}' + 4 f_{N-1}' + f_N' = \frac{3}{h}\left(f_N - f_{N-2}\right)$$

$$2 f_{N-1}' + f_N' = \frac{1}{h}\left(\frac{5}{2} f_N - 2 f_{N-1} - \frac{1}{2} f_{N-2}\right)$$

- 4^{th}-order PADE

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}\left(f_{i+1} - f_{i-1}\right) + O(h^4)$$

  - Why should the formula at the boundary be of lower order?



$$\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \end{bmatrix} = \frac{1}{h} \begin{bmatrix} 3\left(f_2 - f_0\right) \\ \\ \\ \end{bmatrix}$$

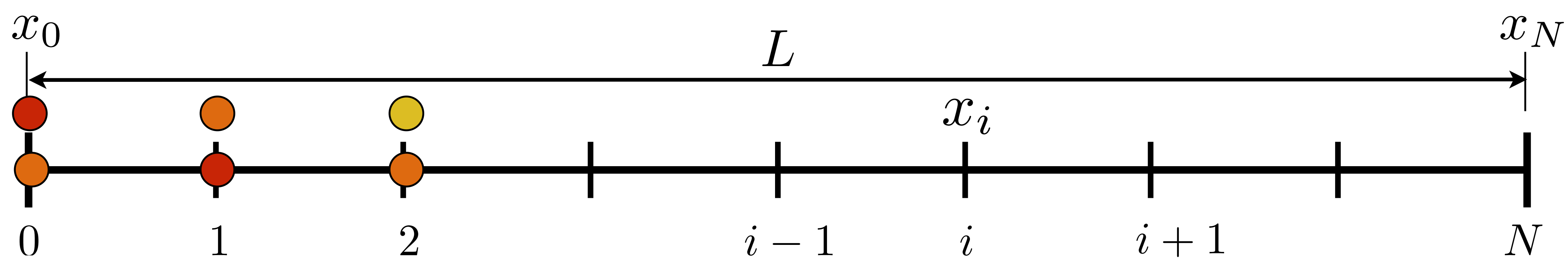
$$f'_0 + 4f'_1 + f'_2 = \frac{3}{h}\left(f_2 - f_0\right)$$

Requirement #1: Maintain structure of matrix (here: tri-diagonal)

- 4^{th}-order PADE

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}\left(f_{i+1} - f_{i-1}\right) + O(h^4)$$

  - Why should the formula at the boundary be of lower order?



$$\begin{bmatrix} \blacksquare & \blacksquare & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & & & & & & & \\ \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \end{bmatrix} = \frac{1}{h} \begin{bmatrix} \\ 3\left(f_2 - f_0\right) \\ \\ \\ \end{bmatrix}$$

$$f'_0 + a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f'_1 = O(?)$$

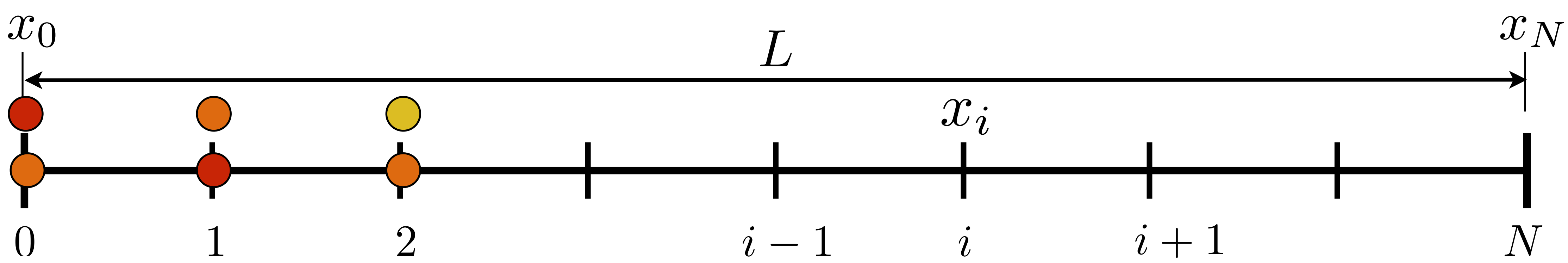
Requirement #1: Maintain structure of matrix (here: tri-diagonal)

Requirement #2: stencil should not be larger than neighbor stencil

- 4^{th}-order PADE

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}\left(f_{i+1} - f_{i-1}\right) + O(h^4)$$

  - Why should the formula at the boundary be of lower order?



$$\begin{bmatrix} \blacksquare & \blacksquare & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2 \\ 3\left(f_2 - f_0\right) \\ \\ \\ \end{bmatrix}$$

$$f'_0 + 2f'_1 = \frac{1}{h}\left(-\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2\right) + O(h^3)$$

Requirement #1: Maintain structure of matrix (here: tri-diagonal)

Requirement #2: stencil should not be larger than neighbor stencil

$$\begin{bmatrix} 1 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 4 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 4 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} f'_0 \\ f'_1 \\ f'_2 \\ \vdots \\ f'_{N-2} \\ f'_{N-1} \\ f'_N \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -\frac{5}{2} f_0 + 2 f_1 + \frac{1}{2} f_2 \\ 3\left(f_2 - f_0\right) \\ 3\left(f_3 - f_1\right) \\ \vdots \\ 3\left(f_{N-1} - f_{N-3}\right) \\ 3\left(f_N - f_{N-2}\right) \\ \frac{5}{2} f_N - 2 f_{N-1} - \frac{1}{2} f_{N-2} \end{bmatrix}$$

- So what do we have now? A tri-diagonal system!
- Was this inevitable? No, it depends on the choice of boundary scheme!
- Recipe:
  - Choose a boundary scheme such that
    a) matrix form is preserved (here tri-diagonal)
    b) boundary stencil ≤ 1st interior stencil
- Does the lower order @ boundary not pollute the higher order in the interior?
  - Depends on the case. In most cases, the additional error remains @ or near the boundary
- How do we solve the resulting tri-diagonal system?

How to solve a tri-diagonal system?

- PxP tri-diagonal matrix A:

$$A\vec{x} = \vec{d}$$

$\vec{x}$: P-dimensional vector of unknowns

$\vec{d}$: P-dimensional vector of given right-hand side

- Gaussian elimination (direct solve)

$$A\vec{x} = \vec{d}$$

1) NEVER store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ \\ \\ \\ \\ \end{bmatrix}$$

$$A\vec{x} = \vec{d}$$

1) **NEVER** store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \\ \\ \\ \end{bmatrix}$$

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$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \\ \\ \\ \\ \end{bmatrix}$$

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$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \\ \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ \\ \\ \end{bmatrix}$$

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$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ \\ \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ \\ \end{bmatrix}$$

$$A\vec{x} = \vec{d}$$

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$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \end{bmatrix}$$

$$A\vec{x} = \vec{d}$$

1) NEVER store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

$$A\vec{x} = \vec{d}$$

1) NEVER store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- store right hand side in vector d_i with length P

$A\vec{x} = \vec{d}$

1) <u>NEVER</u> store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- store right hand side in vector d_i with length P
- store main diagonal in vector b_i with length P

$A\vec{x} = \vec{d}$

1) NEVER store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- store right hand side in vector d_i with length P
- store main diagonal in vector b_i with length P
- store lower diagonal in vector a_i with length?

use length P as well!
ignore the first entry!

$A\vec{x} = \vec{d}$

1) NEVER store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix}
=
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}
$$

- store right hand side in vector d_i with length P
- store main diagonal in vector b_i with length P
- store lower diagonal in vector a_i with length P
- store upper diagonal in vector c_i with length?

use length P as well!
ignore the last entry!

$A\vec{x} = \vec{d}$

1) NEVER store the entire matrix !!!

- instead: store the 3 diagonals in separate P-dim vectors (1D arrays)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- store right hand side in vector d_i with length P
- store main diagonal in vector b_i with length P
- store lower diagonal in vector a_i with length P
- store upper diagonal in vector c_i with length P

2) Elimination

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- 1st pivot b_1: row 2 subtraction multiplier: a_2/b_1

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1} \qquad d_2 \to d_2 - d_1 \frac{a_2}{b_1} \qquad (c_2 \to c_2$$

2) Elimination

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- 1st pivot b_1: row 2 subtraction multiplier: a_2/b_1

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1} \qquad d_2 \to d_2 - d_1 \frac{a_2}{b_1} \qquad (c_2 \to c_2 \qquad a_2 \to 0)$$

- 2nd pivot b_2: row 3 subtraction multiplier: a_3/b_2

$$b_3 \to b_3 - c_2 \frac{a_3}{b_2} \qquad d_3 \to d_3 - d_2 \frac{a_3}{b_2} \qquad (c_3 \to c_3 \qquad a_3 \to 0)$$

2) Elimination

$$\begin{bmatrix} \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{i-1} & b_{i-1} & c_{i-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & a_i & b_i & c_i & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{bmatrix} \vec{f}' = \begin{bmatrix} \vdots \\ d_{i-1} \\ d_i \\ \vdots \end{bmatrix}$$

- Let's look only at the i-1-th and i-th row of the tri-diagonal matrix
  - When we do forward elimination and reach row i, a_{i-1} will be zero!

2) Elimination

$$\begin{bmatrix} \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & b_{i-1} & c_{i-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & b_i & c_i & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \end{bmatrix} \vec{f}' = \begin{bmatrix} \vdots \\ d_{i-1} \\ d_i \\ \vdots \end{bmatrix}$$

- Let’s look only at the i-1-th and i-th row of the tri-diagonal matrix
  - When we do forward elimination and reach row i, a_{i-1} will be zero!
  - Let’s do forward elimination, i.e. generate zeros below the pivot in row i-1 (i.e., b_{i-1})
    - subtract a_i / b_{i-1} * row i-1 from row i

$$a_i \to 0 \qquad b_i \to b_i - c_{i-1}\frac{a_i}{b_{i-1}} \qquad c_i \to c_i \qquad d_i \to d_i - d_{i-1}\frac{a_i}{b_{i-1}}$$

2) Elimination

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{P-2} & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{P-1} & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- 1st pivot b_1: row 2 subtraction multiplier: a_2/b_1

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1} \qquad d_2 \to d_2 - d_1 \frac{a_2}{b_1} \qquad (c_2 \to c_2 \qquad a_2 \to 0)$$

- 2nd pivot b_2: row 3 subtraction multiplier: a_3/b_2

$$b_3 \to b_3 - c_2 \frac{a_3}{b_2} \qquad d_3 \to d_3 - d_2 \frac{a_3}{b_2} \qquad (c_3 \to c_3 \qquad a_3 \to 0)$$

- (i-1)th pivot b_{i-1}: row i subtraction multiplier: a_i/b_{i-1}

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}} \qquad d_i \to d_i - d_{i-1} \frac{a_i}{b_{i-1}} \qquad (c_i \to c_i \qquad a_i \to 0)$$

2) Elimination

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{P-1} & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_P & b_P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- 1st pivot b_1: row 2 subtraction multiplier: a_2/b_1

$$b_2 \to b_2 - c_1 \frac{a_2}{b_1} \qquad d_2 \to d_2 - d_1 \frac{a_2}{b_1} \qquad (c_2 \to c_2 \qquad a_2 \to 0)$$

- 2nd pivot b_2: row 3 subtraction multiplier: a_3/b_2

$$b_3 \to b_3 - c_2 \frac{a_3}{b_2} \qquad d_3 \to d_3 - d_2 \frac{a_3}{b_2} \qquad (c_3 \to c_3 \qquad a_3 \to 0)$$

- (i-1)th pivot b_{i-1}: row i subtraction multiplier: a_i/b_{i-1}

$$b_i \to b_i - c_{i-1} \frac{a_i}{b_{i-1}} \qquad d_i \to d_i - d_{i-1} \frac{a_i}{b_{i-1}} \qquad (c_i \to c_i \qquad a_i \to 0)$$

- (P-1)th pivot b_{P-1}: row P subtraction multiplier: a_P/b_{P-1}

$$b_P \to b_P - c_{P-1} \frac{a_P}{b_{P-1}} \qquad d_P \to d_P - d_{P-1} \frac{a_P}{b_{P-1}} \qquad (\ a_P \to 0)$$

2) Elimination

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & b_{P-2} & c_{P-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{P-1} & c_{P-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_P
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix}
=
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}
$$

- $(i-1)^{th}$ pivot b_{i-1}: row i subtraction multiplier: a_i/b_{i-1}

$$b_i \to b_i - c_{i-1}\frac{a_i}{b_{i-1}} \qquad d_i \to d_i - d_{i-1}\frac{a_i}{b_{i-1}} \qquad (c_i \to c_i \qquad a_i \to 0)$$

  - Translate this to code

```
    b(i) = b(i) - c(i-1)*a(i)/b(i-1)
    d(i) = d(i) - d(i-1)*a(i)/b(i-1)
```

- Forward elimination needs to be done for rows 2 to P
  - put the above into a loop

```
loop from i = 2 to P   # loop over rows to eliminate
    b(i) = b(i) - c(i-1)*a(i)/b(i-1)
    d(i) = d(i) - d(i-1)*a(i)/b(i-1)
end loop i
```

3) Back-Substitution

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & b_{P-2} & c_{P-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{P-1} & c_{P-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_P
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix}
=
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}
$$

- last row:

$$d_P \to \frac{d_P}{b_P} \quad (b_P \to 1)$$

3) Back-Substitution

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & b_{P-2} & c_{P-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{P-1} & c_{P-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix}
=
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}
$$

- last row:

$$
d_P \to \frac{d_P}{b_P} \quad (b_P \to 1)
$$

3) Back-Substitution

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & c_{P-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- last row:

$$d_P \rightarrow \frac{d_P}{b_P} \qquad (b_P \rightarrow 1)$$

- row P-1: subtract c_{P-1} * row P then divide by b_{P-1}

$$d_{P-1} \rightarrow (d_{P-1} - c_{P-1} d_P)\,\frac{1}{b_{P-1}} \qquad (b_{P-1} \rightarrow 1 \qquad c_{P-1} \rightarrow 0)$$

3) Back-Substitution

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{P-2} & c_{P-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- last row:

$$d_P \to \frac{d_P}{b_P} \qquad (b_P \to 1)$$

- row P-1:

$$d_{P-1} \to (d_{P-1} - c_{P-1} d_P) \frac{1}{b_{P-1}} \qquad (b_{P-1} \to 1 \qquad c_{P-1} \to 0)$$

- ith row:

$$d_i \to (d_i - c_i d_{i+1}) \frac{1}{b_i} \qquad (b_i \to 1 \qquad c_i \to 0)$$

3) Back-Substitution

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- last row:

$$d_P \to \frac{d_P}{b_P} \quad (b_P \to 1)$$

- row P-1:

$$d_{P-1} \to (d_{P-1} - c_{P-1} d_P) \frac{1}{b_{P-1}} \qquad (b_{P-1} \to 1 \qquad c_{P-1} \to 0)$$

- ith row:

$$d_i \to (d_i - c_i d_{i+1}) \frac{1}{b_i} \qquad (b_i \to 1 \qquad c_i \to 0)$$

- 1st row:

$$d_1 \to (d_1 - c_1 d_2) \frac{1}{b_1} \qquad (b_1 \to 1 \qquad c_1 \to 0)$$

3) Back-Substitution

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{P-2} \\ x_{P-1} \\ x_P \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{P-2} \\ d_{P-1} \\ d_P \end{bmatrix}$$

- last row:

$$d_P \to \frac{d_P}{b_P} \quad (b_P \to 1)$$

- ith row:

$$d_i \to (d_i - c_i d_{i+1}) \frac{1}{b_i}$$

```
d(P) = d(P)/b(P)
loop from i = P-1 to 1 backwards    # loop over rows
   d(i) = (d(i) - c(i)*d(i+1))/b(i)
end loop i
```

How to solve a tri-diagonal system?

- Gaussian elimination (direct solve)

1) Store diagonals in vectors

2) Elimination

```
loop from i = 2 to P   # loop over rows to eliminate
      b(i) = b(i) - c(i-1)*a(i)/b(i-1)
      d(i) = d(i) - d(i-1)*a(i)/b(i-1)
end loop i
```

3) Back-Substitution

```
d(P) = d(P)/b(P)
loop from i = P-1 to 1 backwards   # loop over rows
      d(i) = (d(i) - c(i)*d(i+1))/b(i)
end loop i
```

Some notes for this algorithm:

- destroys (overwrites) a, b, and d
- solution is in vector d
- implement as function/subroutine
- alternatives can be found in 'Numerical Recipes' online.

Why tri-diagonal solver?

- total operation count for PxP matrix: ~ 3*P operations: O(P)
- for a full matrix, Gaussian elimination operator count: $O(P^3)$
- we will cover alternative methods starting next week