AEE 471 / MAE 561

Final Project – due May 6th, 3pm in ERC 311

(AEE471 Core Course Outcomes # 3, #4 & #5)

The topic of this final project is a mixing chamber problem. The chamber geometry is a simple 2D unit sized square box with no-slip, non-permeable walls on all sides. The left and right wall are at rest ($u = v = 0$), whereas the top wall moves in the $+x$-direction with speed $A(t)$ and the bottom wall moves in the $+x$-direction with speed $B(t)$. Figure 1 summarizes the geometry and the velocity boundary conditions.

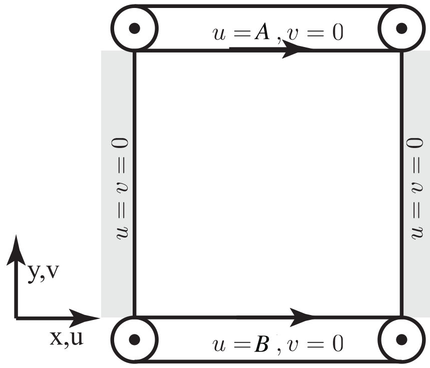

Figure 1: Lid driven cavity.

The flow is governed by the incompressible, 2D Navier Stokes equations,

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0 \\ \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{v}) &= -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{v} \end{aligned}$$

where Re is the Reynolds number.

Problem 1 (100 points)

Determine the steady state solution for $Re = 200$, $A = 1$, and $B = 1/2$ using the **fractional step method on a staggered grid in conservative form**. To calculate the steady state, use standard time advancement as for an unsteady problem until your solution does not change anymore. Within each time step, first solve the viscous Burgers equations on a staggered grid using any spatial and temporal scheme you like. Note that since we are interested in the steady state solution, high temporal accuracy is not that important. In the second step, calculate the Lagrange multiplier φ, by solving a Poisson equation as discussed in class. You must use an iterative method for this, for example Gauss-Seidel or Multigrid. Finally project the solution of step 1 into the subspace of solenoidal velocity fields and repeat the above steps until steady state is reached.

Assignment:

1. (AEE471 Core Course Outcome #3 & #4) Write in index notation, i.e. $u^{n+1}_{i+1/2,j} = \ldots$ etc. **all** the equations you need to solve in the fractional step method, i.e. in the predictor step (Burgers equation solve), Poisson equation solve, and projection step including the boundary conditions used for **each** boundary and **each** step. If you are using ghost cells, write down in index form how these ghost cell values are determined. Provide **all** stability restrictions in index form that your chosen method requires. You can use shorthands if you define them, like $u^n_{i,j} = (u^n_{i+1/2,j} + u^n_{i-1/2,j})/2$. Define the location in the mesh, i.e., cell center, cell face, or cell corner, where to calculate the vorticity using the most compact second order stencil using the face velocities and provide the index formula to calculate the vorticity at that location of the mesh.

2. (AEE471 Core Course Outcome #5) Determine the velocities u and v at the center of the cavity at x=0.5 and y=0.5 with 0.5% accuracy. Document the procedure used to justify your obtained result and accuracy incl. the meshes, numerical schemes, and the convergence criteria used.

3. (AEE471 Core Course Outcome #4) For steady state and a flow that satisfies task 2 of problem 1, plot

   - a contour (iso-line) plot of vorticity using sufficient number of iso-lines to show the features of the flow;
   - the u-velocity along a **vertical line** through the geometric center of the cavity;
   - the v-velocity along a **horizontal line** through the geometric center of the cavity;

4. (Required for MAE561, optional for extra credit for AEE471) (AEE471 Core Course Outcome #4) For steady state and a flow that satisfies task 2 of problem 1, plot a contour (iso-line) plot of the stream function Ψ using sufficient number of iso-lines to show the features of the flow. To calculate the stream function Ψ you **have to solve an additional Poisson equation** as discussed in class. Define the location of Ψ in the mesh, i.e., cell center, cell face, or cell corner, that gives the most compact stencil for the Poisson equation. Add to task 1 of problem 1 in index form all equations and boundary conditions used to determine Ψ.

5. Upload all your code to Blackboard using the SafeAssign mechanism. No credit will be given, if the code is not uploaded. **Furthermore, submit a printout of your code as an appendix of your report**.

Required submission for Problem 1 Task 1:
- name of scheme used;
- index formulas (using half index notation if necessary) for predictor step, incl. **all** *boundary conditions in index form;*
- index formulas (using half index notation if necessary) for Poisson eq., incl. all boundary conditions and residual in index form;
- index formulas (using half index notation if necessary) for projection step, incl. all boundary conditions in index form;
- index formulas for stable time step calculations and any other required stability constraints;
- index formulas (using half index notation if necessary) for vorticity calculation;
- Required for MAE561/bonus AEE471: index formulas (using half index notation if necessary) for stream function Poisson equation, incl. all boundary conditions in index form;

Required submission for Problem 1 Task 2:
- velocities u and v at the center of the cavity at x=0.5 and y=0.5 with 0.5% accuracy;
- solution verification incl. documentation of procedure; used mesh resolutions; CFL or time step size used; convergence criteria for Poisson equation and steady state condition; GCI analysis using at least 3 meshes;
Required submission for Problem 1 Task 3:
- clearly annotated plot of iso-lines of ω for solution satisfying Task 2 using sufficient number of iso-lines to show the features of the flow;
- clearly annotated plot of $u(y)$ for $x = 0.5$ for solution satisfying Task 2;
- clearly annotated plot of $v(x)$ for $y = 0.5$ for solution satisfying Task 2;

Required submission for Problem 1 Task 4 (Required for MAE561/bonus AEE471):
- clearly annotated plot of iso-lines of Ψ for solution satisfying Task 2 using sufficient number of iso-lines to show the features of the flow;

Required submission for Problem 1 Task 5:
- upload of all used code to SafeAssign and printout code in appendix;

Bonus Problem 2 (25 bonus points)

After the flow in the mixing chamber has reached steady state for the conditions defined in problem 1 task 2, a reagent is introduced into the flow at two spots, resulting in an initial condition for the reagents mass fraction Y as

$$Y(x,y,t=0) = \begin{cases} 1 & : \quad 3/8 \le x \le 5/8 \quad \wedge \quad 5/8 \le y \le 7/8 \\ 1 & : \quad 3/8 \le x \le 5/8 \quad \wedge \quad 1/8 \le y \le 3/8 \\ 0 & : \quad \text{otherwise} \end{cases} \tag{1}$$

The transport equation for the reagent mass fraction is

$$\frac{\partial Y}{\partial t} + \nabla \cdot (\boldsymbol{v}Y) = \frac{Sc}{Re}\nabla^2 Y\,. \tag{2}$$

where $Sc = 0.2$ is the Schmidt number and all walls of the chamber act as zero-gradient Neumann conditions for Y. The level of non-mixing in the chamber can be described by the intensity of segregation X, defined as

$$X(t) = \sqrt{\frac{1}{V}\int_V \left(\frac{Y(\boldsymbol{x},t) - \overline{Y}(t)}{\overline{Y}(t)}\right)^2 d\boldsymbol{x}}\,, \tag{3}$$

where V is the area of the mixing chamber and

$$\overline{Y}(t) = \frac{1}{V}\int_V Y(\boldsymbol{x},t)d\boldsymbol{x} \tag{4}$$

is the mean mass fraction of the reagent in the chamber.

Assignment:

1. (AEE471 Core Course Outcome # 2) Write in index notation, i.e. $Y_{i,j}^{n+1} = \ldots$ etc. **all** the equations you are solving for a method of your choosing, where Y has to be defined at the center of each cell. Include index formulas for all boundary conditions using ghost cells and provide **all** stability restrictions in index form that your chosen method requires. You may redefine the location of the velocity from the staggered mesh layout from problem one to any mesh layout you like, for example a cell centered layout. Use bilinear interpolation to obtain the velocities from the staggered mesh at your chosen locations.

2. (AEE471 Core Course Outcome # 2) Calculate the evolution of the reagent mass fraction Y for 6 time units starting with the given initial condition. Plot the intensity of segregation X as a function of time t and plot Y as a contour plot when the intensity of segregation X crosses a value of 2.5, 2.0, 1.5, and 1.0, and state the time when the crossing occurs. Use a mesh that is at least as fine as the one required to solve task 2 of problem 1.

3. (AEE471 Core Course Outcome #5): Determine the time T within 10% accuracy when the mixing chamber reaches an intensity of segregation $X(T) = 1.5$. To do the required solution verification procedure for the solution of the mass fraction equation, use the steady state velocity field from task 2 of problem 1 interpolated to a mesh to solve the Y-equation using bi-linear interpolation.

4. Upload all your code to Blackboard using the SafeAssign mechanism. No credit will be given, if the code is not uploaded. **Furthermore, submit a printout of your code as an appendix of your report**.

Required submission for Bonus Problem 2 Task 1:
- name of scheme used to solve Y-equation;

- *index formulas for finite difference method to solve Y-equation, incl.* **all** *boundary conditions in index form;*
- *index formulas for calculation of velocities from staggered mesh steady state solution of Problem 1;*
- *index formulas for stable time step calculations and any other required stability constraints;*

Required submission for Bonus Problem 2 Task 2:
- *clearly annotated plot of intensity of segregation $X(t)$ as a function of time for $0 \leq t \leq 6$;*
- *4 clearly annotated contour plots of Y at X(t) = 2.5, 2.0, 1.5, and 1.0 incl. their time t;*

Required submission for Bonus Problem 2 Task 3:
- *time T within 10% accuracy when mixing chamber reaches $X(T) = 1.5$;*
- *solution verification incl. documentation of procedure; used mesh resolutions; CFL or time step size used; GCI analysis using at least 3 meshes;*

Required submission for Bonus Problem 2 Task 4:
- *upload of all used code to SafeAssign and printout code in appendix;*

Bonus Problem 3 (10 bonus points)

Repeat Problem 1 Tasks 2-4 with a Reynolds number of $Re = 400$.

Required submission for Bonus Problem 3:
- *see Problem 1*

Bonus Problem 4 (10 bonus points)

Repeat Problem 2 Tasks 2 & 3 with a Reynolds number of $Re = 400$.

Required submission for Bonus Problem 4:
- *see Problem 2*