Muddiest Points from Class 04/05

- *“For the mass fraction, it is a function that ranges between 0 and 1 and at the inlets will be 0 or 1 depending on the substance entering, correct? At the outlet(s) will it have extrapolated conditions like the velocity components.”*
  - Correct. At outlets, you need to use extrapolation. However, linear extrapolation can lead to unphysical ghost cell values (<0 or >1), thus 0-th order extrapolation (= zero gradient Neumann) is preferable.

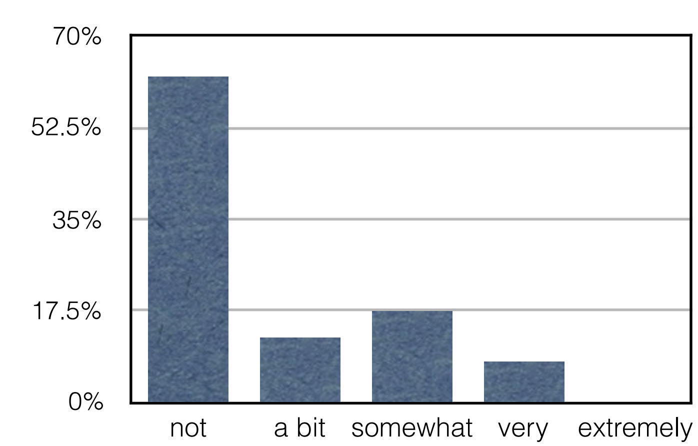
- *“Which order should we apply the methods in to solve each part of the Burger's equation (for 561)?”*
- *“Not sure where to start... how do we apply more than one method?”*
  - Adams Bashforth needs to be calculated first; the result becomes a known, constant in time (for the current time step at least) “source term” in the Crank-Nicholson/ADI solve

- *“I didn't understand the concept of separation of Burger's equation into two parts and solving it. Why are we doing so?”*
  - If your question refers to the hyperbolic and parabolic part: We use different methods in order to use the best 2nd-order methods for each term.

- *“In terms of the coding tips you suggested, how do you add subfunctions to a script in matlab? It seems the code does not allow for that because it is a linear progression unlike in a function script.”*
  - One can call functions from within functions from within functions from within ….

- *“In hw10, Y changes at any new time level, does this mean we are actually solving a new PDE at any new time level?”*
- *“Does the mass fraction need to be solved for on every time step? Or just solved when I want the data visualized?”*
  - No, the PDE stays the same, only Y, u, and v change (the latter are solutions to the viscous Burgers equations)
  - The PDE needs to be solved every time step

- *“No clue as of yet. Too many questions in mind. Especially how to make the movies of the flow…"*
  - Matlab: draw your figure as you normally would, use `getframe` to store the figure and then use `writeVideoFile` to write the movie to disk.



Navier-Stokes

incompressible limit

Recap:

$$\nabla \cdot \vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v}\vec{v}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

in 2D:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

non-conservative:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

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non-dimensionalize:

$$\overline{x} = \frac{x}{L}, \quad \overline{y} = \frac{y}{L}, \quad \overline{u} = \frac{u}{u_{ref}}, \quad \overline{v} = \frac{v}{u_{ref}}, \quad \overline{t} = \frac{t}{L/u_{ref}}, \quad \overline{p} = \frac{p}{\rho u_{ref}^2}$$

drop bars:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = \boxed{-\frac{\partial p}{\partial x}} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = \boxed{-\frac{\partial p}{\partial y}} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\mathrm{Re} = \frac{u_{ref}L}{\nu}$$

so what's new? pressure term & continuity

However: continuity and momentum equations appear decoupled! (?)

for compressible flows, coupling is via density and equation of state!

Vorticity-Streamfunction Formulation

Idea: directly couple/enforce continuity and eliminate pressure

vorticity: $\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{v}$ $\vec{\omega}$: angular velocity

in 2D: $$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

streamfunction ψ: $$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Goal: derive a PDE for vorticity from momentum equations, eliminating pressure

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vorticity-Streamfunction Formulation

Goal: derive a PDE for vorticity from momentum equations, eliminating pressure

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \Big|\frac{\partial}{\partial y}$$

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$$\frac{\partial^2 u}{\partial t \partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial y^3}\right)$$

$$\frac{\partial^2 v}{\partial t \partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial y^2 \partial x}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) =$$

$$\frac{1}{\mathrm{Re}}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right] \qquad |\cdot(-1)$$

$$\boxed{\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)}$$

Vorticity-Streamfunction Formulation

Idea: directly couple/enforce continuity and eliminate pressure

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Goal: derive a PDE for vorticity from momentum equations, eliminating pressure

$$\boxed{\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)}$$

→ vorticity equation: 1 equation, no pressure! But how do we get the velocities?

→ streamfunction! But how do we get the streamfunction?

Vorticity-Streamfunction Formulation

But how do we get the streamfunction?

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

let's just use the definitions and substitute in!

$$\Omega = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \qquad \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega}$$

But what about the continuity equation?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

let's again use the definitions and substitute in!

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

$\Rightarrow$ continuity is automatically satisfied!

Vorticity-Streamfunction Formulation

non-conservative form:

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega$$

conservative form:

$$\frac{\partial \Omega}{\partial t} + \frac{\partial u\Omega}{\partial x} + \frac{\partial v\Omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega$$

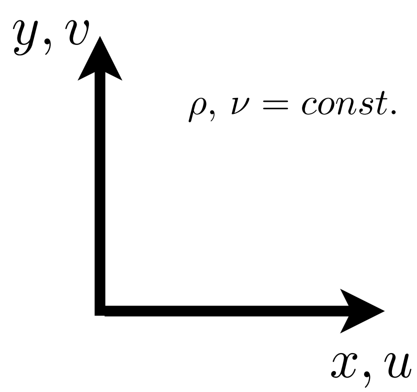
Vorticity-Streamfunction Formulation

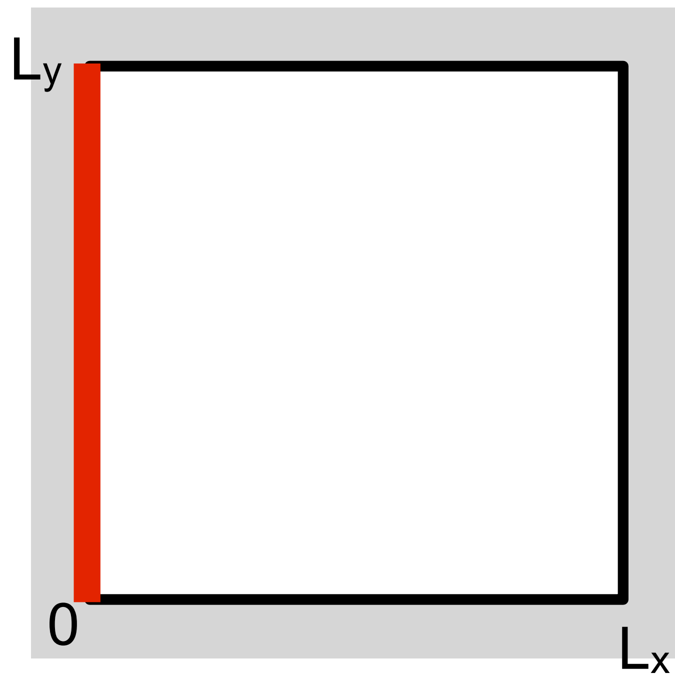
$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

One challenge remains: boundary conditions are usually given in terms of u & v

$\Rightarrow$ need to express as Ω and ψ

Example: Cavity





- all boundaries are impermeable walls
- normal velocity = 0

$\Rightarrow$ walls are streamlines! $\Rightarrow$ $\boldsymbol{\psi = const}$

for example: left boundary: $x = 0 \quad \Rightarrow u = 0$ if wall is stationary: $v = 0$

since $u = \dfrac{\partial \psi}{\partial y} \quad \Rightarrow \left.\dfrac{\partial \psi}{\partial y}\right|_{x=0} = 0 \quad \Rightarrow \psi|_{x=0} = const$

But what about Ω?

Vorticity-Streamfunction Formulation

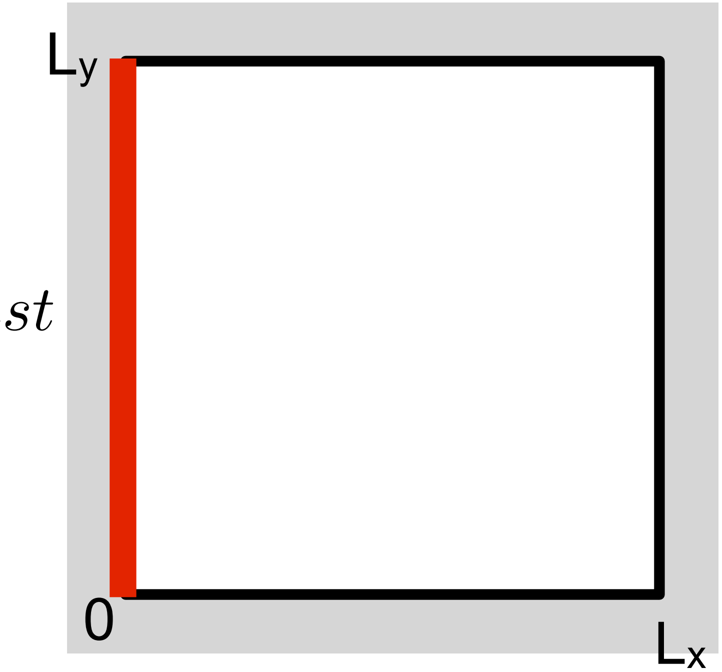
$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

But what about Ω?

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega$$

$$\psi|_{x=0} = const$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{x=0} = 0$$



using node based mesh

Taylor Series for 1st interior point:

$$\psi_{1,j} = \psi_{0,j} + \Delta x \left.\frac{\partial \psi}{\partial x}\right|_{0,j} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{0,j} + \ldots$$

if wall is stationary: $v = 0$

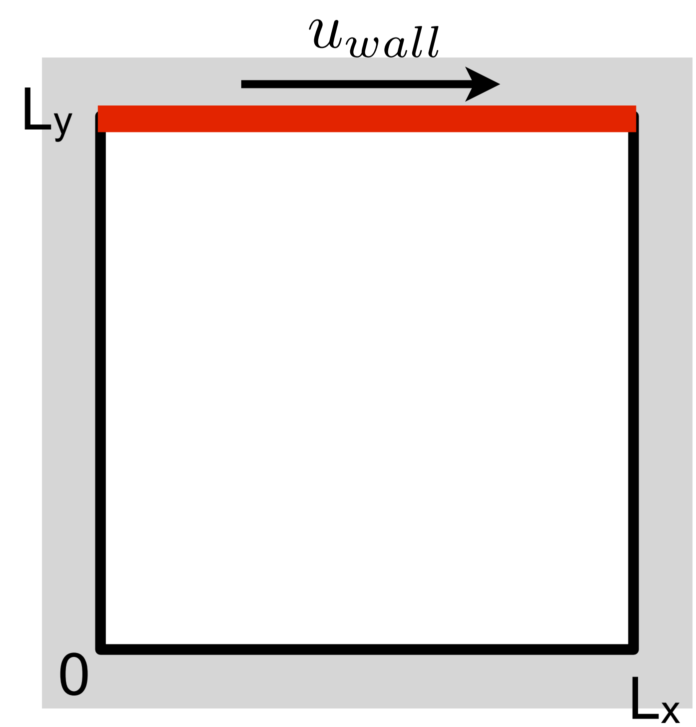
$$\left.\frac{\partial \psi}{\partial x}\right|_{0,j} = -v_{0,j} = 0 \qquad \Rightarrow \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{0,j} = \frac{\psi_{1,j} - \psi_{0,j}}{\frac{\Delta x^2}{2}}$$

$$\Rightarrow \qquad \Omega_{0,j} = \frac{2(\psi_{0,j} - \psi_{1,j})}{\Delta x^2}$$

similar procedure for all other boundaries

Vorticity-Streamfunction Formulation

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$



using node based mesh

What if wall moves tangentially? for example: upper wall

$$u(x, y = L_y) = u_{wall}$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{i,N} = u_{wall} \quad \text{and} \quad \left.\frac{\partial \psi}{\partial x}\right|_{i,N} = 0 \quad \text{(impermeable)}$$

Taylor Series for 1st interior point:

$$\psi_{i,N-1} = \psi_{i,N} - \Delta y \left.\frac{\partial \psi}{\partial y}\right|_{i,N} + \frac{\Delta y^2}{2} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,N} + \dots$$

$$\Rightarrow \quad \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,N} = \frac{2(\psi_{i,N-1} - \psi_{i,N})}{\Delta y^2} + \frac{2}{\Delta y} u_{wall}$$

with

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\Omega \qquad \Rightarrow \qquad \Omega_{i,N} = \frac{2}{\Delta y^2}\left(\psi_{i,N} - \psi_{i,N-1}\right) - \frac{2}{\Delta y} u_{wall}$$

caveat: this is only 1st-order in space! could go 2nd-order:

$$\Omega_{i,N} = \frac{-\psi_{i,N} + 8\psi_{i,N-1} - 7\psi_{i,N-2}}{2\Delta y^2} - \frac{3}{\Delta y} u_{wall}$$

Vorticity-Streamfunction Formulation

Solution Procedure

- initialize Ω
- solve for ψ
- solve boundary conditions for Ω
- advance Ω

this works ok in 2D. But what about 3D?

- need 3D vector potential instead of streamfunction
- need to solve 3 elliptic equations

$\Rightarrow$ very cumbersome and expensive!

$\Rightarrow$ vorticity/streamfunction approach usually used just in 2D