Back to Navier-Stokes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$\leftarrow$ solenoidal velocity field

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$\leftarrow$ need equation for p

in vector form:

$$\frac{\partial \vec{v}}{\partial t} = \underbrace{-\nabla\cdot(\vec{v}\vec{v})}_{N(\vec{v})} - \nabla p + \frac{1}{\mathrm{Re}}\underbrace{\nabla^2\vec{v}}_{L(\vec{v})}$$

$$\frac{\partial \vec{v}}{\partial t} = N(\vec{v}) - \nabla p + \frac{1}{\mathrm{Re}}L(\vec{v})$$

| take $\nabla\cdot$

$$\frac{\partial}{\partial t}(\nabla\cdot\vec{v}) = \nabla\cdot N(\vec{v}) - \nabla^2 p + \frac{1}{\mathrm{Re}}\nabla\cdot L(\vec{v})$$

use continuity: $\nabla\cdot\vec{v} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}(\nabla\cdot\vec{v}) = 0$

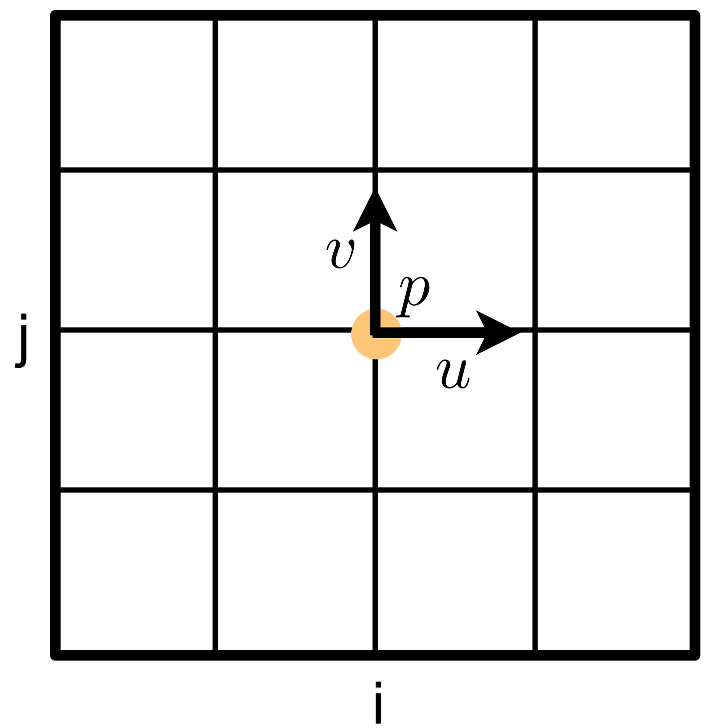
$$\boxed{\nabla^2 p = \nabla\cdot N(\vec{v}) + \frac{1}{\mathrm{Re}}\nabla\cdot L(\vec{v})}$$

pressure Poisson equation!

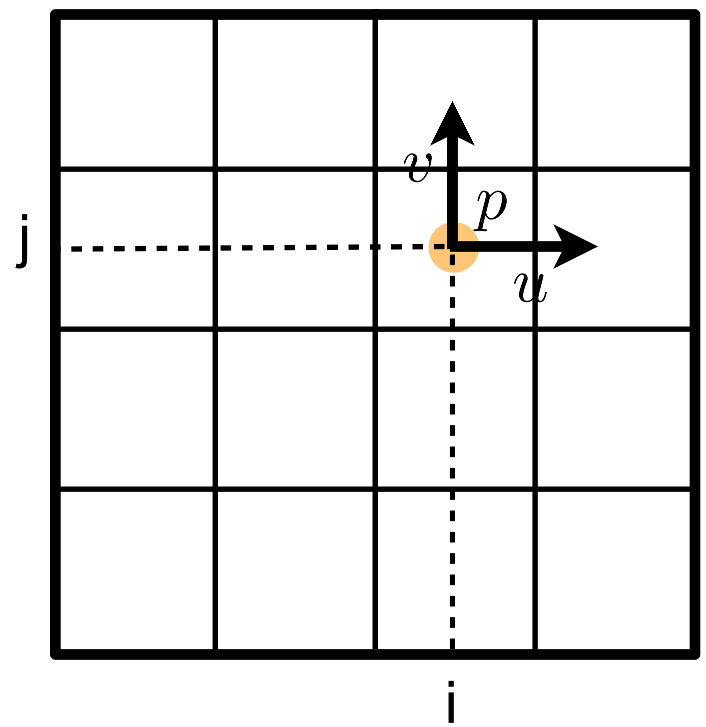
(in principle $\nabla\cdot L(\vec{v}) = 0$, since L is a linear operator, and $\nabla\cdot$ and L commute)

Next: Let's revisit meshing once again

- until now, we have used either one of the following meshes
  - all variables are located at the same location: @ grid intersection lines

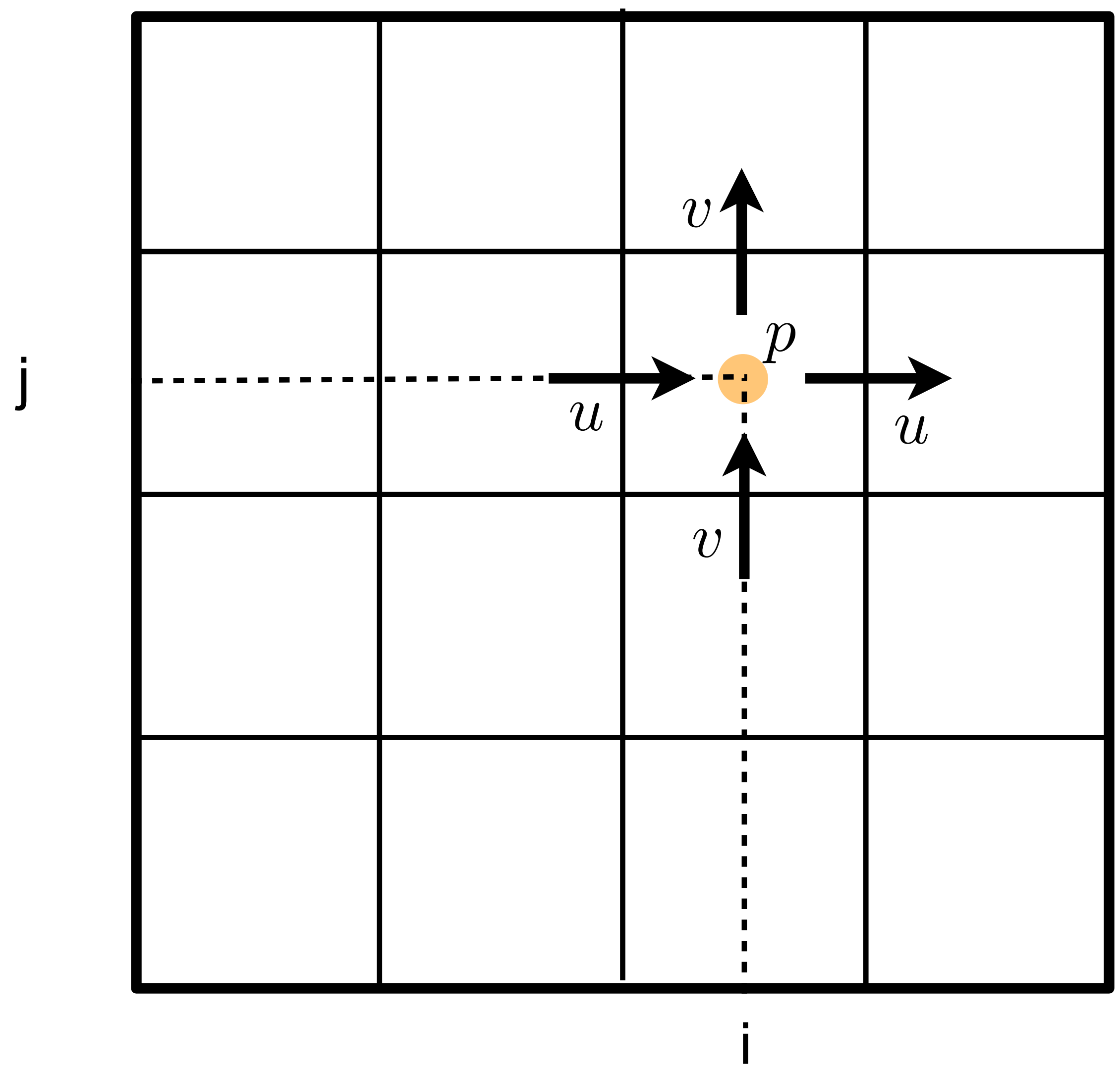


  - all variables are located at the same location: @ cell centers



collocated grids

But: variables need not be located all at the same location! **staggered grids**

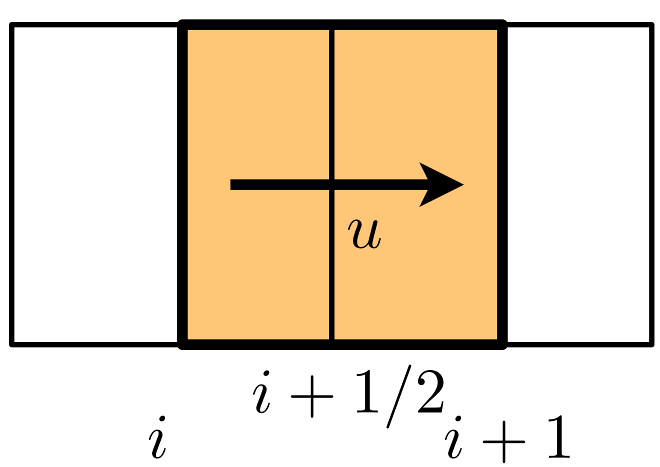


- define i,j to be @ cell centers
  - define p @ cell centers: $p_{i,j}$
  - define u @ x-normal faces: $u_{i\pm1/2,j}$
  - define v @ y-normal faces: $v_{i,j\pm1/2}$
- Why?
  - helps us enforce conservation laws!
  - cell = control volume

change = ∑ fluxes across boundaries + sources

  - but fluxes across boundaries involve only normal velocity components!
  - staggered grids define exactly these!
  - However, the control volume for these velocities are shifted!
  - need to evaluate FDE for u not at (i,j), but at (i+1/2,j)!
  - need all terms at i+1/2,j

cv for $u_{i+1/2}$



Navier-Stokes on Staggered Meshes

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

solve for $u_{i+1/2,j}$:

$$\left.\frac{\partial u}{\partial t}\right|_{i+\frac{1}{2},j} = \frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n}_{i+\frac{1}{2},j}}{\Delta t} + O(\Delta t)$$

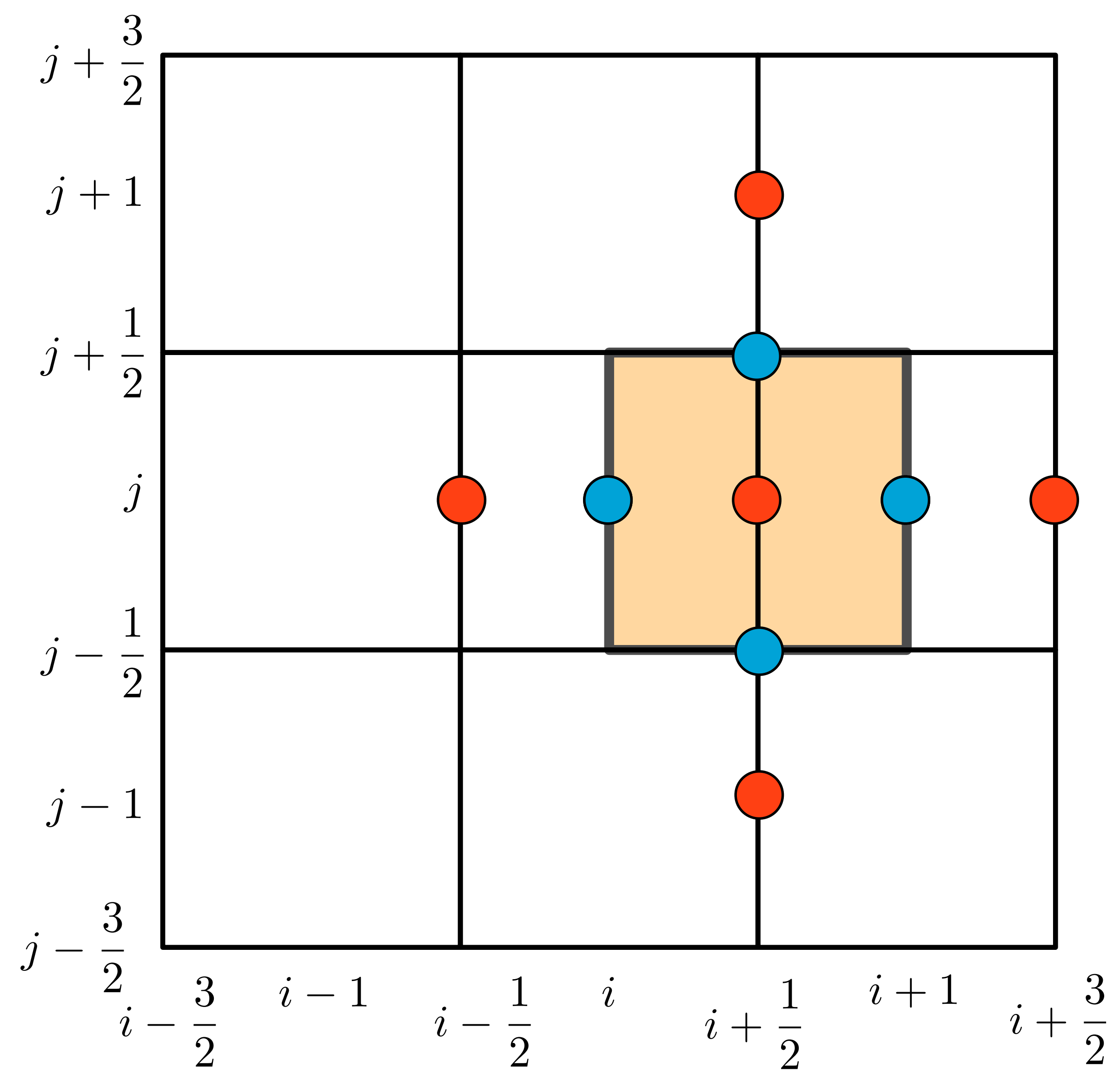
$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} + O(\Delta y^2)$$

$$\left.\frac{\partial p}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x} + O(\Delta x^2)$$



Navier-Stokes on Staggered Meshes

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

solve for $u_{i+1/2,j}$:

$$\left.\frac{\partial u}{\partial t}\right|_{i+\frac{1}{2},j} = \frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n}_{i+\frac{1}{2},j}}{\Delta t} + O(\Delta t)$$

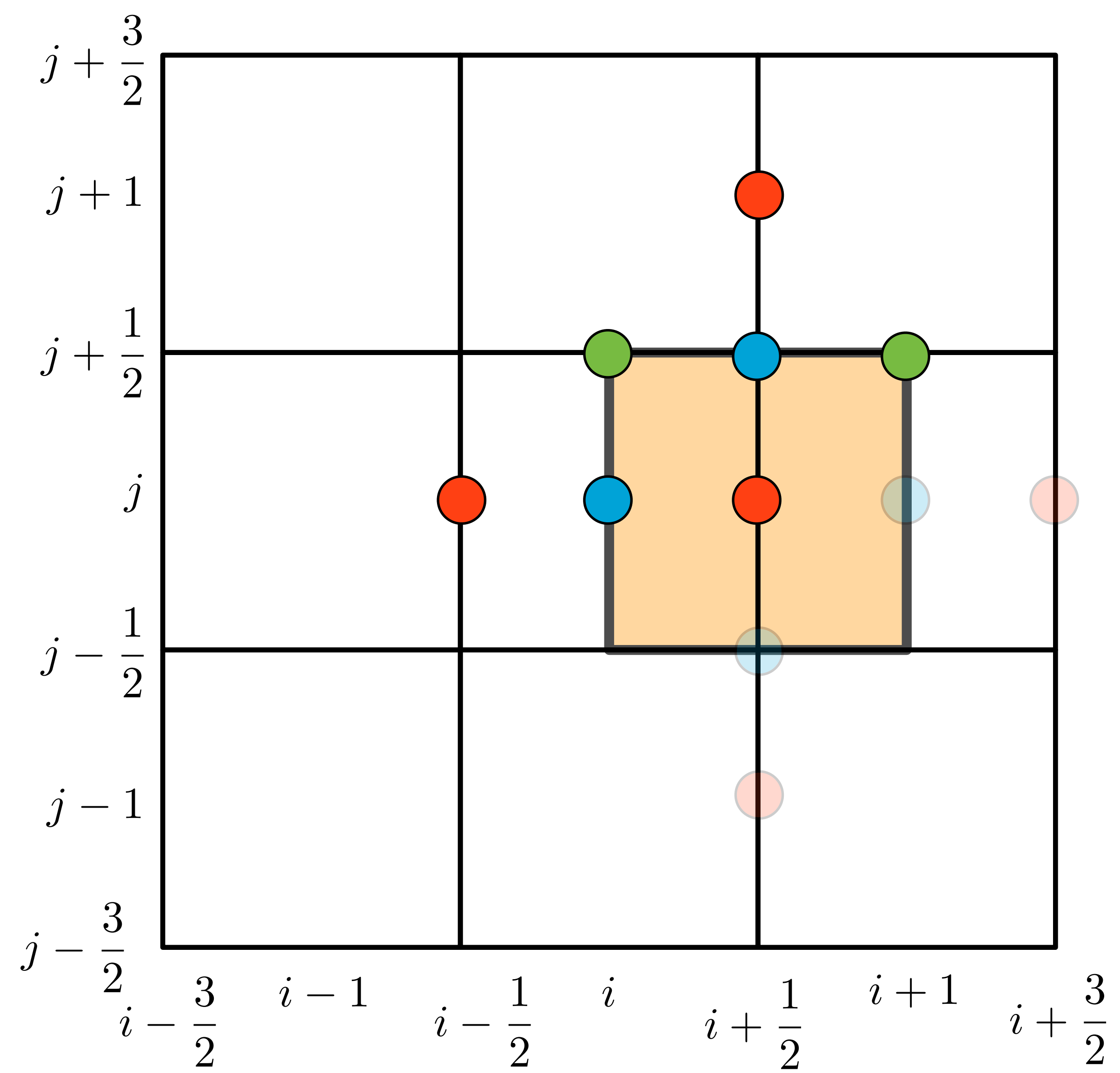
$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} + O(\Delta y^2)$$

$$\left.\frac{\partial p}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x} + O(\Delta x^2)$$



but now need the following:

$$u_{i,j} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)$$

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1}\right)$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}\right)$$

Navier-Stokes on Staggered Meshes

some comments on coding:

- vector/array indices must be whole numbers, i.e. integers ⇒ cannot use i+1/2
- you must decide what `u(i,j)` refers to: $u_{i+\frac{1}{2},j}$ or $u_{i-\frac{1}{2},j}$
- similar considerations apply to `v(i,j)`
- my suggestion:

`u(i,j)` = $u_{i+\frac{1}{2},j}$

`v(i,j)` = $v_{i,j+\frac{1}{2}}$

`p(i,j)` = $p_{i,j}$

let's substitute everything in: **MAC: Marker and Cell Method** (Harlow & Welch 1965)

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{i+\frac{1}{2},j}=\frac{u^{n+1}_{i+\frac{1}{2},j}-u^{n}_{i+\frac{1}{2},j}}{\Delta t}+O(\Delta t) \qquad \left.\frac{\partial p}{\partial x}\right|_{i+\frac{1}{2},j}=\frac{p_{i+1,j}-p_{i,j}}{\Delta x}+O(\Delta x^2)$$

$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j}=\frac{(u_{i+1,j})^2-(u_{i,j})^2}{\Delta x}+O(\Delta x^2) \qquad \left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j}=\frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}+O(\Delta y^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j}=\frac{u_{i+\frac{3}{2},j}-2u_{i+\frac{1}{2},j}+u_{i-\frac{1}{2},j}}{\Delta x^2}+O(\Delta x^2) \qquad \left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j}=\frac{u_{i+\frac{1}{2},j+1}-2u_{i+\frac{1}{2},j}+u_{i+\frac{1}{2},j-1}}{\Delta y^2}+O(\Delta y^2)$$

$$\frac{u^{n+1}_{i+\frac{1}{2},j}-u^{n}_{i+\frac{1}{2},j}}{\Delta t}+\frac{(u^n_{i+1,j})^2-(u^n_{i,j})^2}{\Delta x}+\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}=-\frac{p^n_{i+1,j}-p^n_{i,j}}{\Delta x}+$$
$$+\frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j}-2u^n_{i+\frac{1}{2},j}+u^n_{i-\frac{1}{2},j}}{\Delta x^2}+\frac{u^n_{i+\frac{1}{2},j+1}-2u^n_{i+\frac{1}{2},j}+u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\frac{v^{n+1}_{i,j+\frac{1}{2}}-v^{n}_{i,j+\frac{1}{2}}}{\Delta t}+\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}+\frac{(v^n_{i,j+1})^2-(v^n_{i,j})^2}{\Delta y}=-\frac{p^n_{i,j+1}-p^n_{i,j}}{\Delta y}+$$
$$+\frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}}-2v^n_{i,j+\frac{1}{2}}+v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2}+\frac{v^n_{i,j+\frac{3}{2}}-2v^n_{i,j+\frac{1}{2}}+v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

MAC: Marker and Cell Method

(Harlow & Welch 1965)

$$\frac{u^{n+1}_{i+\frac{1}{2},j}-u^n_{i+\frac{1}{2},j}}{\Delta t}+\frac{(u^n_{i+1,j})^2-(u^n_{i,j})^2}{\Delta x}+\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}=-\frac{p^n_{i+1,j}-p^n_{i,j}}{\Delta x}+$$

$$+\frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j}-2u^n_{i+\frac{1}{2},j}+u^n_{i-\frac{1}{2},j}}{\Delta x^2}+\frac{u^n_{i+\frac{1}{2},j+1}-2u^n_{i+\frac{1}{2},j}+u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\frac{v^{n+1}_{i,j+\frac{1}{2}}-v^n_{i,j+\frac{1}{2}}}{\Delta t}+\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}+\frac{(v^n_{i,j+1})^2-(v^n_{i,j})^2}{\Delta y}=-\frac{p^n_{i,j+1}-p^n_{i,j}}{\Delta y}+$$

$$+\frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}}-2v^n_{i,j+\frac{1}{2}}+v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2}+\frac{v^n_{i,j+\frac{3}{2}}-2v^n_{i,j+\frac{1}{2}}+v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

$$u^{n+1}_{i+\frac{1}{2},j}=u^n_{i+\frac{1}{2},j}+\Delta t\left(\Delta u^n_{i+\frac{1}{2},j}-\frac{p^n_{i+1,j}-p^n_{i,j}}{\Delta x}\right)\qquad v^{n+1}_{i,j+\frac{1}{2}}=v^n_{i,j+\frac{1}{2}}+\Delta t\left(\Delta v^n_{i,j+\frac{1}{2}}-\frac{p^n_{i,j+1}-p^n_{i,j}}{\Delta y}\right)$$

$$\Delta u^n_{i+\frac{1}{2},j}=-\frac{(u^n_{i+1,j})^2-(u^n_{i,j})^2}{\Delta x}-\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)^n_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}+\frac{1}{\mathrm{Re}}\left(\frac{u^n_{i+\frac{3}{2},j}-2u^n_{i+\frac{1}{2},j}+u^n_{i-\frac{1}{2},j}}{\Delta x^2}+\frac{u^n_{i+\frac{1}{2},j+1}-2u^n_{i+\frac{1}{2},j}+u^n_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)$$

$$\Delta v^n_{i,j+\frac{1}{2}}=-\frac{(uv)^n_{i+\frac{1}{2},j+\frac{1}{2}}-(uv)^n_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}-\frac{(v^n_{i,j+1})^2-(v^n_{i,j})^2}{\Delta y}+\frac{1}{\mathrm{Re}}\left(\frac{v^n_{i+1,j+\frac{1}{2}}-2v^n_{i,j+\frac{1}{2}}+v^n_{i-1,j+\frac{1}{2}}}{\Delta x^2}+\frac{v^n_{i,j+\frac{3}{2}}-2v^n_{i,j+\frac{1}{2}}+v^n_{i,j-\frac{1}{2}}}{\Delta y^2}\right)$$

MAC: Marker and Cell Method

(Harlow & Welch 1965)

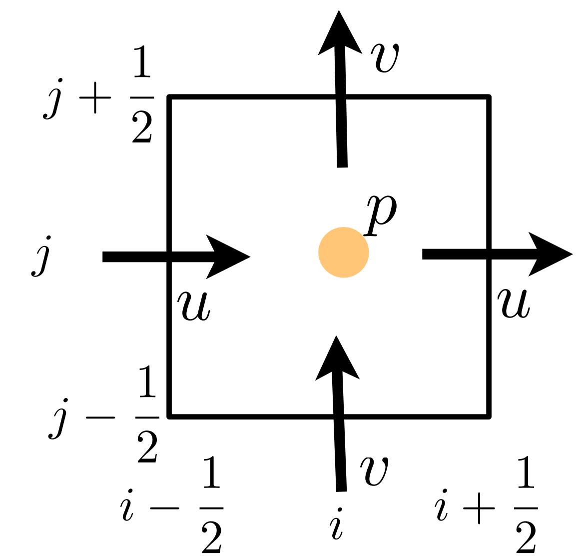
but what about pressure? we need it to solve the equations!

- let's enforce the continuity equation at t^{n+1}

$$(\nabla \cdot \vec{v})^{n+1} = 0$$

- but at what spatial location? @ location of pressure!

$$(\nabla \cdot \vec{v})^{n+1}_{i,j} = 0$$



Board

the result is a Poisson equation!

Solution Procedure

- initialize velocity fields
- solve Poisson equation in interior $\Rightarrow p^n$
- solve momentum equations in interior $\Rightarrow u^{n+1}, v^{n+1}$

But what about boundary conditions?

$$\Rightarrow \frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{\Delta x} + \frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{\Delta y} = 0$$

Shorthand:

$$D_{ij} = (\nabla\cdot\vec{v})_{i,j}$$

Now use (i) and (ii):

$$\frac{u^n_{i+1/2,j} + \Delta t\left(\Delta u^n_{i+1/2,j} - \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x}\right) - u^n_{i-1/2,j} - \Delta t\left(\Delta u^n_{i-1/2,j} - \frac{p^n_{i,j} - p^n_{i-1,j}}{\Delta x}\right)}{\Delta x}$$

$$+ \frac{v^n_{i,j+1/2} + \Delta t\left(\Delta v^n_{i,j+1/2} - \frac{p^n_{i,j+1} - p^n_{i,j}}{\Delta y}\right) - v^n_{i,j-1/2} - \Delta t\left(\Delta v^n_{i,j-1/2} - \frac{p^n_{i,j} - p^n_{i,j-1}}{\Delta y}\right)}{\Delta y} = 0$$

$$\Rightarrow 0 = D^n_{ij} + \Delta t\left(\frac{\Delta u^n_{i+1/2,j} - \Delta u^n_{i-1/2,j}}{\Delta x} + \frac{\Delta v^n_{i,j+1/2} - \Delta v^n_{i,j-1/2}}{\Delta y}\right) - \Delta t\left(\frac{p^n_{i+1,j} - 2p^n_{ij} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{ij} + p^n_{i,j-1}}{\Delta y^2}\right)$$

$$\Rightarrow \frac{\delta_x^2 p^n_{i,j}}{\Delta x^2} + \frac{\delta_y^2 p^n_{i,j}}{\Delta y^2} = \frac{1}{\Delta t} D^n_{i,j} + \frac{1}{\Delta x}\left(-\frac{(u^n_{i+1,j})^2 - (u^n_{i,j})^2}{\Delta x} + \frac{(u^n_{i,j})^2 - (u^n_{i-1,j})^2}{\Delta x}\right)$$

$$+ \frac{1}{\Delta y}\left(-\frac{(v^n_{i,j+1})^2 - (v^n_{i,j})^2}{\Delta y} + \frac{(v^n_{i,j})^2 - (v^n_{i,j-1})^2}{\Delta y}\right)$$

$$- \frac{2}{\Delta x \Delta y}\left((uv)^n_{i+1/2,j+1/2} - (uv)^n_{i+1/2,j-1/2} - (uv)^n_{i-1/2,j+1/2} + (uv)^n_{i-1/2,j-1/2}\right)$$

$$+ \frac{1}{Re}\left(\frac{\delta_x^2}{\Delta x^2}\left(\frac{u^n_{i+1/2,j} - u^n_{i-1/2,j}}{\Delta x}\right) + \frac{\delta_y^2}{\Delta y^2}\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta y}\right) + \frac{\delta_x^2}{\Delta x^2}\left(\frac{v^n_{i,j+1/2} - v^n_{i,j-1/2}}{\Delta y}\right) + \frac{\delta_y^2}{\Delta y^2}\left(\frac{v^n_{i,j+1/2} - v^n_{i,j-1/2}}{\Delta y}\right.\right.$$

MAC: Marker and Cell Method (Harlow & Welch 1965)

need velocity boundary condition @ velocity locations!

Example: tangentially moving bottom wall

- velocity in y-direction:

$$v_{i,\frac{1}{2}} = 0$$

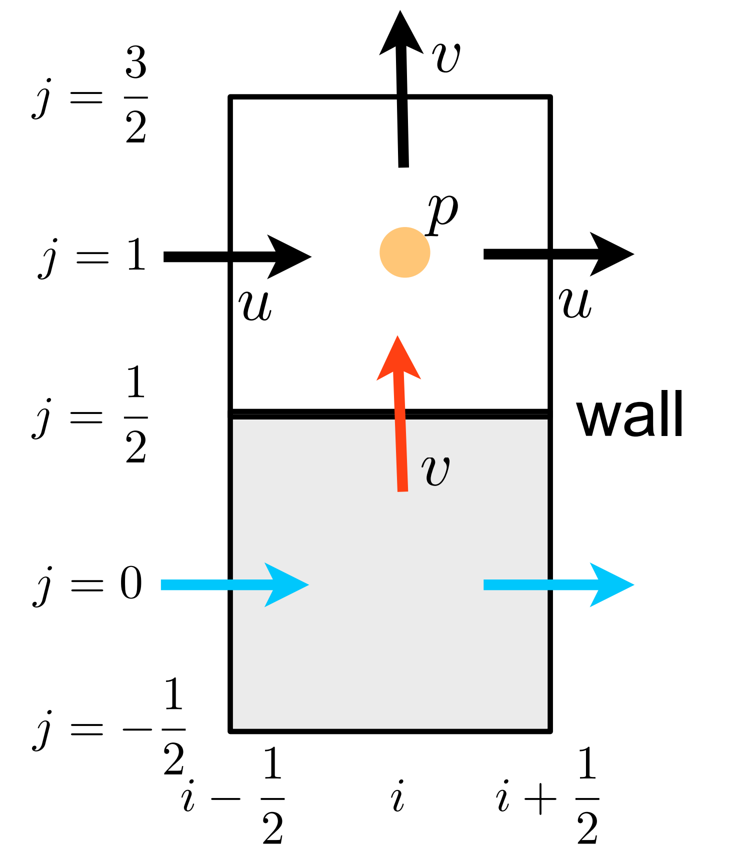
- what about x-direction velocity?
  - we don't have u velocities defined at the wall!
  - idea: let's define ghost cell velocities

$$u_{i+\frac{1}{2},0} = ?$$

  - use ghost cell velocity to determine wall velocity

$$u_{wall} = \frac{1}{2}\left(u_{i+\frac{1}{2},1} + u_{i+\frac{1}{2},0}\right)$$

$$\Rightarrow \quad u_{i+\frac{1}{2},0} = 2u_{wall} - u_{i+\frac{1}{2},1}$$



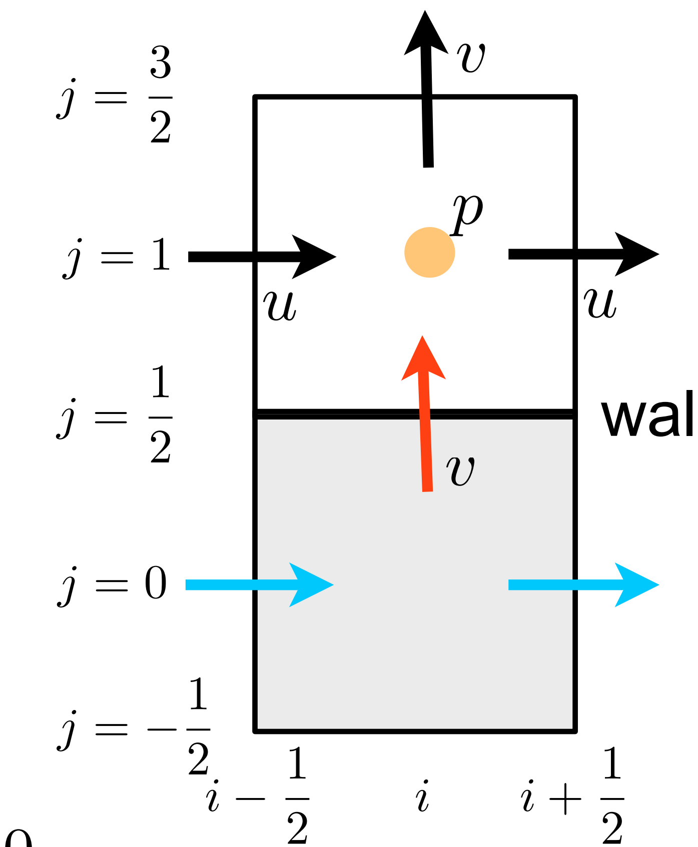
MAC: Marker and Cell Method (Harlow & Welch 1965)

boundary conditions for pressure Poisson equation?

Example: bottom wall

- we want $(\nabla \cdot \vec{v})_{i,1}^{n+1} = 0$

$$\frac{u_{i+\frac{1}{2},1}^{n+1} - u_{i-\frac{1}{2},1}^{n+1}}{\Delta x} + \frac{v_{i,\frac{3}{2}}^{n+1} - v_{i,\frac{1}{2}}^{n+1}}{\Delta y} = 0$$



- use boundary condition for velocity and momentum eqs.

$$\frac{u_{i+\frac{1}{2},1}^{n} - u_{i-\frac{1}{2},1}^{n}}{\Delta x} + \frac{v_{i,\frac{3}{2}}^{n} - v_{i,\frac{1}{2}}^{n+1}}{\Delta y} +$$

$$\Delta t \left[\left(\frac{\Delta u_{i+\frac{1}{2},1}^{n} - \Delta u_{i-\frac{1}{2},1}^{n}}{\Delta x} + \frac{\Delta v_{i,\frac{3}{2}}^{n}}{\Delta y} \right) - \frac{\delta_x^2 p_{i,j}^n}{\Delta x^2} - \frac{p_{i,2}^n - p_{i,1}^n}{\Delta y^2} \right] = 0$$

General approach:

1) write divergence constraint at t^{n+1} for all boundary adjacent interior cells in discrete form
2) for all velocities on the boundary, use boundary conditions
3) for all other velocities use momentum equations

⇒ never need to use ghost cell pressures!

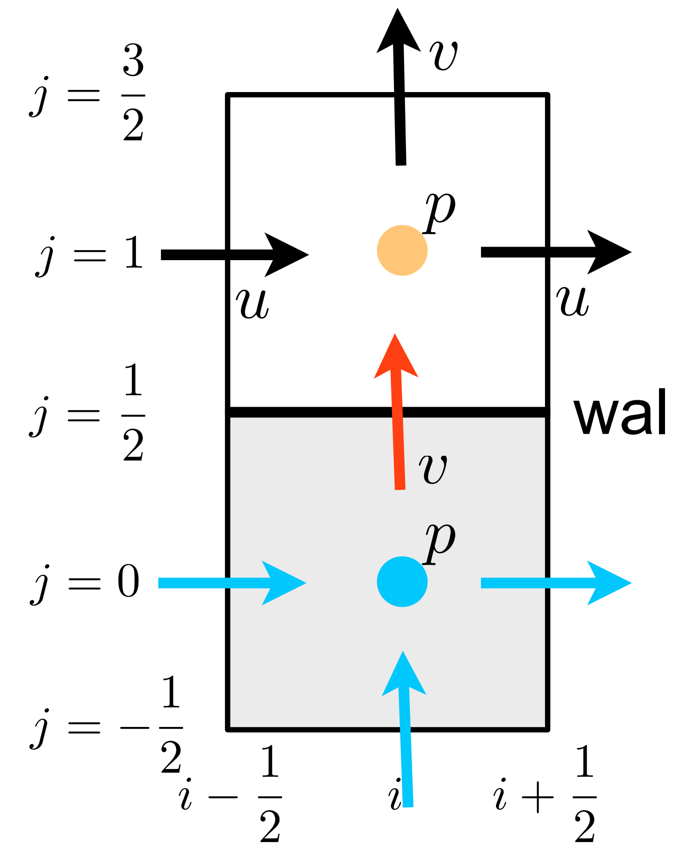
MAC: Marker and Cell Method

(Harlow & Welch 1965)

BUT: this requires a change in the code depending on the location of the cell to be updated

⇒ UGLY ⇒ let's try to avoid this by introducing ghost cell pressures

Idea: project momentum equations onto boundary normal direction ***n***



$$[\ .....\ ] \quad \nabla p \cdot \vec{n} = \frac{1}{\mathrm{Re}} \frac{\partial^2 \vec{v}}{\partial l_n^2} \cdot \vec{n}$$

l_n : normal arclength

Example: bottom wall

$$(\nabla p \cdot \vec{n})_{i,\frac{1}{2}} = \left.\frac{\partial p}{\partial y}\right|_{i,\frac{1}{2}} = \frac{p_{i,1} - p_{i,0}}{\Delta y}$$

$$\frac{1}{\mathrm{Re}} \left(\frac{\partial^2 \vec{v}}{\partial l_n^2} \cdot \vec{n} \right)_{i,\frac{1}{2}} = \frac{1}{\mathrm{Re}} \left.\frac{\partial^2 v}{\partial y^2}\right|_{i,\frac{1}{2}} = \frac{1}{\mathrm{Re}} \frac{v_{i,\frac{3}{2}} - 2v_{i,\frac{1}{2}} + v_{i,-\frac{1}{2}}}{\Delta y^2}$$

- we also know the velocity on the boundary and the divergence free constraint!

$v_{i,\frac{1}{2}} = 0$ and $(\nabla \cdot \vec{v})_{i,\frac{1}{2}} = 0$ $\qquad u \ @ \ wall = const.$

$$\left(\frac{\partial u}{\partial x}\right)_{i,\frac{1}{2}} + \left(\frac{\partial v}{\partial y}\right)_{i,\frac{1}{2}} = 0 \quad \Rightarrow \quad \frac{v_{i,\frac{3}{2}} - v_{i,-\frac{1}{2}}}{2\Delta y} = 0 \quad \Rightarrow \quad v_{i,-\frac{1}{2}} = v_{i,\frac{3}{2}}$$

$$\frac{p_{i,1} - p_{i,0}}{\Delta y} = \frac{1}{\mathrm{Re}} \left(\frac{2v_{i,\frac{3}{2}}}{\Delta y^2} \right) \quad \Rightarrow \quad p_{i,0} = p_{i,1} - \frac{2}{\mathrm{Re}} \frac{v_{i,\frac{3}{2}}}{\Delta y}$$

similar for other boundaries