AEE 471 / MAE 561

Bonus Homework - Due: April 29th, at the beginning of class

Please submit result graphs together with either handwritten or printed out descriptions, equations, and answers at the beginning of class. Combine all code you used to solve the problems into a single text file, and upload the text file to Blackboard using the SafeAssign mechanism. No credit will be given, if your code is not uploaded as a text file using SafeAssign. Also ensure that your code contains adequate comments. Add a printout of all code as an **appendix** to your submission. Please note that the Core Course Outcome bonus points do not count towards your course grade, but solely towards satisfying the AEE471 core course outcome requirements.

Problem 1 (10 bonus points; AEE 471: Core Course Outcome #1 100 bonus outcome points)
Find the most accurate formula for the second derivative of f at x_j utilizing stencil points at x_{j-1}, x_j, and x_{j+2} only. Assume an equidistant mesh. Give the full leading error term and state the order of the method. Document all solution steps.

Required submission: Taylor table, linear system for coefficients, coefficient solution, finite difference formula for f_j'', functional form of leading order error term, order of leading order error term, derivation steps.

Problem 2 (10 bonus points; AEE 471: Core Course Outcome #1 100 bonus outcome points)
Consider the function,

$$f(x) = \frac{\sin x \cos x}{x^3} .$$

Approximate the first derivative of $f(x = 4)$ using the first order accurate forward difference, second-order accurate central difference, and fourth-order accurate central difference ($f_i' = (f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2})/(12h) + O(h^4)$) formulas. Compare the numerical approximation to the exact analytical derivative using grid spacings of $h = 1$, 0.1, 0.01, 0.001, and 0.0001 by plotting the error (defined as the absolute value of the difference from the exact derivative) versus the grid spacing. Use log-log axes and confirm the order of accuracy of each of the formulas. Comment on your results and explain any potential abnormal behavior (there should be two).
Required submission: 1 graph containing all 3 requested curves of error vs. h in log-log scale clearly annotated; confirmation of the order of accuracy for each curve in the aforementioned graph (slopes); identification of deviation from the expected accuracy (2 instances); explanation of the deviation (2 instances); fully commented code uploaded to SafeAssign.

Problem 3 (20 bonus points; AEE 471: Core Course Outcome #2 140 bonus outcome points)
Consider two infinite parallel plates, a distance $H = 0.3\,cm$ apart with the fluid between them initially at rest, $u(y > 0cm, t = 0s) = 0\,m/s$ (y is the coordinate normal to the plates, u is the velocity tangential to the plates). The upper plate is stationary and the lower plate oscillates tangentially according to

$$u(y = 0m, t) = u_0 \cos(1000t) . \tag{1}$$

The governing equation can be derived from the Navier-Stokes equation as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} . \tag{2}$$

Assume the kinematic viscosity is constant, $\nu = 2.17 \cdot 10^{-4}\ m^2/s$, and $u_0 = 40\ m/s$. Discretize the space between the plates by $N + 1 = 31$ equally spaced grid points with $j = 0$ located at the lower plate. The time step size is $\Delta t = 2 \cdot 10^{-5}\ s$.

Use the FTCS explicit scheme to obtain the solution within the domain up to $t = 6.32\ ms$. Print in a table and graph the solution $u(y,t)$ for all spatial locations at time $t =$ 0.0 ms, 1.58 ms, 3.16 ms, 4.74 ms, and 6.32 ms.

Required submission:
- 1 clearly annotated plot containing y as a function of u for t = 0.0 ms, 1.58 ms, 3.16 ms, 4.74 ms, and 6.32 ms;
- 1 table containing grid point locations y and solution u for t = 0.0 ms, 1.58 ms, 3.16 ms, 4.74 ms, and 6.32 ms;
- SafeAssign upload of all used, well commented code.

Problem 4 (30 bonus points; AEE 471: Core Course Outcome #2 210 bonus outcome points)
Consider two infinite parallel plates, a distance $H = 4\,cm$ apart with the fluid between them initially at rest, $u(y > 0cm, t = 0s) = 0\,m/s$. The fluid has a constant kinematic viscosity of $\nu = 2.17 \cdot 10^{-4}\ m^2/s$ and density of $\rho = 800\ kg/m^3$. The upper plate is stationary and the lower plate is suddenly set in motion with a constant tangential velocity of 40 m/s. A constant streamwise pressure gradient of dp/dx is imposed within the domain at the instant the motion starts. The flow's governing equation can be derived from the Navier-Stokes equation and is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial p}{\partial x}\ . \tag{3}$$

In the following use a constant grid spacing of $\Delta y = 1$ mm and a nodal mesh.

(a) Use the FTCS explicit scheme with $\Delta t = 2\,ms$ to compute the velocity within the domain for

(I) $dp/dx = 0$
(II) $dp/dx = 2.0 \cdot 10^4\,N/m^2/m$
(III) $dp/dx = -3.0 \cdot 10^4\,N/m^2/m$

Print in a table the solutions at time $t =$ 0.0s, 0.18s, 0.36s, 0.54s, 0.72s, 0.9s, and 1.08s. Graph the velocity profiles $u(y,t)$ at time $t =$ 0.0s, 0.18s, and 1.08s.

(b) Use the Laasonen implicit scheme (BTCS) to compute the velocity profiles and print in a table and graph them at the same time levels as (a) for

(I) $\Delta t = 0.01s$, $dp/dx = 0.0$
(II) $\Delta t = 0.01s$, $dp/dx = 2.0 \cdot 10^4\,N/m^2/m$
(III) $\Delta t = 0.002s$, $dp/dx = 2.0 \cdot 10^4\,N/m^2/m$

Comment on accuracy and time to solution in comparison to (a).

Required submission:
- 1 clearly annotated plot containing y as a function of u for t = 0.0s, 0.18s, and 1.08s using FTCS and $dp/dx = 0$;
- 1 clearly annotated plot containing y as a function of u for t = 0.0s, 0.18s, and 1.08s using FTCS and $dp/dx = 2.0 \cdot 10^4\,N/m^2/m$;
- 1 clearly annotated plot containing y as a function of u for t = 0.0s, 0.18s, and 1.08s using FTCS and $dp/dx = -3.0 \cdot 10^4\,N/m^2/m$;
- 1 table containing grid point locations y and solution u for t = 0.0s, 0.18s, and 1.08s using FTCS and $dp/dx = 0$;
- 1 table containing grid point locations y and solution u for t = 0.0s, 0.18s, and 1.08s using FTCS and $dp/dx = 2.0 \cdot 10^4\,N/m^2/m$;
- 1 table containing grid point locations y and solution u for t = 0.0s, 0.18s, and 1.08s using FTCS and $dp/dx = -3.0 \cdot 10^4\,N/m^2/m$;
- 1 clearly annotated plot containing y as a function of u for t = 0.0s, 0.18s, and 1.08s using BTCS with $\Delta t = 0.01s$

and $dp/dx = 0$;
- 1 clearly annotated plot containing y as a function of u for t = 0.0s, 0.18s, and 1.08s using BTCS with $\Delta t = 0.01s$ and $dp/dx = 2.0 \cdot 10^4\ N/m^2/m$;
- 1 clearly annotated plot containing y as a function of u for t = 0.0s, 0.18s, and 1.08s using BTCS with $\Delta t = 0.002s$ and $dp/dx = 2.0 \cdot 10^4\ N/m^2/m$;
- 1 table containing grid point locations y and solution u for t = 0.0s, 0.18s, and 1.08s using BTCS with $\Delta t = 0.01s$ and $dp/dx = 0$;
- 1 table containing grid point locations y and solution u for t = 0.0s, 0.18s, and 1.08s using BTCS with $\Delta t = 0.01s$ and $dp/dx = 2.0 \cdot 10^4\ N/m^2/m$;
- 1 table containing grid point locations y and solution u for t = 0.0s, 0.18s, and 1.08s using BTCS with $\Delta t = 0.002s$ and $dp/dx = 2.0 \cdot 10^4\ N/m^2/m$;
- Discussion of cost comparing BTCS cases I and II to FTCS cases I and II, and BTCS case III to FTCS case II.
- Discussion of accuracy comparing BTCS cases I and II to FTCS cases I and II, and BTCS case III to FTCS case II.
- SafeAssign upload of all used, well commented code.