Non-Linear Case

$$E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} + \frac{\partial E}{\partial u}\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

FTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Note: This is **NOT** the same as using FTCS on $\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{\frac{1}{2}(u_{i+1}^n)^2 - \frac{1}{2}(u_{i-1}^n)^2}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

But, we could use $A_i^n = \frac{1}{2}\left(A_{i+1}^n + A_{i-1}^n\right) = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)$

$\Rightarrow$ convective term: $\frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x}$

- Stability: $d \leq \frac{1}{2}$ and $\mathrm{Re}_c \leq \frac{2}{C}$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

FTBCS $u_i^n > 0$

1st-order upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{E_i^n - E_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

or:

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

Du-Fort Frankel

central in time and space with fix in viscous term

$$\boxed{\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} = \nu \frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2}}$$

or:

$$\boxed{\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + A_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu \frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2}}$$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

BTCS

implicit and central in space

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^{n+1}\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = \nu \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

problem: non-linearity! ⇒ need to linearize ⇒ time-lag A

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^{n}\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = \nu \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}}$$

$$-\left(d + A_i^n \frac{\Delta t}{2\Delta x}\right) u_{i-1}^{n+1} + (1+2d)\, u_i^{n+1} - \left(d - A_i^n \frac{\Delta t}{2\Delta x}\right) u_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

RK

same as before, just as right-hand side, use entire

$$-\frac{E^n_{i+1} - E^n_{i-1}}{2\Delta x} + \nu \frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{\Delta x^2}$$

2

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{\Delta x}\left(h^n_{i+1/2} - h^n_{i-1/2}\right) + \frac{\nu \Delta t}{\Delta x^2}\left(u^n_{i+1} - 2u^n_i + u^n_{i-1}\right)$$

with $h^n_{i+1/2} = \frac{1}{2}\left(E^n_{i+1} + E^n_i + \Phi^n_{i+1/2}\right)$

and flux limiter function $\Phi^n_{i+1/2}$ as before

Implicit treatment of the parabolic terms

recap: parabolic PDE: $$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- stability: $$\left(\frac{\alpha \Delta t}{\Delta x^2} + \frac{\alpha \Delta t}{\Delta y^2} \right) \leq \frac{1}{2}$$

- let's define some shorthands: $$d_x = \frac{\alpha \Delta t}{\Delta x^2} \quad d_y = \frac{\alpha \Delta t}{\Delta y^2} \quad \Rightarrow \quad d_x + d_y \leq \frac{1}{2}$$

- stability limit is typically very restrictive ⇒ implicit schemes are preferable!

- BTCS:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha \left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} \right)$$

$$\Rightarrow \quad d_x u_{i+1,j}^{n+1} + d_x u_{i-1,j}^{n+1} + d_y u_{i,j+1}^{n+1} + d_y u_{i,j-1}^{n+1} - (2d_x + 2d_y + 1) u_{i,j}^{n+1} = -u_{i,j}^{n}$$

- penta-diagonal system (block tri-diagonal)
  - not very efficient to solve using Gaussian elimination
  - could use iterative methods, especially multigrid methods

ADI-Methods

Alternating **D**irectional **I**mplicit Methods

$$d_x = \frac{\alpha \Delta t}{\Delta x^2} \quad d_y = \frac{\alpha \Delta t}{\Delta y^2} \qquad d_1 = \frac{d_x}{2} \quad d_2 = \frac{d_y}{2}$$

Idea:

- we don't really need to find the exact solution to the system
  - even the exact solution will have truncation errors!
  - we can introduce modifications, as long as we preserve the formal order of accuracy!

⇒ splitting methods: ADI

$$d_x u_{i+1,j}^{n+1} + d_x u_{i-1,j}^{n+1} + d_y u_{i,j+1}^{n+1} + d_y u_{i,j-1}^{n+1} - (2d_x + 2d_y + 1) u_{i,j}^{n+1} = -u_{i,j}^{n}$$

- Step 1: implicit only in the x-direction for *Δt/2*

$$\frac{d_x}{2} u_{i+1,j}^{n+1/2} + \frac{d_x}{2} u_{i-1,j}^{n+1/2} + \frac{d_y}{2} u_{i,j+1}^{n} + \frac{d_y}{2} u_{i,j-1}^{n} - (2\frac{d_x}{2} + 1) u_{i,j}^{n+1/2} - 2\frac{d_y}{2} u_{i,j}^{n} = -u_{i,j}^{n}$$

$$-d_1 u_{i+1,j}^{n+1/2} + (1 + 2d_1) u_{i,j}^{n+1/2} - d_1 u_{i-1,j}^{n+1/2} = d_2 u_{i,j+1}^{n} + (1 - 2d_2) u_{i,j}^{n} + d_2 u_{i,j-1}^{n}$$

- Step 2: implicit only in the y-direction for *Δt/2*

$$-d_2 u_{i,j+1}^{n+1} + (1 + 2d_2) u_{i,j}^{n+1} - d_2 u_{i,j-1}^{n+1} = d_1 u_{i+1,j}^{n+1/2} + (1 - 2d_1) u_{i,j}^{n+1/2} + d_2 u_{i-1,j}^{n+1/2}$$

ADI-Methods

$$d_x = \frac{\alpha \Delta t}{\Delta x^2} \quad d_y = \frac{\alpha \Delta t}{\Delta y^2} \quad d_1 = \frac{d_x}{2} \quad d_2 = \frac{d_y}{2}$$

- Step 1: implicit only in the x-direction for *Δt/2*

$$-d_1 u_{i+1,j}^{n+1/2} + (1+2d_1) u_{i,j}^{n+1/2} - d_1 u_{i-1,j}^{n+1/2} = d_2 u_{i,j+1}^n + (1-2d_2) u_{i,j}^n + d_2 u_{i,j-1}^n$$

- Step 2: implicit only in the y-direction for *Δt/2*

$$-d_2 u_{i,j+1}^{n+1} + (1+2d_2) u_{i,j}^{n+1} - d_2 u_{i,j-1}^{n+1} = d_1 u_{i+1,j}^{n+1/2} + (1-2d_1) u_{i,j}^{n+1/2} + d_2 u_{i-1,j}^{n+1/2}$$

- introduce two more shorthands for 2nd derivative, 2nd-order central finite difference operators:

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \qquad \delta_y^2 u_{i,j} = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

- Step 1:

$$-d_1 (u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}) + u_{i,j}^{n+1/2} = d_2 (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) + u_{i,j}^n$$

$$(1 - d_1 \delta_x^2) u_{i,j}^{n+1/2} = (1 + d_2 \delta_y^2) u_{i,j}^n$$

Step 1: $(1 - d_1 \delta_x^2) u_{i,j}^{n+1/2} = (1 + d_2 \delta_y^2) u_{i,j}^n$

Step 2: $(1 - d_2 \delta_y^2) u_{i,j}^{n+1} = (1 + d_1 \delta_x^2) u_{i,j}^{n+1/2}$

ADI

Crank-Nicholson

$$d_x = \frac{\alpha \Delta t}{\Delta x^2} \qquad d_y = \frac{\alpha \Delta t}{\Delta y^2} \qquad d_1 = \frac{d_x}{2} \qquad d_2 = \frac{d_y}{2}$$

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \qquad \delta_y^2 u_{i,j} = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{\alpha}{2}\left(\frac{\delta_x^2 u_{i,j}^{n+1}}{\Delta x^2} + \frac{\delta_x^2 u_{i,j}^n}{\Delta x^2} + \frac{\delta_y^2 u_{i,j}^{n+1}}{\Delta y^2} + \frac{\delta_y^2 u_{i,j}^n}{\Delta y^2}\right)$$

$$u_{i,j}^{n+1} - \frac{\alpha \Delta t}{2}\left(\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2}\right) u_{i,j}^{n+1} = u_{i,j}^n + \frac{\alpha \Delta t}{2}\left(\frac{\delta_x^2}{\Delta x^2} + \frac{\delta_y^2}{\Delta y^2}\right) u_{i,j}^n$$

$$\boxed{\left[1 - \left(d_1 \delta_x^2 + d_2 \delta_y^2\right)\right] u_{i,j}^{n+1} = \left[1 + \left(d_1 \delta_x^2 + d_2 \delta_y^2\right)\right] u_{i,j}^n}$$

- How does this compare to ADI? eliminate $u^{n+1/2}$

Step 1: $(1 - d_1 \delta_x^2) u_{i,j}^{n+1/2} = (1 + d_2 \delta_y^2) u_{i,j}^n \quad \Big| \cdot (1 + d_1 \delta_x^2)$

Step 2: $(1 - d_2 \delta_y^2) u_{i,j}^{n+1} = (1 + d_1 \delta_x^2) u_{i,j}^{n+1/2} \quad \Big| \cdot (1 - d_1 \delta_x^2)$

add

$$(1 + d_1 \delta_x^2)(1 - d_1 \delta_x^2) u_{i,j}^{n+1/2} + (1 - d_1 \delta_x^2)(1 - d_2 \delta_y^2) u_{i,j}^{n+1} = (1 + d_1 \delta_x^2)(1 + d_2 \delta_y^2) u_{i,j}^n + (1 - d_1 \delta_x^2)(1 + d_1 \delta_x^2) u_{i,j}^{n+1/2}$$

$$(1 - d_1 \delta_x^2)(1 - d_2 \delta_y^2) u_{i,j}^{n+1} = (1 + d_1 \delta_x^2)(1 + d_2 \delta_y^2) u_{i,j}^n$$

$$(1 - (d_1 \delta_x^2 + d_2 \delta_y^2) + d_1 d_2 \delta_x^2 \delta_y^2) u_{i,j}^{n+1} = (1 + (d_1 \delta_x^2 + d_2 \delta_y^2) + d_1 d_2 \delta_x^2 \delta_y^2) u_{i,j}^n$$

$\Rightarrow$ the same as Crank-Nicholson, except for the $d_1 d_2 \delta_x^2 \delta_y^2$ terms!

but $d_1 d_2 \delta_x^2 \delta_y^2$ is $O(h^4)$, thus a higher order term compared to the truncation error, $O(h^2)$

$\Rightarrow$ we can neglect these terms!

$\Rightarrow$ up to the order of Crank-Nicholson, ADI & Crank-Nicholson are identical!

ADI

- ADI belongs to a class of methods called “Approximate Factorization”
- Idea: split multi-dimensional FDEs into series of FDEs that can be solved in tri-diagonal form

ADI is the approximate factorization of Crank-Nicholson