• Muddiest Points from Class 02/20

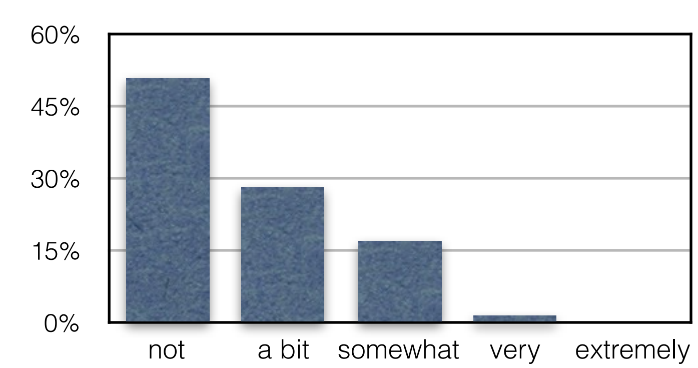
- *"For the FTCS method, why don't we just apply boundary condition first and then calculate the interior points?"*
  - the boundary condition may depend on the interior values
  - even then it would not impact the solution at that time level

- *"What is the benefit of using the BTCS as opposed to just using FTCS with a ghost cell, like we used in the cell-centered mesh?"*
- *"You mentioned […] we need to include the influence in boundary conditions by using the Backward difference for the time derivative. But what happens if there is such a switch BC while formulating a ghost cell system?"*
  - Ghost cells won't change the boundary lagging issue (the values are just at a different spatial location)

- *"Implicit and Explicit systems: are the differences outlined in the slides somewhere that I missed?"*
- *"How are we able to generate 3 values on time step n+1 using only one value from time step n?"*
  - explicit system: the new value depends only on already known values
  - implicit system: the new value depends also on not yet known new values => coupled system of equations

- *"Is the discrete perturbation analysis method always accurate? Like we had an example a few weeks ago where the error (or residual) increased before decreasing"*
  - Yes. Note: the prior example w/ increasing residual/error was for an iterative elliptical system, not a parabolic PDE

- *"Do we code the time as a separate dimension for the matrix. A 3D flaid problem will have 4th dimension for a parabolic equation"*
  - time is a separate dimension, but DO NOT code it as a separate array index (you will run out of memory)
  - compare this to iterative methods for elliptical systems: don't store every iteration



- *"What is the equals sign with a carrot symbol above it? what does it mean?"*
  - mathematical symbol meaning "corresponds to"

- *"Since the FTCS has mathematical drawback, why we still use it, despite of its efficiency?"*
  - because it's simple to code and fast

von Neumann Stability Analysis

FDE: Finite Difference Equation

Limitations:

- influence of boundary conditions is ignored
- valid only for linear FDEs (if non-linear → linearize locally ⇒ results valid locally only)

- How does FDE respond to a certain type of solution?
  - types of solutions to consider?
    - sinusoidals
      (solutions can be decomposed into sum of sinusoidals by Fourier transform)
    - since we analyze linear FDEs only ⇒ superposition
      ⇒ analysis of single mode is sufficient

$$\varphi_j^n = \rho^n e^{ikx_j}$$

$i = \sqrt{-1}$: imaginary number

k : wave number

ρ : amplitude

$$\Rightarrow \varphi_j^{n+1} = \rho^{n+1} e^{ikx_j} \quad \text{and} \quad \varphi_{j\pm1}^n = \rho^n e^{ikx_{j\pm1}}$$

von Neumann Stability Analysis

Example: FTCS

$$\varphi_j^{n+1} = \varphi_j^n + B\left(\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n\right) \qquad B = \frac{\alpha \Delta t}{\Delta x^2}$$

- Substitute in sinusoidal solution:

$$\varphi_j^{n+1} = \rho^{n+1} e^{ikx_j} \qquad \varphi_j^n = \rho^n e^{ikx_j} \qquad \varphi_{j\pm1}^n = \rho^n e^{ikx_{j\pm1}}$$

$$\rho^{n+1} e^{ikx_j} = \rho^n e^{ikx_j} + B\left(\rho^n e^{ikx_{j+1}} - 2\rho^n e^{ikx_j} + \rho^n e^{ikx_{j-1}}\right) \qquad \Big|: e^{ikx_j}$$

$$\rho^{n+1} = \rho^n + B\left(\rho^n e^{ik\Delta x} - 2\rho^n + \rho^n e^{-ik\Delta x}\right) \qquad \Big|: \rho^n$$

$$\frac{\rho^{n+1}}{\rho^n} = 1 + B\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) = 1 - 2B + B\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) = 1 - 2B + 2B\cos(k\Delta x)$$

Stable if $\left|\frac{\rho^{n+1}}{\rho^n}\right| \leq 1$

$$1 + 2B\left(\cos(k\Delta x) - 1\right) \leq 1 \quad \wedge \quad 1 + 2B\left(\cos(k\Delta x) - 1\right) \geq -1$$

$$2B\left(\cos(k\Delta x) - 1\right) \leq 0 \quad \wedge \quad B\left(\cos(k\Delta x) - 1\right) \geq -1 \qquad |\cdot(-1)$$

$$\cos(k\Delta x) \leq 1 \quad \wedge \quad B\underbrace{\left(1 - \cos(k\Delta x)\right)}_{0\ldots2} \leq 1$$

✓

worst case: $B2 \leq 1 \quad \Rightarrow \quad B \leq \frac{1}{2}$

$$\boxed{\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{\alpha}}$$

von Neumann Stability Analysis

Example: Laasonen (BTCS)

$$\varphi_j^{n+1} = \varphi_j^n + B\left(\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1}\right) \qquad B = \frac{\alpha \Delta t}{\Delta x^2}$$

- Substitute in sinusoidal solution:

$$\varphi_j^{n+1} = \rho^{n+1} e^{ikx_j} \qquad \varphi_j^n = \rho^n e^{ikx_j} \qquad \varphi_{j\pm1}^{n+1} = \rho^{n+1} e^{ikx_{j\pm1}}$$

$$\rho^{n+1} e^{ikx_j} = \rho^n e^{ikx_j} + B\left(\rho^{n+1} e^{ikx_{j+1}} - 2\rho^{n+1} e^{ikx_j} + \rho^{n+1} e^{ikx_{j-1}}\right) \qquad |: e^{ikx_j}$$

$$\rho^{n+1} = \rho^n + B\rho^{n+1}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

$$\rho^{n+1}\left(1 + 2B\left(1 - \cos(k\Delta x)\right)\right) = \rho^n \qquad |: \rho^n$$

$$\left|\frac{\rho^{n+1}}{\rho_n}\right| = \left|\frac{1}{\underbrace{1 + 2B\,\underbrace{\left(1 - \cos(k\Delta x)\right)}_{0\ldots2}}_{\geq 1}}\right| \leq 1 \quad ✓$$

Stable if $\left|\frac{\rho^{n+1}}{\rho^n}\right| \leq 1$

$\boxed{\text{unconditionally stable}}$

- no time step limit due to stability
- typical of implicit methods

Parabolic Equations - Explicit Methods

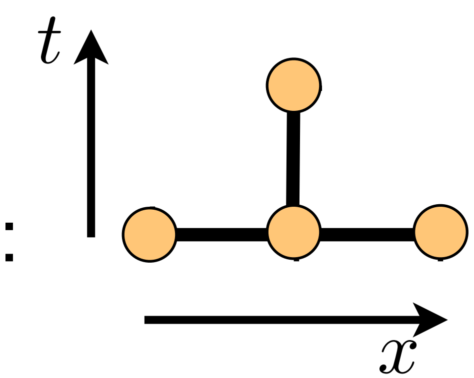
$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Common explicit methods

FTCS

$$\varphi_i^{n+1} = \varphi_i^n + \frac{\alpha \Delta t}{h^2}\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right)$$

stencil:



- truncation errors: $O(\Delta t)$ in time, $O(\Delta x^2)$ in space
- stable for $\frac{\alpha \Delta t}{\Delta x^2} \leq 1$

Richardson Method

Idea: Why not go 2nd-order in time?

From Taylor series in time: $$\left.\frac{\partial \varphi_i}{\partial t}\right|^n = \frac{\varphi_i^{n+1} - \varphi_i^{n-1}}{2\Delta t} + O(\Delta t^2)$$

Parabolic Equations - Explicit Methods

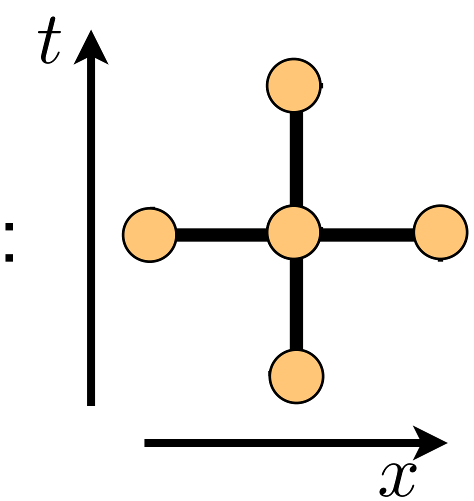
$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Common explicit methods

Richardson Method

$$\varphi_i^{n+1} = \varphi_i^{n-1} + 2\alpha \frac{\Delta t}{h^2} \left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n \right)$$

stencil:



- truncation errors: $O(\Delta t^2)$ in time, $O(\Delta x^2)$ in space

BUT: turns out to be ***always*** unstable

Du-Fort-Frankel

fix Richardson method

$$\frac{\varphi_i^{n+1} - \varphi_i^{n-1}}{2\Delta t} = + \frac{\alpha}{h^2} \left(\varphi_{i+1}^n - 2\overline{\varphi_i^n} + \varphi_{i-1}^n \right) \quad \text{with} \quad \overline{\varphi_i^n} = \frac{1}{2} \left(\varphi_i^{n+1} + \varphi_i^{n-1} \right)$$

$$\frac{\varphi_i^{n+1} - \varphi_i^{n-1}}{2\Delta t} = + \frac{\alpha}{h^2} \left(\varphi_{i+1}^n - \varphi_i^{n-1} - \varphi_i^{n+1} + \varphi_{i-1}^n \right)$$

Is this now implicit?

let's rearrange

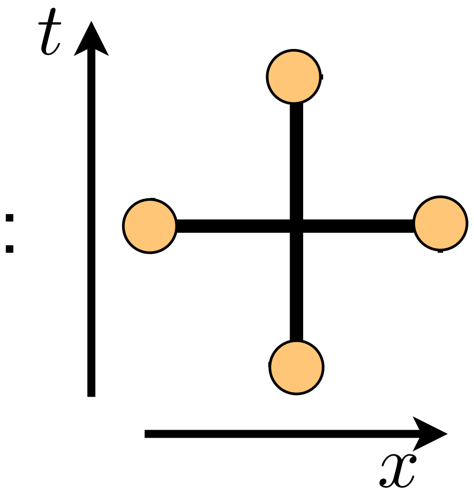
Parabolic Equations - Explicit Methods

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Common explicit methods

Du-Fort-Frankel

stencil:



$$\left(1 + 2\alpha \frac{\Delta t}{h^2}\right) \varphi_i^{n+1} = \left(1 - 2\alpha \frac{\Delta t}{h^2}\right) \varphi_i^{n-1} + 2\alpha \frac{\Delta t}{h^2} \left(\varphi_{i+1}^n + \varphi_{i-1}^n\right)$$

$\Rightarrow$ still explicit

But, does the averaging in time impact the accuracy in time? TS in time!

$$\varphi_i^{n+1} = \varphi_i^n + \Delta t \left.\frac{\partial \varphi_i}{\partial t}\right|^n + \frac{\Delta t^2}{2} \left.\frac{\partial^2 \varphi_i}{\partial t^2}\right|^n + \dots$$

$$+ \quad \varphi_i^{n-1} = \varphi_i^n - \Delta t \left.\frac{\partial \varphi_i}{\partial t}\right|^n + \frac{\Delta t^2}{2} \left.\frac{\partial^2 \varphi_i}{\partial t^2}\right|^n + \dots$$

$$\varphi_i^{n+1} + \varphi_i^{n-1} = 2\varphi_i^n + \Delta t^2 \left.\frac{\partial^2 \varphi_i}{\partial t^2}\right|^n + \dots$$

$$\varphi_i^n = \frac{\varphi_i^{n+1} + \varphi_i^{n-1}}{2} + O(\Delta t^2)$$

$\Rightarrow$ still 2nd order in time

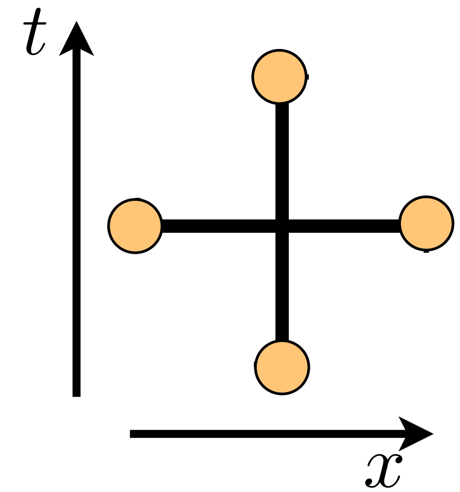
Parabolic Equations - Explicit Methods

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2}$$

Common explicit methods

Du-Fort-Frankel

stencil:



$$\left(1 + 2\alpha \frac{\Delta t}{h^2}\right) \varphi_i^{n+1} = \left(1 - 2\alpha \frac{\Delta t}{h^2}\right) \varphi_i^{n-1} + 2\alpha \frac{\Delta t}{h^2} \left(\varphi_{i+1}^n + \varphi_{i-1}^n\right)$$

$\Rightarrow$ still explicit

But, does the averaging in time impact the accuracy in time?

- truncation errors: $O(\Delta t^2)$ in time, $O(\Delta x^2)$ in space
- Stability? turns out to be always stable!

But, there are some issues:

- must store 3 time levels: $n-1$, n, $n+1$ data
- startup problem: cannot use for $n=0$, since there is no $n=-1$ data!
  fix: start with lower order scheme, e.g. FTCS or BTCS for 1 time step

$$\varphi^0 \xrightarrow{\text{FTCS}} \varphi^1 \xrightarrow{\text{DF}} \varphi^2 \xrightarrow{\text{DF}} \varphi^3 \;\ldots.$$