• Muddiest Points from Class 03/29

- *"What is the physical significance of the viscous burger's equation? "*
  - The viscous Burger's equation is predominantly used as a numerical testbed for numerical methods
  - There's no direct physical significance of the equation (as far as I can tell)

- *"In the 2nd order method where we implement an artificial diffusivity, doesn't the so called switching between 1st and 2nd order methods result in areas of lower formal order of accuracy? Is there a way to determine where these areas are?"*
- *Is dissipation more when there are many frequencies (as in a jump) ?*
- *Why don't we use higher order TVD schemes or we choose to refine the mesh to get finer solution(reduce dissipative errors)?*
  - Flux limiter functions (switches) are designed to revert to first order only in regions where non-TVD behavior would occur, i.e., at shocks/discontinuities
  - one can construct higher order TVD methods, however all are higher order only in smooth regions of the flow and revert through flux limiters to first order schemes at shocks/discontinuities
  - These so-called high-resolution schemes are generally either MUSCL (Monotone Upstream-Centered Schemes for Conservation Laws) with flux limiters (TVD, see last class) or WENO (non-TVD, see earlier class)
  - Note: the concept of order of accuracy breaks down at discontinuities anyways, since the derivatives in the error terms of the Taylor series become infinite and thus are never small

- *"The Courant Number criteria seems to shift from method to method, its dependence can change to include or exclude certain variables too (at least it seems). […] can you touch on the Courant Number in a general sense? What would your definition be?"*
  - linear hyperbolic PDEs

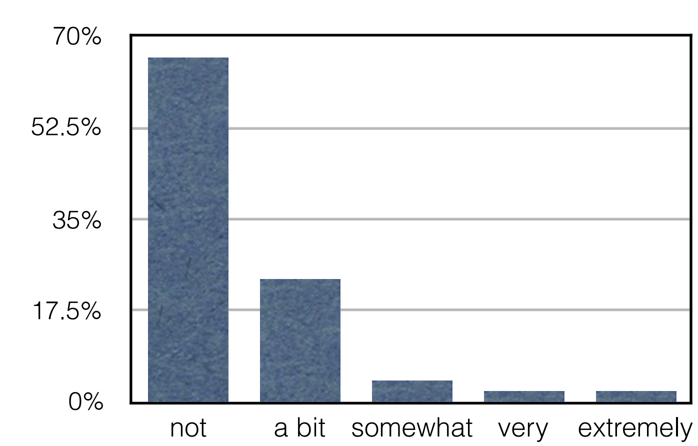
1D: $$C = \frac{\max|a|\Delta t}{\Delta x}$$

2D: $$C = \frac{\max|a|\Delta t}{\Delta x} + \frac{\max|b|\Delta t}{\Delta y}$$

  - TVD methods for non-linear hyperbolic PDEs

1D: $$C = \frac{\max|\alpha_{i+1/2}|\Delta t}{\Delta x}$$

- *"What happened to the density term in the Reynolds number?"*
  - There wasn't a Reynolds number, just the kinematic viscosity that contains the density



FTBCS $a > 0$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

1st-order upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

- large dissipative errors in convective term
- dissipative errors may be larger than physical viscous term!

Remedy:

- could go 2nd-order ➙ dispersive errors
- could go 3rd-order ➙ ENO-3
- could go 5th-order ➙ WENO-5

MacCormack

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

step 1: $$u_i^* = u_i^n - \frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

step 2: $$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{a\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right)\right]$$

- Stability: $\Delta t \leq \dfrac{1}{\dfrac{\max|a|}{\Delta x} + 2\dfrac{\alpha}{(\Delta x)^2}}$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha \Delta t}{\Delta x^2}$$

BTCS

Implicit & central in space

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}}$$

$$\Leftrightarrow \quad -\left(\frac{C}{2} + d\right) u_{i-1}^{n+1} + (1+2d)\, u_i^{n+1} + \left(\frac{C}{2} - d\right) u_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!

BTBCS $a > 0$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

Implicit & upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}} \qquad \text{(1}^{\text{st}}\text{-order)}$$

$$\Leftrightarrow \quad -(C+d)\,u_{i-1}^{n+1} + (1 + C + 2d)\,u_i^{n+1} - du_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!
- but again: dissipative errors may dominate physical viscous forces!
- go 2nd-order upwind for the convective term:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{3u_i^{n+1} - 4u_{i-1}^{n+1} + u_{i-2}^{n+1}}{2\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

$$\Leftrightarrow \quad \frac{C}{2}u_{i-2}^{n+1} - (2C+d)\,u_{i-1}^{n+1} + \left(1 + \frac{3}{2}C + 2d\right)u_i^{n+1} - du_{i-1}^{n+1} = u_i^n$$

- no-longer tri-diagonal! $\Rightarrow$ time-lag u_{i-2} to recover tri-diagonal structure

$$\Leftrightarrow \quad -(2C+d)\,u_{i-1}^{n+1} + \left(1 + \frac{3}{2}C + 2d\right)u_i^{n+1} - du_{i-1}^{n+1} = u_i^n - \frac{C}{2}u_{i-2}^n$$

- go 3rd-order upwind: again time-lag u_{i-2} to recover tri-diagonal structure

Crank-Nicholson

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

shorthands:

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \qquad \delta_x u_i = u_{i+1} - u_{i-1}$$

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2}\left(\frac{\delta_x u_i^n}{2\Delta x} + \frac{\delta_x u_i^{n+1}}{2\Delta x}\right) + \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)}$$

$$\Leftrightarrow \quad u_i^{n+1} + \frac{a\Delta t}{2}\frac{\delta_x u_i^{n+1}}{2\Delta x} - \frac{\alpha\Delta t}{2}\frac{\delta_x^2 u_i^{n+1}}{\Delta x^2} = u_i^n - \frac{a\Delta t}{2}\frac{\delta_x u_i^n}{2\Delta x} + \frac{\alpha\Delta t}{2}\frac{\delta_x^2 u_i^n}{\Delta x^2}$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

Mixed Treatment

Idea: apply different schemes to the individual terms

Goal: $O(\Delta t^2, \Delta x^2)$

- convective terms
  - central in space:

$$\frac{\partial u}{\partial t} = -a\frac{u_{i+1} - u_{i-1}}{2\Delta x} = H_i$$

  - Adams-Bashforth in time:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1}$$

- diffusive terms
  - central in space:

$$\frac{\partial u}{\partial t} = \alpha\frac{\delta_x^2 u_i}{\Delta x^2}$$

  - Crank-Nicholson in time:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)$$

- combine:

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1} + \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)}$$

- since viscous terms are implicit ⇒ no viscous time step restriction!

Non-Linear Case

$$E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} + \frac{\partial E}{\partial u}\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

FTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Note: This is **NOT** the same as using FTCS on $\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{\frac{1}{2}(u_{i+1}^n)^2 - \frac{1}{2}(u_{i-1}^n)^2}{2\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

But, we could use $A_i^n = \frac{1}{2}\left(A_{i+1}^n + A_{i-1}^n\right) = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)$

$\Rightarrow$ convective term: $\frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x}$

- Stability: $d \leq \frac{1}{2}$ and $C \leq 1$ and $\mathrm{Re}_c \leq 2$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad E = \frac{1}{2} u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

FTBCS $u_i^n > 0$

1st-order upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{E_i^n - E_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

or:

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad E = \frac{1}{2}u^2 \qquad A = \frac{\partial E}{\partial u} = u$$

BTCS

implicit and central in space

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^{n+1}\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = \nu \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

problem: non-linearity! ⇒ need to linearize ⇒ time-lag A

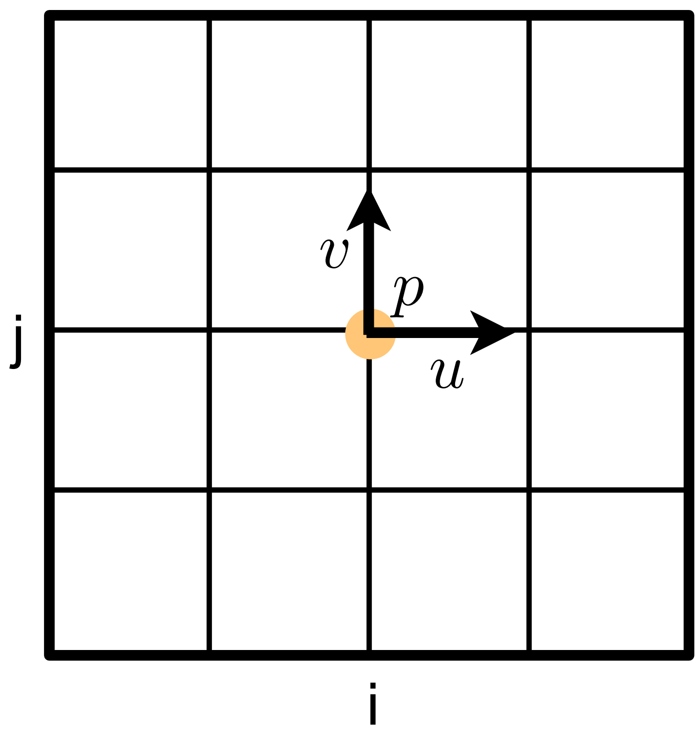
$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i^{n}\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = \nu \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}}$$

$$-\left(d + A_i^n \frac{\Delta t}{2\Delta x}\right) u_{i-1}^{n+1} + (1+2d)\, u_i^{n+1} - \left(d - A_i^n \frac{\Delta t}{2\Delta x}\right) u_{i+1}^{n+1} = u_i^n$$

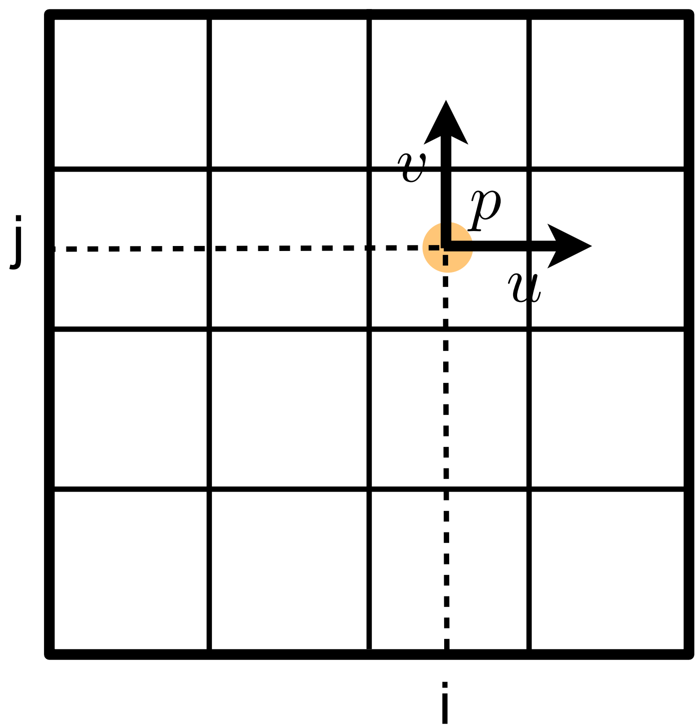
- tri-diagonal!

Before we solve in 2D, let's revisit meshing one last time:

- until now, we have used either one of the following meshes
  - all variables are located at the same location: @ grid intersection lines



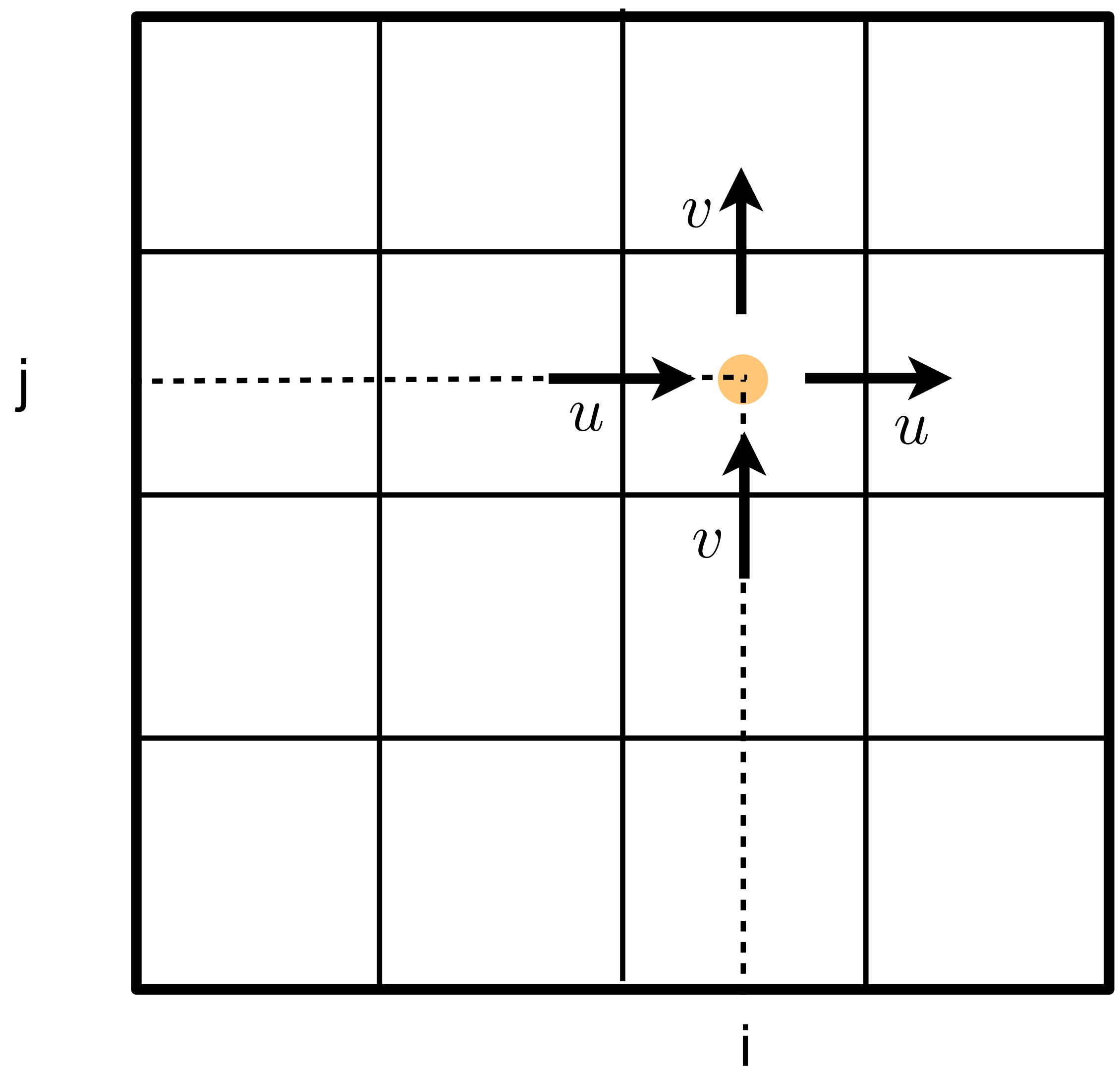
  - all variables are located at the same location: @ cell centers



collocated grids

But: variables need not be located all at the same location!

staggered grids

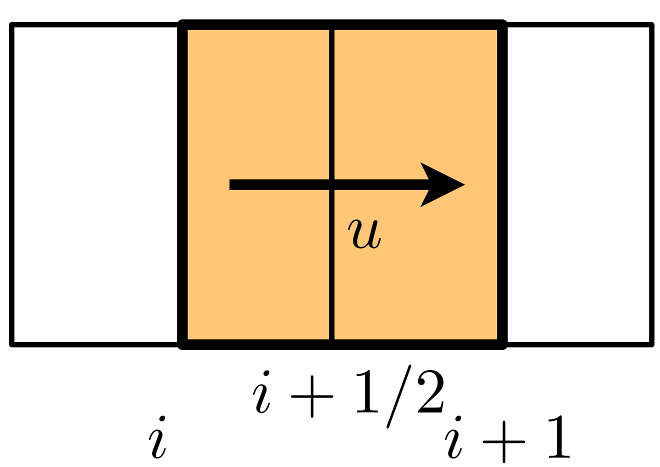


- define i,j to be @ cell centers
  - define u @ x-normal faces: $u_{i\pm 1/2,j}$
  - define v @ y-normal faces: $v_{i,j\pm 1/2}$
- Why?
  - simplifies fractional step method (later)
  - helps us enforce conservation laws!
  - cell = control volume

change = ∑ fluxes across boundaries + sources

- but fluxes across boundaries involve only normal velocity components!
- staggered grids define exactly these!
- However, the control volume for these velocities are shifted!
- need to evaluate FDE for u not at i,j, but at i+1/2,j !
- need all terms at i+1/2,j (or i,j+1/2 for v)

cv for $u_{i+1/2}$



Burgers Equation on Staggered Mesh

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

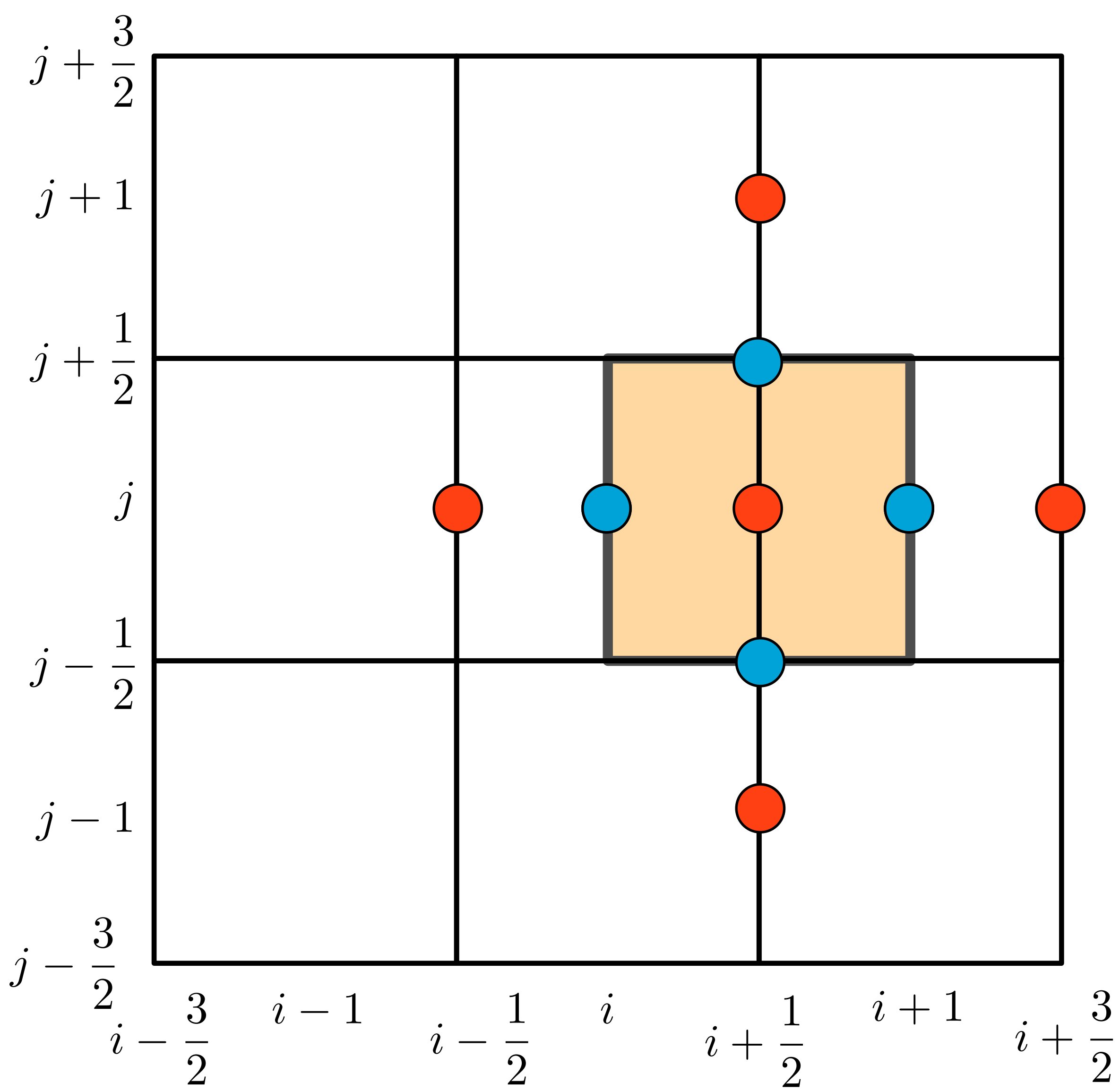
solve for $u_{i+1/2,j}$ with 2nd-order in space:

$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} + O(\Delta y^2)$$



Burgers Equation on Staggered Mesh

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

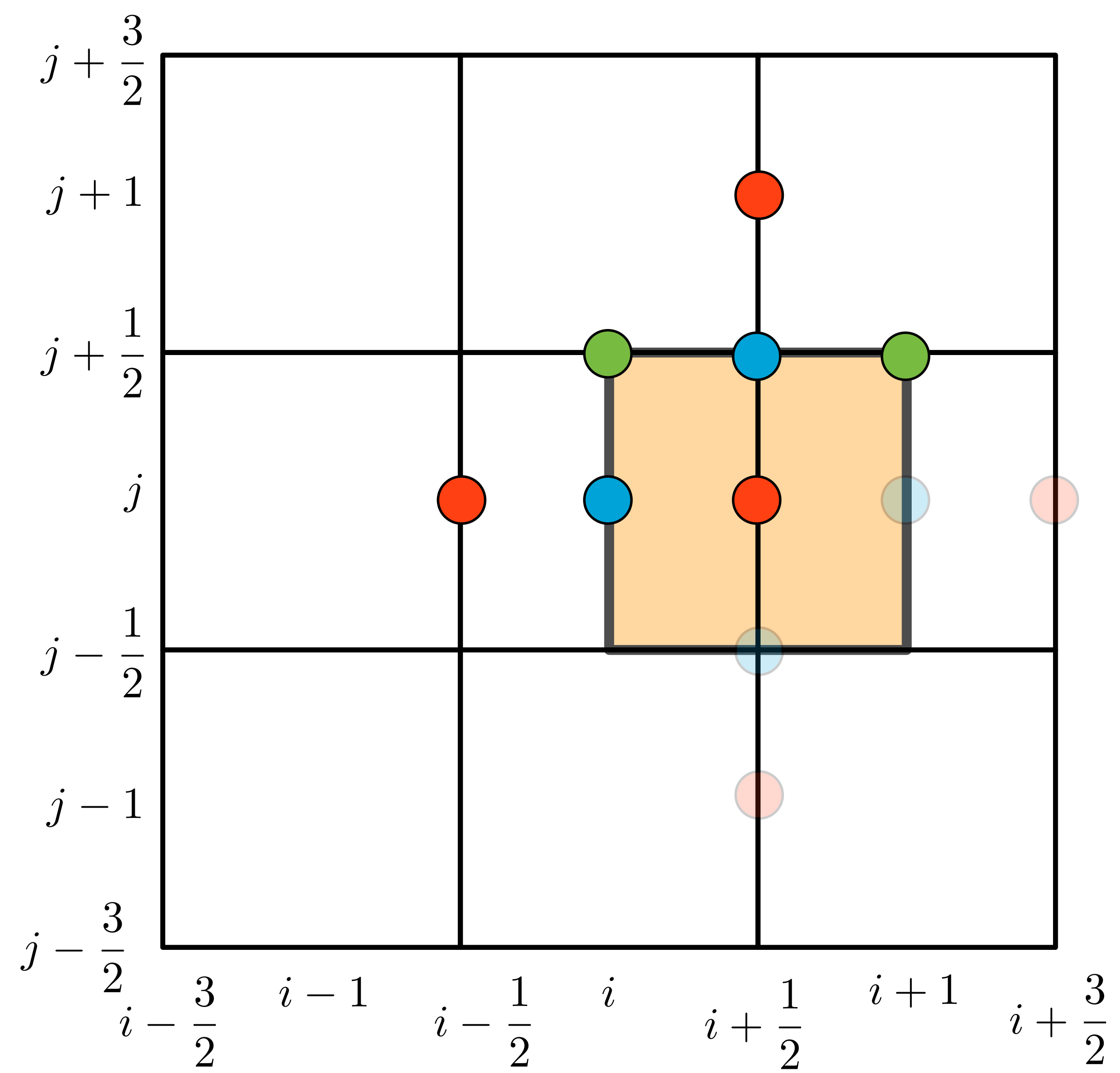
solve for $u_{i+1/2,j}$ with 2nd-order in space:

$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} + O(\Delta y^2)$$



but now need the following:

$$u_{i,j} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)$$

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1}\right)$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}\right)$$

do similar for v-equation

Burgers Equation on Staggered Mesh

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + Q_u$$

FTCS:

$$\frac{u^{n+1}_{i+\frac{1}{2},j} - u^n_{i+\frac{1}{2},j}}{\Delta t} = - \left.\frac{\partial uu}{\partial x}\right|^n_{i+\frac{1}{2},j} - \left.\frac{\partial uv}{\partial y}\right|^n_{i+\frac{1}{2},j}$$

$$+ \nu \left(\left.\frac{\partial^2 u}{\partial x^2}\right|^n_{i+\frac{1}{2},j} + \left.\frac{\partial^2 u}{\partial y^2}\right|^n_{i+\frac{1}{2},j} \right) + Q_u$$

$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x}$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2}$$

$$u_{i,j} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)$$

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1}\right)$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}\right)$$

do similar
for v-equation

Burgers Equation on Staggered Mesh

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + Q_u$$

$$\left.\frac{\partial uu}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{(u_{i+1,j})^2 - (u_{i,j})^2}{\Delta x}$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i+\frac{1}{2},j} = \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i+\frac{1}{2},j} = \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2}$$

$$u_{i,j} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}\right)$$

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1}\right)$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}\right)$$

Adams-Bashforth + Crank-Nicholson:

$$\frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n}_{i+\frac{1}{2},j}}{\Delta t} = \frac{3}{2}\left[- \left.\frac{\partial uu}{\partial x}\right|^{n}_{i+\frac{1}{2},j} - \left.\frac{\partial uv}{\partial y}\right|^{n}_{i+\frac{1}{2},j} \right] - \frac{1}{2}\left[- \left.\frac{\partial uu}{\partial x}\right|^{n-1}_{i+\frac{1}{2},j} - \left.\frac{\partial uv}{\partial y}\right|^{n-1}_{i+\frac{1}{2},j} \right] +$$

$$\frac{\nu}{2}\left(\left.\frac{\partial^2 u}{\partial x^2}\right|^{n}_{i+\frac{1}{2},j} + \left.\frac{\partial^2 u}{\partial x^2}\right|^{n+1}_{i+\frac{1}{2},j} + \left.\frac{\partial^2 u}{\partial y^2}\right|^{n}_{i+\frac{1}{2},j} + \left.\frac{\partial^2 u}{\partial y^2}\right|^{n+1}_{i+\frac{1}{2},j} \right) + Q_u$$

Solve Crank-Nicholson using ADI

Note that Adams-Bashforth part can be calculated at the beginning of each time step and is thus no different from source Qu

But Adams-Bashforth has startup issue: could do FTCS for hyperbolic part in 1st step

Trick: Adams-Bashforth turns into FTCS if one initializes $u^{n-1} = u^{n} = u^{0}$

do similar for v-equation