Advanced Considerations

Ideally, we would like our schemes to conserve mass, momentum, and energy

- for mass: ensure velocity is divergence free ⇒ converge Poisson system
- for momentum: use conservative form
- but what about energy, here kinetic energy?
  - another excursion into Linear Algebra:

$$A\vec{x} = \vec{b}$$

Q: are there any A, where $||\vec{x}|| = ||\vec{b}||$? A: Yes! For example if A is skew-symmetric.

Q: What's the meaning of $||\vec{x}||$?

$$||\vec{x}|| = \sum_i x_i^2 \qquad \text{so if } \vec{x} = \vec{v} \quad \Rightarrow \quad ||\vec{v}|| = 2E_{kin}$$

  - we can write our finite difference methods as $A\vec{v}^n = \vec{v}^{n+1}$

⇒ if A is skew-symmetric, the scheme will conserve kinetic energy!

Advanced Considerations

Skew-symmetric discretization on collocated grids

- let $\frac{\delta f_{i,j}}{\delta x} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$ $\qquad$ $\frac{\delta f_{i,j}}{\delta y} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$

- skipping viscous terms:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\delta(uu)}{\delta x} + \frac{1}{2}\frac{\delta(uv)}{\delta y} + \frac{1}{2}u\frac{\delta u}{\delta x} + \frac{1}{2}v\frac{\delta u}{\delta y} = -\frac{\delta\varphi}{\delta x}$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\delta(uv)}{\delta x} + \frac{1}{2}\frac{\delta(vv)}{\delta y} + \frac{1}{2}u\frac{\delta v}{\delta x} + \frac{1}{2}v\frac{\delta v}{\delta y} = -\frac{\delta\varphi}{\delta y}$$

Advanced Considerations

So what's the problem with collocated grids in the fractional step method?

or

Why are staggered mesh preferable?

- the reason has to do with step 2&3: div and grad operators must be consistent
- Why?
  - let's try different discrete operators: ∇_h and ∇'_h $\quad\quad$ $\nabla_h\cdot$ and $\nabla'_h\cdot$
  - to get to step 2:

$$\frac{\vec{v}^{n+1}-\vec{v}^*}{\Delta t} = -\nabla_h \varphi^{n+1} \qquad \Big| \, \nabla_h \cdot$$

but different on either side

$$\frac{\nabla_h \cdot \vec{v}^{n+1} - \nabla_h \cdot \vec{v}^*}{\Delta t} = -\nabla'_h \cdot \nabla_h \varphi^{n+1}$$

  - use step 3 with different discrete operator: $\vec{v}^{n+1} = \vec{v}^* - \Delta t \nabla'_h \varphi^{n+1}$

$$\frac{\nabla_h \cdot \left(\vec{v}^* - \Delta t \nabla'_h \varphi^{n+1}\right) - \nabla_h \cdot \vec{v}^*}{\Delta t} = -\nabla'_h \cdot \nabla_h \varphi^{n+1}$$

$$\nabla_h \cdot \nabla'_h \varphi^{n+1} = \nabla'_h \cdot \nabla_h \varphi^{n+1}$$

$\Rightarrow$ true, only if $\nabla_h = \nabla'_h$ and $\nabla_h \cdot = \nabla'_h \cdot$

  - as we saw before: this is easy to achieve on staggered meshes

Advanced Considerations

$$\nabla_h \cdot \nabla'_h \varphi^{n+1} = \nabla'_h \cdot \nabla_h \varphi^{n+1}$$ ⇒ true, only if $\nabla_h = \nabla'_h$ and $\nabla_h \cdot = \nabla'_h \cdot$

- as we saw before: this is easy to achieve on **staggered** meshes
  - step 2: $\mathrm{div}\left(\mathrm{grad}\ \varphi^{n+1}\right) = \frac{1}{\Delta t}\mathrm{div}\left(\vec{v}^*\right)$

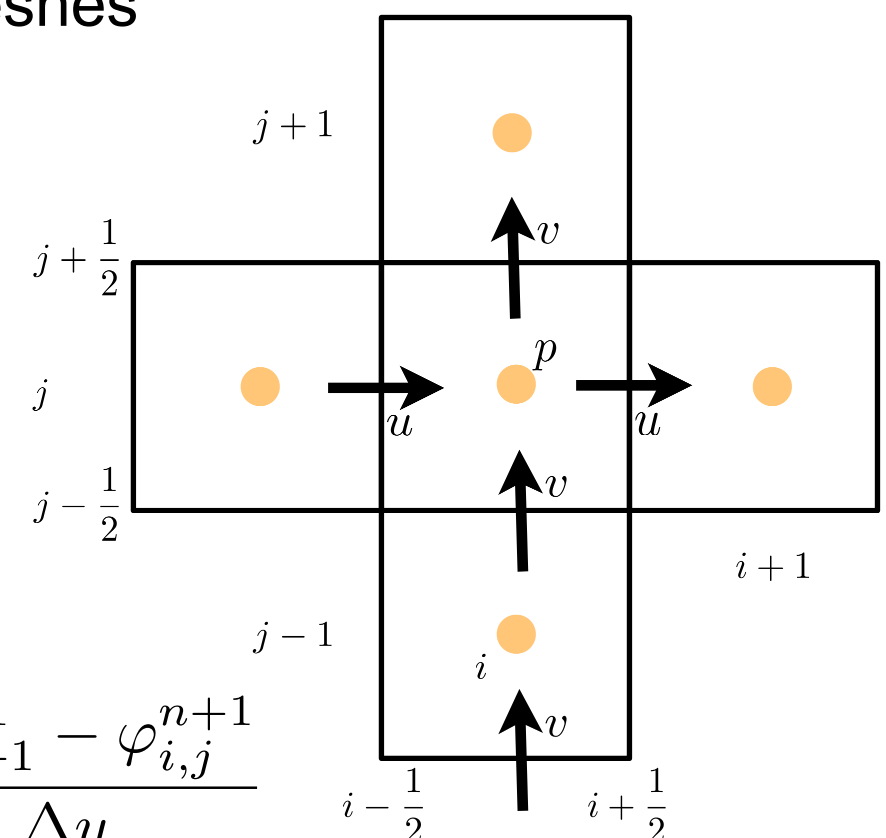
    ⇒ need div at (i,j):

$$\mathrm{div}(\vec{v}^*) = \frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{\Delta y}$$

  - step 3: $\vec{v}^{n+1} = \vec{v}^* - \Delta t\ \mathrm{grad}\ \varphi^{n+1}$

    ⇒ need grad$_x$ at (i+1/2,j) and grad$_y$ at (i,j+1/2):

$$\mathrm{grad}_x(\vec{\varphi}^{n+1}) = \frac{\varphi^{n+1}_{i+1,j} - \varphi^{n+1}_{i,j}}{\Delta x} \qquad \mathrm{grad}_y(\vec{\varphi}^{n+1}) = \frac{\varphi^{n+1}_{i,j+1} - \varphi^{n+1}_{i,j}}{\Delta y}$$



  - thus div(grad) at (i,j) in step 2 is

$$\mathrm{div}(\mathrm{grad}(\vec{\varphi}^{n+1})) = \frac{\mathrm{grad}_x(\varphi)^{n+1}_{i+\frac{1}{2},j} - \mathrm{grad}_x(\varphi)^{n+1}_{i-\frac{1}{2},j}}{\Delta x} + \frac{\mathrm{grad}_y(\varphi)^{n+1}_{i,j+\frac{1}{2}} - \mathrm{grad}_y(\varphi)^{n+1}_{i,j-\frac{1}{2}}}{\Delta y}$$

$$\mathrm{div}(\mathrm{grad}(\vec{\varphi}^{n+1})) = \frac{\frac{\varphi^{n+1}_{i+1,j} - \varphi^{n+1}_{i,j}}{\Delta x} - \frac{\varphi^{n+1}_{i,j} - \varphi^{n+1}_{i-1,j}}{\Delta x}}{\Delta x} + \frac{\frac{\varphi^{n+1}_{i,j+1} - \varphi^{n+1}_{i,j}}{\Delta y} - \frac{\varphi^{n+1}_{i,j} - \varphi^{n+1}_{i,j-1}}{\Delta y}}{\Delta y}$$

$$\mathrm{div}(\mathrm{grad}(\vec{\varphi}^{n+1})) = \frac{\delta_x^2 \varphi^{n+1}_{i,j}}{\Delta x^2} + \frac{\delta_y^2 \varphi^{n+1}_{i,j}}{\Delta y^2}$$

Advanced Considerations

$$\nabla_h \cdot \nabla_h' \varphi^{n+1} = \nabla_h' \cdot \nabla_h \varphi^{n+1}$$ ⇒ true, only if $\nabla_h = \nabla_h'$ and $\nabla_h \cdot = \nabla_h' \cdot$

- let's look at collocated meshes
  - step 2: $\operatorname{div}\left(\operatorname{grad} \varphi^{n+1}\right) = \frac{1}{\Delta t} \operatorname{div}\left(\vec{v}^*\right)$

    ⇒ need div at (i,j):

$$\operatorname{div}(\vec{v}^*) = \frac{u^*_{i+1,j} - u^*_{i-1,j}}{2\Delta x} + \frac{v^*_{i,j+1} - v^*_{i,j-1}}{2\Delta y}$$

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    ⇒ need grad_x and grad_y at (i,j)

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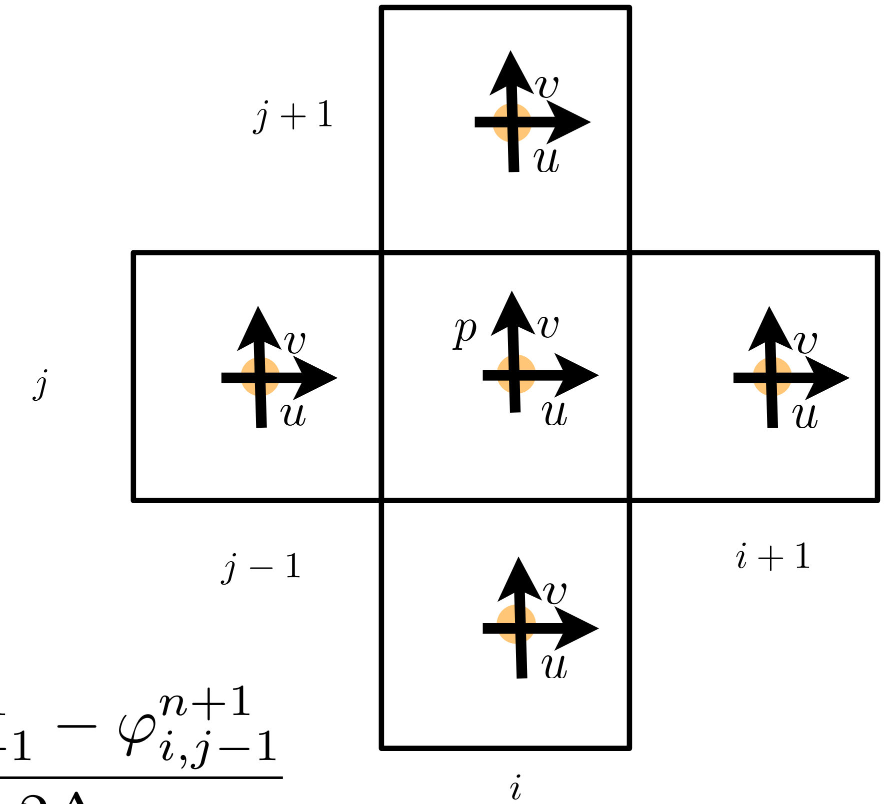

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$$\operatorname{div}(\operatorname{grad}(\vec{\varphi}^{n+1})) = \frac{\frac{\varphi^{n+1}_{i+2,j} - \varphi^{n+1}_{i,j}}{2\Delta x} - \frac{\varphi^{n+1}_{i,j} - \varphi^{n+1}_{i-2,j}}{2\Delta x}}{2\Delta x} + \frac{\frac{\varphi^{n+1}_{i,j+2} - \varphi^{n+1}_{i,j}}{2\Delta y} - \frac{\varphi^{n+1}_{i,j} - \varphi^{n+1}_{i,j-2}}{2\Delta y}}{2\Delta y}$$

$$\operatorname{div}(\operatorname{grad}(\vec{\varphi}^{n+1})) = \frac{\varphi^{n+1}_{i+2,j} - 2\varphi^{n+1}_{i,j} + 2\varphi^{n+1}_{i-2,j}}{4\Delta x^2} + \frac{\varphi^{n+1}_{i,j+2} - 2\varphi^{n+1}_{i,j} + 2\varphi^{n+1}_{i,j+2}}{4\Delta y^2}$$

Advanced Considerations

$$\nabla_h \cdot \nabla'_h \varphi^{n+1} = \nabla'_h \cdot \nabla_h \varphi^{n+1}$$ ⇒ true, only if $\nabla_h = \nabla'_h$ and $\nabla_h \cdot = \nabla'_h \cdot$

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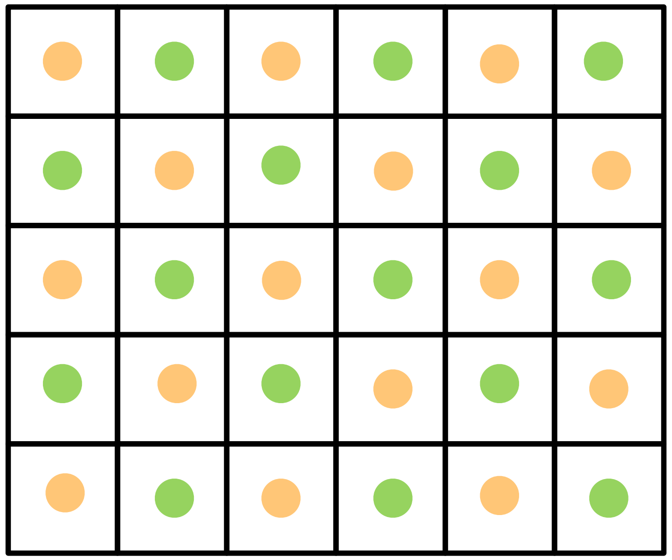
checker boarding!



  - step 3: $\vec{v}^{n+1} = \vec{v}^* - \Delta t\ \mathrm{grad}\ \varphi^{n+1}$

    ⇒ need grad_x and grad_y at (i,j)

    $$\mathrm{grad}_x(\vec{\varphi}^{n+1}) = \frac{\varphi^{n+1}_{i+1,j} - \varphi^{n+1}_{i-1,j}}{2\Delta x} \qquad \mathrm{grad}_y(\vec{\varphi}^{n+1}) = \frac{\varphi^{n+1}_{i,j+1} - \varphi^{n+1}_{i,j-1}}{2\Delta y}$$

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$$\mathrm{div}(\mathrm{grad}(\vec{\varphi}^{n+1})) = \frac{\frac{\varphi^{n+1}_{i+2,j}-\varphi^{n+1}_{i,j}}{2\Delta x} - \frac{\varphi^{n+1}_{i,j}-\varphi^{n+1}_{i-2,j}}{2\Delta x}}{2\Delta x} + \frac{\frac{\varphi^{n+1}_{i,j+2}-\varphi^{n+1}_{i,j}}{2\Delta y} - \frac{\varphi^{n+1}_{i,j}-\varphi^{n+1}_{i,j-2}}{2\Delta y}}{2\Delta y}$$

$$\mathrm{div}(\mathrm{grad}(\vec{\varphi}^{n+1})) = \frac{\varphi^{n+1}_{i+2,j} - 2\varphi^{n+1}_{i,j} + 2\varphi^{n+1}_{i-2,j}}{4\Delta x^2} + \frac{\varphi^{n+1}_{i,j+2} - 2\varphi^{n+1}_{i,j} + 2\varphi^{n+1}_{i,j+2}}{4\Delta y^2}$$

Advanced Considerations

$$\nabla_h \cdot \nabla_h' \varphi^{n+1} = \nabla_h' \cdot \nabla_h \varphi^{n+1}$$ $\Rightarrow$ true, only if $\nabla_h = \nabla_h'$ and $\nabla_h \cdot = \nabla_h' \cdot$

- explanation using linear algebra:
  - step 2: $\mathrm{div}\left(\mathrm{grad}\ \varphi^{n+1}\right) = \frac{1}{\Delta t}\mathrm{div}\left(\vec{v}^*\right)$

    $A\vec{\varphi}^{n+1} = \vec{b}$

    - what's the nullspace of A?

      ➡ for staggered meshes:

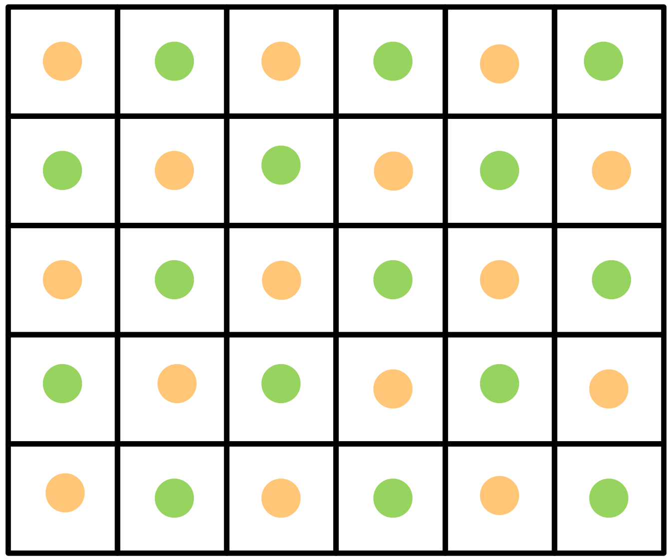
      $N(A) = \alpha\left(1, 1, 1, \ldots, 1\right)$

      ➡ for collocated meshes:

      basis of N(A) in 2D: $\left\{\widehat{\varphi}^0_{i,j} = 1;\ \widehat{\varphi}^1_{i,j} = -1^i;\ \widehat{\varphi}^2_{i,j} = -1^j;\ \widehat{\varphi}^2_{i,j} = -1^{i+j}\right\}$

      $\Rightarrow$ if φ^{n+1} is a solution, so is $\varphi^{n+1} + \sum_{l=0}^{2} a_l \widehat{\varphi}^l_{i,j}$

checker boarding!



➡ strategies to deal with this:

✓ Rhie-Chow interpolation (1983): adds dissipation $\Rightarrow$ destroys kinetic energy conservation

✓ Shashank et al., JCP (2010): find nullspace vector that minimizes local non-smoothness $\Rightarrow$ local least squares projection

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Short Note

A co-located incompressible Navier–Stokes solver with exact mass, momentum and kinetic energy conservation in the inviscid limit

Shashank *, Johan Larsson, Gianluca Iaccarino

Flow Physics and Computational Engineering, Department of Mechanical Engineering, Stanford University, CA, USA

A R T I C L E I N F O

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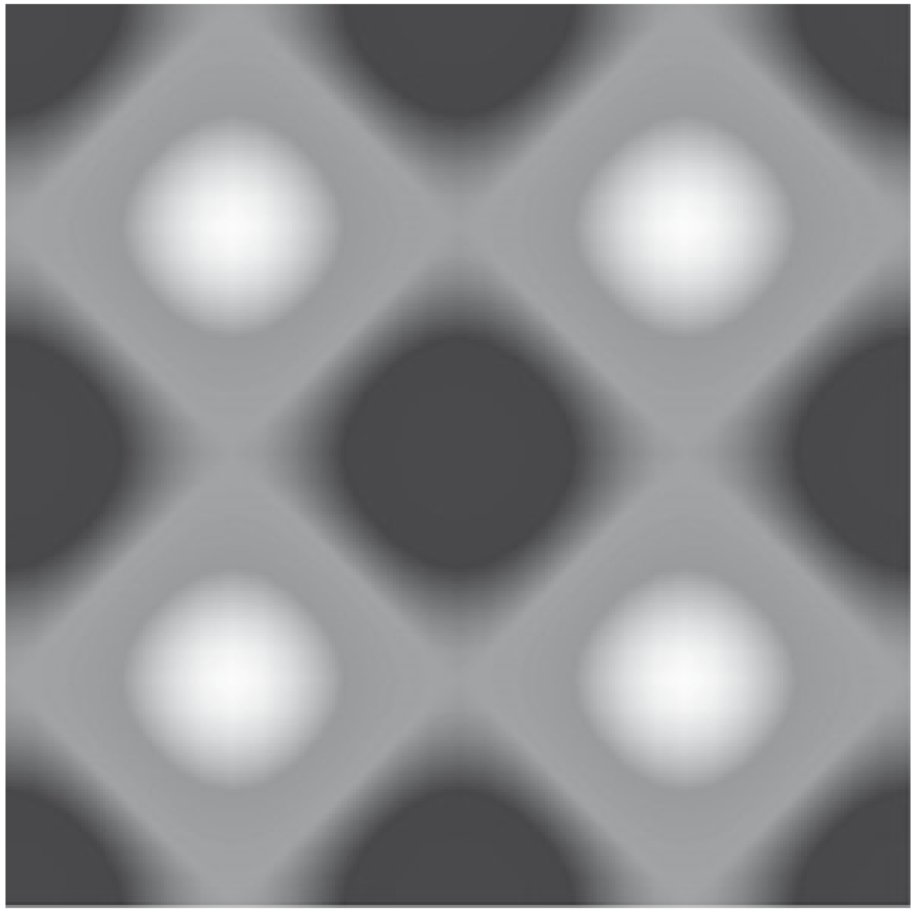
- Taylor Vortex
  - Test case with analytical solution to the 2D inviscid Navier-Stokes equations in periodic domains

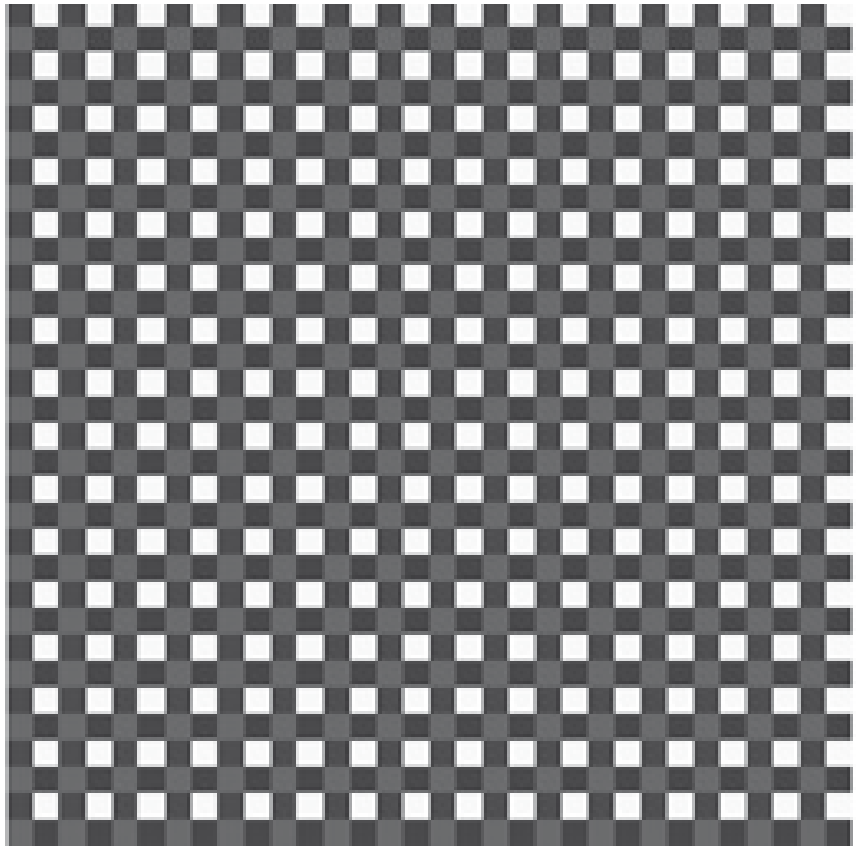
$$u = -\cos(\pi x)\sin(\pi y),$$
$$v = \sin(\pi x)\cos(\pi y),$$
$$p = -\frac{\cos(2\pi x) + \cos(2\pi y)}{4}$$

  - constant mass, momentum, and kinetic energy over time

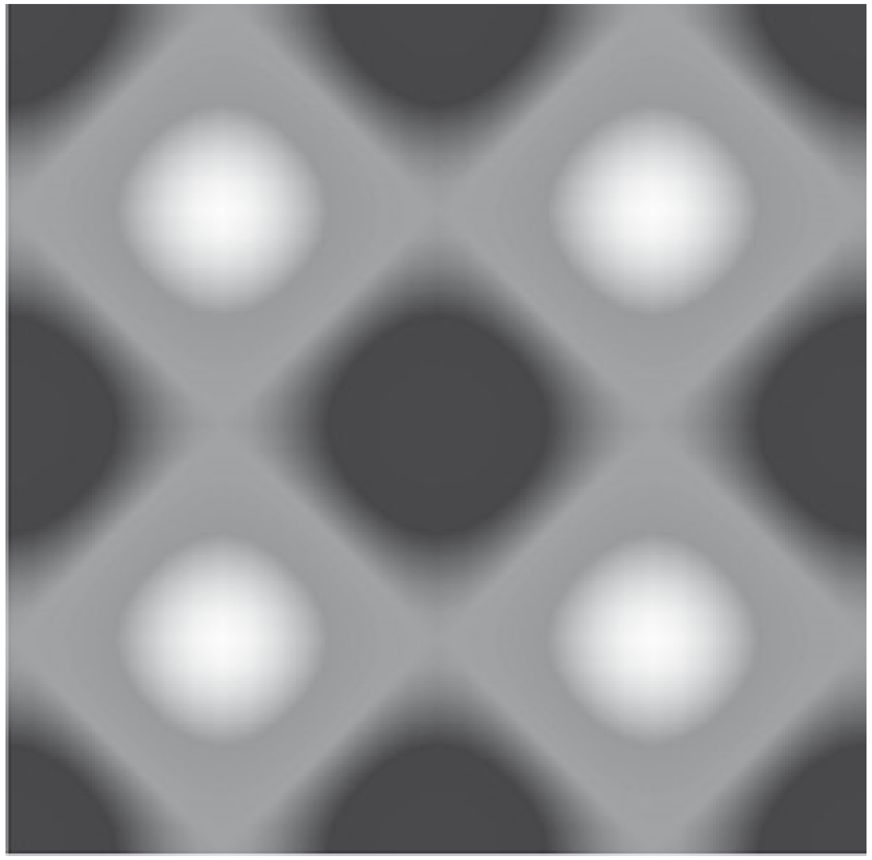
pressure contours



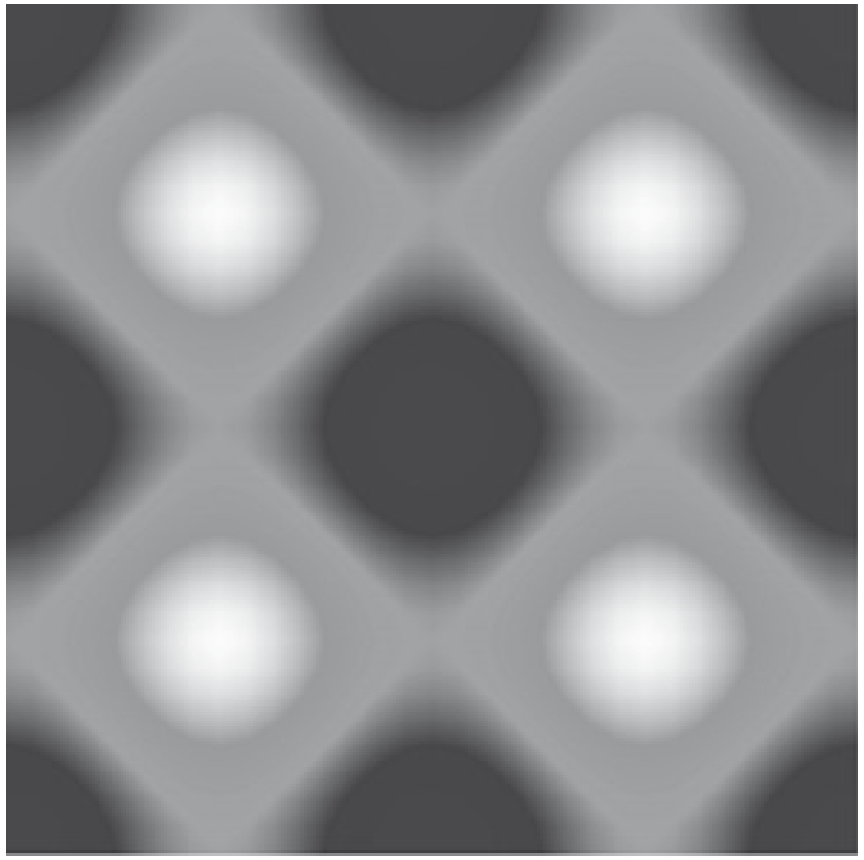
pressure contours



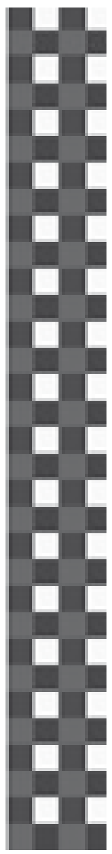
standard

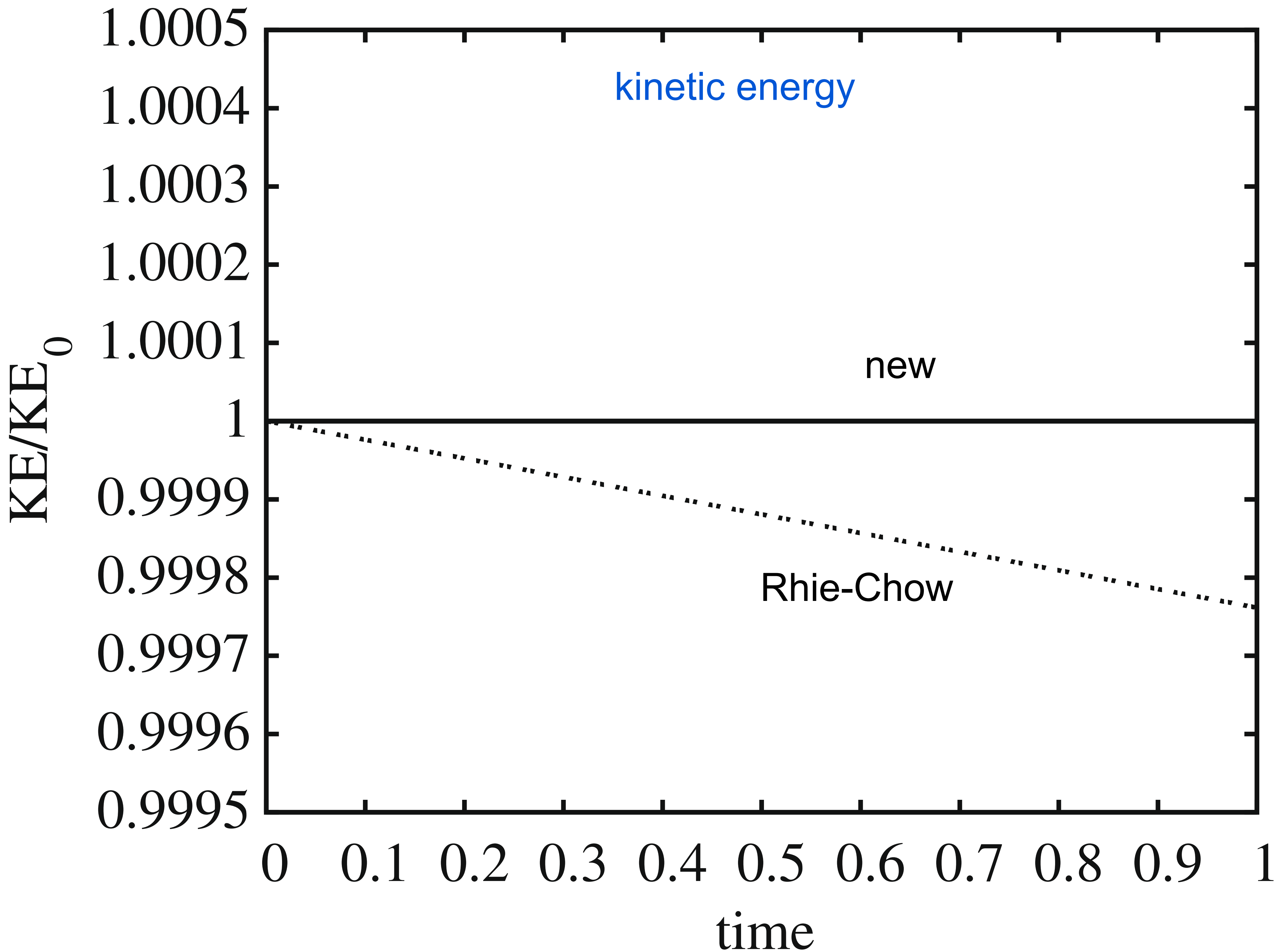


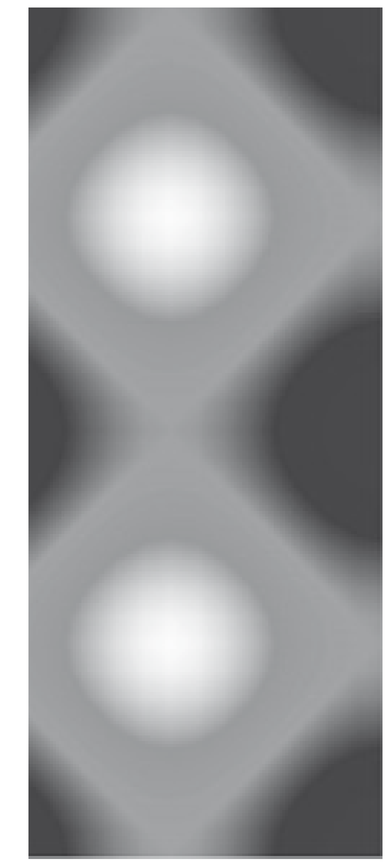
Rhie-Chow



new method

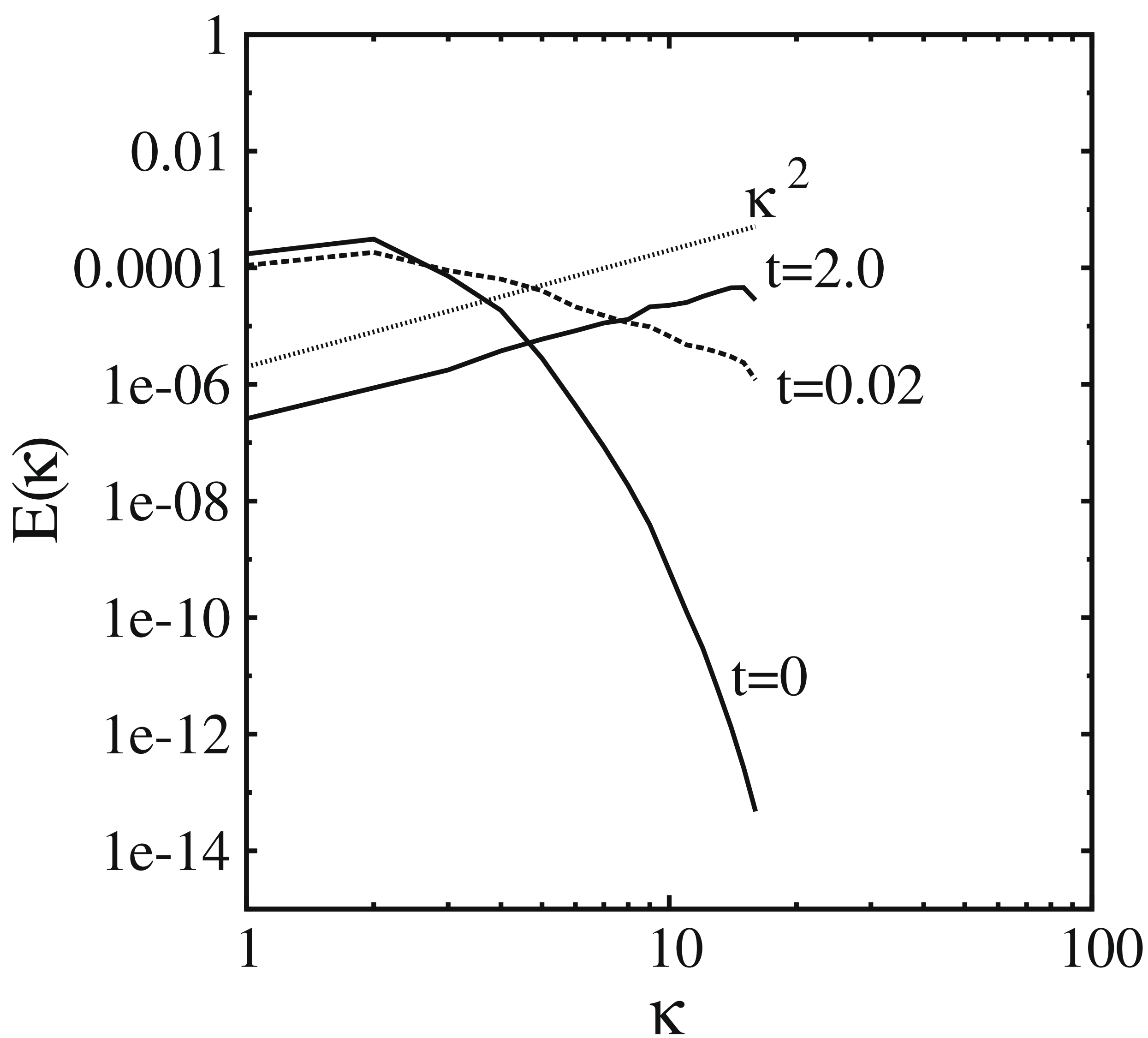



1.0005
1.0004
1.0003
1.0002
1.0001
1
0.9999
0.9998
0.9997
0.9996
0.9995
KE/KE$_0$
kinetic energy
new
Rhie-Chow
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time

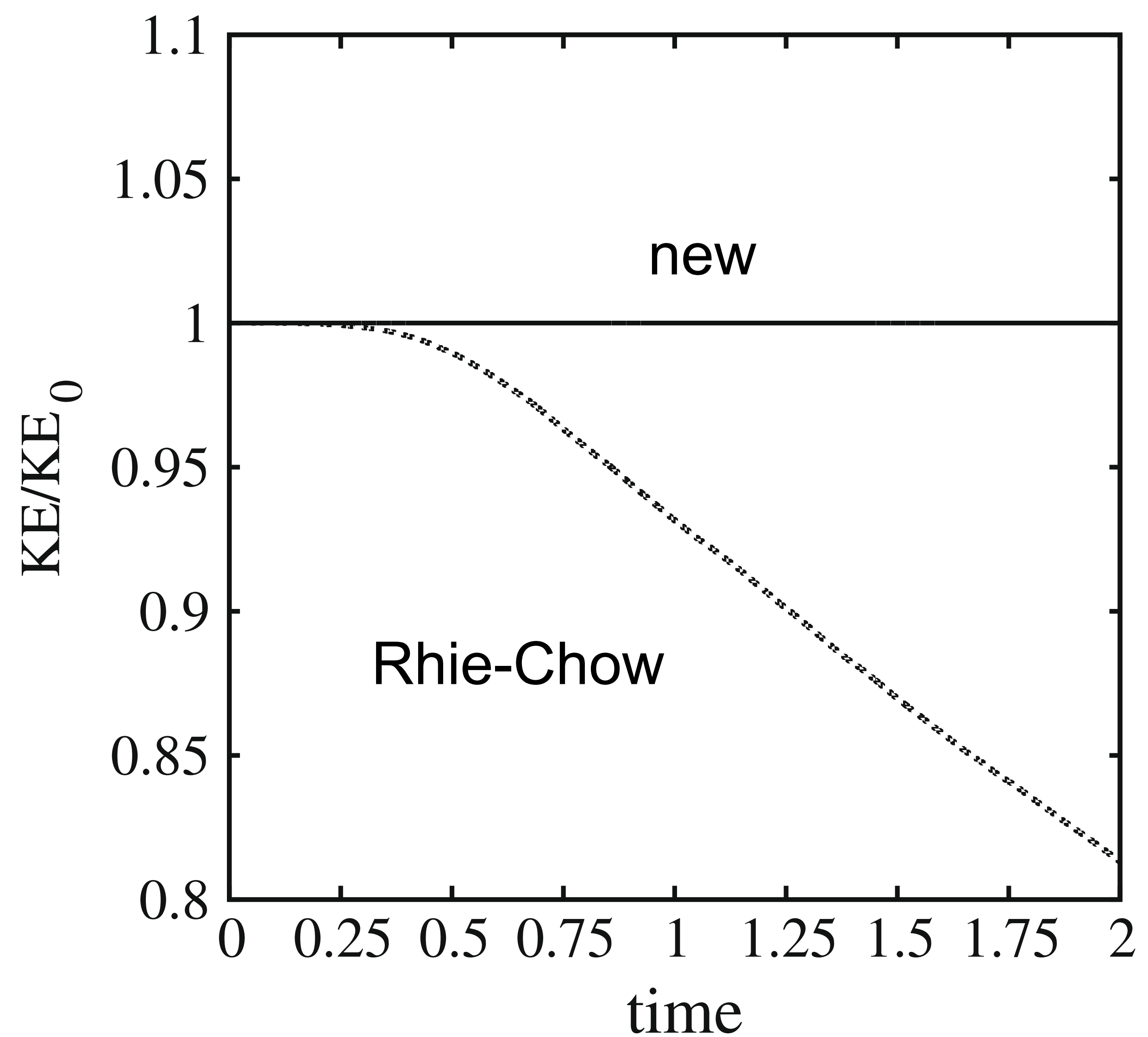


ethod

- Inviscid Turbulence in a 3D Periodic Box
  - Initial condition: Solenoidal random field with $E(\kappa) \propto \kappa^4 e^{-2\kappa^2}$



Temporal evolution of the energy spectrum



Temporal evolution of the total kinetic energy