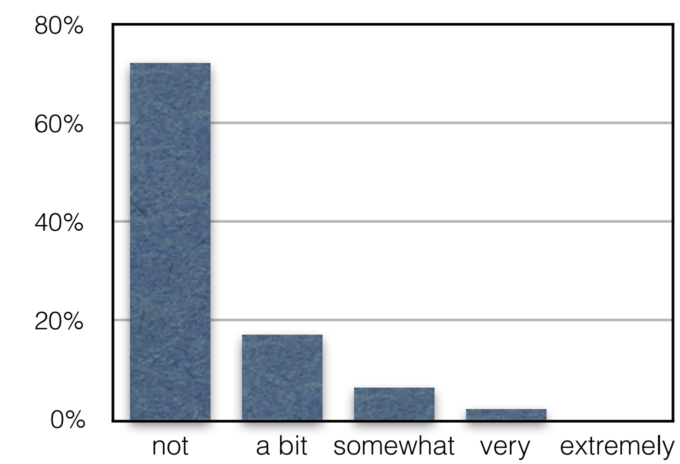
• Muddiest Points from Class 03/27

- *"The dispersive methods are appearing to show some wave properties not unlike a vibrating string (in 2D) when you show us the plots (for certain Courant Numbers). Is there a Courant Number that best correlates with these 'real' behaviors? After a certain point it looks like it loses this property."*
  - Dispersive errors appear if a method that does not propagate wave packets with the correct speeds
  - Constructive and deconstructive interference of waves (Fourier modes) generates the oscillatory behavior
- *"Are we going to learn how to solve 2D parabolic equations? Or we should be able to solve 2D equations with the method we already learned?"*
  - You already have (I hope): see HW 7 and associated class slides
  - If you meant 2D hyperbolic equations: then yes, today (and next week)
- *"Will the an even order error term always imply dissipative effects when using the standard grid points (i.e. not i+1/2 or i-1/2)? And conversely will odd ordered error term always imply dispersive?"*
  - Yes, if this refers to the order of the derivative of the leading order error term in the modified equation
  - Also, this has nothing to do with using half points or not
- *"If 1st Order Upwind methods are some of the most accurate methods to approximate nonlinear hyperbolic equations, does the choice between 1st Order Upwind and MacCormack come down to computation time for large scale simulations or other factors?"*
  - MacCormack is not TVD
  - Plus, there's an even better method: 2nd-order TVD covered today
- *"If we are using a 1st Order TVD Scheme for a non-linear equation that results in 2 solutions, is it guaranteed that one solution will have a decrease in entropy and one will have an increase? If not, is there another immediate way of telling which solution is the correct one?"*
  - Yes, with a so-called entropy fix covered today



1st-order TVD schemes

Is this scheme TVD? $\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - \left(E_i^n + E_{i-1}^n\right) + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

since $\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if } |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if } |\Delta u_{i+1/2}^n| < \epsilon' \end{cases} \Rightarrow E_{i+1}^n - E_i^n = \alpha_{i+1/2}^n \Delta u_{i+1/2}^n$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\alpha_{i+1/2}^n\Delta u_{i+1/2}^n - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n + \alpha_{i-1/2}^n\Delta u_{i-1/2}^n + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

is this TVD? compare to $u_i^{n+1} = u_i + A_{i+1/2}^n\Delta u_{i+1/2}^n - B_{i-1/2}^n\Delta u_{i-1/2}^n$

$A_{i+1/2}^n = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n| - \alpha_{i+1/2}^n\right)$ $\qquad$ $B_{i-1/2}^n = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i-1/2}^n| + \alpha_{i-1/2}^n\right)$

$\Rightarrow A_{i+1/2}^n \geq 0$ $\qquad$ $\Rightarrow B_{i-1/2}^n \geq 0$

$0 \leq A_{i+1/2}^n + B_{i+1/2}^n \leq 1$? $\qquad 0 \leq \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n| - \alpha_{i+1/2}^n + |\alpha_{i+1/2}^n| + \alpha_{i+1/2}^n\right) \leq 1$

$\boxed{0 \leq \frac{\Delta t}{\Delta x}|\alpha_{i+1/2}^n| \leq 1}$ $\Rightarrow$ Courant number requirement!

1st-order TVD schemes

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

or: $$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right)$$

central

dissipation

can implement this as multi-step

Step 1: $$u_i^* = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$$

Step 2: $$u_i^{n+1} = u_i^* - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\Phi_{i+1/2}^n - \Phi_{i-1/2}^n\right)$$

with $\Phi_{i+1/2}^n = -|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n$: flux limiter function

Code: C=0.5, C=0.1, C=1.0

One other issue for non-linear equations:

- weak solutions of the conservation laws may not be unique!
- How to pick the correct one?
  - make use of physics: 2nd law of thermodynamics
    $\Rightarrow$ entropy may not decrease
  - impose the entropy condition to get the physically correct solution

➡ need dissipative mechanism

- but that's exactly what α does in the numerical flux function!

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right)$$

- however, we may have *α = 0*. What then?

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

define $\psi = \begin{cases} |\alpha| & \text{if} \quad |\alpha| \geq \varepsilon \\ \frac{\alpha^2+\varepsilon^2}{2\varepsilon} & \text{if} \quad |\alpha| < \varepsilon \end{cases}$ with $0 \leq \varepsilon \leq \frac{1}{8}$

- use ψ instead of α in the numerical flux function h

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\psi_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

choose $\varepsilon = 0.1$

entropy fix

We now have a 1st-order TVD method, but would prefer at least a 2nd-order TVD method

- idea by Harten: let's modify the flux E by replacing it with $\overline{E} = E + G$
- let's start with a formulation we had earlier (slide 6):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

introduce flux limiter function: $\Phi_{i+1/2}^n = -|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n$

We now have a 1st-order TVD method, but would prefer at least a 2nd-order TVD method

- idea by Harten: let's modify the flux E by replacing it with $\overline{E} = E + G$
- let's start with a formulation we had earlier (slide 6):

$$\boxed{\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right) \\ h_{i+1/2}^n &= \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) + \Phi_{i+1/2}^n\right] \end{aligned}}$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) + \Phi_{i+1/2}^n\right]$$

- Harten-Yee-limiter:

$$\Phi_{i+1/2}^n = (G_{i+1}^n + G_i^n) - \psi(\alpha_{i+1/2}^n + \beta_{i+1/2}^n)\Delta u_{i+1/2}^n$$

with $\psi(y) = \begin{cases} |y| & \text{if} \quad |y| \geq \varepsilon \\ \frac{y^2+\varepsilon^2}{2\varepsilon} & \text{if} \quad |y| < \varepsilon \end{cases}$ $\quad 0 \leq \varepsilon \leq \frac{1}{8}$ (entropy fix function) choose $\varepsilon = 0.1$

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

(same as before)

$$\beta_{i+1/2}^n = \begin{cases} \frac{G_{i+1}^n - G_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ 0 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

but what's G? many many choices!

for example $G_i^n = S \cdot \max(0, \min(\sigma_{i+1/2}|\Delta u_{i+1/2}^n|, S \cdot \sigma_{i-1/2}\Delta u_{i-1/2}^n))$

where $S = \mathrm{sign}(\Delta u_{i+1/2}^n)$ and $\sigma_{i+1/2} = \frac{1}{2}\left[\psi(\alpha_{i+1/2}^n) - \frac{\Delta t}{\Delta x}(\alpha_{i+1/2}^n)^2\right]$

other choices: see book

Code: C=0.5, C=0.1, C=1.0

Higher Dimensions for Hyperbolic Equations

$$C_x = \frac{a\Delta t}{\Delta x} \qquad C_y = \frac{b\Delta t}{\Delta y}$$

- let's start with linear hyperbolic equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \qquad\qquad a > 0\,,\ b > 0$$

- Stability? for example for 1st-order upwind?

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = -a\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} - b\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y}$$

- von Neuman stability analysis

$$\rho^{n+1} = \rho^n - C_x\rho^n\left(1 - e^{-ik\Delta x}\right) - C_y\rho^n\left(1 - e^{-ik\Delta y}\right)$$

$$\Rightarrow \quad G = \frac{\rho^{n+1}}{\rho^n} = 1 - C_x\left(1 - e^{-ik\Delta x}\right) - C_y\left(1 - e^{-ik\Delta y}\right)$$

$$= 1 - C_x\left(1 - \cos(k\Delta x) + i\sin(k\Delta x)\right) - C_y\left(1 - \cos(k\Delta y) + i\sin(k\Delta y)\right)$$

worst case:

$$|G|^2 = \left[1 - C_x\left(1 - \cos(k\Delta x)\right) - C_y\left(1 - \cos(k\Delta x)\right)\right]^2 + \left[C_x\sin(k\Delta x) + C_y\sin(k\Delta y)\right]^2$$

[....] $C_x + C_y \leq 1 \quad \Rightarrow \quad \boxed{\frac{a\Delta t}{\Delta x} + \frac{b\Delta t}{\Delta y} \leq 1}$ (valid for linear equations only) so far, we've done:

- Poisson equation: elliptic equations
- viscous terms: parabolic equations
- convective terms: hyperbolic equations

next:

- combine convective and viscous terms

Viscous Burger's Equation

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$
$$\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (vv)}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

or:

$$\frac{\partial \vec{Q}}{\partial t} + \frac{\partial \vec{E}}{\partial x} + \frac{\partial \vec{F}}{\partial y} = \nu \left(\frac{\partial^2 \vec{Q}}{\partial x^2} + \frac{\partial^2 \vec{Q}}{\partial y^2} \right) \text{ with } \vec{Q} = \begin{bmatrix} u \\ v \end{bmatrix};\ \vec{E} = \begin{bmatrix} u^2 \\ uv \end{bmatrix};\ \vec{F} = \begin{bmatrix} uv \\ v^2 \end{bmatrix}$$

Note: This is **NOT** Navier-Stokes!

In the following, we'll look at 1D cases first, but doing 2D is straightforward but let's start with an even simpler model problem $C = \frac{a\Delta t}{\Delta x}$ $d = \frac{\alpha \Delta t}{\Delta x^2}$

- linear 1D: $\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2}$

FTCS

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

- Stability? $u_i^{n+1} = u_i^n - \frac{C}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$

  - von Neumann stability analysis: $u_i^n = \rho^n e^{ikx_i}$

$$\rho^{n+1} = \rho^n - \frac{C}{2}\rho^n \underbrace{\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}_{2i\sin(k\Delta x)} + d\ \rho^n \underbrace{\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)}_{2\cos(k\Delta x) - 2}$$

$$G = \frac{\rho^{n+1}}{\rho^n} = 1 - iC\sin(k\Delta x) - 2d\left(1 - \cos(k\Delta x)\right)$$

  - stable if $|G|^2 \le 1$: $\left(1 - 2d\left(1 - \cos(k\Delta x)\right)\right)^2 + C^2\sin^2(k\Delta x) \le 1$ [...]

$C \le 1$ and $d \le \frac{1}{2}$ and $C \le 2d$

$\frac{a\Delta t}{\Delta x} \le 2\frac{\alpha\Delta t}{\Delta x^2}$ $\Rightarrow$ $\frac{a\Delta x}{\alpha} \le 2$ cell Reynolds number: $\mathrm{Re}_c = \frac{a\Delta x}{\alpha} = \frac{C}{d}$

but let's start with an even simpler model problem $C = \dfrac{a\Delta t}{\Delta x}$ $d = \dfrac{\alpha \Delta t}{\Delta x^2}$

- linear 1D: $\dfrac{\partial u}{\partial t} = -a\dfrac{\partial u}{\partial x} + \alpha\dfrac{\partial^2 u}{\partial x^2}$

FTCS

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

- Stability?

cell Reynolds number: $\mathrm{Re}_c = \dfrac{a\Delta x}{\alpha} = \dfrac{C}{d}$

$$\boxed{\mathrm{Re}_c \le 2 \quad \text{and} \quad d \le \frac{1}{2} \quad \text{and} \quad C \le 1}$$

- Remember: previously we found that FTCS for hyperbolic equations is **always** unstable!
- Check: pure hyperbolic $\Rightarrow \alpha = 0$ $\Rightarrow Re_c = \infty$ $\Rightarrow$ always unstable

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha \Delta t}{\Delta x^2}$$

FTBCS $a > 0$

1st-order upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x} + \alpha\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

- large dissipative errors in convective term
- dissipative errors may be larger than physical viscous term!

Remedy:

- could go 2nd-order ➙ dispersive errors
- could go 3rd-order ➙ ENO-3
- could go 5th-order ➙ WENO-5

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

Du-Fort Frankel

central in time and space with fix in viscous term

$$\boxed{\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \alpha\frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2}}$$

- Order: $O\left((\Delta t)^2, (\Delta x)^2, \left(\frac{\Delta t}{\Delta x}\right)^2\right)$
- startup problem!

MacCormack

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

step 1: $$u_i^* = u_i^n - \frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

step 2: $$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{a\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right)\right]$$

alternate form:

step 1: $$\Delta u_i^n = -\frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$u_i^* = u_i^n + \Delta u_i^n$$

step 2: $$\Delta u_i^* = -\frac{a\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right)$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* + \Delta u_i^*\right]$$

- Stability: $\Delta t \leq \dfrac{1}{\frac{a}{\Delta x} + \frac{2\alpha}{\Delta x^2}}$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

BTCS

Implicit & central in space

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}}$$

$$\Leftrightarrow \quad -\left(\frac{C}{2} + d\right)u_{i-1}^{n+1} + (1+2d)\,u_i^{n+1} + \left(\frac{C}{2} - d\right)u_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!

BTBCS $a > 0$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

Implicit & upwind for convective, central for viscous terms

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}} \qquad \text{(1}^{\text{st}}\text{-order)}$$

$$\Leftrightarrow \quad -(C+d)\,u_{i-1}^{n+1} + (1 + C + 2d)\,u_i^{n+1} - du_{i+1}^{n+1} = u_i^n$$

- tri-diagonal!
- but again: dissipative errors may dominate physical viscous forces!
- go 2nd-order upwind for the convective term:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{3u_i^{n+1} - 4u_{i-1}^{n+1} + u_{i-2}^{n+1}}{2\Delta x} + \alpha\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

$$\Leftrightarrow \quad \frac{C}{2}u_{i-2}^{n+1} - (2C+d)\,u_{i-1}^{n+1} + \left(1 + \frac{3}{2}C + 2d\right)u_i^{n+1} - du_{i-1}^{n+1} = u_i^n$$

- no-longer tri-diagonal! $\Rightarrow$ time-lag u_{i-2} to recover tri-diagonal structure

$$\Leftrightarrow \quad -(2C+d)\,u_{i-1}^{n+1} + \left(1 + \frac{3}{2}C + 2d\right)u_i^{n+1} - du_{i-1}^{n+1} = u_i^n - \frac{C}{2}u_{i-2}^n$$

- go 3rd-order upwind: again time-lag u_{i-2} to recover tri-diagonal structure

Crank-Nicholson

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha\Delta t}{\Delta x^2}$$

shorthands:

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \qquad \delta_x u_i = u_{i+1} - u_{i-1}$$

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2}\left(\frac{\delta_x u_i^n}{2\Delta x} + \frac{\delta_x u_i^{n+1}}{2\Delta x}\right) + \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)}$$

$$\Leftrightarrow \quad u_i^{n+1} + \frac{a\Delta t}{2}\frac{\delta_x u_i^{n+1}}{2\Delta x} - \frac{\alpha\Delta t}{2}\frac{\delta_x^2 u_i^{n+1}}{\Delta x^2} = u_i^n - \frac{a\Delta t}{2}\frac{\delta_x u_i^n}{2\Delta x} + \frac{\alpha\Delta t}{2}\frac{\delta_x^2 u_i^n}{\Delta x^2}$$

Mixed Treatment

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} \qquad C = \frac{a\Delta t}{\Delta x} \qquad d = \frac{\alpha \Delta t}{\Delta x^2}$$

Idea: apply different schemes to the individual terms

Goal: $O(\Delta t^2, \Delta x^2)$

- convective terms
  - central in space: $$\frac{\partial u}{\partial t} = -a\frac{u_{i+1} - u_{i-1}}{2\Delta x} = H_i$$
  - Adams-Bashforth in time: $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1}$$
- diffusive terms
  - central in space: $$\frac{\partial u}{\partial t} = \alpha\frac{\delta_x^2 u_i}{\Delta x^2}$$
  - Crank-Nicholson in time: $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)$$
- combine: $$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{3}{2}H_i^n - \frac{1}{2}H_i^{n-1} + \frac{\alpha}{2}\left(\frac{\delta_x^2 u_i^n}{\Delta x^2} + \frac{\delta_x^2 u_i^{n+1}}{\Delta x^2}\right)}$$
- since viscous terms are implicit ⇒ no viscous time step restriction!