MAE 471/561

Homework #7 - Due: March 25th, at the beginning of class

Please submit result graphs together with either handwritten or printed out descriptions, equations, and answers at the beginning of class. Combine all code you used to solve the problems into a single text file, and upload the text file to Blackboard using the SafeAssign mechanism. No credit will be given, if your code is not uploaded as a text file using SafeAssign. Also ensure that your code contains adequate comments. Add a printout of all code as an **appendix** to your submission.

Problem 1 (100 points, AEE 471: Core Course Outcome #2)
Consider the following two-dimensional non-dimensional PDE

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + q(x, y, t) \,, \tag{1}$$

defined on a domain of size $[-1/2 \leq x \leq 1/2] \times [-1/2 \leq y \leq 1/2]$ with Neumann boundary conditions $\partial T/\partial n = 0$, where n is the boundary normal direction, initial condition $T(x, y, t = 0) = 1$, and $\alpha = 0.1$. The source term q is given by

$$q(x, y, t) = \begin{cases} \frac{1}{2}\sin(\frac{\pi}{2}t)\left(1 + \cos(4\pi\sqrt{x^2 + y^2})\right) & : \quad \sqrt{x^2 + y^2} \leq \frac{1}{4} \\ 0 & : \quad \sqrt{x^2 + y^2} > \frac{1}{4} \end{cases} \,. \tag{2}$$

1. Write in index form for a cell centered mesh of $M \times M$ elements the equation to solve the PDE using the FTCS method for cell center i, j and time level $n + 1$, i.e. $T^{n+1}_{i,j} = \ldots$, the index equations to calculate ghost cell values, and the equation to determine the maximum stable time step size. No points will be given if the formulas merely appear in code.

2. Using a **cell centered** mesh with $M = 64$ and the FTCS method, calculate and plot as a surface plot $T(x, y)$ at $t = 1, 2, 3, 4$. In an additional figure, plot the maximum temperature versus time for $0 \leq t \leq 12$. Use a time step size Δt that is **half** the maximum stable time step size. To help you debug your code, on Blackboard you will find the file `ftcs.txt` that contains the solution variables for the first 4 time steps using $M = 16$.

3. Write in index form for a cell centered mesh the equation to solve the PDE using the Crank-Nicholson method for cell center i, j and time level $n + 1$. **Do not** use a two-step process. Derive in index form the iterative equation for a Gauss-Seidel method for the resulting system of linear equations in the form $T^{n+1^{(k+1)}}_{i,j} = \ldots$. Use ghost cells for the Gauss Seidel method. Write in index form the equation to calculate the residual at every interior mesh point, making use of ghost cell values. No points will be given if the formulas merely appear in code.

4. **Required for MAE561, Bonus for AEE471**: Using a **cell centered** mesh with $M = 128$ and the Crank-Nicholson method, calculate and plot as a surface plot $T(x, y)$ at $t = 1, 2, 3, 4$. Use a residual convergence criterium of at least 1e-5. Add to the corresponding graph of task 2, the maximum temperature versus time for $0 \leq t \leq 12$ **using only symbols** and not lines. Use a time step size $\Delta t = 0.1$. **Upload your working code to Blackboard using the homework link, in addition to uploading all combined code to SafeAssign**. To help you debug your code, on Blackboard you will find the file `cn.txt` that contains the solution variables for the first 4 time steps using $M = 16$ and a residual convergence criterium of 1e-5.

5. **Bonus for AEE471/MAE561**: Modify the V-cycle multigrid solver from homework 5 to solve the above PDE. You should only need to modify the Gauss-Seidel and residual routines to make this work. Using a **cell centered** mesh with $M = 256$ and the Crank-Nicholson method, calculate and plot as a surface plot $T(x, y)$ at $t = 1, 2, 3, 4$. Add to the corresponding graph of task 2, the maximum temperature versus time for $0 \leq t \leq 12$ **using only symbols** and not lines. Use a time step size $\Delta t = 0.1$.

Required submission:
- Handwritten or typeset FTCS method in index form, ghost cell boundary conditions in index form, maximum time step formula;
- Clearly annotated surface plots of $T(x, y)$ at t = 1, 2, 3, and 4, for $M = 64$ using the FTCS method;
- Clearly annotated plot of maximum temperature vs time t for $0 \leq t \leq 12$ for $M = 64$ using the FTCS method;

- Handwritten or typeset Crank-Nicholson method in index form, iterative equation for Gauss Seidel method of Crank-Nicholson time step in index form, and corresponding residual equation in index form.

- MAE561: Clearly annotated surface plots of $T(x, y)$ *at* $t =$ *1, 2, 3, and 4, for* $M = 128$ *using the Crank-Nicholson method;*

- MAE561: Clearly annotated plot of maximum temperature vs time t *for* $0 \leq t \leq 12$ *for* $M = 64$ *using FTCS (use line) and* $M = 128$ *using the Crank-Nicholson method (must use symbols);*

- MAE561: Upload of working code to Homework assignment link. - Bonus: Clearly annotated surface plots of $T(x, y)$ *at* $t =$ *1, 2, 3, and 4, for* $M = 256$ *using the Crank-Nicholson method solved with V-cycle multigrid;*

- Bonus: Clearly annotated plot of maximum temperature vs time t *for* $0 \leq t \leq 12$ *for* $M = 64$ *using FTCS (use line) and* $M = 128$ *using the Crank-Nicholson method solved using Gauss-Seidel (must use symbols) and* $M = 256$ *using the Crank-Nicholson method solved with V-cycle multigrid (must use symbols);*

- As an Appendix, printout of code;

- SafeAssign upload of all used, well commented code.