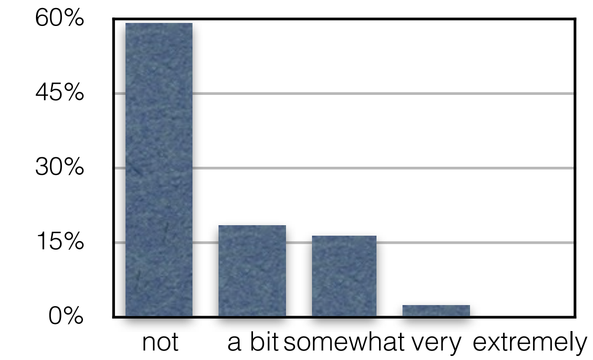
Muddiest Points from Class 03/22

- *"I didn't quite understand the selection of stencil points in the ENO and WENO schemes. Eg: Δ- ui = ui - ui-1 But how do we calculate Δ- Δ- ui?"*
- *"Is the delta+ delta- operator the difference between a point and its next neighbor or a point and its next unused point in the stencil neighbor? Ex. perhaps you can expand the example on class 19 slide 4 to clarify?"*

$$\Delta^-\Delta^+ u_i = \Delta^-\left(\Delta^+ u_i\right) = \Delta^-\left(u_{i+1} - u_i\right) = \Delta^- u_{i+1} - \Delta^- u_i = u_{i+1} - u_i - \left(u_i - u_{i-1}\right)$$

- *"Is it the leading order of truncated error that makes the method to be dispersive or diffusive and if so why do we use odd ordered methods (ENO-3 & WENO-5) and not even ordered methods to reduce the effect of dispersion?"*
  - Yes, it is the leading order error term
  - Because in some cases, diffusive errors are even less desirable
  - Also physical systems usually provide physical diffusion, providing some control of dispersive errors
- *"If you increase the ENO stencil size to 5 will you have more possible stencil combos and thus more calculations to do in WENO?"*
  - Yes, more possible stencils and more calculations, but the results is also a 9th order method
- *"What are some ways to use less memory for variables when coding WENO methods, particularly in 2D cases?"*
  - there are special so-called low storage versions of RK schemes (not covered in this class)
- *"On slide 10. For a > 0 it says we should use a specific psi_WENO formula. Do we use that same formulation of psi_WENO if a < 0?"*
  - Yes
- *"What is the meaning of +/- sign here as a superscript of dϕ/dx? Is it just relative to the sign of a?"*
  - Yes, it is just upwind with respect to sign of a
- *"What is the epsilon in the alpha terms of WENO-5?"*
  - A small number to avoid division by zero. Use 1e-6.



Lax-Method

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = \overline{u_i^n} - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) \quad \text{with} \quad \overline{u_i^n} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)$$

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)}$$

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{4\Delta x}\left(\left(u_{i+1}^n\right)^2 - \left(u_{i-1}^n\right)^2\right)}$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^2 u}{\partial x^2}$ ⇒ dissipative

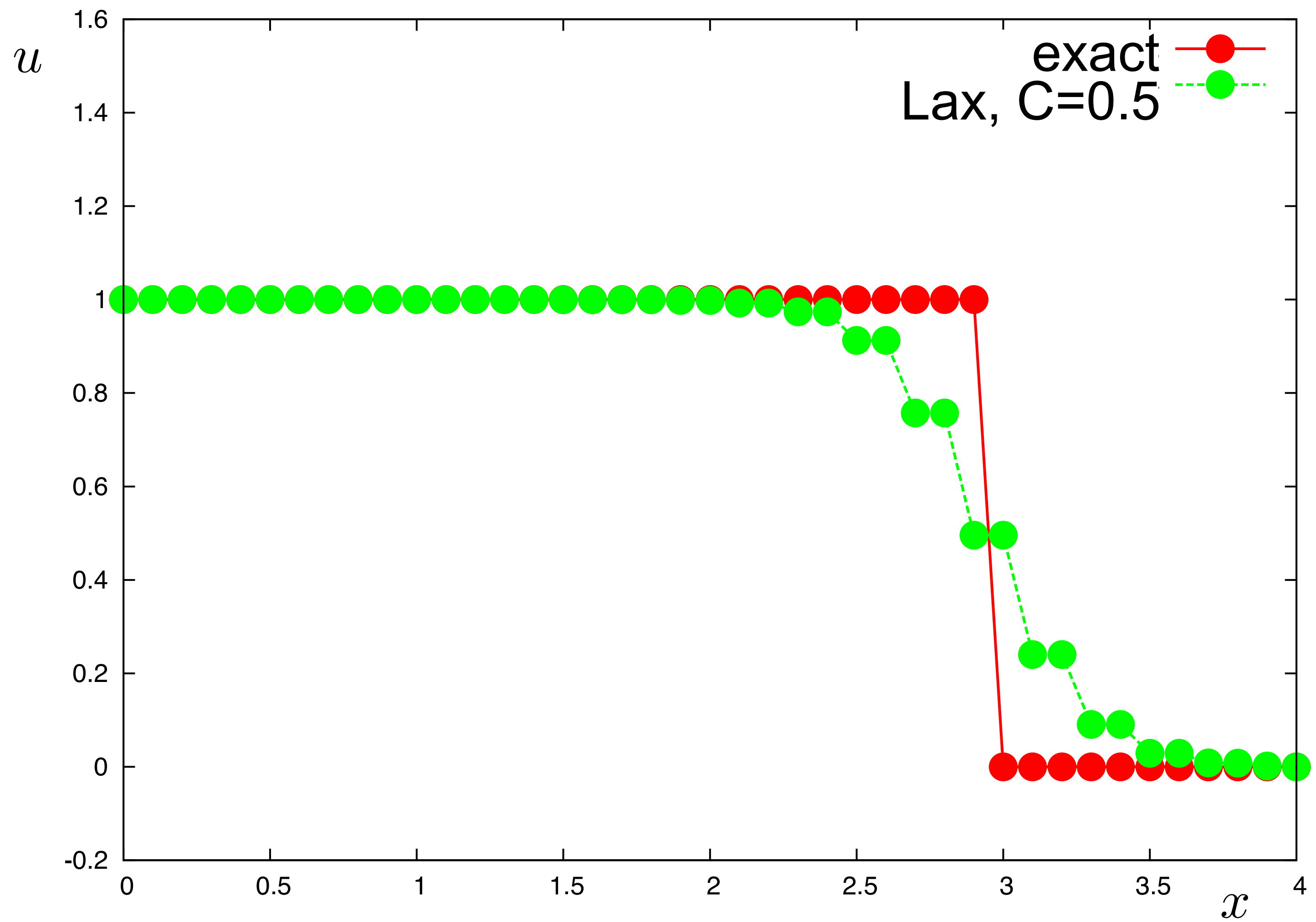
$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

- Example:

$$0 \le x \le 4\,, \quad M = 40$$

initial condition: $u(x, t=0) = \begin{cases} 1 & x \le 2 \\ 0 & x > 2 \end{cases}$

boundary conditions: $u(x=0,t) = 1\,, \quad u(x=4,t) = 0$



Code:
C=0.5,
C=1,
C=0.1

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

Idea: Start from Taylor series at t^n for t^{n+1}

$$u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3)$$

use PDE: $\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x}$ $\qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\frac{\partial E}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t}\right)$

but: $\frac{\partial E}{\partial t} = \frac{\partial E}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial E}{\partial u}\left(-\frac{\partial E}{\partial x}\right) = -A\frac{\partial E}{\partial x}$

define Jacobian as $A = \frac{\partial E}{\partial u}$ $\qquad$ here: $A = \frac{\partial E}{\partial u} = \frac{\partial(\frac{1}{2}u^2)}{\partial u} = u$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right)$$

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \Delta t\frac{\partial E}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right) + O(\Delta t^3)$$

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \Delta t \frac{\partial E}{\partial x} + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial E}{\partial x}\right) + O(\Delta t^3)$$

central (first derivative term); midpoint central (second derivative term)

$$u_i^{n+1} = u_i^n - \Delta t \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \frac{\Delta t^2}{2}\frac{\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n - \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n}{\Delta x} + O(\Delta t^3)$$

$$\left(A\frac{\partial E}{\partial x}\right)_{i+1/2}^n = A_{i+1/2}^n \frac{E_{i+1}^n - E_i^n}{\Delta x} \qquad \left(A\frac{\partial E}{\partial x}\right)_{i-1/2}^n = A_{i-1/2}^n \frac{E_i^n - E_{i-1}^n}{\Delta x}$$

$$A_{i+1/2}^n = \frac{1}{2}\left(A_i^n + A_{i+1}^n\right) = \frac{1}{2}\left(u_i^n + u_{i+1}^n\right) \qquad A_{i-1/2}^n = \frac{1}{2}\left(A_i^n + A_{i-1}^n\right) = \frac{1}{2}\left(u_i^n + u_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\left[\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\right]$$

Lax-Wendroff

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}\left(\left(u_{i+1}^n + u_i^n\right)\left(E_{i+1}^n - E_i^n\right) - \left(u_i^n + u_{i-1}^n\right)\left(E_i^n - E_{i-1}^n\right)\right)$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^3 u}{\partial x^3}$ ⇒ dispersive
- dispersive errors are smallest for $C = \frac{\Delta t}{\Delta x}\max\left(|u|\right) = 1$, they increase for smaller C

Code:
C=0.5,
C=1, C=0.1

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

MacCormack

$$u_i^* = u_i^n + -\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right)$$
$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{\Delta t}{\Delta x}\left(E_i^* - E_{i-1}^*\right)\right]$$

- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: $\frac{\partial^3 u}{\partial x^3}$ $\Rightarrow$ dispersive
- quite good in general!
- dispersive errors are smallest for $C = \frac{\Delta t}{\Delta x}\max\left(|u|\right) = 1$, they increase for smaller C
- different from Lax-Wendroff

Code:
C=0.5,
C=1, C=0.1

Code: cmp. C=0.5

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order Explicit Upwind

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(E_i^n - E_{i-1}^n\right) \quad \text{for} \quad u_i^n > 0$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_i^n\right) \quad \text{for} \quad u_i^n < 0$$

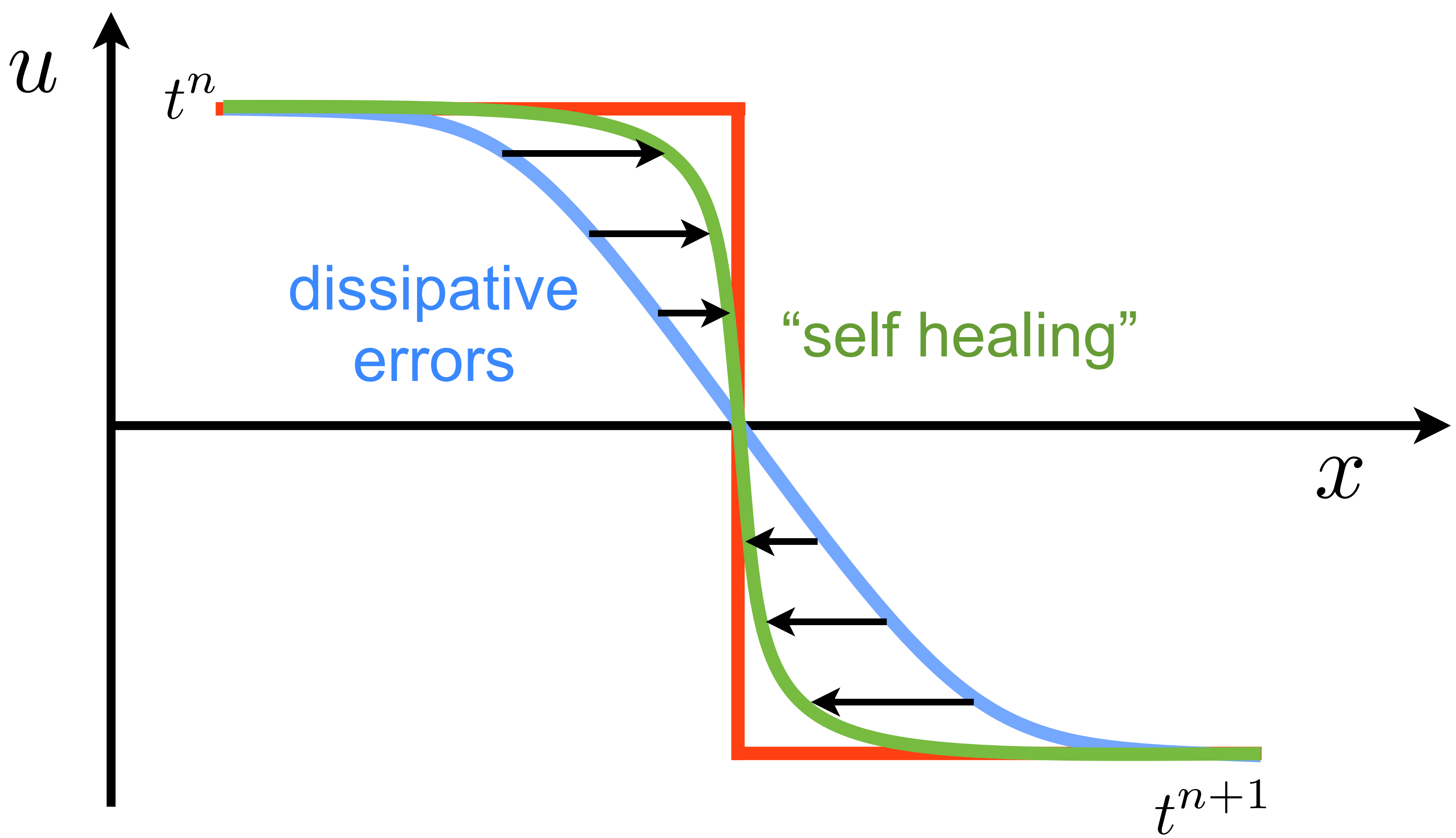
Code:
C=0.5,
C=1, C=0.1

- order: $O(\Delta t)$, $O(\Delta x)$
- stability: stable for $\frac{\Delta t}{\Delta x}\max\left(|u|\right) \leq 1$
- leading order error term: dissipative
- much better here than for wave equation!
- Why?

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order Explicit Upwind

- much better here than for wave equation!
- Why?



$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order Implicit Upwind

if $u_i^n > 0$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{E_i^{n+1} - E_{i-1}^{n+1}}{\Delta x} = -\frac{\frac{(u_i^{n+1})^2}{2} - \frac{(u_{i-1}^{n+1})^2}{2}}{\Delta x}$$

- Problem: non-linear system! ⇒ linearize the non-linear terms!

$$(u_i^{n+1})^2 \approx u_i^n u_i^{n+1} \qquad (u_{i-1}^{n+1})^2 \approx u_{i-1}^n u_{i-1}^{n+1}$$

$$\Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left(u_i^n u_i^{n+1} - u_{i-1}^n u_{i-1}^{n+1}\right)$$

- rearrange: $$\boxed{\left(\frac{\Delta t}{2\Delta x}u_{i-1}^n\right)u_{i-1}^{n+1} - \left(1 - \frac{\Delta t}{2\Delta x}u_i^n\right)u_i^{n+1} = -u_i^n}$$
  - order: $O(\Delta t)$, $O(\Delta x)$
  - stability: unconditionally stable
  - leading order error term: dissipative
  - @ $C=1$: explicit is better than implicit

Code:
C=0.5,
C=1, C=1.5

Observation:

Many of the higher order schemes have large dispersive errors!

How can we address this?

$\Rightarrow$ $\boxed{\text{TVD schemes}}$

TVD schemes

Some definitions:

Monotone schemes

- scheme that does not generate **new** local extrema

  $\Rightarrow$ a local minimum does not decrease

  $\Rightarrow$ a local maximum does not increase

- Consequences
  - oscillation free
  - dissipative
  - **only first order!**

Example: 1st-order upwind

TVD schemes

Some definitions:

Total Variation (TV)

$$TV(u) = \frac{1}{V_\Omega} \int_\Omega \left| \frac{\partial u}{\partial \vec{x}} \right| d\vec{x}$$

for example using 1-sided, 1st-order differences in 1D:

$$TV(u^n) = \frac{1}{N-1} \sum_{i=1}^{N-1} \left| u_{i+1}^n - u_i^n \right|$$

Total Variation Diminishing (TVD)

$$TV(u^{n+1}) \leq TV(u^n)$$

$\Rightarrow$ monotone schemes are TVD!

TVD schemes

Example: 1D wave equation

- a general explicit finite difference method can be written as

$$u_i^{n+1} = u_i + A^n_{i+1/2}\Delta u^n_{i+1/2} - B^n_{i-1/2}\Delta u^n_{i-1/2}$$

where $\Delta u^n_{i+1/2} = u^n_{i+1} - u^n_i$ and $\Delta u^n_{i-1/2} = u^n_i - u^n_{i-1}$

A and B depend on the chosen scheme

A method is TVD if

$A^n_{i+1/2} \geq 0$ and

$B^n_{i-1/2} \geq 0$ and

$A^n_{i+1/2} + B^n_{i+1/2} \leq 1$

1st-order TVD schemes

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

$$\Delta u_{i+1/2}^n = u_{i+1}^n - u_i^n$$

let's revisit explicit 1st-order upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\begin{cases} E_{i+1}^n - E_i^n & \text{if} \quad \alpha_{i+1/2}^n < 0 \\ E_i^n - E_{i-1}^n & \text{if} \quad \alpha_{i-1/2}^n > 0 \end{cases}$$

with $$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

with for example $\epsilon' = 10^{-12}$ for u=O(1)

let's rewrite this into a single equation

use $$sign(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha > 0 \\ -1 & \text{if} \quad \alpha < 0 \end{cases}$$

$$\begin{aligned} u_i^{n+1} = u_i^n &- \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - sign(\alpha_{i+1/2}^n)\right)\left(E_{i+1}^n - E_i^n\right) \\ &- \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + sign(\alpha_{i-1/2}^n)\right)\left(E_i^n - E_{i-1}^n\right) \end{aligned}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

1st-order TVD schemes

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - sign(\alpha_{i+1/2}^n)\right)\left(E_{i+1}^n - E_i^n\right)$$
$$- \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + sign(\alpha_{i-1/2}^n)\right)\left(E_i^n - E_{i-1}^n\right)$$

but also: $sign(\alpha) = \dfrac{|\alpha|}{\alpha}$

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - \frac{|\alpha_{i+1/2}^n|}{\alpha_{i+1/2}^n}\right)\left(E_{i+1}^n - E_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + \frac{|\alpha_{i-1/2}^n|}{\alpha_{i-1/2}^n}\right)\left(E_i^n - E_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 - \frac{|\alpha_{i+1/2}^n|}{\frac{E_{i+1}^n - E_i^n}{\Delta u_{i+1/2}^n}}\right)\left(E_{i+1}^n - E_i^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(1 + \frac{|\alpha_{i-1/2}^n|}{\frac{E_i^n - E_{i-1}^n}{\Delta u_{i-1/2}^n}}\right)\left(E_i^n - E_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - \left(E_i^n + E_{i-1}^n\right) + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right) \quad \text{with} \quad h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

numerical flux function

1st-order TVD schemes

Is this scheme TVD? $\dfrac{\partial u}{\partial t} = -\dfrac{\partial E}{\partial x}$ $\quad E = \dfrac{1}{2}u^2$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - \left(E_i^n + E_{i-1}^n\right) + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

since $\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$ $\Rightarrow$ $E_{i+1}^n - E_i^n = \alpha_{i+1/2}^n \Delta u_{i+1/2}^n$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\alpha_{i+1/2}^n\Delta u_{i+1/2}^n - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n + \alpha_{i-1/2}^n\Delta u_{i-1/2}^n + |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right]$$

is this TVD? compare to $u_i^{n+1} = u_i + A_{i+1/2}^n \Delta u_{i+1/2}^n - B_{i-1/2}^n \Delta u_{i-1/2}^n$

$$A_{i+1/2}^n = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n| - \alpha_{i+1/2}^n\right) \qquad B_{i-1/2}^n = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i-1/2}^n| + \alpha_{i-1/2}^n\right)$$

$$\Rightarrow A_{i+1/2}^n \geq 0 \qquad \Rightarrow B_{i-1/2}^n \geq 0$$

$0 \leq A_{i+1/2}^n + B_{i+1/2}^n \leq 1$? $\quad 0 \leq \dfrac{1}{2}\dfrac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n| - \alpha_{i+1/2}^n + |\alpha_{i+1/2}^n| + \alpha_{i+1/2}^n\right) \leq 1$

$$\boxed{0 \leq \frac{\Delta t}{\Delta x}|\alpha_{i+1/2}^n| \leq 1}$$

$\Rightarrow$ Courant number requirement!

1st-order TVD schemes

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial x} \qquad E = \frac{1}{2}u^2$$

or: $$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right)$$

central

dissipation

can implement this as multi-step

Step 1: $u_i^* = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$

Step 2: $u_i^{n+1} = u_i^* - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\Phi_{i+1/2}^n - \Phi_{i-1/2}^n\right)$

with $\Phi_{i+1/2}^n = -|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n$: flux limiter function

Code: C=0.5, C=0.1, C=1.0

One other issue for non-linear equations:

- weak solutions of the conservation laws may not be unique!
- How to pick the correct one?
  - make use of physics: 2nd law of thermodynamics
    ⇒ entropy may not decrease
  - impose the entropy condition to get the physically correct solution

➡ **need dissipative mechanism**

- but that's exactly what α does in the numerical flux function!

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n - |\alpha_{i-1/2}^n|\Delta u_{i-1/2}^n\right)$$

- however, we may have *α = 0*. What then?

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

define $\psi = \begin{cases} |\alpha| & \text{if} \quad |\alpha| \geq \varepsilon \\ \frac{\alpha^2+\varepsilon^2}{2\varepsilon} & \text{if} \quad |\alpha| < \varepsilon \end{cases}$ with $0 \leq \varepsilon \leq \frac{1}{8}$

- use ψ instead of α in the numerical flux function h

$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\psi_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

choose $\varepsilon = 0.1$

entropy fix

We now have a 1st-order TVD method, but would prefer at least a 2nd-order TVD method

- idea by Harten: let's modify the flux E by replacing it with $\overline{E} = E + G$
- let's start with a formulation we had earlier (slide 6):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$
$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) - |\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n\right]$$

introduce flux limiter function: $\Phi_{i+1/2}^n = -|\alpha_{i+1/2}^n|\Delta u_{i+1/2}^n$

We now have a 1st-order TVD method, but would prefer at least a 2nd-order TVD method

- idea by Harten: let's modify the flux E by replacing it with $\overline{E} = E + G$
- let's start with a formulation we had earlier (slide 6):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right)$$
$$h_{i+1/2}^n = \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) + \Phi_{i+1/2}^n\right]$$

$$\boxed{\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n - h_{i-1/2}^n\right) \\ h_{i+1/2}^n &= \frac{1}{2}\left[\left(E_{i+1}^n + E_i^n\right) + \Phi_{i+1/2}^n\right] \end{aligned}}$$

- Harten-Yee-limiter:

$$\Phi_{i+1/2}^n = (G_{i+1}^n + G_i^n) - \psi(\alpha_{i+1/2}^n + \beta_{i+1/2}^n)\Delta u_{i+1/2}^n$$

with $\psi(y) = \begin{cases} |y| & \text{if} \quad |y| \geq \varepsilon \\ \frac{y^2+\varepsilon^2}{2\varepsilon} & \text{if} \quad |y| < \varepsilon \end{cases}$ $\quad 0 \leq \varepsilon \leq \frac{1}{8}$ (entropy fix function)

choose $\varepsilon = 0.1$

$$\alpha_{i+1/2}^n = \begin{cases} \frac{E_{i+1}^n - E_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ (u_i^n + u_{i+1}^n)/2 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

(same as before)

$$\beta_{i+1/2}^n = \begin{cases} \frac{G_{i+1}^n - G_i^n}{u_{i+1}^n - u_i^n} & \text{if} \quad |\Delta u_{i+1/2}^n| \geq \epsilon' \\ 0 & \text{if} \quad |\Delta u_{i+1/2}^n| < \epsilon' \end{cases}$$

but what's G? many many choices!

for example $G_i^n = S \cdot \max(0, \min(\sigma_{i+1/2}|\Delta u_{i+1/2}^n|, S \cdot \sigma_{i-1/2}\Delta u_{i-1/2}^n))$

where $S = \mathrm{sign}(\Delta u_{i+1/2}^n)$ and $\sigma_{i+1/2} = \frac{1}{2}\left[\psi(\alpha_{i+1/2}^n) - \frac{\Delta t}{\Delta x}(\alpha_{i+1/2}^n)^2\right]$

other choices: see book

Code: C=0.5, C=0.1, C=1.0