The History of Black Holes

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Abstract

Black holes are regions of space-time where gravity is so strong that nothing, including light can escape it's event horizon and originate from the Einstein field equations. We first give a review of special and general relativity, deriving light bending using Newtonian mechanics, and then compare it with the the correct value derived in general relativity. We'll derive the Einstein equations, then present the solutions to the field equations for a spherically symmetric mass with no charge, deriving with it the light bending around an object in Schwarzschild space-time, as well as an AdS solution for an isolated black hole where the cosmological constant is negative. We'll briefly mention the Reissner-Nordström solution for a spherically symmetric charged black hole. During the end of the paper, we'll discuss the quantum mechanical aspects of black holes, in particular Hawking radiation, in which we'll derive the Hawking temperature for Schwarzschild and Reissner-Nordström black hole, making use of Wick rotation to avoid long derivations, consequently finding the relation between entropy and surface area. Each section in the paper will also discuss the history and background of theories presented, as well as highlighting the significance of each physicist's contribution to the field of physics.

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1 Introduction

Newton's law of universal gravitation was published by Sir Isaac Newton in 1687 in his "Philosophiæ Naturalis Principia Mathematica", derived from empirical observation which Newton called inductive reasoning. The law states that every particle attracts every other particle in the universe with a force that is proportional to the product of the two masses and inversely proportional to the square of the distance between their centers.

$$\vec{F_g} = -\frac{GMm}{r^2}\vec{e_r}$$

Where r is the distance between the two centres of mass, $\vec{e_r}$ is the unit-norm vector at the body m pointing towards the body M, and the constant G is the gravitational constant (Newton's constant). We can also express this as:

$$\vec{F_g} = m\vec{g}$$

where $\vec{g} = -\frac{GM}{r^2}\vec{e_r}$ is the gravitational acceleration due to the mass M. Equivalently we can express Newtonian gravity in terms of a gravitational potential Φ, whose gradient is $-\vec{g}$.

$$\vec{g} = -\vec{\nabla}\Phi$$

More generally, if we work with the gravitational potential Φ, the analogue of Newton's law of gravity for a general matter distribution is given by:

$$\boxed{\nabla^2\Phi = 4\pi G\rho}$$

where ρ is the mass density.

This equation is called the Poisson equation, published by French mathematician, Siméon Denis Poisson. The Poisson equation is a generalization of Laplace's second order partial differential equation for the gravitational potential Φ where ∇ is the Laplace operator. The Laplace operator expressed in Cartesian co-ordinates is given by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Newton's law still continues to be used as an excellent approximation for the effects of gravity in most applications, however Newton's law was later superseded by Albert Einstein's theory of general relativity, providing a unified description of gravity through the Einstein equations, as a geometric property of space and time. In particular the curvature of space-time is related to the energy and momentum of whatever matter and radiation is present. The first solutions to the Einstein equations were found by Karl Schwarzschild, describing the gravitational field outside a compact gravitating body, predicting the existence of a spherically symmetric and static black hole.

John Archibald Wheeler was an American physicist who popularised the term "Black hole" in the 1960s, the formation of which happens when the centre of a massive star at the end of it's life cycle collapses in on itself. The first black-hole ever discovered was Cygnus X-1, located within the Milky Way in the constellation of Cygnus, the Swan. Astronomers saw the first signs of the black hole in 1964 when a sounding rocket detected a celestial source of X-rays.

1.1 Unifying Celestial and Terrestrial Mechanics

As an introduction, let's examine how we can take Newtonian mechanics and approximate an angle by which light will be deflected by a massive body, and compare this approximation with that of the general relativity prediction derived later. Prior to this we have to cover the unification of Celestial and Terrestrial mechanics to lead us to the equations that we can use to derive an expression for the bending angle θ.

In 1687, Newton unified celestial and terrestrial mechanics, which were previously thought of as two different areas of physics, he postulated that all objects, celestial or terrestrial, act on each other from a distance in uniform fashion. Let's imagine a planet approximately described as a point particle of mass m that goes around its sun of mass $M \gg m$. The force between the two objects is given by the equation:

$$m\frac{d^2\vec{r}}{dt^2} = -GMm\frac{\vec{r}}{r^3} \tag{1}$$

where we use the notation $\vec{r}$ = (x,y,z) and $r = \sqrt{x^2+y^2+z^2}$.

From (1), the m cancels out, so that $\ddot{\vec{r}} = -k\frac{\vec{r}}{r^3}$ with k $\equiv$ GM. Since the force points in the direction of $\vec{r}$, a simple symmetry argument shows that the motion is confined to a plane, which we take to be the (x,y) plane and set z=0.

$$\ddot{x} = -k\frac{x}{r^3} \text{ and } \ddot{y} = -k\frac{y}{r^3} \tag{2}$$

Evidently we should switch from Cartesian coordinates (x,y) to Polar co-ordinates (r, θ).

$$x = r\cos\theta \text{ and } y = r\sin\theta \tag{3}$$

Taking the first and second derivatives of (3):

$$\begin{aligned}\ddot{x} &= \ddot{r}\cos\theta - 2\dot{r}\sin\theta\dot{\theta} - r\cos\theta\dot{\theta}^2 - r\sin\theta\ddot{\theta}\\ \ddot{y} &= \ddot{r}\sin\theta + 2\dot{r}\cos\theta\dot{\theta} - r\sin\theta\dot{\theta}^2 + r\cos\theta\ddot{\theta}\end{aligned} \tag{4}$$

we can then multiply the first equation in (2) by $\cos\theta$, the second by $\sin\theta$ and take their sum, using (4) to obtain:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{r^2} \tag{5}$$

likewise, multiplying the first equation in (2) by $\sin\theta$, the second by $\cos\theta$ and subtracting, again using (4):

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{6}$$

After some thought, we can re-write (6).

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{7}$$

Which implies that (for some constant l):

$$\dot{\theta} = \frac{l}{r^2} \tag{8}$$

Inserting into (5):

$$\ddot{r} = \frac{l^2}{r^3} - \frac{k}{r^2} = -\frac{dv(r)}{dr} \tag{9}$$

where we have defined:

$$v(r) = \frac{l^2}{2r^2} - \frac{k}{r} \tag{10}$$

Multiplying (9) by $\dot{r}$ and integrating over t:

$$\int dt \frac{1}{2}\frac{d}{dt}\dot{r}^2 = \int dt \dot{r}\ddot{r} = -\int dt \frac{dr}{dt}\frac{dv(r)}{dr} = -\int dr \frac{dv(r)}{dr} \tag{11}$$

so that finally we have:

$$\frac{1}{2}\dot{r}^2 + v(r) = \epsilon \tag{12}$$

with ϵ as an integration constant.

This equation describes a unit mass particle moving in the potential v(r) with energy ϵ. If ϵ is equal to the minimum of the potential $v_{min} = -\frac{k^2}{2l^2}$, then you find $\dot{r} = 0$ and r stays constant, so the planet would follow a circular orbit. However, if $\epsilon > v_{min}$ the orbit is elliptical, with r varying between r_{min} and r_{max} defined by the solutions to $\epsilon = v(r)$.

1.2 Deflection of Light in Newtonian Mechanics

In 1704, Newton proposed a corpuscular theory of light which states that light is made up of tiny particles called "corpuscles" that always travel in a straight line. This theory was used by Johann Georg von Soldner to derive an expression for the deflection of light by a massive body in his paper written in 1801. Soldner noted that if it were possible to observe fixed stars in close distance to the sun, it might be important to take this effect into account. However, due to the time period such observations were impossible and so these effects can be neglected.

As the particle goes from r_{min} to r_{max} and then back to r_{min}, θ changes by precisely 2π. To verify this we solve (12) for $\dot{r}$ and divide by (8):

$$\frac{dr}{d\theta} = \pm(\frac{r^2}{l})\sqrt{2(\epsilon - v(r)} \tag{13}$$

Changing variable from r to u = 1/r, using (10), that $2(\epsilon - v(r))$ becomes the quadratic polynomial $2\epsilon - l^2u^2 + 2ku$, which we can write in terms of its two roots as $l^2(u_{max} - u)(u - u_{min})$. Since u varies between u_{min} and u_{max}, we can make another change of variables from $u = u_{min} + (u_{max} - u_{min})\sin^2\delta$, to δ, so that δ ranges from 0 to $\pi/2$. Thus as the particle completes one round trip in r, the polar angle changes by:

$$\begin{aligned}\Delta\theta =& 2\int_{r_{min}}^{r_{max}} \frac{ldr}{r^2\sqrt{2(\epsilon - v(r))}} = 2\int_{u_{min}}^{u_{max}} \frac{ldu}{\sqrt{2\epsilon - l^2u^2 + 2ku}} \\ =& 2\int_{u_{min}}^{u_{max}} \frac{du}{(u_{max} - u)(u - u_{min})} = 4\int_0^{\pi/2} d\delta = 2\pi\end{aligned} \tag{14}$$

Using Newton's corpuscular theory, take the case of (12) with $\epsilon > 0$, we can still use (14) except that the root r_{min} is now negative, which is not physical since $u = 1/r > 0$. To get around this, simply set $u = 0$ for our second coordinate transformation yielding a minimum value for δ to be used as the lower limit for the integral:

$$\sin^2 \delta_{min} = -\frac{u_{min}}{u_{max} - u_{min}} = 1/2 - \frac{k}{2\sqrt{2\epsilon l^2 + k^2}} \tag{15}$$

where δ_{min} is determined by Taylor expanding (15) in leading orders of k.

$$1/2 - \frac{k}{2\sqrt{2\epsilon l^2 + k^2}} \approx 1/2 - \frac{k}{2\sqrt{2\epsilon l^2}} \approx \pi/2 - \frac{k}{2\sqrt{2\epsilon l^2}} \tag{16}$$

Plugging (16) back into (14):

$$\Delta\theta = 4\int_{\delta_{min}}^{\pi/2} d\delta = \frac{2k}{l\sqrt{2\epsilon}} \tag{17}$$

where $\Delta\theta = \frac{2k}{l\sqrt{2\epsilon}}$ can be expressed in terms of the impact parameter b defined by saying that as $x \to \infty$, the light ray moves along a path specified by $y = b$. Translating into polar coordinates, we have as $r \to \infty$, $b \approx r\theta$, $\dot{r}^2 \approx 2\epsilon$ and $\dot{r} \approx -\sqrt{2\epsilon}$.

Using:

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \text{ and } \dot{\theta} = \frac{l}{r^2}$$

we find that:

$$l = b\sqrt{2\epsilon} \tag{18}$$

Newton did not know the value of the speed of light, but if we set $\dot{r}^2 = c^2$ so that $\epsilon = \frac{c^2}{2}$, the Newtonian result for the deflection of light by a massive body:

$$\Delta\theta = \frac{2GM}{c^2 b} \tag{19}$$

However, there is a problematic aspect to this prediction, because it's based on the assumption that particles of light can be accelerated or decelerated just like ordinary matter, if this was the case it would be very difficult to explain why all the light we observe travels at the same speed. The idea that gravity might bend light rays was largely discounted in Newtonian physics, but was later revived by Einstein in his 1911 paper "On the Influence of Gravitation on the Propagation of Light".

2 Overview of Special Relativity

2.1 Galilean Relativity

Before Newton, in 1632, Galileo Galilei was the first scientist to hypothesize a universal principle of relativity which states that the form of the equations of motion is independent of the choice of inertial reference frames. This principle was implemented into Newtonian mechanics where these inertial reference frames are related by the Galilean Transformations, most generally given by:

$$t' = t \text{ and } \vec{x'} = R(\vec{x} - \vec{v}t - \vec{x_*})$$

Note that time is invariant under these transformations. To quote Newton himself, "Absolute, true and mathematical time, of itself, and from it's own nature flows equably without regard to anything external". Absolute time exists independently of any observer and progresses at a consistent pace within the universe.

We'll come to recognise that this notion of absolute time does not work for relativistic speeds, and therefore requires a new theory to describe how motion is related between observers. However there are two important mathematical concepts we have to cover before proceeding.

2.1.1 Rotation

Imagine two separate observers measuring the same point on an plane.The first observer will measure a point P with co-ordinates (x,y). The second observer records the same point with co-ordinates (x',y'), however their co-ordinate axis has been rotated by an angle θ with respect to the first observer, but sharing the same origin O. Given this angle θ, the equations relating the two observers are given by:

$$x' = x\cos\theta + y\sin\theta \text{ and } y' = -x\sin\theta + y\cos\theta \tag{20}$$

The magnitude of the distances that both observers calculate are equal, this is confirmed by the Pythagorean Theorem, which requires $\sqrt{x'^2 + y'^2} = \sqrt{x^2 + y^2}$.

Lets rewrite (20) in vector form, with column vectors:

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \vec{r'} = \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{21}$$

with the rotation through θ being written as a matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{22}$$

Where now the relation between the two observers is given in vector form as $\vec{r'} = R(\theta)\vec{r}$. Given (21), the scalar or dot product is left invariant under rotations - this would imply:

$$\vec{r}^T \cdot \vec{r} = \vec{r'}^T \cdot \vec{r'}$$

where $\vec{r}^T$ is the transpose of the vector $\vec{r}$.

Using $\vec{r'} = R(\theta)\vec{r}$ and suppressing the θ dependence in R(θ):

$$\vec{r}^T \cdot \vec{r} = (R\vec{r})^T \cdot (R\vec{r}) = \vec{r}^T \cdot (R^T R)\vec{r}$$

For this to hold:

$$R^T R = I \tag{23}$$

where I denotes the identity matrix.

2.1.2 Index Notation

Let's introduce a new notation to rewrite coordinates with indices that will be useful in formalising vectors and tensors. We introduce an explicit index to label the components x^i, where i = 1,2,3. Hence we can rewrite co-ordinates as:

$$x^i = (x^1, x^2, x^3) = (x, y, z)$$

The total magnitude of the vector can be written as:

$$| \vec{x} |^2 = \sum_{i=1}^{3} x^i x^i \tag{24}$$

The notation extends to any 3-dimensional vector $\vec{V}$ as $V^i = (V^1, V^2, V^3)$ where the index of the vector are the components x,y,z. For the inner product between any two vectors:

$$\vec{V} \cdot \vec{W} = \sum_{i=1}^{3} V^i W^i$$

We can now apply this to rotations and rewrite (22) using index notation, with $\vec{r'} = R(\theta)\vec{r}$ becoming:

$$x'^i = \sum_{i=1}^{3} R^i_j x^j$$

Where the rotation matrix R is rewritten with two indices i and j, pointing each component of the corresponding matrix.

2.2 Notion of a Metric

Although the foundations for special relativity were laid out by the likes of Poincaré and Lorentz, in 1905, Einstein published his special relativity paper that included new definitions of space and time. Einstein showed that the Lorentz transformations are not the result of interactions between matter and aether, but rather concern the nature of space and time itself. Moving forward we'll extend our coordinate system to include a time component, I'll refer to these objects as four-vectors or tensors.

Let's use the index notation introduced in the previous section and write coordinates with an upper index μ:

$$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

Since t has the dimension of time, we multiply it by the speed of light so that it has dimensions of length.

The basic features of special relativity include the speed of light being independent of the choice of reference frame and Newtonian mechanics being a valid approximation for small velocities. Unlike Galilean Relativity, inertial observers in special relativity do not agree on time or length intervals, but instead agree on space-time intervals. The distance between two space-time points is given by:

$$s^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{25}$$

where $\Delta t = t_A - t_B$ and $\Delta x = x_A - x_B$ etc. The space-time interval Δs^2 is an extension of the notion of the length squared. However Δs^2 can take on negative values, therefore describing different types of trajectories.

Using index notation let's rewrite (25) using the formalism in (24):

$$s^2 = -c^2(\Delta t)^2 + \sum_{i=1}^{3} \Delta x^i \Delta x^i = -c^2(\Delta t)^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \Delta x^i \Delta x^j \tag{26}$$

where the δ_{ij} represents the Kronecker delta, a function of two variables i and j, which equals one if the variables are equal and zero otherwise.

Let's rewrite (26) using space-time coordinates, replacing the indices i and j with μ and v:

$$s^2 = \sum_{\mu=0}^{3}\sum_{v=0}^{3} \eta_{\mu v} \Delta x^\mu \Delta x^v \tag{27}$$

Where we set:

$$\eta_{\mu v} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

This matrix represents the metric, a notion used to measure distances in space. This in particular is called the Minkowski metric or tensor, used to determine distances in Minkowski Space. Real Minkowski Spacetime was introduced by Hermann Minkowski in 1908 to understand special relativity using a four dimensional space.

A simple example can be a metric for three-dimensional surface in Cartesian coordinates:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Where ds can be referred to as the infinitesimal line element, using the same formalism as (26) but ignoring the time component. The line element takes the form:

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} dx^i dx^j \tag{29}$$

We can now determine that the metric for Euclidean Space is the Kronecker delta δ_{ij}, which can also be written as a 3x3 Identity matrix:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{30}$$

Einstein Summation Convention : Whenever we have a pair of indices that are summed over, we suppress the sum symbol for notation simplicity. This notation was introduced by Einstein in 1916 and allows us to simplify (27):

$$s^2 = \eta_{\mu v} \Delta x^\mu \Delta x^v \tag{31}$$

2.3 Lorentz Transformations

In 1904, Hendrik Lorentz set out to create a set of transformations under which Maxwell's equations are invariant. Later in 1905, Henri Poincaré submitted a summary of work which closed the existing gaps of Lorentz work. He corrected Lorentz formulas, and gave them the symmetrical form used to this day. A Lorentz boost in the x-direction is given by the following:

$$ct' = \frac{ct - vx/c}{\sqrt{1 - v^2/c^2}} \text{ and } x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{32}$$

since the boost is only in the x-direction all other spatial coordinates are left unchanged, y' = y and z' = z.

Lets introduce two recurring terms in the transformations, labelled β and γ:

$$\beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1 - \beta^2}} \tag{33}$$

Using (33), and setting the speed of light to be equal to one we can rewrite (32)in a much more simplified form:

$$t' = \gamma(t - \beta x) \text{ and } x' = \gamma(x - t\beta) \tag{34}$$

We can write the Lorentz transformations from (34) as a 2x2 matrix L^μ_v, with two indices μ and v.

$$L^{\mu}_{v} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

A Lorentz Boost in the x-direction can therefore be written as a matrix product given by:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{36}$$

More compactly we can generalise any Lorentz transformations and write it using index notation where $x'^{\mu} = L^{\mu}_{v} x^{v}$, using μ and v as space-time indices. Lets take two separate space-time points given by:

$$x'^{\mu} = L^{\mu}_{\alpha} x^{\alpha} \text{ and } x'^{v} = L^{v}_{\beta} x^{\beta} \tag{37}$$

As stated before, two observers in special relativity will agree on space-time intervals, the Lorentz transformations preserve space-time intervals, similar to how rotations preserve the length between two observers, so lets equate the space-time intervals between two observers using (31):

$$\eta_{\alpha\beta} x^{\alpha} x^{\beta} = \eta_{\mu v} x'^{\mu} x'^{v} \tag{38}$$

Substituting (37) into (38):

$$\begin{aligned} \eta_{\alpha\beta} x^{\alpha} x^{\beta} &= (L^{\mu}_{\alpha} x^{\alpha}) \eta_{\mu v} (L^{v}_{\beta} x^{\beta}) \\ &= (L^{\mu}_{\alpha} \eta_{\mu v} L^{v}_{\beta}) x^{\alpha} x^{\beta} \end{aligned}$$

For this to hold:

$$(L^{\mu}_{\alpha} \eta_{\mu v} L^{v}_{\beta}) = \eta_{\alpha\beta}$$

L^{μ}_{α} can be written as $(L^{T})^{\mu}_{\alpha}$ which in matrix notation can be written in the form below.

$$L^{T} \eta L = \eta \tag{39}$$

When compared with (23), it's interesting to notice that we have arrived at a similar identity as with rotations but using space-time indices, we can take the Lorentz transformations to be analogous of how rotations preserve lengths, except here the quantity preserved is the space-time interval.

2.4 Electromagnetism

Historically, the Maxwell equations of electromagnetism were discovered before the theory of special relativity, it was thought that light waves must be oscillations of some substance which fills a space, dubbed the aether. However the negative outcome of the Michelson and Morley experiment in 1887 suggested that the aether did not exist, a finding that was confirmed in subsequent experiments in the 1920s. Instead Maxwell's equations hold in all inertial frames and are the first equations of physics which are consistent with the laws of special relativity, ultimately it's the study of these equations that lead Lorentz to develop the form of the Lorentz transformations, consequently laying down Einstein's vision for space and time.

The electromagnetic field is made of the electric field $\vec{E}$ and magnetic field $\vec{B}$, and the charged particles have the charge density ρ_e and current density $\vec{j_e}$. The Maxwell equations are given by:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} = \rho_e \ , \ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j_e} \\ \vec{\nabla} \cdot \vec{B} = 0 \ , \ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{aligned} \tag{40}$$

where the constants μ_0, ϵ_0 and c have all been set to 1.

We want to rewrite the Maxwell equations in tonsorial form to know how each object transforms under a Lorentz transformation. A few possible attempts can be made such as upgrading $\vec{E}$ and $\vec{B}$ into two 4-vector fields E^μ and B^v, but this doesn't work due to the Lorentz transformation mixing both components $\vec{E}$ and $\vec{B}$. Ultimately, the final solution is to conclude that $\vec{E}$ and $\vec{B}$ must form a single tensor, the earliest instance of this tensor was in a paper written by Minkowski in 1908. We've come to recognise this tensor as the electromagnetic field-strength tensor with a total of six components written as a 4x4 anti-symmetric matrix:

$$F_{\mu v} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{v\mu} \tag{41}$$

where we denote $\vec{E} = (E_1, E_2, E_3)$ and $\vec{B} = (B_1, B_2, B_3)$.

How do we interpret the terms ρ_e and $\vec{j_e}$ in tensor form? The answer is to write them as a 4-vector:

$$j^\mu = (\rho_e, j_e^i)$$

so we can finally write the Maxwell equations as:

$$\partial^v F_{\mu v} = j_\mu \tag{42}$$

$$\partial_\lambda F_{\mu v} + \partial_\mu F_{v\lambda} + \partial_v F_{\lambda\mu} = 0 \tag{43}$$

where (42) and (43) refer to the homogeneous and non-homogeneous Maxwell equations. Here we used a common notation for the derivatives that makes the expressions look tidier (note $\eta^{\mu v}$ is the inverse of the Minkowski metric $\eta_{\mu v}$).

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \text{ and } \partial^\mu = \eta^{\mu v}\partial_v \tag{44}$$

2.5 Causality and Kinematics

In this section, I want to briefly expand on the trajectories of particles in space-time, as well as introduce new notation for velocities and accelerations in space-time. Let's recall (31), the space-time interval between two space-time points.

$$s^2 = \eta_{\mu v}\Delta x^\mu \Delta x^v$$

A fundamental property of the space-time interval is that:

$$\boxed{s^2 = 0 \text{ for light-separated space-time points}}$$

This introduces a concept of causality, which tells us which events may affect other events. For a given space-time point x^μ, we would like to know which events may affect it, be affected by it and neither affect nor be affected by it. This can also be determined by the light-cone of x^μ.

- $s^2 \leq 0$ (inside or on the light-cone): $\Delta t > 0$ is the future of x^μ, $\Delta t < 0$ is the past of of x^μ
- $s^2 > 0$ (outside of the light-cone): casually disconnected from x^μ

2.5.1 Massive Particles

The physical trajectory of a massive particle is parameterised by the proper time τ, so when describing a trajectory of $x^\mu(\tau)$, it can be written as:

$$x^\mu(\tau) = (t(\tau), x^i(\tau)) \tag{45}$$

where the proper time is $d\tau$ defined such that $d\tau^2 = -ds^2 < 0$, it's physical meaning is the time in the particles rest frame. Using the Minkowski metric we can define proper time.

$$
\begin{aligned}
d\tau &= \sqrt{-ds^2} \\
&= \sqrt{-\eta_{\mu v} dx^\mu dx^v}
\end{aligned} \tag{46}
$$

With the trajectory described as $x^\mu(\tau)$, we can introduce the four-velocity, momentum and acceleration of the particle.

The four-velocity:

$$
u^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau} = \frac{dt(\tau)}{d\tau}(1, \frac{dx^i(t)}{dt}) = \gamma(1, v^i) \tag{47}
$$

where we used $d\tau = \gamma dt$.

The four-momentum:

$$
p^\mu(\tau) = mu^\mu(\tau) = m\gamma(1, v^i) = (m\gamma, p^i) \tag{48}
$$

where $E = \gamma m$.

The four acceleration:

$$
a^\mu(\tau) = \frac{du^\mu(\tau)}{d\tau} = \frac{d^2 x^\mu(\tau)}{d\tau^2} \tag{49}
$$

2.5.2 Mass-less Particles

For mass-less particles, the proper time is along a null trajectory with $ds^2 = 0$, that is mass-less particle don't experience time, hence when we talk about observers in physics, they always follow time-like trajectories. We can take the expression for the four-momentum and $p^\mu = \gamma m(1, v^i)$, we can consider a finite limit where $\gamma \to \infty$ and $m \to 0$.

$$
\frac{dx^\mu(\tau)}{d\lambda} = p^\mu(\lambda) \tag{50}
$$

where λ is some parameter along the null trajectory.

The time component of this equation tells us that $d\lambda = dt/E$. Notice also that:

$$
\eta_{\mu v} p^\mu p^v = \eta_{\mu v} \frac{dx^\mu}{d\lambda} \frac{dx^v}{d\lambda} = \frac{ds^2}{d\lambda^2} = 0
$$

for the null trajectory, from which:

$$
E = \mid \vec{p} \mid \tag{51}
$$

2.6 Energy-Momentum Tensor

Our journey so far has been building up the knowledge and notation for General Relativity, now it is time to examine one of the components of the Einstein equations - namely the energy-momentum tensor which describes the energy and momentum in space-time. Consider ordinary matter at rest, the energy density is given by:

$$
\rho = \frac{m}{V}
$$

where m is the mass inside the an infinitesimally small volume V.

If we imagine matter at rest in a Lorentz frame, we can perform a boost which will both increase the energy of the matter and decrease the volume, therefore in a boosted frame, the energy density can be given by $\gamma^2 \rho$, leading us to conclude that the energy-density is not a Lorentz scalar, and instead must be the component of a tensor. Recalling (35), since we need two factors of γ, we conclude that this tensor is a rank two one - dubbed the Energy-Momentum (E-M) tensor. The form of the E-M tensor depends on the type of matter, but is always symmetric $T^{\mu v} = T^{v\mu}$, the total E-M tensor is the sum of contributions from each type, an example is a fluid obeying:

$$T^{\mu v}(x) = \rho(x)u^{\mu}(x)u^{v}(x) \tag{52}$$

where $\rho(t,\vec{x})$ is the rest frame energy density and $u^{\mu}(t,\vec{x})$ is the local four-velocity of the fluid.

The conservation of energy and momentum law can therefore be expressed as:

$$\partial_{\mu}T^{\mu v} = 0 \tag{53}$$

Einstein's 1905 special relativity paper describes how lengths in space and duration's of time are different between observers moving at different speeds, however even when Einstein was developing his theory he himself knew that the theory was incomplete; this incompleteness stemmed from the fact that it was hard to reconcile special relativity with the Newtonian theory of gravity, leaving us with a few unanswered questions.

3 Overview of General Relativity

3.1 General Coordinate Transformations

Shortly after publishing his special theory of relativity, Einstein began to work towards creating an even more complete and far-reaching theory of space and time, it took him decades, but eventually Einstein came up with an expanded and a completely general form of his theory. The main principle that Einstein based his new theory on is the equivalence principle which he introduced in 1907, the ability to not distinguish between a gravitational force experienced locally while standing on a massive body such as Earth, and a pseudo-force experienced by an observer in a non-inertial accelerated frame. To expand our knowledge to lead into the Einstein equations there are a few important concepts within differential geometry that we must discuss prior.

In the previous section we worked under the assumption that the laws of physics take the same form under Lorentz transformations, here we introduce the principle of general covariance. Rather than the laws of physics taking the same form under Lorentz transformations, they take the same form under any general co-ordinate transformation. For a scalar function $\phi(x)$ of some arbitrary co-ordinates x^{μ}, $\phi(x)$ is a function that doesn't change under any general co-ordinate transformation:

$$\phi(x) = \phi'(x') \tag{54}$$

for $x^{\mu} \to x'^{\mu}$.

For a co-vector $A_{\mu}(x')$, a vector function of some arbitrary co-ordinates x^{μ} that undergoes a co-ordinate transformation transforms as the gradient of a scalar function $\phi(x)$:

$$A'_{\mu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}A_{\alpha}(x) \tag{55}$$

For a contra-variant vector B^{μ}, it transforms as the inverse of a co-vector.

$$B'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}B^{\alpha}(x) \tag{56}$$

A covariant tensor $T_{\mu v}$ transforms as the product of two co-vectors.

$$T'_{\mu v}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{v}}T_{\alpha\beta}(x) \tag{57}$$

A contra-variant tensor transforms as the product of two contra-vectors.

$$T'^{\mu v} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{v}}{\partial x^{\beta}} T^{\alpha\beta}(x) \tag{58}$$

We can generalise any transformation of any ranked tensor as:

$$T'^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}} \frac{\partial x^{v_1}}{\partial x'^{\beta_1}} \frac{\partial x^{v_2}}{\partial x'^{\beta_2}} T^{\mu_1\mu_2}_{v_1 v_2} \tag{59}$$

It's important to define a differential operator in tensor form, but we can't use the derivative ∂_μ introduced earlier to define $\partial_\mu V^v$ due to not working under general co-ordinate transformations, instead we define a covariant derivative ∇_μ such that $\nabla_\mu V^v$ is a tensor:

$$\boxed{\nabla_\mu V^v = \partial_\mu V^v + \Gamma^v_{\mu\lambda} V^\lambda}$$

where $\Gamma^v_{\mu\lambda}$ are the Christoffel symbols.

$$\Gamma^v_{\mu\lambda} = \frac{1}{2} g^{v\alpha}(\partial_\mu g_{\lambda\alpha} + \partial_\lambda g_{\mu\alpha} - \partial_\alpha g_{\mu\lambda}) = \Gamma^v_{\lambda\mu} \tag{60}$$

Introduced by Elwin Bruno Christoffel in his 1869 paper on equivalence problem for differential forms in n variables, the Christoffel symbols were used to express the components of the Levi-Civita connection with respect to a system of local co-ordinates. His ideas were greatly developed by Gregorio Ricci-Curbastro and his student Tullio Levi-Civita, who turned them into a concept of tensors which formed the mathematical basis of the general theory of relativity. Here they're introduced as a correction term for the transformation of the derivative of a vector or tensor and are also symmetric in the two lower indices, i.e $\Gamma^\alpha_{\lambda\mu} = \Gamma^\alpha_{\mu\lambda}$ and $\Gamma^\alpha_{\lambda v} = \Gamma^\alpha_{v\lambda}$.

We can now define the covariant derivative for co-vectors, contra-vectors and tensors.

$$\begin{aligned} \nabla_\mu A_v &= \frac{\partial A_v}{\partial x^\mu} - \Gamma^\alpha_{\mu v} A_\alpha \\ \nabla_\mu B^\lambda &= \frac{\partial B^\lambda}{\partial x^\mu} - \Gamma^\lambda_{\mu\alpha} B^\alpha \\ \nabla_\mu T^{\alpha_1\alpha_2}_{\beta_1\beta_2} &= \frac{\partial T^{\alpha_1\alpha_2}_{\beta_1\beta_2}}{\partial x^\mu} - \Gamma^v_{\mu\beta_1} T^{\alpha_1\alpha_2}_{v\beta_1} \ldots\ldots \end{aligned} \tag{61}$$

The term $g_{\mu v}$ in (60) is known as a metric tensor, used to encode geometric information about a mathematical space called a manifold. A manifold can be thought of as a space that locally resembles flat space. A line element for a general manifold can be written as:

$$ds^2 = g_{\mu v} dx^\mu dx^v \tag{62}$$

For a flat manifold, the space will resemble Euclidean space, the metric tensor is constant and doesn't depend on the co-ordinates and thus the derivative of the metric tensor vanishes, this translates to the covariant derivative that also vanishes:

$$\nabla_\lambda g_{\mu v} = \frac{\partial}{\partial x^\lambda} g_{\mu v} - \Gamma^\alpha_{\lambda\mu} g_{\alpha v} - \Gamma^\alpha_{\lambda v} g_{\mu\alpha} = 0 \tag{63}$$

thus we can define the Christoffel symbols in terms of the metric tensor in (60).

3.2 Curvature

Benhard Riemann was a German mathematician who made profound contributions to differential geometry, delivering his first lecture in 1854, which founded the field of Riemannian geometry, laying the ground work for the mathematics used by Einstein in general relativity. Riemann developed his theory of higher dimensions and delivered his lecture in Göttingen in 1854, but his work wasn't published until twelve years later, after his death. In his work, Riemann found the correct way to extend into n dimensions the differential geometry of surfaces, which Carl Gauss himself proved in his theorema egregium. The fundamental object introduced in the work is the Riemann curvature tensor, used to express the curvature of a manifold.

$$R^{\lambda}_{\rho\mu v} = \partial_\mu \Gamma^{\lambda}_{v\rho} - \partial_v \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{v\rho} - \Gamma^{\lambda}_{v\sigma}\Gamma^{\sigma}_{\mu\rho} \tag{64}$$

A key note is that the Christoffel symbols aren't tensors, but can be used to construct this curvature tensor. If (64) vanishes everywhere, the manifold is flat, an example of this can be seen in Minkowski space-time where the metric tensor becomes $g_{\mu v} = \eta_{\mu v}$ and the Christoffel symbols vanish along with the curvature tensor vanish as expected.

From the Curvature tensor we can define the Ricci tensor, developed by Gregorio Ricci-Cubastro, defined as:

$$R_{\rho v} = R^{\lambda}_{\mu\lambda v} \tag{65}$$

which can also be used to define the Ricci scalar, by contracting it with the inverse metric tensor.

$$R = g^{\mu v} R_{\mu v} \tag{66}$$

For curvature in a two-dimensional space using co-ordinates $x^\mu = (x^1, x^2)$, the symmetries of the Riemann tensor has only one non-vanishing component.

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}$$

Since there is a single independent component, there is a single non-trivial function expressing the curvature, which can be taken to be the Ricci scalar. So in two-dimensions, the Riemann and Ricci tensors are fully determined by the Ricci scalar and the metric tensor:

$$\begin{aligned} R_{\mu v} &= \frac{R}{2} g_{\mu v} \\ R_{\lambda\rho\mu v} &= \frac{R}{2}(g_{\lambda\mu} g_{\rho v} - g_{\lambda v} g_{\rho\mu}) \end{aligned} \tag{67}$$

3.3 Geodesic Equation

Newton formulated the laws of motion in his 1687 volumes in Principia, however Newton's development was geometrical and is not how we see classical dynamics. The laws of mechanics are now more considered to be in the realm of analytical mechanics, in which classical dynamics is presented in a more elegant way, it's based on principles of variations, whose foundations began with the work of Euler and Lagrange with Euler coining the term calculus of variations in 1756. The shortest distance between two points is a straight line, this statement is true for Euclidean space, but what about two points on a sphere or on any general manifold?

Recall the covariant derivative of a vector:

$$\nabla_\mu V^v = \partial_\mu V^v + \Gamma^{v}_{\mu\lambda} V^\lambda$$

We may choose for a vector T^μ the tangent vector to a path $x^\mu(\sigma)$ with parameter σ, that is, $T^\mu = dx^\mu/d\sigma$. The covariant derivative along the path is therefore defined as:

$$\frac{DV^\mu}{D\sigma} = \frac{dx^v}{d\sigma}\nabla_v V^\mu = \frac{dx^v}{d\sigma}\partial_v V^\mu + \Gamma^{\mu}_{v\lambda}\frac{dx^v}{d\sigma}V^\lambda = \frac{dV^\mu}{d\sigma} + \Gamma^{\mu}_{v\lambda}\frac{dx^v}{d\sigma}V^\lambda \tag{68}$$

which gives us the rate of change of the vector along the path. A geodesic is a path where the vector tangent to the path doesn't change along the path, if the vector does not change along the path we obtain:

$$\boxed{\text{parallel transport: } \frac{DV^\mu}{D\sigma} = 0}$$

Geodesic in Riemannian Manifold
The natural parameter for a path in a Riemannian manifold is the length, therefore we describe the path as $x^\mu(s)$ with:

$$\int ds = \int \sqrt{g_{\mu v} dx^\mu dx^v} = \int d\sigma \sqrt{g_{\mu v} \frac{dx^\mu}{d\sigma} \frac{dx^v}{d\sigma}}$$

where we have written it in terms of an arbitrary parameter σ. With the length parameter, the tangent vector to the path is dx^μ/ds, the geodesic equation is therefore defined as:

$$\frac{D}{Ds}\frac{dx^\mu}{ds} = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{v\lambda}\frac{dx^v}{ds}\frac{dx^\lambda}{ds} = 0 \tag{69}$$

Time-like Geodesics in Lorentzian Manifolds
Massive particles follow time-like trajectories in the Lorentzian manifold that is space-time. The natural parameter for a time-like trajectory is the proper time τ, we describe the trajectory of $x^\mu(\tau)$ as:

$$\int d\tau = \int \sqrt{-g_{\mu v} dx^\mu dx^v} = \int d\sigma \sqrt{-g_{\mu v} \frac{dx^\mu}{d\sigma} \frac{dx^v}{d\sigma}}$$

where the tangent vector to the trajectory has a physical interpretation as the four-velocity $u^\mu = dx^v/d\tau$. The geodesic equation is defined for a free massive particle following a time-like trajectory with a vanishing four-acceleration.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{v\lambda}\frac{dx^v}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{70}$$

Null Geodesics in Lorentzian Manifolds
Mass-less particles follow null trajectories since $ds^2 = 0$, for this reason the proper time cannot be used as a parameter. Instead, we use a parameter λ such that the four-momentum is the tangent vector $p^\mu = dx^\mu/d\lambda$, the geodesic equation for this case is where the four-momentum is transported in parallel along the trajectory.

$$\frac{Dp^\mu}{D\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{v\rho}\frac{dx^v}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \tag{71}$$

In the three cases above, the geodesic equation took exactly the same form, the only distinction is the interpretation of the natural parameter along the path.

3.4 Einstein Field Equations

3.4.1 The Einstein Tensor

In 1912, Einstein visited an old classmate Marcel Grossmann, who introduced him to Riemannian geometry, or more generally, to differential geometry. On the recommendation of Levi-Civita, Einstein began exploring the principle of general covariance for his gravitational theory, and by late 1915, had published his general theory of relativity in the form we use today. This theory explains gravitation as the distortion of the structure of space-time by matter, affecting the inertial motion of other matter.

Italian mathematician, Luigi Bianchi, in 1902 rediscovered what are now called the Bianchi Identities for the Riemann tensor, which play an even more important role in general relativity, expressed by:

$$\nabla_\gamma R^\lambda_{\rho\alpha\beta} + \nabla_\alpha R^\lambda_{\rho\beta\gamma} + \nabla_\beta R^\lambda_{\rho\gamma\alpha} = 0 \tag{72}$$

where the indices $\gamma\alpha\beta$ are cyclically permuted. Suppose we contract (72) with δ^α_λ.

$$\nabla_\gamma R_{\rho\beta} + \nabla_\lambda R^\lambda_{\rho\beta\gamma} - \nabla_\beta R_{\rho\gamma} = 0$$

Here we used the definition of the Ricci tensor from (65). Let's simplify this equation further by contracting two more indices with $g^{\rho\gamma}$ so we end with:

$$\nabla^\gamma R_{\gamma\beta} + \nabla_\lambda R^\lambda_\beta - \nabla_\beta R = 2\nabla^\gamma R_{\gamma\beta} - \nabla_\beta R = 0$$

where we used the definition of the Ricci scalar from (67), let's further re-write this expression using the metric tensor $g_{\mu v}$ so it takes the following form.

$$\nabla^\mu (R_{\mu v} - \frac{1}{2} g_{\mu v} R) = \nabla^\mu R_{\mu v} - \frac{1}{2}\nabla_v R = 0 \tag{73}$$

We can conclude that the the tensor that can satisfy (73) can be written as:

$$G_{\mu v} = R_{\mu v} - \frac{1}{2} g_{\mu v} R \tag{74}$$

and satisfies the identity.

$$\boxed{\nabla^\mu G_{\mu v} = 0}$$

The rank-two curvature tensor $G_{\mu v}$ is known as the Einstein tensor, used to express the curvature of a pseudo-Riemannian manifold.

3.4.2 Newtonian Limit

In order to arrive at the Einstein equations, we must consider in detail the limit in which we expect general relativity to reduce to Newtonian gravity. This comes in the form of the Newtonian limit, the approximation in which Newtonian physics is valid. It consists of three main parts, each of which we will use as constraints when formulating the Einstein Equations.

Weak Gravitational Field: The metric has to be close to Minkowski space-time, so we can split the curved metric $g_{\mu v}$ between the background Minkowski metric $\eta_{\mu v}$ and a small perturbation $h_{\mu v}$:

$$g_{\mu v} = \eta_{\mu v} + h_{\mu v} \tag{75}$$

and for the inverse metric, we keep the contributions which have at most one power of $h_{\mu v}$ and take a linear approximation.

$$g^{\mu v} \approx \eta^{\mu v} - h^{\mu v} \tag{76}$$

Where $h^{\mu v} = \eta^{\mu\alpha}\eta^{\mu\beta} h_{\alpha\beta}$.

Equation (75) allows us to refine the correspondence between the gravitational potential Φ and the metric to be between Φ and the metric perturbation $h_{\mu v}$ which introduces curvature.

Slowly varying Gravitational field: Since time derivatives of Φ don't appear in Newtonian equations of motion, any time derivatives of $h_{\mu v}$ must be suppressed in the Newtonian limit.

$$\boxed{\partial_t h_{\mu v} \approx 0}$$

Small velocities of Matter: Any kind of matter must have much smaller speeds than the speed of light for the approximation to hold. Using the relation between co-ordinate time t in the Minkowski background and the proper time τ of a massive particle with speed $v << 1$:

$$dt = \gamma dt = \frac{d\tau}{\sqrt{1 - v^2}} \approx d\tau$$

so that $dt = d\tau$, hence we obtain absolute time, which all observers will agree on.

Freely Falling massive particle in the Newtonian Limit

Recalling (70) for a free massive particle following a time-like geodesic, for the Newtonian limit the four-velocity is defined as:

$$\frac{dx^\mu}{d\tau} \approx \frac{dx^\mu}{dt} = (1, \frac{dx^i}{dt})$$

with the spatial components of the geodesic equation for small velocities, $v^i << 1$.

$$\frac{d^2x^i}{dt^2} \approx -\Gamma^i_{v\lambda}\frac{dx^v}{dt}\frac{dx^\lambda}{dt} \approx -\Gamma^i_{00} \tag{77}$$

The Christoffel symbols are here re-written using (75) and (76) to take the following form.

$$\Gamma^v_{\mu\lambda} = \frac{1}{2}g^{v\alpha}(\partial_\mu h_{\lambda\alpha} + \partial_\lambda h_{\mu\alpha} - \partial_\alpha h_{\mu\lambda}) \approx \frac{1}{2}\eta^{v\alpha}(\partial_\mu h_{\lambda\alpha} + \partial_\lambda h_{\mu\alpha} - \partial_\alpha h_{\mu\lambda})$$

Where we used $\partial_\mu g_{\lambda\alpha} = \partial_\mu h_{\lambda\alpha}$, and we neglected the sub-leading term in $g^{v\alpha}$ because it would lead to terms in $\Gamma^v_{\mu\lambda}$ which are quadratic in the metric perturbation. Taking $\partial_0 h_{\mu v} \approx 0$:

$$\Gamma^i_{00} \approx -\frac{1}{2}\partial_i h_{00} \tag{78}$$

and substituting it back into (77):

$$\frac{d^2x^i}{dt^2} \approx \frac{1}{2}\partial_i h_{00}$$

leads us to the relation between between the gravitational potential Φ and the metric perturbation.

$$h_{00} \approx -2\Phi \tag{79}$$

Ricci Tensor in the Newtonian Limit

For the linear approximation, the Ricci Tensor takes the form:

$$R_{\mu v} = R^\lambda_{\mu\lambda v} \approx \frac{1}{2}(\partial_\mu\partial^\alpha h_{\alpha v} + \partial_v\partial^\alpha h_{\alpha\mu} - \partial^\alpha\partial_\alpha h_{\mu v} - \partial_\mu\partial_v h^\alpha_\alpha)$$

hence R_{00} with $\partial_0 h_{\mu v} \approx 0$ gives:

$$R_{00} \approx -\frac{1}{2}\nabla^2 h_{00}$$

and can be re-written to incorporate the gravitational potential Φ.

$$R_{00} \approx \nabla^2\Phi \tag{80}$$

Energy-Momentum Tensor in the Newtonian Limit

For small velocities of matter, the dominant contribution of the energy-momentum tensor is the energy density, and the latter is approximately the same as the mass density:

$$T_{\mu v} \approx \rho u_\mu u_v$$

with $u^\mu \approx (1, v^i)$, which gives T_{00}.

$$T_{00} \approx \rho \tag{81}$$

Also note the identity:

$$\boxed{T = g^{\mu v}T_{\mu v} \approx \rho g^{\mu v}u_\mu u_v = -\rho}$$

3.4.3 From Newton to Einstein

After listing a number of important quantities in the Newtonian limit, we are now ready to introduce the Einstein equations, the goal here is to upgrade the Poisson equation of Newtonian gravity using the tensors we introduced.

$$\nabla^2\Phi = 4\pi G\rho$$

Using (80) and (81), we can see that this equation can be interpreted as the Newtonian limit of:

$$R_{00} = 4\pi G T_{00} \tag{82}$$

where we remind ourselves that the energy-momentum tensor has to satisfy the following identity.

$$\nabla^\mu T_{\mu v} = 0$$

For a generic space-time, the Ricci tensor does not satisfy this relation, however the Einstein tensor introduced earlier does, leading (82) to take the following form:

$$R_{\mu v} - \frac{1}{2}g_{\mu v}R = kT_{\mu v} \tag{83}$$

where k is a constant to be determined.

To determine the Ricci scalar, we can contract (83) with the inverse metric tensor:

$$g^{\mu v}(R_{\mu v} - \frac{1}{2}g_{\mu v}R) = g^{\mu v}T_{\mu v}$$

which simplifies to:

$$R - \frac{1}{2}4R = kT$$

giving us R.

$$R = -kT$$

Where the constant k is to be determined by the Newtonian limit, focusing on the time-time component.

$$R_{00} = k(T_{00} - \frac{1}{2}g_{00}T) = \frac{k}{2}\rho$$

Matching the Newtonian Poisson equation fixes k to equal:

$$k = 8\pi G$$

and so we can finally write (83) as the **Einstein Equations**.

$$R_{\mu v} - \frac{1}{2}g_{\mu v}R = 8\pi G T_{\mu v} \tag{84}$$

Justice wouldn't be served unless I mentioned that although Einstein is credited with finding the field equations, the German mathematician David Hilbert published them in an article before Einstein's publication. These equations tell us how gravity works in our universe; it relates the geometry of space-time with the distribution of matter within it and in 1915, Einstein presented these field equations to the Prussian Academy of Sciences. With his publication of the general relativity theory in the same year, it sparked a new field of research, one of solving the equations for various cases and interpreting their solutions, and this is where the second half of our journey takes us.

3.5 Deflection of Light in General Relativity

Let's take a brief detour and head back to the deflection light by a massive body to compare with the classical value proposed by Soldner in the introduction. In Einstein's "On the Influence of Gravitation on the Propagation of Light", he derives a value for the deflection angle, however later noted that when working on his general relativity theory during 1915, the value he calculated was off by a factor of two. We'll go through both the incorrect and correct derivations of the deflection angle so we can see what Einstein missed.

The case for deflection of light in itself is complicated due to the involvement of the variation of not just the proper time, but also the speed of light. Let's use a metric that resembles the Minkowski metric, except for the time-time component, g_{tt}, is different and can be derived from motion of a test particle in a general manifold using a perturbed Minkowski metric, we'll discuss this in [4.2.2].

The line element can be written as:

$$ds^2 = (1 - \frac{2m}{r})(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{85}$$

where $m = GM/c^2$ is the geometric mass.

For a light ray, $ds = 0$ for a null trajectory which implies that the speed of light moving in the x-direction is:

$$c(r) = \frac{dx}{dt} = \sqrt{1 - \frac{2m}{r}} \approx 1 - \frac{m}{r} \tag{86}$$

where we Taylor expanded (86) to the first order.

We can hypothesize that the direction of the wave-front changes approximately by the partial derivative of the speed of light with respect to the y-axis. Note the total deflection is extremely small, so we can just consider the x-component of it's motion:

$$\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}(1 - \frac{m}{\sqrt{x^2 + y^2}}) = \frac{m}{r^3}y \tag{87}$$

where $r^2 = x^2 + y^2$.

To compute the total-deflection of the wave front along the path, Einstein simply integrated over the entire path, his calculation relies on the fact that nearly all the deflection occurs within some reasonable proximity of the gravitational body, this can be defined as an impact parameter or the point of closest approach. We can set $y = R$ and integrate over all of space to find the deflection angle θ.

$$\theta = mR \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + R^2}^{\frac{3}{2}}} dx = 2\frac{m}{R} \tag{88}$$

This is the same value as predicted in Einsteins 1911 paper:

$$\alpha = \frac{1}{c^2} \int_{\theta=-\frac{\pi}{2}}^{\theta=+\frac{\pi}{2}} \frac{kM}{r^2} \cos(\theta) ds = \frac{2kM}{c^2 \Delta}$$

where k denotes the gravitational constant G and Δ being the point of closest approach.

The reason for Einsteins later corrected value was that the full theory of general relativity takes into account not just the variation of the time-time component of the metric as in (85), but also the variation of spatial components, the line element for a spherically symmetrical gravitational field is given by:

$$(ds)^2 = (1 - \frac{2m}{r})(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 - \frac{1}{r^2}(\frac{2m}{r - 2m})(xdx + ydy + zdz)^2 \tag{89}$$

where the metric is known as the Schwarzschild metric, which we will look at in detail in [4.1].

The g_{tt} component is the same, however we will take into account the spatial curvature in our assessment of the deflection of light rays. In Einstein's review article in 1916, he filled in the details of his calculation, which is essentially the repetition of (87), except the spatial components of the metric are taken into account. We still take $ds = 0$ and set $dy = dz = 0$ along the path so the speed of light is:

$$c(r) = \frac{dx}{dt} = \sqrt{-\frac{g_{tt}}{g_{xx}}} \tag{90}$$

where g_{xx} and g_{tt} are:

$$\begin{aligned} g_{tt} &= (1 - \frac{2m}{r}) \\ g_{xx} &= -1 - \frac{x^2}{r^2}(\frac{2m}{r-2m}) \end{aligned} \tag{91}$$

Plugging (91) into (90) and Taylor expanding to the first order:

$$c(r) = \sqrt{\frac{1 - \frac{2m}{r}}{1 + \frac{2m}{r}\frac{x^2}{r^2}(\frac{1}{1-2m/r})}} \approx (1 - \frac{m}{r})(1 - \frac{m}{r}\frac{x^2}{r^2}) \approx 1 - \frac{m}{r}(1 + \frac{x^2}{r^2})$$

where the partial derivative of c with respect to y becomes:

$$\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}(1 - \frac{m}{\sqrt{x^2+y^2}}(1 + \frac{x^2}{x^2+y^2})) = \frac{m(4x^2+y^2)}{r^5}y \tag{92}$$

Recalling that $y = R$, Einstein integrated along the entire path to give the total deflection.

$$\theta = mR\int_{-\infty}^{\infty} \frac{4x^2+R^2}{(x^2+R^2)^5/2}dx = 4\frac{m}{R} \tag{93}$$

Plugging $m = GM/c^2$ back into (93) and replacing the impact parameter R with b yields the total deflection angle

$$\theta = \frac{4GM}{c^2 b} \tag{94}$$

Comparing with (19), the GR value is double the Newtonian approximation. The deflection of light was one of the tests used to confirm the theory of general relativity, with an expedition carried out by Arthur Eddington and Frank Dyson in 1919 to observe a solar eclipse, allowing the observation of stars near the sun. The experimental observations confirmed Einstein's prediction and made the front pages of the newspapers, catapulting Einstein to celebrity status.

4 Schwarzschild Black Hole

4.1 Schwarzschild Solution

The second half of our journey begins with Karl Schwarzschild, a German physicist and astronomer who provided the first exact solution to the Einstein field equations. He made his publication in 1914, the same year that Einstein published the general relativity theory, in his paper, his solution describes the vacuum region of a space-time whose curvature is sourced by a spherically-symmetric matter distribution. His solutions lead to the description of a region of space having a gravitational field so intense that no matter or radiation can escape, the name of such object I will reveal once we have seen the solution.

Let us consider the the restriction to a region of space-time where there is a vacuum leading to the energy-momentum tensor vanishing.

$$\boxed{T_{\mu v} = 0}$$

In a vacuum, since $T_{qmuv} = 0$, the Ricci scalar also vanishes, leading to the **Vacuum Einstein Equations**.

$$R_{\mu v} = 0 \tag{95}$$

Let's describe the metric of a spherically-symmetric body of mass M, assuming that outside the body we have the vacuum Einstein equations. We assume the solution is static and spherically symmetric, which restricts the line element to:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{96}$$

where we have switched to spherical co-ordinates and set the speed of light to equal one.

$$\begin{aligned} x &= r\cos(\phi)\sin(\theta) \\ y &= r\sin(\phi)\sin(\theta) \\ z &= r\cos(\theta) \end{aligned} \tag{97}$$

Introducing an ansatz into the vacuum Einstein equations, we obtain differential equations for A(r) and B(r), whose solutions are:

$$A(r) = A(1 + \frac{B}{r});\text{ and; } B(r) = (1 + \frac{B}{r})^{-1} \tag{98}$$

where A and B are constants to be determined. To find A and B, we require that the solution is consistent with the Newtonian limit, using (75):

$$g_{tt} = \eta_{tt} + h_{tt} = -1 - 2\Phi = -1 + \frac{2GM}{r}$$

Since $g_{tt} = A(r)$, we obtain $A = 1$ and $B = -2GM$, leading us to the **Schwarzschild Solution**.

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{99}$$

where the Schwarzschild metrci can be expressed as:

$$g_{\mu v} = \begin{pmatrix} -(1 - \frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \tag{100}$$

We can notice that this solution is said to be asymptotically flat, as $r \to \infty$, we obtain the Minkowski line element, additionally we can see that the metric components become singular at two values of the radius. At $r = 2GM$, where $g_{rr} = (1 - \frac{2GM}{r})^-1 = \infty$, we obtain a co-ordinate singularity, while at $r = 0$, where $g_{tt} = -(1 - \frac{2GM}{r}) = \infty$, we obtain a physical singularity in which the metric is not singular for any co-ordinate system.

Since the radius $r = 2GM$ is only a co-ordinate singularity, which can be eliminated by using a different co-ordinate system, there is nothing special occurring locally at this radius in space-time. However this radius is special and is called the Schwarzschild radius, often denoted as $R_S = 2GM$, for an object that lies inside this radius, we obtain an object called a **Black Hole**.

Having introduced the the concept of a light cone in special relativity, which defines the causal structure of space-time. For the Schwarzschild space-time, let us consider for simplicity a radially-directed light rays, obeying $d\theta = d\phi = 0$.

$$0 = ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}}$$

leading to:

$$\frac{dr}{dt} = \pm(1 - \frac{2GM}{r})$$

For $r > 2GM$, we pick the plus sign for an outgoing ray and the minus sign for an in-going ray. At large radius $r >> 2GM$, the light-cones are approximately like in Minkowski space-time where $dr \approx \pm dt$. As we move from a large radius towards $r = 2GM$, the light-cone obeys $\mid dr/dt \mid < 1$, this implies that for an observer that falls within the Schwarzschild radius or beyond the event horizon, the orientation of the observers light-cone flips, the future direction becomes that of decresing r, so the singularity is the future of the observer and there is no going back, this also applies for light that crosses the event horizon.

4.2 Motion of a Test Particle

4.2.1 Minkowski Space-time

Let's consider the motion of a test particle in Minkowski space-time, we'll introduce the the notion of action, and use the Euler-Lagrange equations to find the equations of motion. We'll begin with recalling the line element for Minkowski space-time.

$$ds^2 = \eta_{\mu v} dx^\mu dx^v$$

We can re-write the infinitesimal four-vector using the chain rule:

$$dx^\mu = \frac{dx^\mu}{d\tau} d\tau = \dot{x}^\mu d\tau \tag{101}$$

and τ is the proper time so the line element in terms of (101) becomes:

$$ds^2 = \eta_{\mu v} \dot{x}^\mu \dot{x}^v d\tau^2 \tag{102}$$

where we assume the proper time τ equals the universal time t.

$$\begin{aligned} ds^2 &= -c^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ ds^2 &= (-c^2 + v^2) dt^2 \end{aligned} \tag{103}$$

We can derive the geodesic equation via the action principle between two time-like separated events:

$$S = \int ds$$

where ds is the line element in the context of motion along a geodesic, minimising the action S.

$$\begin{aligned} S &= -mc \int ds \\ S &= -mc \int_{\tau_1}^{\tau_2} \sqrt{-\eta_{\mu v} \dot{x}^\mu \dot{x}^v} d\tau \end{aligned} \tag{104}$$

Plugging (103) into (104) yields:

$$
\begin{aligned}
S &= -mc\int_{t_1}^{t_2}\sqrt{c^2-v^2}dt \\
&= -mc\int_{t_1}^{t_2}c\sqrt{1-\frac{v^2}{c^2}}dt \\
&= -mc^2\int_{t_1}^{t_2}\sqrt{1-\frac{v^2}{c^2}}dt
\end{aligned}
\tag{105}
$$

where we can perform a Binomial expansion of (105) to first order assuming $v << c$:

$$
\sqrt{1-\frac{v^2}{c^2}} \approx \left(1-\frac{1}{2}\frac{v^2}{c^2}\right)
$$

and plug into (105).

$$
S = \int_{t_1}^{t_2}\left(-mc^2+\frac{1}{2}mv^2\right)dt \tag{106}
$$

The first term in the in the integrand is the rest energy of the particle and the second is the term for the kinetic energy. Using the Euler-Lagrange equations we can find the equations of motion of the test particle.

$$
\begin{aligned}
\frac{\partial\zeta}{\partial x}-\frac{d}{dt}\left(\frac{\partial\zeta}{\partial\dot{x}}\right) &= 0 \\
0-\frac{d}{dt}(m\dot{x}) &= 0 \\
m\ddot{x} &= 0
\end{aligned}
\tag{107}
$$

The solution to (107) is given by:

$$
x = kt + c
$$

which is the equation of a straight line.

4.2.2 General Manifold

For any general manifold, let's use some general metric, which can be written as a perturbed Minkowski metric.

$$
g_{\mu v} = \begin{pmatrix} -c^2+\Phi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{108}
$$

We can follow the same method as in the previous section to compute the line element.

$$
ds^2 = g_{\mu v}\dot{x}^\mu\dot{x}^v dt^2 = (-c^2+\Phi+v^2)dt^2 \tag{109}
$$

where $v^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$ and the action from (103) becomes:

$$
S = -mc^2\int_{t_1}^{t_2}\sqrt{1-\frac{v^2}{c^2}-\frac{\Phi}{c^2}}dt \tag{110}
$$

in which we can take the binomial expansion to the first order and follow from (106).

$$S = \int_{t_1}^{t_2} \left(-mc^2 + \frac{1}{2}mv^2 + \frac{1}{2}m\Phi \right) dt \tag{111}$$

Let's fix Φ by comparing (111) to the action of a particle moving in a gravitational potential given by:

$$S = \int_{t_1}^{t_2} \left(-mc^2 + \frac{1}{2}mv^2 + \frac{GMm}{r} \right) dt$$

where we can easily deduce what Φ is.

$$\Phi = \frac{2GM}{r}$$

So the final metric is given as:

$$g_{\mu v} = \begin{pmatrix} -(1-\frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{112}$$

where g_{tt} is the time-time component of the metric as seen in the GR deflection of light.

$$g_{tt} = -(1 - \frac{2GM}{r})$$

4.3 Deflection of Light in Schwarzschild Space-time

Let's once again look at the deflection of light, but by a black hole in Schwarzschild space-time using a long derivation incorporating Lagrangian mechanics and a particular form of the metric to extend it to multi-dimensional Schwarzschild and Ads black holes. Recall the action for a massive particle:

$$S_{massive} = -m \int \sqrt{-g_{\mu v} dx^\mu dx^v} \tag{113}$$

where we set $c = 1$.

We'll rewrite (113) introducing a factor of $\sigma(\zeta)$ parameterised by ζ so we can make the jump to mass-less action for a light ray.

$$S_{massive} = -\frac{1}{2} \int_{\zeta_1}^{\zeta_2} d\zeta \left(\sigma(\zeta) \left(\frac{dx}{d\zeta} \right)^2 + \frac{m^2}{\sigma} \right) \tag{114}$$

where

$$\left(\frac{dx}{d\zeta} \right)^2 = -g_{\mu v} \frac{dx^\mu}{d\zeta} \frac{dx^v}{d\zeta}$$

The term σ appears without derivatives in the Euler-Lagrange equations:

$$\frac{\partial \zeta}{\partial \sigma} = \frac{d}{d\zeta} \frac{\partial \zeta}{\partial (d\sigma / d\zeta)} = 0$$

leading to:

$$\left(\frac{dx}{d\zeta} \right) - \frac{m^2}{\sigma^2} = 0$$

For a massive particle:

$$\sigma = \frac{m}{\sqrt{(dx/d\zeta)^2}} \tag{115}$$

where we can obtain the original action for a massive particle. Setting $\sigma = 1$, we obtain the action for a mass-less particle.

$$S_{mass-less} = \frac{1}{2}\int_{\zeta_1}^{\zeta_2} d\zeta g_{\mu v}\frac{dx^\mu}{d\zeta}\frac{dx^v}{d\zeta} \tag{116}$$

The Lagrangian of a mass-less particle that we will use is given by:

$$\zeta = \frac{1}{2}\left(-A(r)\dot{t}^2 + B(r)\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) \tag{117}$$

where $\dot{t} = \frac{dt}{d\zeta}$ and so on. Using the Euler Lagrange equations and considering a light ray in the $\theta = \frac{\pi}{2}$ (to confine the motion in the x-y plane) we obtain the non vanishing quantities, suppressing the r dependence in A(r) and B(r):

$$\begin{aligned} \dot{t} &= \frac{\epsilon}{A} \\ \phi &= \frac{l}{r^2} \end{aligned} \tag{118}$$

where the terms ϵ and l relate to the energy and angular momentum respectively which don't change with respect to ζ and are conserved. Substituting (118) into (117) with $\theta = \pi/2$ yields and expression for a light-like trajectory:

$$\begin{aligned} -A\dot{t}^2 + B\dot{r}^2 + r^2\dot{\phi}^2 &= 0 \\ -A\frac{\epsilon^2}{A^2} + B\dot{r}^2 + r^2\frac{l^2}{r^4} &= 0 \\ \dot{r}^2 + \frac{1}{B}\left(\frac{l^2}{r^2} - \frac{\epsilon^2}{A}\right) &= 0 \end{aligned} \tag{119}$$

where we can once again define an impact parameter b and express it in terms of l and ϵ with $(c = 1)$.

$$\begin{aligned} l &= bp \\ \epsilon &= p \\ b &= \frac{l}{\epsilon} \end{aligned} \tag{120}$$

To find the bending angle $\Delta\phi$ we can reformulate $\dot{r}$ in terms of ϕ using the chain rule.

$$\begin{aligned} \frac{dr}{d\phi} &= \frac{dr}{d\zeta}\frac{\zeta}{d\phi} = \dot{r}\frac{d\zeta}{d\phi} \\ \dot{r} &= \frac{dr}{d\phi}\frac{d\phi}{d\zeta} = \frac{dr}{d\phi}\frac{l}{r^2} \end{aligned} \tag{121}$$

Let's combine (120), (121) and (119):

$$
\begin{aligned}
\left(\frac{dr}{d\phi}\right)^2 \frac{l^2}{r^4} + \frac{Al^2}{r^2} - \epsilon^2 = 0 \\
\left(\frac{dr}{d\phi}\right)^2 + Ar^2 - \frac{\epsilon^2 r^4}{l^2} = 0 \\
\left(\frac{dr}{d\phi}\right)^2 + Ar^2 - \frac{r^4}{b^2} = 0
\end{aligned}
\tag{122}
$$

where, assuming a light ray approaching a mass from an infinite distance gets reflected at the distance of closest approach r_0 and shoots off to infinity; we define a deflection angle $\Delta\phi$.

$$\Delta\phi = \int_{r_0}^{\infty} \frac{dr}{\sqrt{\frac{r^4}{b^2} - Ar^2}}$$

Using the Schwarzschild metric and introducing a factor of two to account for how the light ray approaching an infinite distance away is reflected at r_0 and shooting off to infinity, the integral thus becomes:

$$\Delta\phi = 2\int_{r_0}^{\infty} \frac{dr}{\sqrt{\frac{r^4}{b^2} + r^2 - 2GMr}} \tag{123}$$

where we can simplify the integral with the substitution $r = r_0/u$ and $dr = -r_0/u^2 du$.

$$\Delta\phi = 2\int_{0}^{1} \frac{r_0/u^2 du}{\sqrt{\frac{r_0^4}{u^4 b^2} - \frac{r_0^2}{u^2} - 2GMr}} \tag{124}$$

We adjusted the limits of the integral to obtain a positive integrand, r_0 is the point of closest approach where the radial velocity equals zero implying:

$$\frac{dr}{d\phi} = 0$$

leading to:

$$\frac{r_0^2}{b^2} = -1 + \frac{2GM}{r_0} \tag{125}$$

where substituting into (124) gives the final integral defining the deflection angle.

$$\Delta\phi = 2\int_{0}^{1} \frac{du}{\sqrt{1 - u^2 - \frac{2GM}{r_0(1+u^3)}}} \tag{126}$$

To determine the bending angle, we find the real solution r_0 for the cubic polynomial:

$$\frac{r_0^3}{b^2} + r_0 + 2GM = 0$$

and take the Taylor expansion assuming $GM > 0$ and $b > 0$ in leading orders of GM.

$$\approx b - GM - \frac{3M^2G^2}{2b} - \frac{4M^3G^3}{b^2} \ldots.$$

We can also take the Taylor expansion of the integrand assuming $2GM/r_0 > 0$ in leading orders of $2GM/r_0$:

$$\approx \frac{1}{\sqrt{1-u^2}} - \frac{2GM(-1+u^3)}{2r_0(1-u^2)^{3/2}} \ldots.$$

where we can finally solve the integral, re-introducing c into the expression and subtracting π from the result(if the light ray is not deflected the result would still be π for the light ray travelling in a straight line).

$$\Delta\phi = \frac{4GM}{c^2 b} \tag{127}$$

Which matches the result from (94).

5 More Black Hole Solutions

5.1 Anti-de Sitter Black Hole

In the Einstein equations, $\nabla^\mu G_{\mu v} = 0$ is the geometrical counterpart of the conservation law $\nabla^\mu T_{\mu v} = 0$, notice if the metric is covariantly constant then we can write:

$$\nabla^\mu (G_{\mu v} + \Lambda g_{\mu v}) = 0$$

where Λ is a constant which we can use to to motivate an extension of the Einstein equations.

$$R_{\mu v} - \frac{1}{2} g_{\mu v} R + \Lambda g_{\mu v} = 8\pi G T_{\mu v}$$

Λ is known as the Cosmological constant, introduced by Einstein in 1917 to counterbalance the effect of gravity and achieve a static universe. It was later abandoned after Edwin Hubble's confirmation that the universe was expanding, however the discovery that the universe's expansion is accelerating implied that the cosmological constant could have a positive value. There are three cases to be made for the cosmological constant, each resulting in a different space.

- $\Lambda > 0$: de-Sitter space-time
- $\Lambda = 0$: Minkowksi space-time
- $\Lambda < 0$: Anti-de-Sitter space-time

$\Lambda > 0$ describes de-Sitter space (dS), after Willem de Sitter, a Dutch physicist, who came up with the concept as a solution for the Einstein equations in which there is no matter. de-Sitter space describes space with a positive curvature and can be thought of as a hyper-sphere (the generalisation of a circle in n-dimensions). $\Lambda < 0$ describes Anti-de-Sitter space(AdS) describes a negative curvature, visualised as a saddle shape, both Ads and dS have as many symmetries as flat Minkowski space-time.

Let's derive a metric for black hole in AdS space-time to then use to solve the Einstein equations. Firstly we'll embed the AdS_n in a $(n+1)$ dimensional flat space using cartesian co-ordinates $(X_1, X_2, X_3....., X_{n+1})$ by the following constraint:

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -\alpha^2 \tag{128}$$

where α is the AdS radius. X_1 and X_2 represent a normal tw-dimensional space and X_{-1} and X_0 represent some other arbitrary two-dimensional space.

$$-\rho^2 + r^2 = -\alpha^2$$

where we can now switch to spherical polars and write the co-ordinates for the AdS_2.

$$\begin{aligned}
X_{-1} &= \sqrt{\alpha^2 + r^2} \cos\tau \\
X_0 &= \sqrt{\alpha^2 + r^2} \sin\tau \\
X_1 &= r\cos\phi \\
X_2 &= r\sin\phi
\end{aligned}$$

The line element thus becomes:

$$\begin{aligned} ds^2 &= -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2 \\ ds^2 &= -(\alpha^2 + r^2)dt^2 + \frac{\alpha^2}{\alpha^2 + r^2}dr^2 + r^2 d\phi^2 \end{aligned} \tag{129}$$

5.1.1 Null Geodesic in Ads Space-time

One of the most fascinating results from Ads space-time stems from the a null geodesic using the Ads metric where we assume $ds^2 = 0$.

$$(\alpha^2 + r^2)dt^2 = \frac{\alpha^2}{\alpha^2 + r^2}dr^2$$

simplifying to:

$$dt = \frac{\alpha}{\alpha^2 + r^2}dr \tag{130}$$

where we can now perform an integral to determine the amount of time it would take a light wave to travel an infinite distance in Ads space.

$$\Delta T = \int_0^\infty \frac{\alpha}{\alpha^2 + r^2}dr = \alpha\frac{\pi}{2} \tag{131}$$

This is a very interesting result that is completely counter intuitive from what we know about Minkowski and Riemannian geometry. A light beam will take a finite amount of time to travel an infinite distance in Ads space-time. Ads space generalises to any number of space dimensions, in higher dimensions Ads space plays a huge role in the Ads/CFT correspondence, which suggests that it's possible to describe a force in quantum mechanics in a certain number of dimensions with a string theory where reality is made up of infinitesimal vibrating strings, smaller than atoms, electrons or quarks populating an anti-de Sitter space, with one additional dimension.

5.2 Reissner-Nordström Black Hole

Physicist John Archibald Wheeler proposed the no-hair theorem, which states that all stationary black holes can be completely characterised by three external observables, mass, electric charge and angular momentum. All other information about the matter that formed the black hole disappears behind the event horizon, making it not accessible to external observers. Wheeler expressed this idea with the phrase, "black holes have no hair", which was the origin of the name.

In 1916, Gunnar Nordström, a Finnish theoretical physicist solved the Einstein equations outside a spherically symmetric charged body. The metric is obtained by coupling the Maxwell Equations with the Einstein field equations.

$$ds^2 = -dt^2\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^- 1dr^2 + r^2 d\Omega^2 \tag{132}$$

The solution was also found by Hans Reissner, Hermann Weyl and George Barker Jeffrey, hence the metric nowadays is dubbed as the Reissner-Nordström metric. We can see the black hole solution for this metric as:

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) = 0$$

where the solution for r is:

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{133}$$

The Reissner-Nordström black hole has two horizons, one of the horizons is the familiar event horizon and the other is an internal charge horizon. Where depending on the value of M:

- $M > Q$: regular Reissner-Nordström black hole
- $M = Q$: the horizons coincide and you have an external black hole
- $M < Q$: no real horizon and so there is a 'naked' singularity

A naked singularity could allow scientists to observe an infinitely dense material, which would under normal circumstances be impossible by the cosmic censorship hypothesis, which states that a gravitational singularity would remain hidden hidden by an event horizon. Without a horizon, some speculate that naked singularities could actually emit light.

6 Black Hole Thermodynamics

6.1 Wick Rotation

We are now near the end of our journey, we have so far found the solutions from the Einstein equations that described different types of black holes, here we'll consider the thermodynamic properties of black holes, mainly temperature and entropy.

We'll first cover the topic of Wick rotation, introduced by Gian Carlo Wick, an Italian physicist, who developed a method to find the solutions to mathematical problems in Minkowski space from a solution related to a problem in Euclidean space by means of a transformation that substitutes an imaginary number variable for a real number variable.

We start with the Schrödinger equation, postulated by Erwin Schrödinger in 1925, defined as a linear partial differential equation that governs the wave function of a quantum-mechanical system.

$$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

If the Hamiltonian is time dependent we can write:

$$\psi(x,t) = e^{-i\frac{t}{m}\hat{H}}\psi(x,0) \tag{134}$$

where $e^{-i\frac{t}{m}\hat{H}}$ is the time evolution operator that evolves a system from one state to another. If the energy eigenvalues of the Hamiltonian are E_n we can define the return wave-function at a time $t > 0$.

$$e^{-i\frac{t}{m}\hat{H}}\psi_n(x) = e^{-itE_n/\hbar}$$

Using Wick rotation, we can make time complex defining $t \to -i\tau$ so our evolution operator becomes:

$$e^{-i(-i)\frac{\tau}{m}\hat{H}} = e^{-\frac{1}{m}\tau\hat{H}}$$

applying this to (134) yields:

$$e^{-i(-i\tau)E_n/\hbar} = e^{-\tau E_n/\hbar} \tag{135}$$

showing the periodicity of the system since it returns the same state after a complex time $i\tau$. We can compare the exponential in (135) to the Boltzman factor from statistical mechanics:

$$p_i \propto e^{E_n/\beta}$$

where $\beta = 1/kT$, and k_b is the Boltzman constant.

6.2 Hawking Temperature via Euclidean Gravity

Euclidean gravity introduces a complex time co-ordinate, so the sign of the metric changes from $(-,+,+,+)$ to $(+,+,+,+)$. Let's apply a Wick rotation to our general metric and ignore the angular momentum parts:

$$ds^2 = A(r)d\tau^2 + B(r)dr^2 \tag{136}$$

where for a Schwarzschild black hole $A(r_s) = 0$. However, if the first derivative is non-zero, we can take the Taylor expansion to the first order in leading orders of r_s.

$$A(r_s) \approx A(r_s) + A'(r_s)(r - r_s) \approx A'(r_s)(r - r_s)$$

Let's define a new co-ordinate $\rho = (r - r_s)$ and substituting it into a Euclidean metric:

$$\begin{aligned} ds^2 &= A'(r_s)\rho^2 d\tau^2 + \frac{4}{A'(r_s)d\rho^2} \\ &= \frac{4}{A'(r_s)}) \left(\frac{A'(r_s)^2\rho^2}{4} d\tau^2 + d\rho^2 \right) \end{aligned} \tag{137}$$

which is the metric for a cone, describing the Euclidean time t_E and radial directions which form a so called 'cigar' geometry which is smooth at the tip $r = r_s$.

To avoid a singularity appearing at $r = r_s$, we can fix the metric to match that of plane polar co-ordinates.

$$\phi = \frac{A'(r_s)}{2}\tau \approx \phi + 2\pi$$

$$\tau \approx \tau + \frac{4\pi}{A'(r_s)}$$

Using the fact that $\tau = 1/T$ where $k_b = 1$, we obtain the Hawking temperature.

$$T = \frac{A'(r_s)}{4\pi} \tag{138}$$

Substituting the Schwarzschild metric $A(r) = 1 - \frac{2GM}{r}$ yields:

$$\begin{aligned} A'(r_s) &= \frac{1}{2GM} \\ T_H &= \frac{1}{8\pi GM} \end{aligned} \tag{139}$$

where we can perform some dimensional analysis to re-introduce constant, giving the temperature of a Schwarzschild black hole in four-dimensions.

$$T_H = \frac{\hbar c^3}{8\pi GMk_B} \tag{140}$$

T_H is the Hawking temperature, introduced by British physicist Stephen Hawking. In 1974, Hawking used quantum field theory in curved space-time to explain how a black hole would emit radiation from the interaction of anti-matter and matter fields.

6.3 Hawking Radiation as Quantum Mechanical Tunneling

6.3.1 Hawking Radiation from Schwarzschild Black Holes

In the next two sections we'll derive the Hawking temperature for a Schwarzschild and Reissner-Nordström black hole setting $\hbar = G = 1$, closely following a paper published by Maulik Parikh and Frank Wilczek titled "Hawking Radiation as Tunneling.

Let's imagine a particle with some energy determined by the Hamiltonian, trying to escape the event horizon of a Schwarzschild black hole, due to the potential being higher than the value of the particle's energy, the only way that the particle can escape is through quantum mechanical tunneling, a phenomenon in which a particle is able to penetrate through a potential enegy barrier that is higher than the particles energy. Since the particle is trying to tunnel we can compute the imaginary part of the action for an outgoing wave in the radial direction:

$$ImS = Im \int L(r, \dot{r}) dt \tag{141}$$

where $L = \dot{r}p_r - H$, substituting into (141) and dropping the Hamiltonian due to it being real and conserved:

$$ImS = Im \int \dot{r} p_r dt \tag{142}$$

we obtain:

$$\begin{aligned} ImS &= Im \int_{r_{in}}^{out} p_r dr \\ &= Im \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r' dr \end{aligned} \tag{143}$$

where we used $\dot{r}dt = dr$ and re-wrote p_r as an integral. Let's manipulate the integral further, assuming the black hole's mass is fixed.

$$\begin{aligned} \dot{r} &= \frac{dH}{dp_r} \\ H &= M - w' \\ dH &= -dw' \\ dp_r' &= \frac{dH}{\dot{r}} = -\frac{dw'}{\dot{r}} \end{aligned} \tag{144}$$

Consider the the Schwarzschild metric, which for this derivation won't be convenient because of the co-ordinate singularity at $r = 2GM$, we can perform a co-ordinate transformation by first introducing a time co-ordinate:

$$t = t_s + 2\sqrt{2Mr} + 2Mln\frac{sqrtr - \sqrt{2M}}{\sqrt{r} - \sqrt{2M}} \tag{145}$$

where t_s is the Schwarzschild time. With this choice, the line element reads:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dtdr + dr^2 + r^2 d\Omega^2 \tag{146}$$

We can also introduce Painlevé time co-ordinate as:

$$\begin{aligned} t &= t' + f(r) \\ dt &= dt' + f'(r) dr \end{aligned} \tag{147}$$

where the line element becomes:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right) 2f'(r)dt'dr - \left(1 - \frac{2M}{r}\right) f'^2 dr^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2\Omega^2 \tag{148}$$

in which we can substitue (144).

$$ImS = Im \int_0^2 \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}}(-dw') \tag{149}$$

To progress, let's demand that the last three terms in (148) to be $dr^2 + r^2\Omega^2$ to solve for $f'(r)$.

$$\begin{aligned} -\left(1 - \frac{2M}{r}\right) f'(r)^2 + \frac{1}{1 - \frac{2M}{r}} &= 1 \\ f'(r) &= \frac{\sqrt{2M/r}}{1 - \frac{2M}{r}} \end{aligned} \tag{150}$$

Substitute (150) into (148):

$$ImS = Im \int_0^w \int_{r_{in}}^{r_{out}} \frac{dr}{1 - \sqrt{\frac{2(M-w')}{r}}}(-dw') \tag{151}$$

where the metric is defined for a particle's self gravitation.

$$ds^2 = -\left(1 - \frac{2(M-w)}{r}\right) dt^2 + 2\sqrt{\frac{2(M-w)}{r}} dtdr + dr^2 + r^2 d\Omega^2 \tag{152}$$

We can now find the null geodesics as:

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2(M-w)}{r}} \tag{153}$$

The limits r_{in} is the black hole's Schwarzschild radius before the particle tunnels out, once the particle tunnels it loses w energy and so $r_{out} = 2(M-w)$. The integral in (151) is not well defined due to the denominator vanishing between r_{in} and r_{out}. We can solve this integral by deforming the contour, using $w' = w' - i\epsilon$ where we make w complex. By doing this we ensure positive energy solutions decay in time:

$$e^{-\frac{iEt}{\hbar}} = e^{-}iw't - \epsilon t \tag{154}$$

where we also make use of the substitutions into (154).

$$\begin{aligned} r &= U^2 \\ dr &= 2udu \\ U_{in} &= \sqrt{2M} \\ U_{out} &= \sqrt{2(M-w)} \end{aligned} \tag{155}$$

Pulling out a factor of $i\epsilon$ out of the square root.

$$ImS = Im \int_0^2 \int_{U_{in}}^{U_{out}} \frac{2U^2 dU}{U - \sqrt{2(M-w') - i\epsilon}}(-dw') \tag{156}$$

To solve the integral lets use the substitution $x = U - \sqrt{2(M-w')}$ which carries out the integral, leading us to find a principle value that is real.

$$
\begin{aligned}
ImS &= \pi \int_0^w \int_{U_{in}}^{U_{out}} 2U^2 dU \delta(U - \sqrt{2(M-w')})(-dw') \\
ImS &= 4\pi w(M - \frac{w}{2})
\end{aligned} \tag{157}
$$

The tunneling rate is given by:

$$
\Gamma \approx e^{-2ImS} = e^{-8\pi w(M-\frac{w}{2})} \tag{158}
$$

where we compare (158) to the Boltzman factor where $E = w$ to extract the temperature(ignoring $\frac{w}{2}$).

$$
T_H = \frac{1}{8\pi M k_B} \tag{159}
$$

In SI units we obtain the same result as in (140). Let's also extract the entropy of a black hole, by using the fact that the tunneling rate is given by the ratio of the number of states of the black hole before and after tunneling.

$$
\begin{aligned}
\Delta S &= ln\left(\frac{W_{afrer}}{W_{before}}\right) \\
e^{\Delta S} &= \frac{W_{after}}{W_{before}}
\end{aligned}
$$

Leading to:

$$
\Gamma = e^{-8\pi w(M-\frac{w}{2})} = e^{\Delta S}
$$

where we can determine the change in entropy ΔS.

$$
\Delta S = -8\pi w M \tag{160}
$$

Finally, let's compute the change in energy/mass of the black hole as $dM = dE = -w$, substituting the Schwarzschild radius $r_s = 2GM$ into the expression.

$$
\begin{aligned}
dS &= 8\pi M dM \\
S &= 4\pi M^2 \\
S &= \frac{4\pi r_s^2}{4} = \frac{A}{4}
\end{aligned} \tag{161}
$$

Where we have derived the realtion between the surface area and the entropy of the black hole, a bizzare property considering the the entropy usually depends on the volume as it's an extensive thermodynamic variable.

6.3.2 Hawking Radiation from Reissner-Nordström Black Holes

Let's repeat the same derivation in the previous section but for a Reissner-Nordström black hole. Let's again change the metric to get rid off the singularity at $r_s = 2GM$, we'll use Painlevé co-ordinates given by:

$$
t = t_r + 2\sqrt{2Mr - Q^2} + M ln\left(\frac{r - \sqrt{2Mr - Q^2}}{r + \sqrt{2Mr - Q^2}}\right) \tag{162}
$$

where the Reissner-Nordström metric reads:

$$
ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} dtdr + dr^2 + r^2 d\Omega^2 \tag{163}
$$

and once again, taking account the particle's self gravitation modifies the metric.

$$ds^2 = -\left(1 - \frac{2(M-w')}{r} + \frac{Q^2}{r^2}\right)dt^2 + 2\sqrt{\frac{2(M-w')}{r} - \frac{Q^2}{r^2}}dtdr + dr^2 + r^2 d\Omega^2 \tag{164}$$

Let's solve for the null radial geodesic, setting $ds^2 = 0$ and $d\Omega^2 = 0$ and solving $\dot{r} = dr/dt$.

$$\dot{r} = 1 \pm \sqrt{\frac{2(M-w')}{r} - \frac{Q^2}{r^2}}$$

Let's take the outgoing geodesic and substituting into (152):

$$ImS = Im \int_0^w \int_{r_{in}}^{r_{out}} \frac{rdr}{r - \sqrt{2(M-w')r - Q^2 - i\epsilon}}(-dw') \tag{165}$$

where we used $w' \to w' - i\epsilon$ and the substitution:

$$\begin{aligned} U &= \sqrt{2(M-w')r - Q^2} \\ r &= \frac{U^2 + Q^2}{2(M-w')} \\ dr &= \frac{UdU}{M-w'} \end{aligned} \tag{166}$$

$$ImS = Im \int_0^w \int_{U_{in}}^{U_{out}} \frac{\frac{U(U^2+Q^2)}{M-w'}}{U^2 - 2(M-w')U + Q^2 - e\epsilon}dU(-dw') \tag{167}$$

Let's remind ourselves of the Reissner-Nordström black hole horizon's:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

where we'll consider the outer horizon because it's where all the necessary 'action' is happening, after tunneling, both r_{in} and r_{out} take on different values.

$$\begin{aligned} r_{in} &= r_+ = M + \sqrt{M^2 - Q^2} \\ r_{out} &= r_- = (M-w') + \sqrt{(M-w')^2 - Q^2} \end{aligned} \tag{168}$$

We pick the pole at the outer horizon $U = U_+$, which can be derived from the denominator of the integrand in (168).

$$\begin{aligned} U^2 - 2(M-w')U + Q^2 &= (U - U_-)(U - U_+) \\ U_+ &= (M-w') \pm \sqrt{(M-w')^2 - Q^2} \end{aligned} \tag{169}$$

The integral in (168) now becomes:

$$ImS = Im \int_0^2 \int_{U_{in}}^{U_{out}} \frac{1}{2\sqrt{(M-w')^2 - Q^2}} \frac{\frac{U(U^2+Q^2)}{M-w'}}{U - U_1 - i\epsilon}dU(-dw') \tag{170}$$

where we rewrote the denominator in terms of (170). Now using the fact that:

$$\int_{-\infty}^{\infty} \frac{1}{u - U_1 - i\epsilon} = i\pi\delta(U - U_1)$$

$$ImS = Im \int_0^w \int_{U_{in}}^{U_{out}} \frac{1}{2\sqrt{(M-w')^2 - Q^2)}} \frac{\frac{U(U^2+Q^2)}{M-w'}}{U - U_1 - i\epsilon}\delta(U - U_+)dU(-dw') \tag{171}$$

where we can use the property of the Dirac-delta function and achieve the correct sign due to $U_{in} > U_{out}$.

$$Ims = \pi \int_0^2 \frac{1}{2\sqrt{(M-w')^2 - Q^2}} \frac{U_+(U_+^2 + Q^2)}{M-w'}dU(dw')$$

Let's simplify the expression by substituting U_+ to arrive at the solution:

$$ImS \approx 2\pi w(M - \frac{w}{2}) - (M-w)\sqrt{(M-w)^2 - Q^2} + M\sqrt{M^2 - Q^2} \tag{172}$$

where the tunneling rate can now be found.

$$\Gamma \approx e^{-2ImS} = e^{-2\pi(2w(M-w/2)-(M-w)\sqrt{(M-w)^2-Q^2}+M\sqrt{M^2-Q^2})} \tag{173}$$

In order to extract the the Hawking temperature, let's Taylor expand (174) around $w = 0$:

$$\Gamma \approx e^{\frac{-2\pi w(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}}} \tag{174}$$

where we can finally extract an approximate result for the temperature by comparing (176) to the Boltzmann factor where $E = w$.

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2} \tag{175}$$

To approximate the entropy, we use $dE = dM = -w$:

$$\Delta S = \frac{-2\pi w(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}}$$

$$dS = \frac{2\pi dM(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}}$$

results in:

$$S = \int \frac{2\pi dM(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}} = \pi(\sqrt{M^2 - Q^2} + M^2)^2 = \frac{A}{4} \tag{176}$$

which matches the result from (161).

This concludes the end of our journey through the history of Black holes. We started with Newtonian mechanics, and with the introduction of space-time, expanded into a whole new world of physics, deriving the Einstein equations and predicting the existence of black holes, as well as exploring the properties and different types of black holes that could exist. I'd like to express that the theory of relativity didn't come from Einstein alone, but physicist's and mathematician's we've discussed who introduced new mathematics through three centuries, all played a considerable role in the development of special and general relativity.

7 Conclusion

In this paper, I wanted to structure it more like a story, starting from the Newtonian theory of gravity, approximating light bending using the corpuscular theory defined by an impact parameter, and comparing it with the general relativity derivation. Deriving the Einstein equations and presenting the Schwarzschild solutions, describing a metric outside a static, spherically symmetric mass, along with the light bending in Schwarzschild space-time. We've also presented the solutions for the AdS black hole, proving that it takes a light ray a finite amount of time to travel an infinite amount of distance, which has lead scientists to use AdS for string theory. It was shown that Wick rotation can be used to extract thermodynamic properties from black holes, additionally using quantum mechanical tunneling to extract the Hawking temperature and entropy. In the final chapter, we derived an expression for Hawking temperature and entropy for Schwarzschild and Reissner-Nordström black holes, relating the entropy to the surface area of the black holes respectively. In addition I wanted to really stress the historical background to these topics and give credit to the people that have dedicated their lives to physics research.

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