Numerical Approximation
NUMA12, Spring, 2017
Homework 4

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Task 1. Calculate the coefficients w_0, w_1, x_0, x_1, that make the approximation:

$$\int_0^1 xf(x)dx \approx w_0 f(x_0) + w_1 f(x_1)$$

Exact when $f \in \mathcal{P}_3$. Verify your solution by letting f be a general cubic polynomial.

Solution. This exercise scream Gaussian quadrature as we have a evaluated f in $(1+1)$ points and want it to be true for polynomials of degree $(2 \times 1 + 1)$. Theorem 12.3 with the weight equal to x fits our problem perfectly.

We thus need to calculate a polynomial $\phi_2 = a + bx + x^2$ such that it is orthogonal to all polynomials in $\mathcal{P}_1$. We choose to look for a monic polynomial for conveniences giving us a unique solution. As $span(\{1, x\}) = \mathcal{P}_1$ we get:

$$\begin{cases} (\phi_2, 1) = 0 \\ (\phi_2, x) = 0 \end{cases} \leftrightarrow \begin{cases} \int_0^1 x(a + bx + x^2)dx = 0 \\ \int_0^1 x^2(a + bx + x^2)dx = 0 \end{cases} \leftrightarrow$$

$$\begin{cases} \frac{a}{2} + \frac{b}{3} + \frac{1}{4} = 0 \\ \frac{a}{3} + \frac{b}{4} + \frac{1}{5} = 0 \end{cases} \leftrightarrow \begin{cases} a = \frac{3}{10} \\ b = -\frac{6}{5} \end{cases} \leftrightarrow \phi_2 = \frac{3}{10} - \frac{6}{5}x + x^2$$

Now that we have ϕ_2 we set x_0 and x_1 as its roots. To get the w_0 and w_1 we need to evaluate the integral $w_i = \int_0^1 x l_i dx$, where l_i are the Lagrange interpolation polynomials.

$$x_0 = -\frac{\sqrt{6}}{10} + \frac{3}{5} \qquad w_0 = -\frac{\sqrt{6}}{36} + \frac{1}{4}$$

$$x_1 = \frac{\sqrt{6}}{10} + \frac{3}{5} \qquad w_1 = \frac{\sqrt{6}}{36} + \frac{1}{4}$$

To check that this result is correct we compute the integral for a general polynomial $f = a + bx + cx^2$.

$$
\begin{aligned}
& w_0 f(x_0) + w_1 f(x_1) = \\
&= \left(-\frac{\sqrt{6}}{36} + \frac{1}{4}\right)\left(a + b\left(-\frac{\sqrt{6}}{10} + \frac{3}{5}\right) + c\left(-\frac{\sqrt{6}}{10} + \frac{3}{5}\right)^2 + d\left(-\frac{\sqrt{6}}{10} + \frac{3}{5}\right)^3\right) + \\
& \left(\frac{\sqrt{6}}{36} + \frac{1}{4}\right)\left(a + b\left(\frac{\sqrt{6}}{10} + \frac{3}{5}\right) + c\left(\frac{\sqrt{6}}{10} + \frac{3}{5}\right)^2 + d\left(\frac{\sqrt{6}}{10} + \frac{3}{5}\right)^3\right) = \\
& = \frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{d}{5} = \int_0^1 x(a + bx + cx^2 + dx^3)dx
\end{aligned}
$$

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Task 2. Construct a program to perform Gaussian integration, with the following input:

- Interval of integration
- Function to be integrated
- Number of terms in the quadrature formula

According to Ramanujan the number of numbers between a and b that are either squares or sums of two squares is given approximately by

$$0.764 \int_a^b \frac{\mathrm{dx}}{\sqrt{\ln x}}$$

Use your program to test this statement for $a = 1$ and $b = 30$

Solution. First of all, let us outline the basic theory behind the algorithm that we have implemented. The general formula for the Gauss quadrature is:

$$\int_{-1}^{1} f(x)\,\mathrm{dx} \approx \sum_{i=1}^{n} f(x_i) w_i$$

We can change the interval to whatever we like by:

$$\int_a^b f(x)\,\mathrm{dx} \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2} x_i + \frac{a+b}{2}\right) \tag{1}$$

To calculate the nodes x_i we take them as the roots of the Legendre polynomial of degree n, which is defined as:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (x-1)^{n-k}(x+1)^k$$

Then, we construct the Lagrange polynomials l_i and compute the weighs as:

$$w_i = \int_{-1}^{1} l_i \quad \forall i = 0, \ldots, n$$

Finally we put all of this together in equation 1 to get the approximation of the integral.

The algorithm seems simple at first but the actual implementation posed a few challenges. We first tried to find the roots of the Legendre polynomials exactly, but this turned to be disappointingly slow for degree greater than 5, we overcame this by turning to `numpy.roots` to find the roots numerically. We also did not succeed in finding a closed form for the integrals of the Lagrange polynomials, thus we integrated them symbolically as needed using the python library SymPy. After these workarounds (and some other minor ones) we finally got a working script and calculated the integral, which turned out to be, with 15 terms in the quadrature formula:

$$0.764 \int_{1}^{30} \frac{\mathrm{dx}}{\sqrt{\ln x}} \approx 15.1713138457160$$

There exist 15 numbers in $1, 2, \ldots, 30$ which are either squares or sums of two squares. This suggests that our method is correct and that Ramanujan did a good job with his formula. Finally, here is our code:

```
import sympy as sp
import scipy.special
import math
import numpy



def gauss(intervall, f, k):
    a, b = intervall;
    x = sp.symbols('x');
```

```
    #Legendre polynomial
    Lk = 0;
    for i in range(k+1):
        Lk += scipy.special.binom(k, i)**2 * (x - 1)**(k - i) * (x + 1)**i;
    Lk *= 2**(-k);

    #We need to find the coefficients to use numpy.roots
    coefficients = []
    for i in range(k):
        coefficients.append(sp.expand(Lk).coeff(x**(k-i)))
    coefficients.append(sp.expand(Lk).subs(x, 0).evalf())
    print(coefficients);

    roots = numpy.roots(coefficients);

    lagpols = [];
    for i in range(len(roots)):
        li = 1
        for j in range(len(roots)):
            if (i != j):
                li *= (x - roots[j]) / (roots[i] - roots[j])
                li = sp.simplify(li)
        lagpols.append(li)



    #Integrating the lagrange polynomials to find the weighs
    weights = [];
    for lag in lagpols:
        weights.append(sp.integrate(lag, (x, -1, 1)).evalf());

    res = 0;
    for i in range(k):
        res += weights[i] * f((b - a) * roots[i] / 2 + (a + b) / 2)
    return (b - a ) / 2 * res;

def func(val):
```

```
    #Definition of the function to integrate
    return 0.764 * math.sqrt(math.log(val))**(-1);

if (__name__ == "__main__"):
    print(gauss([1, 30], func, 15));
    x = sp.symbols('x');
    #integrate with sympy to compare results
    print(sp.integrate(0.764 / sp.sqrt(sp.log(x)), (x, 1, 30)).evalf())
```

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Task 3. Find the best L_1 approximation to the function $f(x) = \sin(x) + \cos(x)$ in the interval $[\pi/3, 5\pi/3]$ from the space of polynomials of degree 2. Show that your result is indeed the unique best L_1 approximation to f.

Solution. By theorem (14.5 in Powell's book) if the approximation space is the space of polynomials of degree at most n, if e^* has exactly $n+1$ zeros, then they are the extrema of the Chebyshev polynomial T_{n+2}. Therefore, by changing the interval from $[\pi/3, 5\pi/3]$ to $[-1, 1]$ and finding the extrema, we get that the zeros of the error function are:

$$\xi_0 = \frac{\pi(3-\sqrt{2})}{3}, \quad \xi_1 = \pi, \quad \xi_2 = \frac{\pi(3+\sqrt{2})}{3}.$$

The polynomial we are trying to calculate has the form $ax^2 + bx + c$, by interpolating at the values calculated above ($p^*(\xi_i) = f(\xi_i)$) and solving the linear system, we should get the best approximation:

$$a\left(\frac{\pi(3-\sqrt{2})}{3}\right)^2 + b\left(\frac{\pi(3-\sqrt{2})}{3}\right) + c = \sin\left(\frac{\pi(3-\sqrt{2})}{3}\right) + \cos\left(\frac{\pi(3-\sqrt{2})}{3}\right)$$
$$a\pi + b\pi + c = -1$$
$$a\left(\frac{\pi(3+\sqrt{2})}{3}\right)^2 + b\left(\frac{\pi(3+\sqrt{2})}{3}\right) + c = \sin\left(\frac{\pi(3+\sqrt{2})}{3}\right) + \cos\left(\frac{\pi(3+\sqrt{2})}{3}\right)$$

This gives us the values $a = 0.4150, b = -3.2803, c = 5.2091$, thus the best approximation polynomial is:

$$p^*(x) = 0.4150x^2 - 3.2803x + 5.2091$$

To verify that this approximation is a best approximation we need to be convinced that the error function only has 3 zeros. This is hard to do analytically so the plot of the error function in figure 1 will have to do. Fortunately the plot reveal that there is exactly 3 zeros which fulfills the demands of theorem 14.5 and thus we can conclude that p^* is a best approximation.

As for uniqueness, we are approximating from the space of polynomials of degree at most two, which is a Haar space, therefore by theorem (14.3 in Powell's book) our best approximation is unique. ∎

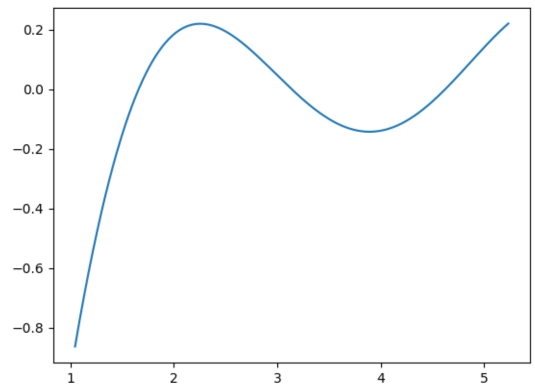

Figure 1: Plot of $f - p^*$. Note that there is exactly 3 zeros!.

Task 4. Let $\mathcal{A}$ be the 3-dimensional space of functions on $[-1, 1]$ composed of two straight line segments joined at $x = 0$. Calculate the element of $\mathcal{A}$ that minimizes

$$\int_{-1}^{1} |x^2 - p(x)| \, \mathrm{dx}, \quad p \in \mathcal{A}. \tag{2}$$

Solution. $\mathcal{A}$ is a Haar space as if a function in $\mathcal{A}$ has more than 2 zeros it is identically zero. Which can be taken as the definition of a Haar space.

We then note that equation 2 is the 1-norm and the element in $\mathcal{A}$ we are searching for is the best L_1 approximation from $\mathcal{A}$ to x^2. Now we use our dear theorem 14.5 again, giving us the zeros of the error function. These are given by equation 3.

$$\zeta_k = \cos\left(\pi\left(-\frac{k}{4} + \frac{3}{4}\right)\right) \Leftrightarrow \begin{cases} \zeta_0 = -\frac{\sqrt{2}}{2} \\ \zeta_1 = 0 \\ \zeta_2 = \frac{\sqrt{2}}{2} \end{cases} \tag{3}$$

By demanding that the error function have these three zeros we get a candidate for a best approximation p^* in equation 4.

$$p^*(x) = \begin{cases} -\frac{\sqrt{2}x}{2} & \text{for } x \in [-1, 0) \\ \frac{\sqrt{2}x}{2} & \text{for } x \in [0, 1] \end{cases} \tag{4}$$

$$x^2 - p^*(x) = \begin{cases} x^2 + \frac{\sqrt{2}x}{2} & \text{for } x \in [-1, 0) \\ x^2 - \frac{\sqrt{2}x}{2} & \text{for } x \in [0, 1] \end{cases} \tag{5}$$

The only thing remaining is to check that the criteria, from 14.4, of there being exactly 3 zeros in the error function is meet. In this case solving for all the zeros becomes solving a piece wise defined second order polynomial which is indeed possible. One need not however solve the equations explicitly. Instead we note that second order polynomials can have at most 2 zeros and both parts of the error in equation 5 are second order polynomials with one root in common ($x = 0$). We thus conclude that the error function has at most 3 zeros and we know by construction that it has 3 zeros. Therefore we know it has **exactly** three zeros. Thus we have found an element of $\mathcal{A}$ that minimizes equation 2. Further as $\mathcal{A}$ is a Haar space the best approximation is unique.

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Task 5. Construct a program to calculate the best L_1 approximation in $\mathcal{P}_1$ to the data

Table 1: Data to be approximated

x_i	0	1	2	3	4	5	6
$f(x_i)$	-35	-56	0	-16	-3	4	10

Take all weights equal to one. Plot your approximation together with the data.

Solution. In this task, theorem 15.3 is our best friend. The theorem states the existence of a best approximation such that an element of $\mathcal{A}$ that is zero in all the points the error is zero will be identically equal to zero. Theorem 15.3 tells us that the a best approximation will go through two of the points. There is only a finite number of points and each pair of points have a unique polynomial

$p \in \mathcal{P}_1$ shown in equation 6 that interpolates f in those point.

$$p(x) = \frac{f(x_j)x_i - f(x_i)x_j}{x_i - x_j} + \frac{f(x_i) - f(x_j)}{x_i - x_j} \times x \tag{6}$$

Our algorithm test every combination of points to interpolate with and finds the one with the smallest $\mathcal{L}_1$ error. The resulting polynomial is shown in equation 7. The approximating data points in table 2. Finally a plot of the approximation function with the data can be found in figure 2.

$$p^*(x) = 7.8 * x + -35.0 \tag{7}$$

Table 2: Approximation data compared to the actual

$p^*(x_i)$	-35	-27	-19	-12	-3.8	4	11
$f(x_i)$	-35	-56	0	-16	-3	4	10

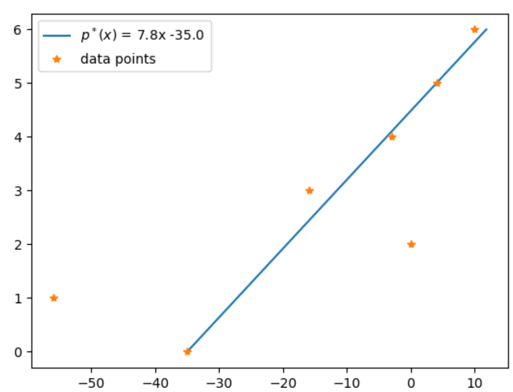

Figure 2: The data in 1 together with the best approximation.

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