Numerical Approximation
NUMA12, Spring, 2017
Homework 4

Anton Makarov and Emil Johansson
Lund University

Task 1. Show that T_n and T_{n1} have no common zeros.

Solution. First we assume there is an number θ such that equation 1 holds. If there is such a θ then there is such a $x = \cos\theta$ so that T_n and T_{n1} have common zeros. We will thus need to show that this leads to a contradiction.

$$\begin{cases} T_n(\cos\theta) = \cos n\theta = 0 \\ T_{n+1}(\cos\theta) = \cos(n+1)\theta = 0 \end{cases} \Leftrightarrow \begin{cases} n\theta = \frac{\pi}{2} + 2\pi k \\ (n+1)\theta = \frac{\pi}{2} + 2\pi k' \end{cases} \tag{1}$$

$$\Leftrightarrow 2\pi k = 2\pi k' - \theta \Leftrightarrow \theta = 2\pi(k' - k)$$

$$\Rightarrow \cos n\theta = (-1)^{n(k'-k)} \neq 0 \tag{2}$$

The result of the assumption of existence leads to the contradiction in equation 2. Thus our assumption must be false there is no θ such that it is a root to both T_n and T_{n1}. Note that not all x can be written as $x = \cos\theta$ but only $x \in [-1, 1]$. However this is the interval in which Chebyshev polynomials are studied and what happens outside is of little importance. ■

Task 2. Calculate the quadratic polynomial that minimizes the expression

$$\max_i |f(\xi_i) - p(\xi_i)|, \quad p \in \mathcal{P}_2 \tag{3}$$

for $f(x) = |x - 1/2|, x \in [0, 1]$ and the reference $\{0, 1/4, 1/2, 1\}$. Plot the function and its approximation. Plot the error function. Is this a best minimax approximation?

Solution. We want to solve the equation system given by:

$$f(\xi_i) = \sum_{j=0}^{n} (\lambda_j \Phi_j(\xi)) + (-1)^i h \tag{4}$$

We use the monomial basis $\{1, x, x^2\}$ and substituting our data in equation 4 we get

$$f(\xi_i) = \lambda_0 + \lambda_1 \xi_i + \lambda_2 \xi_i^2 + (-1)^i h \tag{5}$$

This leads to the system of equations:

$$\begin{aligned} \frac{1}{2} &= \lambda_0 + h \\ \frac{1}{4} &= \lambda_0 + \frac{\lambda_1}{4} + \frac{\lambda_2}{16} - h \\ 0 &= \lambda_0 + \frac{\lambda_1}{2} + \frac{\lambda_2}{4} + h \\ \frac{1}{2} &= \lambda_0 + \frac{\lambda_1}{2} + \frac{\lambda_2}{4} - h \end{aligned}$$

Note that this system is always solvable as we know that the best approximation exists. Using MATLAB's `backslash` we obtain the solution $[0.5556, -1.8889, 1.7778, -0.0556]$ where the last element is h. Therefore we have that our polynomial is $p(x) = 0.5556 - 1.8889x + 1.7778x^2$. As we can see in the Figure 1 where we plot the function, the approximation and the error, this is not yet the best approximation. To get a better result, we can take a step of the one point exchange

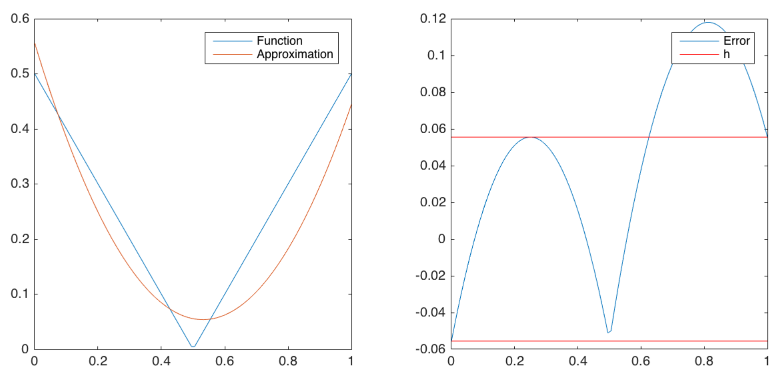

Figure 1: Function, approximation and error for the reference {0, 1/4, 1/2, 1}

algorithm. We can see that $\eta \approx 0.75$, therefore, following the algorithm (substitute η with the ξ that is closest and has the same sign), we have to change the reference to $\{0, 1/4, 1/2, 0.75\}$ obtaining the polynomial $p(x) = 0.5625 - 2x + 2x^2$ and the plots in Figure 2. Now this, if not the best

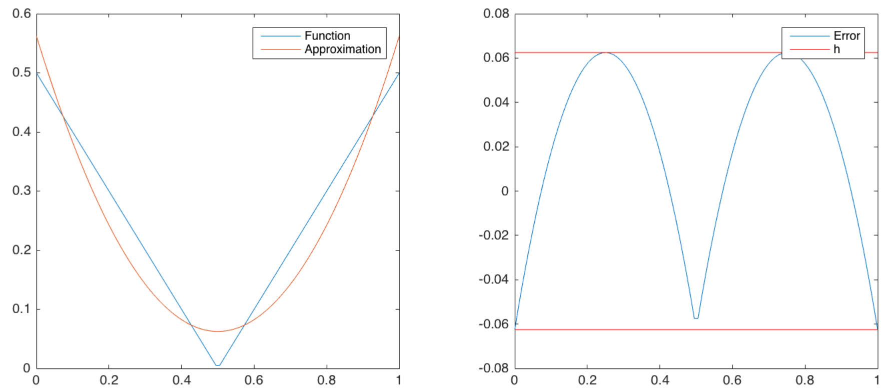

Figure 2: Function, approximation and error for the reference $\{0, 1/4, 1/2, 0.75\}$

approximation, is very close to what we would want in practice. ■

Task 3. Consider the space spanned by $\Phi_0(x) = 1$ and $\Phi_1(x) = x^2$, $0 \leq x1$. Is this a Haar space? Use Theorem 7.4 to find a best approximation on a reference of your choice.

Solution.

Haar space? Let $\mathcal{A} = span(\{1, x^2\})$. Then $\dim(\mathcal{A}) = 2$. Also $p(x) = 1 - x^2 \in \mathcal{A}$ but p have two roots $x = \pm 1$. This violates the Haar condition that any element with more than $\dim(\mathcal{A}) - 1$ zeros are identically zero.

Find best approximation As $\mathcal{A}$ is not a Haar space theorem 7.4 does not apply. But there is nothing stopping us from trying anyway. Lets choose the function $f(x) = \sin x$ on the interval $[0, \pi]$ with the reference

$\{0, \frac{\pi}{2}, 1\}$. Using 7.4 becomes solving the equation system in equation 6.

$$\lambda_0 + \lambda_1\zeta_i^2 + (-1)^i h = f(\zeta_i) \Leftrightarrow \begin{cases} \lambda_0 + \lambda_1 0^2 + h = \sin 0 \\ \lambda_0 + \lambda_1 \frac{\pi^2}{4} - h = \sin \frac{\pi}{2} \\ \lambda_0 + \lambda_1 \pi^2 + h = \sin \pi \end{cases} \tag{6}$$

$$\Leftrightarrow \begin{cases} \lambda_0 = -h \\ \lambda_0 + \lambda_1 \frac{\pi^2}{4} - h = 1 \\ \lambda_0 + \lambda_1 \pi^2 + h = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = -h \\ h = -\frac{1}{2} \\ \lambda_1 = 0 \end{cases}$$

$$\Rightarrow p(x) = \frac{1}{2} \tag{7}$$

At first the result in equation 7 seem weird. But when considering that any quadratic behavior would be centered around $x = 0$ it is clear this actually is a best approximation, despite that theorem 7.4 do not apply.

■

Task 4. Show that the best approximation from $\mathcal{P}_2$ to the function $f(x) = 144/(3x+5)$, $x \in [-1, 1]$, is $p(x) = 27 - 24x + 18x^2$, and that the error is maximal at the points -1, -2/3, 1/3 and 1.

Solution. We construct the error function as $f(x) - p(x)$ obtaining:

$$e^*(x) = \frac{144}{3x+5} - 27 + 24x - 18x^2$$

Then to find the critical points we take the derivative of $e^*(x)$ and make it equal to zero:

$$-\frac{432}{(3x+5)^2} - 36x + 24 = 0$$

Obtaining the solutions $x = -2/3, x = 1/3$ and $x = -7/3$. We can discard the last solution as it is outside the given range. Finally we have to consider the end points $x = \pm 1$ which have the same functional value as the other solutions, therefore they are also extrema of the error function, that is, the absolute value of the error is maximal at these four points. Note that this also shows that $p(x)$ is the best approximation from $\mathcal{P}_2$ to $f(x)$ by the sign condition theorem. We have 4 points in $Z_M = \{-1, -2/3, 1/3, 1\}$, where Z_M is the set where the error function attains its maximum. The error function has degree 3 and changes its sign three times, thus we can not find a polynomial in $\mathcal{P}_2$ with 3 sign changes since a polynomial of degree n can only have n sign changes. Therefore by the theorem, $p(x)$ is the best approximation. ■

Task 5. In the exchange algorithm it is necessary to find the point η that satisfies the equation

$$|f(\eta)p(\eta)| = \|fp\|_\infty \tag{8}$$

but in practice it is inefficient to try to calculate extrema of functions exactly. Investigate useful ways to approximate η and discuss your results.

Solution. There are two different approaches to finding this maxima. Firstly, one can use one of all the fancy optimization methods in existence. The other is two do a quick and dirty solution. Sacrificing precision for speed and simplicity.

The most simple quick-and-dirty method is to calculate the error function on a grid, Choosing η as the point on the grid that corresponds to the highest value of the points tested. The grid spacing is be chosen to "resolve" the changes in the function. What is important is to find a value close to the actual maxima, not to accurately find it.

The more sophisticated approach is to use one of the many optimization algorithms in existence. These methods have in common that they are able to locate the maxima at a much higher accuracy. However, they also often get stuck at a local maxima which is of much larger concern than inaccuracy in position. However this do not exclude the existence of less naive brute force methods.

The brute-force grid method sacrifice inaccuracy for simplicity and avoidance of getting stuck in local maxima. The maxima is not a significant result in itself but simply just a guess, not an answer, to the choice of reference. As the choice is of the maxima as the next point is an approximation, the precision of the actual maxima becomes less important. ■