Noname manuscript No.
(will be inserted by the editor)

Quasi-structure in the dual Hessian for distributed MPC with non-delayed couplings

Emil Klintberg · **Sebastien Gros**

Received: date / Accepted: date

Abstract Bla bla bla

Keywords First keyword · Second keyword · More

1 Introduction

Bla bla bla

2 Prelimenaries

2.1 Motivation: Distributed MPC with non-delayed couplings

We consider M discrete time linear systems given by:

$$x_{1,i+1} = A_{1,i}x_{1,i} + B_{1,i}u_{1,i}, \quad i = 0, \dots, N \tag{1a}$$

$$\vdots \tag{1b}$$

$$x_{M,i+1} = A_{M,i}x_{M,i} + B_{M,i}u_{M,i}, \quad i = 0, \dots, N \tag{1c}$$

where $x_{k,i} \in \mathbb{R}^{p_k}$ and $u_{k,i} \in \mathbb{R}^{m_k}$ represents the state and input of system k at time instance i. We also assume state and input constraints:

$$x_{k,i} \in \mathcal{X}_{k,i} \subseteq \mathbb{R}^{p_k} \tag{2a}$$

$$u_{k,i} \in \mathcal{U}_{k,i} \subseteq \mathbb{R}^{m_k} \tag{2b}$$

Emil Klintberg
first address
Tel.: +123-45-678910
Fax: +123-45-678910
E-mail: fauthor@example.com

Sebastien Gros
second address

Moreover, we assume non-delayed couplings between the systems:

$$h_i(x_{1,i}, u_{1,i}, \ldots, x_{M,i}, u_{M,i}) = 0, \quad i = 0, \ldots, N \tag{3}$$

Observe that non-delayed couplings implies that states and inputs for a subsystem at time i can only have a direct impact on states and inputs of another subsystem at time i. Note that this can always be fulfilled by introducing extra states to handle the delays locally even when coupling delays are present in the original formulation.

Accordingly, we can state the MPC problem over the horizon N as:

$$\begin{aligned}
\min_{x_{k,i}, u_{k,i}} \quad & \sum_{k=1}^{M} \left(\sum_{i=0}^{N-1} l(x_{k,i}, u_{k,i}) + l_f(x_{k,N}) \right) && \text{(4a)}\\
\text{s.t.} \quad & x_{k,i+1} = A_{k,i} x_{k,i} + B_{k,i} u_{k,i} && \text{(4b)}\\
& h_i(x_{1,i}, u_{1,i}, \ldots, x_{M,i}, u_{M,i}) = 0 && \text{(4c)}\\
& x_{k,i} \in \mathcal{X}_{k,i}, \quad u_{k,i} \in \mathcal{U}_{k,i} && \text{(4d)}
\end{aligned}$$

Furthermore, let us assume that the non-delayed couplings (4c) are affine, constraints (4d) are convex polytopes and that the objective function (4a) is quadratic. We introduce the following notation: $z_{k,i} = [x_{k,i}^T \quad u_{k,i}^T]^T \in \mathbb{R}^{n_k}$, representing the optimization variables. The MPC problem can then be stated as:

$$\begin{aligned}
\min_{z} \quad & \sum_{k=1}^{M} \sum_{i=0}^{N} \frac{1}{2} z_{k,i}^T H_{k,i} z_{k,i} + c_{k,i}^T z_{k,i} && \text{(5a)}\\
\text{s.t.} \quad & \sum_{k=1}^{M} F_{k,i} z_{k,i} = e_i && \text{(5b)}\\
& C_{k,i} z_{k,i} + D_{k,i+1} z_{k,i+1} = d_{k,i} && \text{(5c)}\\
& G_{k,i} z_{k,i} \leq f_{k,i} && \text{(5d)}
\end{aligned}$$

where $H_{k,i} \in \mathbb{S}_{++}^{n_k \times n_k}$, $C_{k,i} \in \mathbb{R}^{l_k \times n_k}$, $D_{k,i+1} \in \mathbb{R}^{l_k \times n_k}$ and $d_{k,i} \in \mathbb{R}^{l_k}$ form the dynamics, $F_{k,i} \in \mathbb{R}^{r_i \times n_k}$ and $e_i \in \mathbb{R}^{r_i}$ yield the coupling constraints, $G_{k,i} \in \mathbb{R}^{t_k \times n_k}$ and $f_{k,i} \in \mathbb{R}^{t_k}$ form the local constraints.

Additionaly, to avoid an unnecessary heavy notation at places where we are only dealing with decomposition in space, we introduce the following augmented notations: $z_k = [z_{k,0}^T \ldots, z_{k,N}^T]^T \in \mathbb{R}^{(N+1)n_k}$, which represents the collection of states and inputs of system k over the horizon N. The MPC problem (5) can then be expressed as:

$$\min_{z} \quad \sum_{k=1}^{M} \frac{1}{2} z_k^T H_k z_k + c_k^T z_k \tag{6a}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} F_k z_k = e \tag{6b}$$

$$C_k z_k = d_k \tag{6c}$$

$$G_k z_k \leq f_k \tag{6d}$$

where $H_k \in \mathbb{S}_{++}^{(N+1)n_k \times (N+1)n_k}$, $C_k \in \mathbb{R}^{Nl_k \times (N+1)n_k}$ and $d_k \in \mathbb{R}^{Nl_k}$, $F_k \in \mathbb{R}^{(N+1)r \times (N+1)n_k}$ and $e \in \mathbb{R}^{(N+1)r}$, $G_k \in \mathbb{R}^{(N+1)t_k \times (N+1)n_k}$ and $f_k \in \mathbb{R}^{(N+1)t_k}$ and the matrices are accordingly given by:

$$H_k = \begin{bmatrix} H_{k,0} & & \\ & \ddots & \\ & & H_{k,N} \end{bmatrix},$$

$$C_k = \begin{bmatrix} C_{k,0} & D_{k,1} & & & \\ & C_{k,1} & D_{k,2} & & \\ & & \ddots & \ddots & \\ & & & C_{k,N-1} & D_{k,N} \end{bmatrix},$$

$$A_k = \begin{bmatrix} A_{k,0} & & \\ & \ddots & \\ & & A_{k,N} \end{bmatrix}.$$

2.2 Dual decomposition with second-order information

We introduce the dual variables $\lambda \in \mathbb{R}^{(N+1)r}$ corresponding to the coupling constraints (6b) and define the Lagrange function as:

$$\mathcal{L}(z, \lambda) = \sum_{k=1}^{M} (\frac{1}{2} z_k^T H_k z_k + c_k^T z_k) + \lambda^T (\sum_{k=1}^{M} F_k z_k - e) \tag{8}$$

Note that $\mathcal{L}(z, \lambda)$ is separable in z, i.e.

$$\mathcal{L}(z, \lambda) = \sum_{k=1}^{M} \mathcal{L}_k(z_k, \lambda) \tag{9}$$

with

$$\mathcal{L}_k(z_k, \lambda) = \frac{1}{2} z_k^T H_k z_k + c_k^T z_k + \lambda^T (F_k z_k - \frac{1}{M} e) \tag{10}$$

The dual function $d(\lambda) = -\min_{z \in \mathcal{Z}} \mathcal{L}(z, \lambda)$ can thus be evaluated in parallel as:

$$d(\lambda) = -\sum_{k=1}^{M} \min_{z_k \in \mathcal{Z}_k} \mathcal{L}_k(z_k, \lambda) \tag{11}$$

Since (6) is strictly convex, (11) is convex and continuously differentiable, but not twice differentiable. However, the Hessian of $d(\lambda)$ is a piecewise constant matrix and change with the active-set of (5) [9].

The non-smoothness implies that $d(\lambda)$ is not self-concordant and the solution to the dual problem is hence not easily tracked with Newton's method. However, if we relax the inequality constraints (6d) with a self-concordant log-barrier, according to:

$$\begin{aligned}
\min_{z} \quad & \sum_{k=1}^{M} \frac{1}{2} z_k^T H_k z_k + c_k^T z_k - \tau \sum_{i=1}^{(N+1)t_k} \log([s_k]_i) && \text{(12a)} \\
\text{s.t.} \quad & \sum_{k=1}^{M} F_k z_k = e && \text{(12b)} \\
& C_k z_k = d_k && \text{(12c)} \\
& G_k z_k + s_k = f_k && \text{(12d)}
\end{aligned}$$

where $\tau > 0$ will be referred to as the *barrier parameter*, the resulting *relaxed dual function* $d(\lambda, \tau)$ is self-concordant [10]. This opens for the possibility of solving a sequence of dual problems $\{\min_\lambda d(\lambda, \tau)\}_{\tau \to 0}$ where each problem is self-concordant and therefore easily solved with Newton's method.

In contrast to methods based on the *augmented Lagrangian* [8], the separability of the dual function is preserved, and the relaxed dual function can be evaluated in parallel as:

$$d(\lambda, \tau) = -\sum_{k=1}^{M} \min_{z_k \in \mathcal{Z}_k} \left(\mathcal{L}_k(z_k, \lambda) - \tau \sum_{i=1}^{m_k} \log([s_k]_i) \right) \tag{13}$$

Observe that evaluating (13) involves solving M subproblems of the form:

$$\begin{aligned}
\min_{z_k} \quad & \frac{1}{2} z_k^T H_k z_k + c_k^T z_k + \lambda^T F_k z_k - \tau \sum_{i=1}^{m_k} \log([s_k]_i) \\
\text{s.t.} \quad & C_k z_k = d_k \\
& G_k z_k + s_k = f_k \\
& s_k \geq 0
\end{aligned} \tag{14}$$

Moreover, strict convexity of (13) implies that the gradient of $d(\lambda, \tau)$ with respect to λ is given by the residual of the coupling constraints [2], i.e.

$$\nabla d(\lambda, \tau) = -\sum_{k=1}^{N} F_k z_k^*(\lambda, \tau) + e \tag{15}$$

where $z_k^*(\lambda, \tau)$ is the solution to (14). The dual Hessian is then given by:

$$\nabla^2 d(\lambda, \tau) = -\sum_{k=1}^{N} F_k \frac{\partial z_k^*(\lambda, \tau)}{\partial \lambda} \tag{16}$$

A Newton direction $\Delta\lambda$ in the dual space can then be obtained as a solution to the Newton system:

$$\nabla^2 d(\lambda, \tau)\Delta\lambda + \nabla d(\lambda, \tau) = 0 \tag{17}$$

3 Structure in the dual Hessian

3.1 Dual Hessian

By introducing $y_k = \tau s_k^{-1} \in \mathbb{R}^{t_k(N+1)}$, the KKT conditions to (14) are given by:

$$0 = H_k z_k^* + c_k + F_k^T \lambda + C_k^T \mu_k^* + G_k^T y_k^* \tag{18a}$$
$$0 = C_k z_k^* - d_k \tag{18b}$$
$$0 = G_k z_k^* + s_k^* - f_k \tag{18c}$$
$$0 = Y_k^* s_k^* - \tau \mathbf{1} \tag{18d}$$
$$s_k^* > 0, \quad y_k^* > 0 \tag{18e}$$

As we have seen in the previous section, that in order to form the dual Hessian we need to compute $\frac{\partial z_k^*(\lambda)}{\partial \lambda}$, where $z_k^*(\lambda)$ is the optimal primal solution and is hence fulfilling (18). By differentiating (18), the following linear system is obtained [7]:

$$\begin{bmatrix} H_k & C_k^T & G_k^T & 0 \\ C_k & 0 & 0 & 0 \\ G_k & 0 & 0 & I \\ 0 & 0 & S_k & Y_k \end{bmatrix} \begin{bmatrix} \frac{\partial z_k^*}{\partial \lambda} \\ \frac{\partial \mu_k^*}{\partial \lambda} \\ \frac{\partial y_k^*}{\partial \lambda} \\ \frac{\partial s_k^*}{\partial \lambda} \end{bmatrix} = - \begin{bmatrix} F_k^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

If block elimination of (19) is used, the *normal equations*, equivalent to (19), can be formed [13]:

$$\Lambda_k(s_k, y_k) \frac{\partial \mu_k^*}{\partial \lambda} = -C_k \Phi_k^{-1} F_k^T \tag{20a}$$
$$\Phi_k(s_k, y_k) \frac{\partial z_k^*}{\partial \lambda} = -F_k^T - C_k^T \frac{\partial \mu_k^*}{\partial \lambda} \tag{20b}$$
$$\frac{\partial s_k^*}{\partial \lambda} = -G_k \frac{\partial z_k^*}{\partial \lambda} \tag{20c}$$
$$\frac{\partial y_k^*}{\partial \lambda} = -S^{-1} Y \frac{\partial s_k^*}{\partial \lambda} \tag{20d}$$

where

$$\Phi_k(s_k, y_k) = H_k + G_k^T S_k^{-1} Y_k G_k \in \mathbb{S}_{++}^{(N+1)n_k \times (N+1)n_k} \tag{21a}$$

$$\Lambda_k(s_k, y_k) = C_k \Phi_k^{-1} C_k^T \in \mathbb{S}_{++}^{Nl_k \times Nl_k} \tag{21b}$$

In the following we omit the arguments and use $\Phi_k = \Phi_k(s_k, y_k)$ and $\Lambda_k = \Lambda_k(s_k, y_k)$. Moreover, observe that according to (21a), Φ_k is block diagonal with blocks $\Phi_{k,i} = H_{k,i} + G_{i,k}^T S_{k,i}^{-1} Y_{k,i} G_{k,i}$. As a consequence, Λ_k has a block tridiagonal structure given by:

$$\Lambda_k = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & & \\ \Lambda_{12}^T & \Lambda_{22} & \ddots & \\ & \ddots & \ddots & \Lambda_{N-1,N} \\ & & \Lambda_{N-1,N}^T & \Lambda_{N,N} \end{bmatrix} \in \mathbb{S}_{++}^{Nl_k \times Nl_k} \tag{22}$$

where

$$\Lambda_{i,i} = C_{k,i-1} \Phi_{k,i-1}^{-1} C_{k,i-1}^T + D_{k,i} \Phi_{k,i}^{-1} D_{k,i}^T \in \mathbb{R}^{l_k, \times l_k} \tag{23a}$$

$$\Lambda_{i,i+1} = D_{k,i} \Phi_{k,i}^{-1} C_{k,i}^T \in \mathbb{R}^{l_k \times l_k} \tag{23b}$$

By using (20a) and (20b), it can be obtained that:

$$F_k \frac{\partial z_k}{\partial \lambda} = -F_k (\Phi_k^{-1} - \Phi_k^{-1} C_k^T \Lambda_k^{-1} C_k \Phi_k^{-1}) F_k^T \tag{24}$$

As a consequence, the dual Hessian (16) can be written as

$$\nabla^2 d(\lambda, \tau) = \sum_{k=1}^{M} F_k (\Phi_k^{-1} - \Phi_k^{-1} C_k^T \Lambda_k^{-1} C_k \Phi_k^{-1}) F_k^T \tag{25}$$

Accordingly, all matrices in (25) are banded except Λ_k^{-1} which in general is dense.

3.2 Quasi-structure the dual Hessian

Let us first assume that the problem data is bounded.

Assumption 1 *The row and column absolute sums of Jacobians of equality constraints (i.e. (6b) and (6c)) are bounded. Hence:*

1. $\|C_{k,i}\|_\bullet \leq \gamma$
2. $\|D_{k,i}\|_\bullet \leq \gamma$
3. $\|F_{k,i}\|_\bullet \leq \gamma$

where $\bullet$ *represent* ∞ *and* 1.

Remark 1 It should be observed that Assumption 1 is not by any means restrictive, since any solver would struggle with a problem where it is not fulfilled [].

Furthermore, we assume boundedness of Φ_k^{-1}:

Assumption 2 *The row and column absolute sums of Φ_k^{-1} are bounded. Hence:*

$$\|\Phi_{k,i}^{-1}\|_\infty = \|\Phi_{k,i}^{-1}\|_1 \leq \gamma_{\Phi_k^{-1}}, \quad \forall i \tag{26}$$

Remark 2 $\|\Phi_{k,i}^{-1}\|_\infty = \|\Phi_{k,i}^{-1}\|_1$ is not an assumption, but a result of Φ_k being symmetric.

Remark 3 Possible problems with interior-point methods are related to numerical difficulties due to ill-conditioning [12]. Indeed, since elements in $S_k^{-1}Y_k$ can be close to zero in late iterations when τ is small. However, since we assume that (5) is strongly convex, the eigenvalues of Φ_k are lower bounded by the smallest eigenvalue of H_k, even when $S_k^{-1}Y_k$ is singular. According to this reasoning it is not restrictive to assume boundedness of Φ_k^{-1}.

Inverses of sparse matrices are in general dense, but individual elements are often small in absolute value [1]. Since Λ_k is banded, symmetric and positive definite, we will use the following classical result:

Lemma 1 *If A is Hermitian positive definite and m-banded, i.e. $[A]_{ij} = 0$ if $|i-j| > m$, the entries of A^{-1} satisfy the following bound:*

$$|[A^{-1}]_{ij}| < K\omega^{|i-j|}, \quad \forall i,j \tag{27}$$

where $\sigma_{\min}(A)$ and $\kappa(A)$ are the the smallest singular value and the condition number of A respectively, $K = \max\{\sigma_{\min}^{-1}(A), K_0\}$, $K_0 = (1+\sqrt{\kappa(A)})$, $\omega = \left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^{1/m}$.

Proof A proof is given in [4]. □

Remark 4 The entries in A^{-1} are bounded by an exponentially decaying function along each row or column. However, the bound depends on the condition number and the bandwidth of the matrix. Matrices with a high condition number and/or a high bandwidth can therefore result in a large K and $\omega \approx 1$, leading to a slow decay. The opposite, i.e. a low condition number and a small band, would result in a rapid decay.

Remark 5 Observe that due to the block tridiagonal structure of Λ_k, it is $3l_k$-banded, i.e. $m = 3l_k$ for Λ_k.

In the following, if we introduce the notation

$$\Lambda_k^{-1} = \begin{bmatrix} T_{11} & T_{21}^T & \dots & T_{N+1,1}^T \\ T_{21} & T_{22} & \dots & T_{N+1,2}^T \\ \vdots & \vdots & \ddots & \vdots \\ T_{N+1,1} & T_{N+1,2} & \dots & T_{N+1,N+1} \end{bmatrix} \tag{28}$$

where $T_{i,j} \in \mathbb{R}^{l_k \times l_k}$, we can establish the following proposition:

Proposition 1 *The off-diagonal blocks (i.e. $T_{i,j}$ where $i-j>0$) in Λ_k^{-1} satisfy the following bounds:*

$$\|T_{i,j}\|_\bullet \leq K_{\Lambda_k}\omega_{\Lambda_k}^{i-j} \tag{29}$$

where $\bullet$ represents ∞ and 1, $\sigma_{\min}(\Lambda_k)$ and $\kappa(\Lambda_k)$ are the the smallest singular value and the condition number of Λ_k respectively, $K_{\Lambda_k} = \max\{\sigma_{\min}(\Lambda_k)^{-1}, 1+\sqrt{\kappa(\Lambda_k)}\}l_k\omega_{\Lambda_k}^{1/l_k}$ and $\omega_{\Lambda_k} = \left(\frac{\sqrt{\kappa(\Lambda_k)}-1}{\sqrt{\kappa(\Lambda_k)}+1}\right)^{\frac{1}{3}}$.

Proof According to Lemma 1, the element in $T_{i,j}$ with the largest bound is located in the top-right corner, and is hence the element $[\Lambda_k^{-1}]_{il_k+1,jl_k}$ in Λ_k^{-1}. By directly applying Lemma 1 it follows that:

$$\max|[T_{ij}]| \leq \max\{\sigma_{min}(\Lambda_k)^{-1}, 1+\sqrt{\kappa(\Lambda_k)}\}\left(\frac{\sqrt{\kappa(\Lambda_k)}-1}{\sqrt{\kappa(\Lambda_k)}+1}\right)^{\frac{1}{3l_k}((i-j)l_k+1)} \tag{30}$$

where $\max|[T_{ij}]|$ refers to the maximum absolute value of the components in T_{ij}. Moreover, since there are l_k elements in each row or column of a block $T_{i,j}$, we obtain the bounds given in (29). □

Remark 6 The constants K_{Λ_k} and ω_{Λ_k} in depends heavily on the conditioning of Λ_k, which for interior point methods in general can be very high for small values of τ. Our experience is however that this is not a major issue in practice, which is also supported by commercial interior-point implementations [12] [5]. Moreover, it should be understood that strong convexity of (5), should improve the worst case conditioning of Φ_k and hence also of Λ_k.

We have concluded that Λ_k^{-1} is decaying exponentially towards the off-diagonal corners and that all other matrices in (25) are banded. This suggests that also $\nabla^2 d(\lambda,\tau)$ should decay towards the off-diagonal corners. Let us introduce the notation:

$$C_k^T\Lambda_k^{-1}C_k = \begin{bmatrix} V_{11} & V_{21}^T & \dots & V_{N+1,1}^T \\ V_{21} & V_{22} & \dots & V_{N+1,2}^T \\ \vdots & \vdots & \ddots & \vdots \\ V_{N+1,1} & V_{N+1,2} & \dots & V_{N+1,N+1} \end{bmatrix} \tag{31}$$

where $V_{ij}\in\mathbb{R}^{l_k\times l_k}$, and look at the decay of $C_k^T\Lambda_k^{-1}C_k$.

Proposition 2 *The off diagonal blocks (i.e. $V_{i,j}$ where $i-j>0$) of $C_k^T\Lambda_k^{-1}C_k$ are bounded by:*

$$\|V_{i,j}\|_\bullet \leq \gamma^2\bar{K}_{\Lambda_k}\omega_{\Lambda_k}^{i-j} \tag{32a}$$

$$\tag{32b}$$

where $\bar{K}_{\Lambda_k} = (2+\omega_{\Lambda_k}+\omega_{\Lambda_k}^{-1})K_{\Lambda_k}$ and $\bullet$ represents ∞ and 1.

Proof By performing the multiplication $C_k^T \Lambda_k^{-1} C_k$, we find that:

$$V_{i,j} = D_{k,i-1}^T T_{i,j-1} D_{k,j-1} + C_{k,i-1}^T T_{i-1,j-1} D_{k,j-1} + \\ + D_{k,i-1}^T T_{i-1,j} C_{k,j-1} + C_{k,i-1}^T T_{i,j} C_{k,j-1} \tag{33}$$

for $i - j > 0$. Furthermore, by using Assumption 1 and (29), we can bound establish the following bound:

$$\|V_{i,j}\|_\bullet \leq \gamma^2 (2 + \omega_{\Lambda_k} + \omega_{\Lambda_k}^{-1}) K_{\Lambda_k} \omega_{\Lambda_k}^{i-j} \tag{34}$$

□

Let us now introduce the notation:

$$\nabla^2 d(\lambda, \tau) = \begin{bmatrix} W_{11} & W_{21}^T & \dots & W_{N+1,1}^T \\ W_{21} & W_{22} & \dots & W_{N+1,2}^T \\ \vdots & \vdots & \ddots & \vdots \\ W_{N+1,1} & W_{N+1,2} & \dots & W_{N+1,N+1} \end{bmatrix} \tag{35}$$

where $W_{i,j} \in \mathbb{R}^{r_i \times r_j}$. We can now establish a decay towards the off-diagonal corners of $\nabla^2 d(\lambda, \tau)$.

Proposition 3 *The off-diagonal blocks (i.e. $W_{i,j}$ where $i - j > 0$) of $\nabla^2 d(\lambda, \tau)$ satisfy the following bounds:*

$$\|W_{i,j}\|_\bullet \leq \sum_{k=1}^{M} \gamma^4 \gamma_{\Phi_k^{-1}}^2 \bar{K}_{\Lambda_k} \omega_{\Lambda_k}^{i-j} \tag{36}$$

where $\bullet$ represents ∞ and 1.

Proof Since Φ_k^{-1} is block-diagonal, the off-diagonal blocks of $F_k(\Phi_k^{-1} - \Phi_k^{-1} C_k^T \Lambda_k^{-1} C_k \Phi_k^{-1}) F_k^T$ and $-F_k \Phi_k^{-1} C_k^T \Lambda_k^{-1} C_k \Phi_k^{-1} F_k^T$ are identical. Accordingly, by using Assumption 1, Assumption 2, Proposition 2 and adding the contributions from all subproblems together, brings us to (36). □

Let $\lfloor \nabla^2 d(\lambda, \tau) \rceil_{\mathcal{M}}$ represent the diagonal $\mathcal{M}$-block band of (35), which in the following will be referred to as the *clipped dual Hessian*. Moreover, recall Gershgorin's circle theorem:

Theorem 1 *For $A \in \mathbb{R}^{n \times n}$ with elements a_{ij}, let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the row i. Let $D(a_{ii}, R_i)$ be the closed disc centered in a_{ii} with radius R_i, then every eigenvalue of A lies within at least one of the discs $D(a_{ii}, R_i)$.*

Proof A proof is given in [6]. □

We can now establish our main result:

Lemma 2 *The following bound holds:*

$$\|\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)\|_2 \leq \left(N - \frac{\mathcal{M}-1}{2}\right)\sum_{k=1}^{M}\gamma^4\gamma^2_{\Phi_k^{-1}}\tilde{K}_{\Lambda_k}\omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}} \tag{37}$$

where $\tilde{K}_{\Lambda_k} = \bar{K}_{\Lambda_k}\omega_{\Lambda_k} = (2 + \omega_{\Lambda_k} + \omega_{\Lambda_k}^{-1})K_{\Lambda_k}$.

Proof Since $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)$ is symmetric, its singular values are equal to the absolute value of its eigenvalues. The problem of finding a bound on the 2-norm is hence reduced to the problem of bounding the magnitude of the largest eigenvalue.

First, observe that all diagonal elements of $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)$ are zero, all Gershgorin discs will hence be centered at the origin. A bound on the largest eigenvalue can then be found by finding the largest radius of the Gershgorin discs.

According to (36), the blocks $W_{i,j}$ in $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)$ with the largest bound on the maximum absolute row sum are located next to the diagonal $\mathcal{M}$-block band of zeros, i.e. where $i - j = \frac{\mathcal{M}-1}{2} + 1$. Furthermore, there is at least $N - \frac{\mathcal{M}-1}{2}$ nonzero blocks at each block row. This means that we can establish a bound on the radius of the Gershgorin discs, and hence the following:

$$\|\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)\|_2 \leq \left(N - \frac{\mathcal{M}-1}{2}\right)\sum_{k=1}^{M}\gamma^4\gamma^2_{\Phi_k^{-1}}\bar{K}_{\Lambda_k}\omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}+1} \tag{38}$$

Equation (37) is then directly obtained by introducing $\tilde{K}_{\Lambda_k} = \bar{K}_{\Lambda_k}\omega_{\Lambda_k}$. □

It is well known that the use of an inexact Hessian will degrade the convergence of Newton's method to a linear rate provided that the inexact Hessian has full rank [11]. Recall the following corollary that follows from Weyl's inequality [6]:

Corollary 1 *Let* $A, B \in \mathbb{S}^{n\times n}$, *then*

$$\lambda_{\min}(A) + \lambda_{\min}(B) \leq \lambda_{\min}(A + B) \tag{39}$$

where $\lambda_{\min}$ *represents the minimum eigenvalue.*

Proof A proof is given in [6].

We can now establish a criterion for non-singularity of $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}}$:

Lemma 3 *If*

$$\lambda_{\min}(\nabla^2 d(\lambda,\tau)) > \left(N - \frac{\mathcal{M}-1}{2}\right)\sum_{k=1}^{M}\gamma^4\gamma^2_{\Phi_k^{-1}}\tilde{K}_{\Lambda_k}\omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}} \tag{40}$$

$\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}}$ *is positive definite.*

Proof Observe that

$$\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} = \nabla^2 d(\lambda,\tau) + \left(\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau) \right) \tag{41}$$

From (39), we can realize that:

$$\lambda_{\min}(\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}}) \geq \lambda_{\min}(\nabla^2 d(\lambda,\tau)) + \lambda_{min}(\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)) \tag{42}$$

Accordingly, $\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}}$ is nonsingular if:

$$\lambda_{\min}(\nabla^2 d(\lambda,\tau)) > |\lambda_{min}(\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau))| \tag{43}$$

We have already established that the eigenvalues of $\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau))$ are located close to the origin and are not more negative then the largest Gershgorin radius given by (38). Hence, we realize that

$$\lambda_{min}(\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)) \geq - \left(N - \frac{\mathcal{M}-1}{2} \right) \sum_{k=1}^{M} \gamma^4 \gamma^2_{\Phi_k^{-1}} \tilde{K}_{\Lambda_k} \omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}} \tag{44}$$

Using (43) and (44) gives (40). □

Remark 7 It should be noted that $\lambda_{\min}(\nabla^2 d(\lambda,\tau)) > 0, \ \forall \lambda$ [7].

Remark 8 Observe that (40) is a sufficient and not a necessary condition. According to our experiments, singularity is not a problem in practice, and (40) is hence very conservative.

3.3 Selection of bandwidth

In the previous section, we have seen that the clipped dual Hessian is positive definite and close, in an euclidean sense, to the exact Hessian provided that the band $\mathcal{M}$ is wide enough. Here, we aim at formalizing criterions for selecting the bandwidth.

Theorem 2 *If the dual function fulfills:*

$$\left\| \int_0^1 (\nabla^2 d(\lambda + t\Delta\lambda,\tau) - \nabla^2 d) dt \ \lfloor \nabla^2 d \rceil_{\mathcal{M}}^{-1} \right\|_2 \leq \omega_1 \|\nabla d\|_2 \tag{45}$$

then the following bound holds:

$$\|\nabla d(\lambda^+,\tau)\|_2 \leq \omega_1 \|\nabla d(\lambda,\tau)\|_2^2 + f(\mathcal{M}) \|\nabla d(\lambda,\tau)\|_2 \tag{46}$$

where

$$f(\mathcal{M}) = \|(\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)) \lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}}^{-1}\|_2 \tag{47}$$

Proof Recall that an approximate Newton step computed with the clipped dual Hessian is given by the following linear system:

$$\lfloor \nabla^2 d(\lambda,\tau) \rceil_{\mathcal{M}} \Delta\lambda + \nabla d(\lambda,\tau) = 0 \tag{48}$$

Moreover, to avoid a heavy notation, we will omit arguments when they are obvious from the context and introduce the the notation $\nabla d^+ = \nabla d(\lambda + \Delta\lambda, \tau)$. We can then express:

$$\begin{aligned}
\|\nabla d^+\|_2 =& \|\nabla d^+ - \nabla d - \lfloor \nabla^2 d \rceil_{\mathcal{M}} \Delta\lambda\|_2 = \\
&= \left\| \int_0^1 (\nabla^2 d(\lambda + t\Delta\lambda, \tau)\Delta\lambda)\mathrm{dt} - \lfloor \nabla^2 d \rceil_{\mathcal{M}} \Delta\lambda \right\|_2 = \\
&= \left\| \left(\int_0^1 (\nabla^2 d(\lambda + t\Delta\lambda, \tau) - \nabla^2 d)\mathrm{dt} + \nabla^2 d - \lfloor \nabla^2 d \rceil_{\mathcal{M}} \right) \Delta\lambda \right\|_2 = \\
&= \left\| \left(\int_0^1 (\nabla^2 d(\lambda + t\Delta\lambda, \tau) - \nabla^2 d)\mathrm{dt} + \nabla^2 d - \lfloor \nabla^2 d \rceil_{\mathcal{M}} \right) \lfloor \nabla^2 d \rceil_{\mathcal{M}}^{-1} \nabla d \right\|_2
\end{aligned} \tag{49}$$

where we have used (48) and the following result from calculus:

$$\int_0^1 \nabla^2 d(\lambda + t\Delta\lambda, \tau)\Delta\lambda \ \mathrm{dt} = \nabla d^+ - \nabla d \tag{50}$$

Consequently, we can bound (49) according to:

$$\|\nabla d^+\|_2 \le \omega_1 \|\nabla d\|_2^2 + f(\mathcal{M}) \, \|\nabla d\|_2 \tag{51}$$

where we have introduced:

$$\left\| \int_0^1 (\nabla^2 d(\lambda + t\Delta\lambda, \tau) - \nabla^2 d) \ \mathrm{dt} \lfloor \nabla^2 d \rceil_{\mathcal{M}}^{-1} \right\|_2 \le \omega_1 \|\nabla d\|_2 \tag{52}$$

and

$$f(\mathcal{M}) = \left\| \left(\nabla^2 d - \lfloor \nabla^2 d \rceil_{\mathcal{M}} \right) \lfloor \nabla^2 d \rceil_{\mathcal{M}}^{-1} \right\|_2 \tag{53}$$

□

Remark 9 Observe the $f(\mathcal{M}) \ge 0$ is a function which value we in principle can choose.

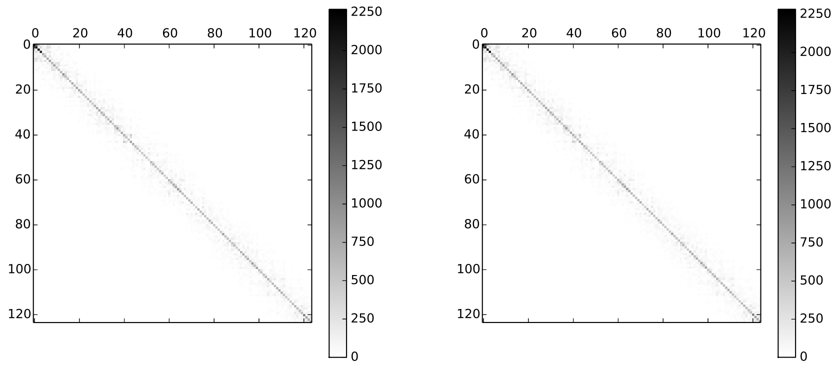
In terms of iterations, the optimal choice would be to use the full dual Hessian and hence make $f(\mathcal{M}) = 0$. However, since a dense factorization is more costly than its banded counterpart, the overall complexity of solving a problem can be lower if a nonzero $f(\mathcal{M})$ is used.

4 Numerical experiments

4.1 An example

To investigate the consistency of the off-diagonal decay for a given problem, we look at a randomly generated problem, of the form (5), with $M = 4$ subproblems each with a horizon $N = 30$, 6 states, 4 controls, and $4(N+1) = 124$ inequality constraints. Each time instance the subproblems are coupled together with 4 coupling constraints. Moreover, the elements are generated with an even distribution in the interval $[-1000, 1000]$.

The absolute values of the elements in the dual Hessian evaluated at the solution $\lambda^*(\tau)$ for $\tau = 1$, are visualized in Figure 1a. It can be seen that elements with a large magnitude are present along the diagonal, while elements towards the off-diagonal corners are small in absolute value.



a) $\tau = 1$ b) $\tau = 10^{-6}$

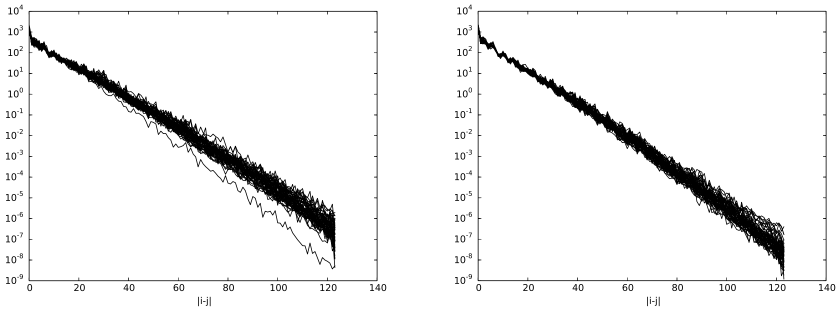
Fig. 1: Illustration of the dual Hessian evaluated at $\lambda^*(\tau)$.

Let us investigate the consistency of the decay with respect to λ. The maximal absolute value of the elements in $\nabla^2 d(\lambda, \tau)$ for given distances from its diagonal, are visualized in Figure 2a for 50 randomly selected values of λ. More specifically, the metric used in Figure 2a is:

$$\begin{aligned} \max_{i,j} \quad & |[\nabla^2 d(\lambda, \tau)]_{i,j}| \\ \text{s.t.} \quad & |i - j| = \text{constant} \end{aligned} \tag{54}$$

Observe that for this example, at all evaluated points λ, the dual Hessian has a clear decaying trend towards its off-diagonal corners.

As discussed previously, the conditioning of Λ_k affects the decaying property and small values of τ would therefore possibly lead to a slower decay.



a) $\tau = 1$ b) $\tau = 10^{-6}$

Fig. 2: Illustration of the off-diagonal decay. Each line corresponds to different λ.

Corresponding figures to Figure 1a and Figure 2a, for $\tau = 10^{-6}$, are shown in Figure 1b and Figure 2b respectively. Observe that, for this example, there is no significant difference in the decay between the different values of τ. This is also consistent with what we have observed in experiments with other problems.

The parameter, that according to our experiments, has the strongest impact on the decaying is the length of the horizon. To illustrate this, we generate another random problem with $N = 10$, and all other dimensions being the same as before. The dual Hessian at the solution $\lambda^*(\tau)$, for $\tau = 1$ is visualized in Figure 3. Observe that the dual Hessian also in this case has a clear decay towards the off-diagonal corners. The band with elements of a significant magnitude is however wider compared to the size of the matrix. Moreover, the decaying property is visualized for 50 different values of λ in Figure 4. Note that in this figure it is easier to see that there is a smaller difference between the diagonal and the off-diagonal elements.

References

1. Benzi, M.: Preconditioning Techniques for Large Linear Systems: A Survey. Journal of Computational Physics **182**, 418–477 (2002)
2. Bertsekas, D., Tsitsiklis, J.N.: Parallel and distributed computation: Numerical methods. Prentice Hall (1989)
3. Boyd, S., Vandenberghe, L.: Convex Optimization. University Press, Cambridge (2004)
4. Demko, S., Moss, W., Smith, P.: Decay rates for inverses of band matrices. Mathemathics of Computation **43**, 491–499 (1986)
5. Domahidi, A., Zgraggen, A., Zeilinger, M., Morari, M., Jones, C.: Efficient Interior Point Methods for Multistage Problems Arising in Receding Horizon Control. In: IEEE Conference on Decision and Control (CDC), pp. 668 – 674. Maui, HI, USA (2012)
6. Horn, R.A., Johnson, C.R.: Matrix Analysis - 2nd ed. Cambridge University Press (2013)
7. Klintberg, E., Gros, S.: A primal-dual newton method for distributed quadratic programming. In: Conference on Decision and Control (2014)

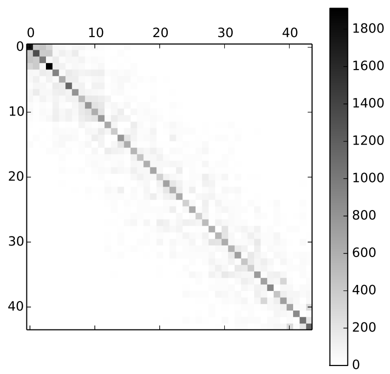


Fig. 3: Illustration of the dual Hessian evaluated at $\lambda^*(\tau)$.

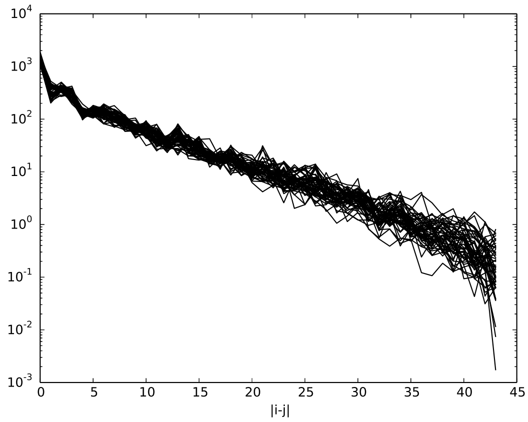


Fig. 4: Illustration of the off-diagonal decay. Each line corresponds to different λ.

8. Kozma, A., Conte, C., Diehl, M.: Benchmarking large scale distributed convex quadratic programming algorithms. Optimization Methods & Software (2013). (submitted for publication)
9. Kozma, A., Klintberg, E., Gros, S., Diehl, M.: An improved distributed dual Newton-CG method for convex quadratic programming problems. In: American Control Conference (2014)
10. Necoara, I., Suykens, J.: An interior-point Lagrangian decomposition method for separable convex optimization. J. Optim. Theory and Appl. **143**(3), 567–588 (2009)
11. Nocedal, J., Wright, S.: Numerical Optimization, 2 edn. Springer Series in Operations Research and Financial Engineering. Springer (2006)
12. Wright, M.: Ill-conditioning and computational error in interior methods for nonlinear programming. SIAM Journal on Optimization **9**, 84–111 (1998)
13. Wright, S.: Primal-Dual Interior-Point Methods. SIAM Publications, Philadelphia (1997)