Noname manuscript No.
(will be inserted by the editor)

Quasi-structure in the dual Hessian for distributed MPC with non-delayed couplings

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Received: date / Accepted: date

Abstract Bla bla bla

Keywords First keyword · Second keyword · More

1 Introduction

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1.1 Motivation: Distributed MPC with non-delayed couplings

We consider M discrete time linear systems given by:

$$x_{1,i+1} = A_{1,i}x_{1,i} + B_{1,i}u_{1,i} \tag{1a}$$

$$\vdots \tag{1b}$$

$$x_{M,i+1} = A_{M,i}x_{M,i} + B_{M,i}u_{M,i} \tag{1c}$$

where $x_{k,i} \in \mathbb{R}^{p_k}$ and $u_{k,i} \in \mathbb{R}^{m_k}$ represents the state and input of system k. We also assume state and input constraints:

$$x_{k,i} \in X_{k,i} \subseteq \mathbb{R}^{p_k} \tag{2a}$$

$$u_{k,i} \in U_{k,i} \subseteq \mathbb{R}^{m_k} \tag{2b}$$

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Moreover, we assume non-delayed couplings between the systems:

$$h_i(x_{1,i}, u_{1,i}, \dots, x_{M,i}, u_{M,i}) = 0, \quad i = 0, \dots, N-1 \tag{3}$$

This means that we can state the MPC problem over the horizon N as:

$$\min_{x_{k,i}, u_{k,i}} \quad \sum_{k=1}^{M} \left(\sum_{i=0}^{N-1} l(x_{k,i}, u_{k,i}) + l_f(x_{k,N}) \right) \tag{4a}$$

$$\text{s.t.} \quad x_{k+1,i} = A_{k,i} x_{k,i} + B_{k,i} u_{k,i} \tag{4b}$$

$$h_k(x_{k,1}, u_{k,1}, \dots, x_{k,M}, u_{k,M}) = 0 \tag{4c}$$

$$x_{k,i} \in X_{k,i}, \quad u_{k,i} \in U_{k,i} \tag{4d}$$

Furthermore, let us assume that the non-delayed couplings (4c) are linear, constraints (4d) are polyhedral and that the objective function (4a) is quadratic, and introduce the following notation: $z_{k,i} = [x_{k,i}^T \quad u_{k,i}^T]^T \in \mathbb{R}^{n_k}$. The MPC problem can then be stated as:

$$\min_{z} \quad \sum_{k=1}^{M} \sum_{i=0}^{N} \frac{1}{2} z_{k,i}^T H_{k,i} z_{k,i} + c_{k,i}^T z_{k,i} \tag{5a}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} F_{k,i} z_{k,i} = e_i \tag{5b}$$

$$C_{k,i} z_{k,i} + D_{k,i+1} z_{k,i+1} = d_{k,i} \tag{5c}$$

$$G_{k,i} z_{k,i} \leq f_{k,i} \tag{5d}$$

where $H_{k,i} \in \mathbb{S}_{++}^{n_k \times n_k}$, $C_{k,i} \in \mathbb{R}^{l_k \times n_k}$, $D_{k,i+1} \in \mathbb{R}^{l_k \times n_k}$ and $d_{k,i} \in \mathbb{R}^{l_k}$ form the dynamics, $F_{k,i} \in \mathbb{R}^{r_i \times n_k}$ and $e_i \in \mathbb{R}^{r_i}$ yield the coupling constraints, $G_{k,i} \in \mathbb{R}^{t_k \times n_k}$ and $f_{k,i} \in \mathbb{R}^{t_k}$ form the local constraints.

Additionaly, to avoid an unnecessary heavy notation at places where we are only dealing with decomposition in space, we introduce the following augmented notations: $z_k = [z_{k,0}^T \dots, z_{k,N-1}^T]^T \in \mathbb{R}^{n_k}$. The MPC problem (5) can then be expressed as:

$$\min_{z} \quad \sum_{k=1}^{M} \frac{1}{2} z_k^T H_k z_k + c_k^T z_k \tag{6a}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} F_k z_k = e \tag{6b}$$

$$C_k z_k = d_k \tag{6c}$$

$$G_k z_k \leq f_k \tag{6d}$$

where $H_k \in \mathbb{S}_{++}^{(N+1)n_k \times (N+1)n_k}$, $C_k \in \mathbb{R}^{Nl_k \times (N+1)n_k}$ and $d_k \in \mathbb{R}^{Nl_k}$, $F_k \in \mathbb{R}^{(N+1)r \times (N+1)n_k}$ and $e \in \mathbb{R}^{(N+1)r}$, $G_k \in \mathbb{R}^{(N+1)t_k \times (N+1)n_k}$ and $f_k \in \mathbb{R}^{(N+1)t_k}$

and the matrices are accordingly given by:

$$H_k = \begin{bmatrix} H_{k,0} & & \\ & \ddots & \\ & & H_{k,N} \end{bmatrix},$$

$$C_k = \begin{bmatrix} C_{k,0} & D_{k,1} & & & \\ & C_{k,1} & D_{k,2} & & \\ & & \ddots & \ddots & \\ & & & C_{k,N-1} & D_{k,N} \end{bmatrix},$$

$$A_k = \begin{bmatrix} A_{k,0} & & \\ & \ddots & \\ & & A_{k,N} \end{bmatrix}.$$

2 Prelimenaries

2.1 Dual decomposition with second-order information

We introduce the dual variables $\lambda \in \mathbb{R}^{(N+1)r}$ corresponding to the coupling constraints (6b) and define the Lagrange function as

$$\mathcal{L}(z,\lambda) = \sum_{k=1}^{M} (\frac{1}{2} z_k^T H_k z_k + c_k^T z_k) + \lambda^T (\sum_{k=1}^{M} F_k z_k - e) \tag{8}$$

Note that $\mathcal{L}(z,\lambda)$ is separable in z, i.e.

$$\mathcal{L}(z,\lambda) = \sum_{k=1}^{M} \mathcal{L}_k(z_k,\lambda) \tag{9}$$

with

$$\mathcal{L}_k(z_k,\lambda) = \frac{1}{2} z_k^T H_k z_k + c_k^T z_k + \lambda^T (C_k z_k - \frac{1}{M} d) \tag{10}$$

The dual function $d(\lambda) = -\min_{z \in \mathcal{Z}} \mathcal{L}(z,\lambda)$ can thus be evaluated in parallel as:

$$d(\lambda) = -\sum_{k=1}^{M} \min_{z_k \in \mathcal{Z}_k} \mathcal{L}_k(z_k,\lambda) \tag{11}$$

Since (6) is strictly convex, (11) is convex and continuously differentiable, but not twice differentiable. However, the Hessian of $d(\lambda)$ is a piecewise constant matrix and change with the active-set [5].

The non-smoothness implies that $d(\lambda)$ is not self-concordant and the solution to the dual problem is hence not easily tracked with Newton's method.

However, if we relax the inequality constraints (6d) with a self-concordant log-barrier, according to:

$$\begin{aligned}
\min_{z} \quad & \sum_{k=1}^{M} \frac{1}{2} z_k^T H_k z_k + c_k^T z_k - \tau \sum_{i=1}^{(N+1)t_k} \log([s_k]_i) && \text{(12a)}\\
\text{s.t.} \quad & \sum_{k=1}^{M} F_k z_k = e && \text{(12b)}\\
& C_k z_k = d_k && \text{(12c)}\\
& G_k z_k + s_k = f_k && \text{(12d)}
\end{aligned}$$

where $\tau > 0$ will be referred to as the *barrier parameter*, the resulting *relaxed dual function* $d(\lambda, \tau)$ is self-concordant [6]. This opens for the possibility of solving a sequence of dual problems $\{\min_\lambda d(\lambda, \tau)\}_{\tau \to 0}$ where each problem is self-concordant and therefore easily solved with Newton's method.

The relaxed dual function is separable and can be evaluated in parallel as

$$d(\lambda, \tau) = -\sum_{k=1}^{N} \min_{z_k \in \mathcal{Z}_k} \left(\mathcal{L}_k(z_k, \lambda) - \tau \sum_{i=1}^{m_k} \log([s_k]_i) \right) \tag{13}$$

Hence, evaluating (13) involves solving local subproblems of the form

$$\begin{aligned}
\min_{z_k} \quad & \frac{1}{2} z_k^T H_k z_k + c_k^T z_k + \lambda^T F_k z_k - \tau \sum_{i=1}^{m_k} \log([s_k]_i)\\
\text{s.t.} \quad & C_k z_k = d_k\\
& G_k z_k + s_k = f_k\\
& s_k \geq 0
\end{aligned} \tag{14}$$

The relaxed dual problem then reads

$$\min_{\lambda} d(\lambda, \tau) \tag{15}$$

from which solution, the solution to (12) can be recovered according to strong duality [2].

Strict convexity also implies that the gradient of $d(\lambda, \tau)$ is given by the residual of the coupling constraints [1], i.e.

$$\nabla d(\lambda, \tau) = -\sum_{k=1}^{N} F_k z_k^*(\lambda, \tau) + e \tag{16}$$

where $z_k^*(\lambda, \tau) = \arg\min_{z_k \in \mathcal{Z}_k} \mathcal{L}_k(z_k, s_k, \lambda, \tau)$. The dual Hessian is then given by

$$\nabla^2 d(\lambda, \tau) = -\sum_{k=1}^{N} F_k \frac{\partial z_k^*(\lambda, \tau)}{\partial \lambda} \tag{17}$$

A Newton direction $\Delta\lambda$ in the dual space can then be obtained as a solution to the Newton system

$$\nabla^2 d(\lambda, \tau) \Delta\lambda + \nabla d(\lambda, \tau) = 0 \tag{18}$$

3 Structure in the dual Hessian

3.1 Dual Hessian

By introducing $y_k = \tau / s_k \in \mathbb{R}^{m_k}$, the KKT conditions to (14) are given by

$$r_k(w_k^*, \lambda, \tau) = \begin{bmatrix} r_{Dk}(w_k^*, \lambda) \\ r_{Ek}(w_k^*) \\ r_{Ik}(w_k^*) \\ r_{Ck}(w_k^*, \tau) \end{bmatrix} = 0 \tag{19a}$$

$$s_k^* > 0, \quad y_k^* > 0 \tag{19b}$$

where we use the notation $w_k = [z_k^T, \mu_k^T, y_k^T, s_k^T]^T$ for the local primal-dual variables and $r_k(w_k, \lambda, \tau)$ is given by:

$$r_{D_k}(w_k, \lambda) = H_k z_k + c_k + F_k^T \lambda + C_k^T \mu_k + G_k^T y_k \tag{20a}$$

$$r_{Ek}(w_k) = C_k z_k - d_k \tag{20b}$$

$$r_{Ik}(w_k) = G_k z_k + s_k - f_k \tag{20c}$$

$$r_{Ck}(w_k, \tau) = Y_k s_k - \tau \mathbf{1} \tag{20d}$$

As we have seen in previous section, in order to form the dual Hessian we need to compute $\frac{\partial z_k^*}{\partial \lambda}$, where $z_k^* = z_k^*(\lambda)$ is the optimal primal solution and is hence fulfilling (19). By differentiating (19), the following linear system is obtained:

$$\begin{bmatrix} H_k & C_k^T & G_k^T & 0 \\ C_k & 0 & 0 & 0 \\ G_k & 0 & 0 & I \\ 0 & 0 & S_k & Y_k \end{bmatrix} \begin{bmatrix} \frac{\partial z_k^*}{\partial \lambda} \\ \frac{\partial \mu_k^*}{\partial \lambda} \\ \frac{\partial y_k^*}{\partial \lambda} \\ \frac{\partial s_k^*}{\partial \lambda} \end{bmatrix} = - \begin{bmatrix} F_k^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

If block elimination of (21) is used, the *normal equations* can be formed [7],

$$\Lambda_k \frac{\partial \mu_k^*}{\partial \lambda} = -C_k \Phi_k^{-1} F_k^T \tag{22a}$$

$$\Phi_k \frac{\partial z_k^*}{\partial \lambda} = -F_k^T - C_k^T \frac{\partial \mu_k^*}{\partial \lambda} \tag{22b}$$

$$\frac{\partial s_k^*}{\partial \lambda} = -G_k \frac{\partial z_k^*}{\partial \lambda} \tag{22c}$$

$$\frac{\partial y_k^*}{\partial \lambda} = -S^{-1} Y \frac{\partial s_k^*}{\partial \lambda} \tag{22d}$$

where $\Phi_k = H_k + G_k^T S_k^{-1} Y_k G_k \in \mathbb{S}_{++}^{(N+1)n_k \times (N+1)n_k}$ and $\Lambda_k = C_k \Phi_k^{-1} C_k^T \in \mathbb{S}_{++}^{Nl_k \times Nl_k}$. By using (22a) and (22b), it can be obtained that

$$F_k \frac{\partial z_k}{\partial \lambda} = -F_k (\Phi_k^{-1} - \Phi_k^{-1} C_k^T \Lambda_k^{-1} C_k \Phi_k^{-1}) F_k^T \tag{23}$$

This means that the dual Hessian (17) can be written as

$$\nabla^2 d(\lambda,\tau) = \sum_{k=1}^{M} F_k(\Phi_k^{-1} - \Phi_k^{-1} C_k^T \Lambda_k^{-1} C_k \Phi_k^{-1}) F_k^T \tag{24}$$

Let us now turn our attention to the structure coming from the time domain. We have already seen that C_k is block bidiagonal, and it can trivially be realized that Φ_k is block diagonal. This means that Λ_k has a block tridiagonal structure given by:

$$\Lambda_k = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & & \\ \Lambda_{12}^T & \Lambda_{22} & \ddots & \\ & \ddots & \ddots & \Lambda_{N-1,N} \\ & & \Lambda_{N-1,N}^T & \Lambda_{N,N} \end{bmatrix} \in \mathbb{S}_{++}^{Nl_k \times Nl_k} \tag{25}$$

where

$$\Lambda_{i,i} = C_{k,i-1}\Phi_{k,i-1}^{-1}C_{k,i-1}^T + D_{k,i}\Phi_{k,i}^{-1}D_{k,i}^T \in \mathbb{R}^{l_k, \times l_k} \tag{26a}$$

$$\Lambda_{i,i+1} = D_{k,i}\Phi_{k,i}^{-1}C_{k,i}^T \in \mathbb{R}^{l_k \times l_k} \tag{26b}$$

Accordingly, all matrices in (24) are banded except Λ_k^{-1} which in general is dense.

3.2 Decaying of the dual Hessian

Let us first assume that the problem data is bounded. In other words, let us assume the following:

Assumption 1 *The row and column absolute sums of Jacobians of equality constraints (i.e. (6b) and (6c)) are bounded. Hence:*

1. $\|C_{k,i}\|_\infty \leq \gamma$ *and* $\|C_{k,i}\|_1 \leq \gamma$
2. $\|D_{k,i}\|_\infty \leq \gamma$ *and* $\|D_{k,i}\|_1 \leq \gamma$
3. $\|F_{k,i}\|_\infty \leq \gamma$ *and* $\|F_{k,i}\|_1 \leq \gamma$

It should be observed that Assumption 1 is not by any means restrictive, since any solver would struggle with a problem where it is not fulfilled.

Furthermore, we assume boundedness of Φ_k^{-1}:

Assumption 2 *The row and column absolute sums of* Φ_k^{-1} *are bounded. Hence:*

$$\|\Phi_{k,i}^{-1}\|_\infty = \|\Phi_{k,i}^{-1}\|_1 \leq \gamma_{\Phi_k^{-1}}, \quad \forall i \tag{27}$$

Possible problems with interior-point methods are related to numerical difficulties due to ill-conditioning []. This might occur since elements in $S_k^{-1}Y_k$ can be close to zero in late iterations when τ is small. Our experience is however that this is not a major issue in practice, which is also supported by commercial interior-point implementations []. Moreover, since we assume that (5) is strongly convex, the eigenvalues of Φ_k are lower bounded by the smallest eigenvalue of H_k, even when $S_k^{-1}Y_k$ is singular. According to this reasoning it is not restrictive to assume boundedness of Φ_k^{-1}.

Inverses of sparse matrices are in general dense, but individual elements are often small in absolute value. Since Λ_k is banded, symmetric and positive definite, we will relay our analysis on the following classical result:

Lemma 1 *If A is Hermitian positive definite and m-banded ($[A]_{ij} = 0$ if $|i-j| > m$), the entries of A^{-1} satisfy the following bound:*

$$|[A^{-1}]_{ij}| < K\omega^{|i-j|}, \quad \forall i,j \tag{28}$$

where $[a,b]$ is the smallest interval containing the spectrum $\sigma(A)$ of A, $K = \max\{a^{-1}, K_0\}$, $K_0 = (1+\sqrt{\kappa})$, $\omega = q^{1/m}$, $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, $\kappa = \frac{b}{a}$.

Proof A proof is given in []. □

This means that the entries of A^{-1} are bounded by an exponentially decaying function along each row or column. However, the bound depends on the condition number and the bandwidth of the matrix. Matrices with a high condition number and/or a high bandwidth can accordingly result in a large K and $\omega \approx 1$, leading to a slow decay. The opposite, i.e. a low condition number and a small band, would result in a rapid decay.

Observe that due to the block tridiagonal structure of Λ_k, it is $3l_k$-banded. Moreover, if we introduce the notation

$$\Lambda_k^{-1} = \begin{bmatrix} T_{11} & T_{21}^T & \dots & T_{N+1,1}^T \\ T_{21} & T_{22} & \dots & T_{N+1,2}^T \\ \vdots & \vdots & \ddots & \vdots \\ T_{N+1,1} & T_{N+1,2} & \dots & T_{N+1,N+1} \end{bmatrix} \tag{29}$$

where $T_{i,j} \in \mathbb{R}^{l_k \times l_k}$, we can establish the following proposition:

Proposition 1 *The off-diagonal blocks (i.e. $T_{i,j}$ where $i-j>0$) in Λ_k^{-1} satisfy the following bounds:*

$$\|T_{i,j}\|_\bullet \le K_{\Lambda_k}\omega_{\Lambda_k}^{i-j} \tag{30a}$$

$$\tag{30b}$$

where $\bullet$ represents ∞ and 1, $\sigma_{\min}(\Lambda_k)$ and κ_{Λ_k} are the the smallest singular value and the condition number of Λ_k respectively, $K_{\Lambda_k} = \max\{\sigma_{\min}(\Lambda_k)^{-1}, 1+\sqrt{\kappa_{\Lambda_k}}\}l_k\omega_{\Lambda_k}^{1/l_k}$ and $\omega_{\Lambda_k} = \left(\frac{\sqrt{\kappa_{\Lambda_k}}-1}{\sqrt{\kappa_{\Lambda_k}}+1}\right)^{\frac{1}{3}}$.

Proof According to Lemma 1, the element in $T_{i,j}$ with the largest bound is located in the top-right corner, and is hence element $[\Lambda_k^{-1}]_{il_k+1,jl_k}$. By directly applying Lemma 1 it follows that:

$$\max|[T_{ij}]| \le \max\{\sigma_{min}(\Lambda_k)^{-1}, 1+\sqrt{\kappa_{\Lambda_k}}\}\left(\frac{\sqrt{\kappa_{\Lambda_k}}-1}{\sqrt{\kappa_{\Lambda_k}}+1}\right)^{\frac{1}{3l_k}((i-j)l_k+1)} \tag{31}$$

where $\max|[T_{ij}]|$ refers to the maximum absolute value of the components in T_{ij}. Moreover, since there are l_k elements in each row or column of a block $T_{i,j}$, we obtain the bounds given in (30). □

The constants K_{Λ_k} and ω_{Λ_k} in Proposition 1 depends heavily on the conditioning of Λ_k, which indeed for interior point methods in general can be very high for small values of τ. It should however be understood that strong convexity of (5), should improve the worst case conditioning of Φ_k and hence also of Λ_k.

To gain some perspective, we have concluded that Λ_k^{-1} is decaying exponentially towards the off-diagonal corners and that all other matrices in (24) are banded. This suggests that also $\nabla^2_{\lambda\lambda} d(\lambda, \tau)$ should decay towards the off-diagonal corners. To maintain a simple reasoning, one more stepping stone will be used before arriving at the main results of this section. Therefore, let us introduce the notation:

$$C_k^T \Lambda_k^{-1} C_k = \begin{bmatrix} V_{11} & V_{21}^T & \dots & V_{N+1,1}^T \\ V_{21} & V_{22} & \dots & V_{N+1,2}^T \\ \vdots & \vdots & \ddots & \vdots \\ V_{N+1,1} & V_{N+1,2} & \dots & V_{N+1,N+1} \end{bmatrix} \tag{32}$$

where $V_{ij} \in \mathbb{R}^{l_k \times l_k}$, and look at the decay of $C_k^T \Lambda_k^{-1} C_k$.

Proposition 2 *The off diagonal blocks (i.e. $V_{i,j}$ where $i-j > 0$) of $C_k^T \Lambda_k^{-1} C_k$ satisfy the following bounds:*

$$\|V_{i,j}\|_\bullet \leq \gamma^2 \bar{K}_{\Lambda_k} \omega_{\Lambda_k}^{i-j} \tag{33a}$$

$$\tag{33b}$$

where $\bar{K}_{\Lambda_k} = (2 + \omega_{\Lambda_k} + \omega_{\Lambda_k}^{-1})\omega_{\Lambda_k}^{-1} K_{\Lambda_k}$ and $\bullet$ represents ∞ and 1.

Proof Bla bla bla □

Let us now introduce the notation:

$$\nabla^2 d(\lambda, \tau) = \begin{bmatrix} W_{11} & W_{21}^T & \dots & W_{N+1,1}^T \\ W_{21} & W_{22} & \dots & W_{N+1,2}^T \\ \vdots & \vdots & \ddots & \vdots \\ W_{N+1,1} & W_{N+1,2} & \dots & W_{N+1,N+1} \end{bmatrix} \tag{34}$$

where $W_{i,j} \in \mathbb{R}^{r_i \times r_j}$. We can now establish a decay towards the off-diagonal corners of $\nabla^2 d(\lambda, \tau)$.

Proposition 3 *The off-diagonal blocks (i.e. $W_{i,j}$ where $i-j > 0$) of $\nabla^2 d(\lambda, \tau)$ satisfy the following bounds:*

$$\|W_{i,j}\|_\bullet \leq \sum_{k=1}^{M} \gamma^4 \gamma^2_{\Phi_k^{-1}} \bar{K}_{\Lambda_k} \omega_{\Lambda_k}^{i-j} \tag{35}$$

where $\bullet$ represents ∞ and 1.

Proof Since Φ_k^{-1} is block-diagonal, the off-diagonal blocks of $F_k(\Phi_k^{-1}-\Phi_k^{-1}C_k^T\Lambda_k^{-1}C_k\Phi_k^{-1})F_k^T$ and $-F_k\Phi_k^{-1}C_k^T\Lambda_k^{-1}C_k\Phi_k^{-1}F_k^T$ are identical. Accordingly, by using Assumption 1, Assumption 2, Proposition 2 and adding the contributions from all subproblems together, brings us to (35).

Moreover, let us continue by finding a bound on the euclidean distance between $\nabla^2 d(\lambda,\tau)$ and a band along its diagonal. To do so, we start by recalling Gershgorin's circle theorem:

Theorem 1 *For $A \in \mathbb{R}^{n\times n}$ with elements a_{ij}, let $R_i = \sum_{j\neq i}|a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the row i. Let $D(a_{ii},R_i)$ be the closed disc centered in a_{ii} with radius R_i, then every eigenvalue of A lies within at least one of the discs $D(a_{ii},R_i)$.*

Proof A proof is given in []. □

Finally, we can establish our main result:

Lemma 2 *Let $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}}$ represent the diagonal $\mathcal{M}$-block band of $\nabla^2 d(\lambda,\tau)$. The following bound holds:*

$$\|\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)\|_2 \le \left(N - \frac{\mathcal{M}-1}{2}\right)\sum_{k=1}^{M}\gamma^4\gamma^2_{\Phi_k^{-1}}\tilde{K}_{\Lambda_k}\omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}} \tag{36}$$

where $\tilde{K}_{\Lambda_k} = \bar{K}_{\Lambda_k}\omega_{\Lambda_k} = (2+\omega_{\Lambda_k}+\omega_{\Lambda_k}^{-1})K_{\Lambda_k}$.

Proof Since $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)$ is symmetric, its singular values collapse into the absolute value of its eigenvalues. The problem of finding a bound on the 2-norm is hence reduced to the problem of bounding the magnitude of the largest eigenvalue.

First, observe that all diagonal elements of $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)$ are zero. This means that every Gershgorin disc will be centered in the origin, and a bound on the largest eigenvalue can be found by finding the largest radius of a Gershgorin disc.

The blocks $W_{i,j}$ in $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)$ with the largest bound is located next to the diagonal band of zeros, i.e. where $i-j = \frac{\mathcal{M}-1}{2}+1$. Furthermore, there is at least $N-\frac{\mathcal{M}-1}{2}$ nonzero blocks at each block row. Each block is upper bounded by Proposition 3. This means that we can establish a bound on the radius of the Gershgorin discs, and hence the following:

$$\|\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}} - \nabla^2 d(\lambda,\tau)\|_2 \le \left(N - \frac{\mathcal{M}-1}{2}\right)\sum_{k=1}^{M}\gamma^4\gamma^2_{\Phi_k^{-1}}\bar{K}_{\Lambda_k}\omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}+1} \tag{37}$$

Equation (36) is then directly obtained by introducing $\tilde{K}_{\Lambda_k} = \bar{K}_{\Lambda_k}\omega_{\Lambda_k}$. □

It is well known that the use of an inexact Hessian will degrade the convergence of Newton's method to a linear rate []. However, let us still look at the relative error in the Newton direction that we get from using $\lfloor\nabla^2 d(\lambda,\tau)\rceil_{\mathcal{M}}$ instead of $\nabla^2 d(\lambda,\tau)$. First, recall the following lemma:

Lemma 3 *If x is a solution to $Ax = b$ and $\hat{x}$ is a solution to the pertubed system $(A+F)\hat{x} = b + f$, then the following bound holds:*

$$\frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq \kappa(A)(\rho_A - \rho_b) \tag{38}$$

where $\kappa(A)$ represents the condition number of A, $\rho_A = \frac{\|F\|_2}{\|A\|_2}$ and $\rho_b = \frac{\|f\|_2}{\|b\|_2}$.

Proof A proof is given in [].

Accordingly, by combining Lemma 2 and Lemma 3 we can establish the following:

Lemma 4 *If $\Delta\hat{\lambda}$ is a solution to $\lfloor \nabla^2 d(\lambda, \tau) \rceil_{\mathcal{M}} \Delta\hat{\lambda} = -\nabla d(\lambda, \tau)$, the following bound on the relative error compared to the true Newton direction $\Delta\lambda$ holds:*

$$\frac{\|\Delta\hat{\lambda} - \Delta\lambda\|_2}{\|\Delta\lambda\|_2} \leq \|\nabla^2 d(\lambda, \tau)^{-1}\|_2 (N - \frac{\mathcal{M} - 1}{2}) \sum_{k=1}^{M} \gamma^4 \gamma^2_{\Phi_k^{-1}} \tilde{K}_{\Lambda_k} \omega_{\Lambda_k}^{\frac{\mathcal{M}-1}{2}} \tag{39}$$

Proof We can view $\Delta\hat{\lambda}$ as a solution to a perturbed system:

$$\left(\nabla^2 d(\lambda, \tau) + \left(\lfloor \nabla^2 d(\lambda, \tau) \rceil_{\mathcal{M}} - \nabla^2 d(\lambda, \tau)\right)\right) \Delta\hat{\lambda} = -\nabla d(\lambda, \tau) \tag{40}$$

Equation (39) follows then directly from Lemma 2, Lemma 3 and (40). □

4 Numerical experiments

4.1 Decaying of the dual Hessian

4.1.1 A small problem

As an example of a small problem, we use a randomly generated problem with 4 subproblems (i.e. $M = 4$), where each subproblem has $N = 10$, 6 states and 4 controls and hence $n_k = 10$, and each time instance has 4 coupling constraints.

According to Section 3.2, we expect a slower decay when the conditioning of $S_k^{-1} Y_k$ gets worse, i.e. when we reduce τ. Surprisingly, this is something that does not seem to have a big influence in practice. To illustrate this, Figure 2 shows the dual Hessian for the same problem as in Figure 1, but with $\tau = 10^{-5}$.

4.1.2 A large problem

4.2 Newton steps with inexact Hessian

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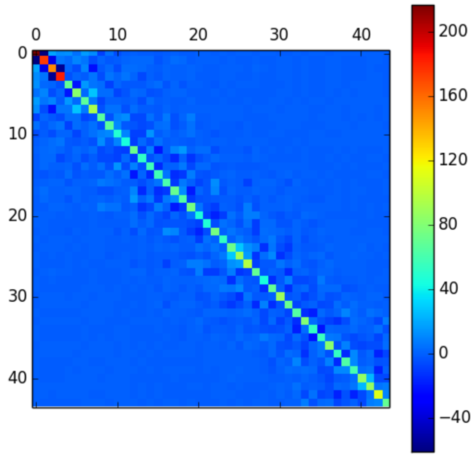

Fig. 1 $\tau = 1$

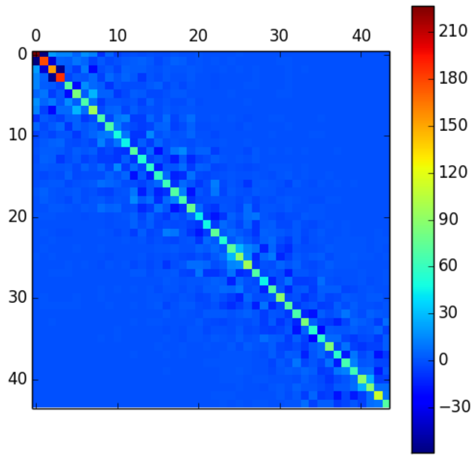

Fig. 2 $\tau = 10^{-5}$

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