

General moments of the inverse real Wishart distribution and orthogonal Weingarten functions

Sho Matsumoto

Abstract

We study a random positive definite symmetric matrix distributed according to a real Wishart distribution. We compute general moments of the random matrix and of its inverse explicitly. To do so, we employ the orthogonal Weingarten function, which was recently introduced in the study for Haar-distributed orthogonal matrices. As applications, we give formulas for moments of traces of a Wishart matrix and its inverse.¹

1 Introduction

1.1 Wishart distributions

Let d be a positive integer. Let $\text{Sym}(d)$ be the \mathbb{R} -linear space of $d \times d$ real symmetric matrices, and $\Omega = \text{Sym}^+(d)$ the open convex cone of all positive definite matrices in $\text{Sym}(d)$. Let $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d} \in \Omega$, and let

$$\beta \in \left\{ \frac{1}{2}, \frac{2}{2}, \dots, \frac{d-1}{2} \right\} \sqcup \left(\frac{d-1}{2}, +\infty \right).$$

Then there exists a probability measure $\mathfrak{W}_{d, \beta, \sigma}$ on Ω such that its moment generating function (or its Laplace transform) is given by

$$\int_{\Omega} e^{\text{tr}(\theta w)} \mathfrak{W}_{d, \beta, \sigma}(dw) = \det(I_d - \theta \sigma)^{-\beta},$$

where θ is any $d \times d$ symmetric matrix such that $\sigma^{-1} - \theta \in \Omega$. We call $\mathfrak{W}_{d, \beta, \sigma}$ the *real Wishart distribution* on Ω with parameters (β, σ) .

We call a random matrix $W \in \Omega$ a *real Wishart matrix* associated with parameters (β, σ) and write $W \sim W_d(\beta, \sigma; \mathbb{R})$ if its distribution is $\mathfrak{W}_{d, \beta, \sigma}$. Thus the moment generating function for W is given by

$$\mathbb{E}[e^{\text{tr}(\theta W)}] = \det(I_d - \theta \sigma)^{-\beta},$$

with θ being as above. Here \mathbb{E} stands for the average.

¹ MSC 2000 subject classifications: Primary 15A52; secondary 60E05, 20C30.
Keywords: Wishart distribution, Gelfand pair, zonal polynomial, orthogonal group, hafnian.

If 2β is a positive integer, $p = 2\beta$ say, then a Wishart matrix W is expressed as follows. Let X_1, \dots, X_p be d -dimensional random column vectors distributed independently according to the Gaussian distribution $N_d(0, \frac{1}{2}\sigma)$. Then

$$W = X_1 X_1^t + \dots + X_p X_p^t,$$

where X_i^t is the transpose of X_i , i.e., a row vector.

If $\beta > \frac{d-1}{2}$ (not necessarily an integer), the distribution $\mathfrak{W}_{d,\beta,\sigma}$ has the expression

$$\mathfrak{W}_{d,\beta,\sigma}(dw) = f(w; d, \beta, \sigma) \mathfrak{L}(dw),$$

where $f(w; d, \beta, \sigma)$ is the density function given by

$$(1.1) \quad f(w; d, \beta, \sigma) = \Gamma_d(\beta)^{-1} (\det \sigma)^{-\beta} (\det w)^{\beta - \frac{d+1}{2}} e^{-\text{tr}(\sigma^{-1}w)} \quad (w \in \Omega)$$

with the multivariate gamma function

$$\Gamma_d(\beta) = \pi^{d(d-1)/4} \prod_{j=1}^d \Gamma\left(\beta - \frac{1}{2}(j-1)\right).$$

Here \mathfrak{L} is the Lebesgue measure on $\text{Sym}(d)$ defined by

$$\mathfrak{L}(dw) = \prod_{1 \leq i \leq j \leq d} dw_{ij} \quad \text{with } w = (w_{ij})_{1 \leq i, j \leq d}.$$

Likewise, a *complex* Wishart distribution is defined on the set of all $d \times d$ positive definite hermitian complex matrices. Given a Wishart matrix W , the distribution of the inverse matrix W^{-1} is called the *inverse* (or *inverted*) Wishart distribution. We denote by W_{ij} and W^{ij} the (i, j) -entry of W and W^{-1} , respectively.

The Wishart distributions are fundamental distributions in multivariate statistical analysis. We refer to [Mu]. The structure of Wishart distributions has been studied for a long time, nevertheless, a lot of results are obtained only recently. We are interested in moments of the forms $\mathbb{E}[P(W)]$ and $\mathbb{E}[P(W^{-1})]$, where $P(A)$ is a polynomial in entries A_{ij} of a matrix A . In particular, we would like to compute *general moments*

$$\mathbb{E}[W_{i_1 j_1} W_{i_2 j_2} \dots W_{i_k j_k}] \quad \text{and} \quad \mathbb{E}[W^{i_1 j_1} W^{i_2 j_2} \dots W^{i_k j_k}]$$

for W and W^{-1} , respectively.

Von Rosen [Vo] computed the general moments of low orders for W^{-1} . Lu and Richards [LR] gave formulas for W by applying MacMahon's master theorem. Graczyk et al. [GLM1] gave formulas for $W^{\pm 1}$ in the complex case by using representation theory of symmetric groups, while they [GLM2] gave results for only W (not W^{-1}) in the real case by using representation theory of hyperoctahedral groups. Letac and Massam [LM1] computed the moments $\mathbb{E}[P(W)]$ and $\mathbb{E}[P(W^{-1})]$ in both real and complex cases, where the P is a polynomial depending only on the eigenvalues of a matrix. Furthermore, a *noncentral* Wishart distribution was also studied, see [LM2] and [KN1].

1.2 Results

Our main purpose in the present paper is to compute the general moment

$$\mathbb{E}[W^{i_1 j_1} W^{i_2 j_2} \dots W^{i_k j_k}]$$

for an *inverse real* Wishart matrix $W^{-1} = (W^{ij})$. As we described, in the complex case Graczyk et al. [GLM1] obtained formulas for such a moment by the representation-theoretic approach. Our main results are precisely their counterparts for the real case, which had been unsolved.

To describe our main result, we recall *perfect matchings*. Let n be a positive integer and put $[n] = \{1, 2, \dots, n\}$. A perfect matching \mathbf{m} on the $2n$ -set $[2n]$ is an unordered pairing of letters $1, 2, \dots, 2n$. Denote by $\mathcal{M}(2n)$ the set of all such perfect matchings. For example, $\mathcal{M}(4)$ consists of three elements: $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$.

Given a perfect matching $\mathbf{m} \in \mathcal{M}(2n)$, we attach a (undirected) graph $G = G(\mathbf{m})$ defined as follows. The vertex set of G is $[2n]$. The edge set of G is

$$\{\{2k-1, 2k\} \mid k \in [n]\} \sqcup \{\{p, q\} \mid \{p, q\} \in \mathbf{m}\}.$$

Then each vertex has just two edges, and each connected component of G has even vertices. We denote by $\kappa(\mathbf{m})$ the number of connected components in $G(\mathbf{m})$. For example, given $\mathbf{m} = \{\{1, 3\}, \{2, 7\}, \{4, 8\}, \{5, 6\}\} \in \mathcal{M}(8)$, the graph $G(\mathbf{m})$ has two connected components (where one component has vertices $1, 2, 3, 4, 7, 8$ and the other has $5, 6$), and therefore $\kappa(\mathbf{m}) = 2$.

Now we give a formula of the general moments for W . It would be reader-friendly to describe the results of both the real Wishart law itself (Theorem 1) and its inverse (Theorem 2) on display.

Theorem 1. *Let $W = (W_{ij})_{1 \leq i, j \leq d} \sim W_d(\beta, \sigma; \mathbb{R})$. Given indices k_1, k_2, \dots, k_{2n} from $\{1, \dots, d\}$, we have*

$$(1.2) \quad \mathbb{E}[W_{k_1 k_2} W_{k_3 k_4} \dots W_{k_{2n-1} k_{2n}}] = 2^{-n} \sum_{\mathbf{m} \in \mathcal{M}(2n)} (2\beta)^{\kappa(\mathbf{m})} \prod_{\{p, q\} \in \mathbf{m}} \sigma_{k_p k_q}.$$

For example, since $\kappa(\{\{1, 2\}, \{3, 4\}\}) = 2$ and $\kappa(\{\{1, 3\}, \{2, 4\}\}) = \kappa(\{\{1, 4\}, \{2, 3\}\}) = 1$ we have

$$(1.3) \quad \mathbb{E}[W_{k_1 k_2} W_{k_3 k_4}] = \beta^2 \sigma_{k_1 k_2} \sigma_{k_3 k_4} + \frac{\beta}{2} \sigma_{k_1 k_3} \sigma_{k_2 k_4} + \frac{\beta}{2} \sigma_{k_1 k_4} \sigma_{k_2 k_3}.$$

Theorem 1 is not new. Indeed, it is equivalent to Theorem 10 in [GLM2]. Moreover, Kuriki and Numata [KN1] extended it to noncentral Wishart distributions very recently. We revisit it in the framework of *alpha-hafnians*. We develop a theory of the alpha-hafnians in Section 2, and apply it to the proof of Theorem 1 in Section 3.

The following is our main result. Let σ^{ij} be the (i, j) -entry of the inverse matrix σ^{-1} .

Theorem 2. *Let $W \sim W_d(\beta, \sigma; \mathbb{R})$. Put $\gamma = \beta - \frac{d+1}{2}$ and suppose $\gamma > n - 1$. Given indices k_1, k_2, \dots, k_{2n} from $\{1, \dots, d\}$, we have*

$$(1.4) \quad \mathbb{E}[W^{k_1 k_2} W^{k_3 k_4} \dots W^{k_{2n-1} k_{2n}}] = \sum_{\mathbf{m} \in \mathcal{M}(2n)} \widetilde{\text{Wg}}(\mathbf{m}; \gamma) \prod_{\{p, q\} \in \mathbf{m}} \sigma^{k_p k_q}.$$

Here $\widetilde{\text{Wg}}(\mathbf{m}; \gamma)$ is defined in Section 5 below.

For example, for $\gamma > 1$ we will see

$$\begin{aligned}\widetilde{\text{Wg}}(\{\{1, 2\}, \{3, 4\}\}; \gamma) &= \frac{2\gamma - 1}{\gamma(\gamma - 1)(2\gamma + 1)}, \\ \widetilde{\text{Wg}}(\{\{1, 3\}, \{2, 4\}\}; \gamma) &= \widetilde{\text{Wg}}(\{\{1, 4\}, \{2, 3\}\}; \gamma) = \frac{1}{\gamma(\gamma - 1)(2\gamma + 1)},\end{aligned}$$

and therefore we have

$$\mathbb{E}[W^{k_1 k_2} W^{k_3 k_4}] = \frac{1}{\gamma(\gamma - 1)(2\gamma + 1)} ((2\gamma - 1)\sigma^{k_1 k_2} \sigma^{k_3 k_4} + \sigma^{k_1 k_3} \sigma^{k_2 k_4} + \sigma^{k_1 k_4} \sigma^{k_2 k_3}).$$

The quantity $\widetilde{\text{Wg}}(\mathbf{m}; \gamma)$ is a slight deformation of the *orthogonal Weingarten function*. The function was introduced by Collins and his coauthors [CM, CS], in order to compute the general moments for a Haar-distributed orthogonal matrix. In general, $\widetilde{\text{Wg}}(\mathbf{m}; \gamma)$ ($\mathbf{m} \in \mathcal{M}(2n)$) is given by a sum over partitions of n , and derived from the harmonic analysis of the Gelfand pair (S_{2n}, H_n) , where S_{2n} is the symmetric group and H_n is the hyperoctahedral group. Amazingly, the same function thus appears in two different random matrix systems. In Section 4, we review the theory of the Weingarten function developed in [CM, Mat2], and, in Section 5, we prove Theorem 2.

In Section 6, we give applications of Theorem 1 and Theorem 2. In particular, we obtain results of Letac and Massam [LM1] as corollaries of Theorem 1 and Theorem 2.

2 Alpha-hafnians

2.1 An expansion formula for alpha-hafnians

Let A be a $2n \times 2n$ symmetric matrix $A = (A_{pq})_{p,q \in [2n]}$. Let α be a complex number. We define an α -*hafnian* of A (see [KN2]) by

$$\text{hf}_\alpha(A) = \sum_{\mathbf{m} \in \mathcal{M}(2n)} \alpha^{\kappa(\mathbf{m})} \prod_{\{p,q\} \in \mathbf{m}} A_{pq}.$$

The ordinary hafnian of A is nothing but $\text{hf}_1(A)$. For example, if $n = 2$,

$$\text{hf}_\alpha(A) = \alpha^2 A_{12} A_{34} + \alpha A_{13} A_{24} + \alpha A_{14} A_{23}.$$

We remark that $\text{hf}_\alpha(A)$ does not depend on diagonal entries $A_{11}, A_{22}, \dots, A_{2n, 2n}$. Note that the right hand side in (1.2) is equal to $2^{-n} \text{hf}_{2\beta}(\sigma_{k_p k_q})_{p,q \in [2n]}$.

Proposition 1. *Let $A = (A_{pq})_{p,q \in [2n]}$ be a symmetric matrix. Let $D = (A_{pq})_{p,q \in [2n-2]}$. For each $j = 1, 2, \dots, 2n - 2$, let $B^{(j)}$ be the symmetric matrix obtained by replacing the j th row/column of D by the $(2n - 1)$ th row/column of A . In formulas, $B^{(j)} = (B_{pq}^{(j)})_{p,q \in [2n-2]}$ is given by*

$$B_{pq}^{(j)} = \begin{cases} A_{2n-1, 2n-1} & \text{if } p = j \text{ and } q = j, \\ A_{2n-1, q} & \text{if } p = j \text{ and } q \neq j, \\ A_{p, 2n-1} & \text{if } p \neq j \text{ and } q = j, \\ A_{p, q} & \text{if } p \neq j \text{ and } q \neq j. \end{cases}$$

Then we have

$$(2.1) \quad \text{hf}_\alpha(A) = \sum_{j=1}^{2n-2} A_{j,2n} \text{hf}_\alpha(B^{(j)}) + \alpha A_{2n-1,2n} \text{hf}_\alpha(D).$$

We call (2.1) an *expansion formula for an α -hafnian* with respect to the $(2n)$ th row/column.

Proof. For each $j = 1, 2, \dots, 2n-1$, we set

$$\mathcal{M}_j(2n) = \{\mathbf{m} \in \mathcal{M}(2n) \mid \{j, 2n\} \in \mathbf{m}\}.$$

Then $\mathcal{M}(2n) = \bigsqcup_{j=1}^{2n-1} \mathcal{M}_j(2n)$. We define a one-to-one map $\mathbf{m} \mapsto \mathbf{n}$ from $\mathcal{M}_j(2n)$ to $\mathcal{M}(2n-2)$ as follows.

First, suppose $j = 2n-1$. Given $\mathbf{m} \in \mathcal{M}_{2n-1}(2n)$, we let $\mathbf{n} \in \mathcal{M}(2n-2)$ to be the perfect matching obtained from \mathbf{m} by removing the block $\{2n-1, 2n\}$. It is clear that the mapping $\mathcal{M}_{2n-1}(2n) \ni \mathbf{m} \mapsto \mathbf{n} \in \mathcal{M}(2n-2)$ is bijective and that $\kappa(\mathbf{m}) = \kappa(\mathbf{n}) + 1$.

Next, suppose $j \in [2n-2]$. Given $\mathbf{m} \in \mathcal{M}_j(2n)$, we let $\mathbf{n} \in \mathcal{M}(2n-2)$ to be obtained by removing the block $\{j, 2n\}$ and a block $\{i, 2n-1\}$ (with some $i \in [2n-2]$) and by adding $\{i, j\}$. It is easy to see that this mapping $\mathcal{M}_j(2n) \ni \mathbf{m} \mapsto \mathbf{n} \in \mathcal{M}(2n-2)$ is bijective, $\kappa(\mathbf{m}) = \kappa(\mathbf{n})$, and $\prod_{\{p,q\} \in \mathbf{m}} A_{pq} = A_{j,2n} \prod_{\{p,q\} \in \mathbf{n}} B_{pq}^{(j)}$.

For example, consider $\mathbf{m} = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$. Then $\mathbf{m} \in \mathcal{M}_3(6)$, and we obtain $\mathbf{n} = \{\{1, 4\}, \{2, 3\}\} \in \mathcal{M}(4)$. Therefore, we have $\kappa(\mathbf{m}) = 1 = \kappa(\mathbf{n})$ and $\prod_{\{p,q\} \in \mathbf{m}} A_{pq} = A_{14}A_{25}A_{36} = A_{36}B_{14}^{(3)}B_{23}^{(3)} = A_{36} \prod_{\{p,q\} \in \mathbf{n}} B_{pq}^{(3)}$.

Using the correspondence $\mathcal{M}_j(2n) \ni \mathbf{m} \leftrightarrow \mathbf{n} \in \mathcal{M}(2n-2)$ with $j = 1, 2, \dots, 2n-1$, it follows that

$$\begin{aligned} \text{hf}_\alpha(A) &= \sum_{j=1}^{2n-1} A_{j,2n} \sum_{\mathbf{m} \in \mathcal{M}_j(2n)} \alpha^{\kappa(\mathbf{m})} \prod_{\substack{\{p,q\} \in \mathbf{m} \\ \{p,q\} \neq \{j,2n\}}} A_{pq} \\ &= A_{2n-1,2n} \sum_{\mathbf{n} \in \mathcal{M}(2n-2)} \alpha^{\kappa(\mathbf{n})+1} \prod_{\{p,q\} \in \mathbf{n}} A_{pq} \\ &\quad + \sum_{j=1}^{2n-2} A_{j,2n} \sum_{\mathbf{n} \in \mathcal{M}(2n-2)} \alpha^{\kappa(\mathbf{n})} \prod_{\{p,q\} \in \mathbf{n}} B_{pq}^{(j)}, \end{aligned}$$

which is equal to $A_{2n-1,2n} \alpha \cdot \text{hf}_\alpha(D) + \sum_{j=1}^{2n-2} A_{j,2n} \text{hf}_\alpha(B^{(j)})$. □

2.2 Another expression for α -hafnians

Let S_n be the symmetric group on $[n]$. Each permutation π is uniquely decomposed into a product of cycles. For example, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 4 & 3 & 2 \end{pmatrix} \in S_6$ is expressed as $\pi = (1 \rightarrow 5 \rightarrow 3 \rightarrow 1)(2 \rightarrow 6 \rightarrow 2)(4 \rightarrow 4)$. Denote by $C(\pi)$ the set of all cycles of π , and let $\nu(\pi)$ be the number of cycles of π : $\nu(\pi) = |C(\pi)|$.

Let $A = (A_{pq})_{p,q \in [2n]}$ be a symmetric matrix. For each $k, l \in [n]$, we denote by $A[k, l]$ the 2×2 matrix

$$A[k, l] = \begin{pmatrix} A_{2k-1, 2l-1} & A_{2k-1, 2l} \\ A_{2k, 2l-1} & A_{2k, 2l} \end{pmatrix}.$$

For a cycle $c = (c_r \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r)$ on $\{1, \dots, n\}$, we put

$$P_c(A) = \text{tr}(A[c_1, c_2]JA[c_2, c_3]J \cdots A[c_r, c_1]J), \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, $P_{(c_1 \rightarrow c_1)}(A) = \text{tr}(A[c_1, c_1]J) = 2A_{2c_1-1, 2c_1}$ for a 1-cycle $(c_1 \rightarrow c_1)$. It is easy to see that $P_c(A)$ can be written

$$(2.2) \quad P_c(A) = \sum_{j_1, j_2, \dots, j_{2r}} A_{j_{2r}, j_1} A_{j_2, j_3} \cdots A_{j_{2r-2}, j_{2r-1}}$$

summed over $(j_{2k-1}, j_{2k}) \in \{(2c_k - 1, 2c_k), (2c_k, 2c_k - 1)\}$ ($k = 1, 2, \dots, r$). For a permutation $\pi \in S_n$, we define

$$P_\pi(A) = \prod_{c \in C(\pi)} P_c(A).$$

Similarly, given an r -cycle $c = (c_r \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r)$, we let c_r to be the largest number among $\{c_1, c_2, \dots, c_r\}$. We define $Q_c(A)$ as follows: If $r = 1$ then $Q_c(A) = A_{2c_1-1, 2c_1}$; if $r \geq 2$ then

$$Q_c(A) = \sum_{(j_1, j_2)} \cdots \sum_{(j_{2r-3}, j_{2r-2})} A_{2c_r-1, j_1} A_{j_2, j_3} A_{j_4, j_5} \cdots A_{j_{2r-2}, 2c_r},$$

summed over $(j_{2k-1}, j_{2k}) \in \{(2c_k - 1, 2c_k), (2c_k, 2c_k - 1)\}$ ($k = 1, 2, \dots, r-1$). As $P_\pi(A)$, we define

$$Q_\pi(A) = \prod_{c \in C(\pi)} Q_c(A).$$

For example, for a cycle $(3 \rightarrow 2 \rightarrow 1 \rightarrow 3)$, we have

$$\begin{aligned} Q_c(A) &= \sum_{(j_1, j_2) \in \{(3, 4), (4, 3)\}} \sum_{(j_3, j_4) \in \{(1, 2), (2, 1)\}} A_{5j_1} A_{j_2, j_3} A_{j_4, 6} \\ &= A_{53} A_{41} A_{26} + A_{54} A_{31} A_{26} + A_{53} A_{42} A_{16} + A_{54} A_{32} A_{16}. \end{aligned}$$

Lemma 2. Let $c = (c_r \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r)$ be a cycle. Then

$$P_c(A) = Q_c(A) + Q_{c^{-1}}(A),$$

where $c^{-1} = (c_r \rightarrow \cdots \rightarrow c_2 \rightarrow c_1 \rightarrow c_r)$.

Proof. Suppose c_r is the largest number in $\{c_1, \dots, c_r\}$. We can express

$$P_c(A) = \sum_{j_1, j_2, \dots, j_{2r-2}} A_{2c_r-1, j_1} A_{j_2, j_3} \cdots A_{j_{2r-2}, 2c_r} + \sum_{j_1, j_2, \dots, j_{2r-2}} A_{2c_r, j_1} A_{j_2, j_3} \cdots A_{j_{2r-2}, 2c_r-1},$$

summed over $(j_{2k-1}, j_{2k}) \in \{(2c_k - 1, 2c_k), (2c_k, 2c_k - 1)\}$ ($k = 1, 2, \dots, r-1$). Here the first sum coincides with $Q_c(A)$, while the second one does with $Q_{c^{-1}}(A)$. \square

Proposition 3. Let $A = (A_{pq})_{p,q \in [2n]}$ be a symmetric matrix. Then

$$\text{hf}_\alpha(A) = \sum_{\pi \in S_n} \left(\frac{\alpha}{2}\right)^{\nu(\pi)} P_\pi(A) = \sum_{\pi \in S_n} \alpha^{\nu(\pi)} Q_\pi(A).$$

This is a key lemma in our proofs of Theorem 1 and Theorem 2. We show this proposition in the next subsection.

Remark 1. Let $A = (A_{ij})_{1 \leq i,j \leq n}$ be a complex matrix and α a complex number. An α -permanent of A is defined by

$$\text{per}_\alpha(A) = \sum_{\pi \in S_n} \alpha^{\nu(\pi)} \prod_{i=1}^n A_{i\pi(i)}.$$

It intertwines the permanent and determinant:

$$\text{per}_1(A) = \text{per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^n A_{i\pi(i)} \quad \text{and} \quad \text{per}_{-1}(A) = (-1)^n \det(A).$$

It is also called an α -determinant. See [Ve] and also [Sh]. Alpha-hafnians are generalizations of the alpha-permanents in the following sense. Given a matrix $A = (A_{ij})_{1 \leq i,j \leq n}$, we define the $2n \times 2n$ symmetric matrix $B = (B_{pq})_{1 \leq p,q \leq 2n}$ by

$$B_{2i-1,2j-1} = B_{2i,2j} = 0 \quad \text{and} \quad B_{2i-1,2j} = B_{2j,2i-1} = A_{ij} \quad \text{for all } i, j = 1, 2, \dots, n.$$

Then, since $Q_c(B) = A_{c_r, c_1} A_{c_1, c_2} \dots A_{c_{r-1}, c_r}$ for $c = (c_r \rightarrow c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_r)$, it follows from Proposition 3 that $\text{hf}_\alpha(B) = \text{per}_\alpha(A)$. Thus any α -permanent can be given by an α -hafnian.

Remark 2. Let $B = (B_{pq})_{p,q \in [2n]}$ be a skew-symmetric matrix and let α be a complex number. In [Mat1], an α -pfaffian of B was defined. In a similar way to the proof of Proposition 3, we can see that the definition in [Mat1] is equivalent to the expression

$$\text{pf}_\alpha(B) = \sum_{\mathbf{m} \in \mathcal{M}(2n)} (-\alpha)^{\kappa(\mathbf{m})} \text{sgn}(\mathbf{m}) \prod_{\{p,q\} \in \mathcal{M}(2n)} B_{pq}.$$

Here, for $\mathbf{m} = \{\{\mathbf{m}(1), \mathbf{m}(2)\}, \dots, \{\mathbf{m}(2n-1), \mathbf{m}(2n)\}\}$ we define

$$\text{sgn}(\mathbf{m}) \prod_{\{p,q\} \in \mathcal{M}(2n)} B_{pq} = \text{sgn} \begin{pmatrix} 1 & 2 & \dots & 2n \\ \mathbf{m}(1) & \mathbf{m}(2) & \dots & \mathbf{m}(2n) \end{pmatrix} \cdot B_{\mathbf{m}(1)\mathbf{m}(2)} \dots B_{\mathbf{m}(2n-1)\mathbf{m}(2n)}.$$

When $\alpha = -1$, the α -pfaffian is exactly the ordinary pfaffian. Moreover, as α -hafnians are so, the α -pfaffians are generalizations of α -permanents.

2.3 Proof of Proposition 3

Put

$$\tilde{\text{hf}}_\alpha(A) = \sum_{\pi \in S_n} \left(\frac{\alpha}{2}\right)^{\nu(\pi)} P_\pi(A) = \sum_{\pi \in S_n} \alpha^{\nu(\pi)} Q_\pi(A)$$

for any $n \geq 1$ and any symmetric matrix A of size $2n$. Here the second equality follows from Lemma 2.

Let $B^{(1)}, B^{(2)}, \dots, B^{(2n-2)}, D$ be as in Proposition 1. In order to obtain Proposition 3, it is enough to show the recurrence formula

$$(2.3) \quad \tilde{\text{hf}}_\alpha(A) = \sum_{j=1}^{2n-2} A_{j,2n} \tilde{\text{hf}}_\alpha(B^{(j)}) + \alpha A_{2n-1,2n} \tilde{\text{hf}}_\alpha(D).$$

To see (2.3), we will show a recurrence formula involving $Q_c(A)$ and $P_c(A)$. For each $k \in [n]$, we denote by $S_n^{(k)}$ the subset of permutations in S_n such that $\pi(k) = n$. Note $S_n = \bigsqcup_{k=1}^n S_n^{(k)}$.

Let $k \in [n-1]$ and let $\pi \in S_n^{(k)}$. Let $u_n(\pi) \in C(\pi)$ be the cycle including the letter n , which is of the form

$$u_n(\pi) = (n \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r \rightarrow k \rightarrow n),$$

with (possibly empty) distinct $c_1, \dots, c_r \in [n] \setminus \{k, n\}$. Then, define

$$\tilde{u}_n(\pi) = (n \rightarrow c_r \rightarrow \cdots \rightarrow c_2 \rightarrow c_1 \rightarrow k \rightarrow n),$$

and let $\tilde{\pi}$ be the permutation obtained by replacing $u_n(\pi)$ in π by $\tilde{u}_n(\pi)$. Note that $u_n(\tilde{\pi}) = \tilde{u}_n(\pi)$, and that $\tilde{\pi} = \pi$ if and only if $u_n(\pi)$ is a 2 or 3-cycle. The map $\pi \mapsto \tilde{\pi}$ is an involution on $S_n^{(k)}$. For example, given $\pi = (7 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 7)(6 \rightarrow 4 \rightarrow 6)(5 \rightarrow 5) \in S_7$, we have $\tilde{\pi} = (7 \rightarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 7)(6 \rightarrow 4 \rightarrow 6)(5 \rightarrow 5)$.

In general, for the cycle $u_n(\pi) = (n \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r \rightarrow k \rightarrow n)$ with $k \neq n$, we see that

$$\begin{aligned} Q_{u_n(\pi)}(A) &= \sum_{(j_1, j_2)} \cdots \sum_{(j_{2r-1}, j_{2r})} (A_{2n-1, j_1} A_{j_2, j_3} \cdots A_{j_{2r}, 2k-1} A_{2k, 2n} + A_{2n-1, j_1} A_{j_2, j_3} \cdots A_{j_{2r}, 2k} A_{2k-1, 2n}) \\ &= \sum_{(j_1, j_2)} \cdots \sum_{(j_{2r-1}, j_{2r})} (B_{2k, j_1}^{(2k)} B_{j_2, j_3}^{(2k)} \cdots B_{j_{2r}, 2k-1}^{(2k)} A_{2k, 2n} + B_{2k-1, j_1}^{(2k-1)} B_{j_2, j_3}^{(2k-1)} \cdots B_{j_{2r}, 2k}^{(2k-1)} A_{2k-1, 2n}) \end{aligned}$$

summed over

$$(j_{2p-1}, j_{2p}) \in \{(2c_p - 1, 2c_p), (2c_p, 2c_p - 1)\} \quad (p = 1, 2, \dots, r).$$

Similarly,

$$\begin{aligned} Q_{\tilde{u}_n(\pi)}(A) &= \sum_{(j_1, j_2)} \cdots \sum_{(j_{2r-1}, j_{2r})} (A_{2n-1, j_{2r}} \cdots A_{j_3, j_2} A_{j_1, 2k-1} A_{2k, 2n} + A_{2n-1, j_{2r}} \cdots A_{j_3, j_2} A_{j_1, 2k} A_{2k-1, 2n}) \\ &= \sum_{(j_1, j_2)} \cdots \sum_{(j_{2r-1}, j_{2r})} (B_{2k, j_{2r}}^{(2k)} \cdots B_{j_3, j_2}^{(2k)} B_{j_1, 2k-1}^{(2k)} A_{2k, 2n} + B_{2k-1, j_{2r}}^{(2k-1)} \cdots B_{j_3, j_2}^{(2k-1)} B_{j_1, 2k}^{(2k-1)} A_{2k-1, 2n}). \end{aligned}$$

Therefore, we have

$$(2.4) \quad Q_{u_n(\pi)}(A) + Q_{u_n(\tilde{\pi})}(A) = A_{2k,2n}P_{u'_n(\pi)}(B^{(2k)}) + A_{2k-1,2n}P_{u'_n(\pi)}(B^{(2k-1)}).$$

Here $u'_n(\pi)$ is the cycle obtained from $u_n(\pi)$ by removing the letter n : $u'_n(\pi) = (k \rightarrow c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_r \rightarrow k)$. We note that the mapping $\pi \mapsto \pi' := u'_n(\pi) \prod_{c \in C(\pi) \setminus \{u_n(\pi)\}} c$ is the bijective map from $S_n^{(k)}$ to S_{n-1} , and that $\nu(\pi) = \nu(\pi')$.

Now we go back to the proof of (2.3). We rewrite

$$\widetilde{\text{hf}}_\alpha(A) = \sum_{\pi \in S_n^{(n)}} \alpha^{\nu(\pi)} Q_\pi(A) + \sum_{k=1}^{n-1} \sum_{\pi \in S_n^{(k)}} \alpha^{\nu(\pi)} Q_{u_n(\pi)}(A) 2^{-(\nu(\pi)-1)} \prod_{c \in C(\pi) \setminus \{u_n(\pi)\}} P_c(A).$$

The first sum is equal to

$$\sum_{\pi' \in S_n} \alpha^{\nu(\pi')+1} Q_{\pi'}(A) Q_{(n)}(A) = \alpha A_{2n-1,2n} \widetilde{\text{hf}}_\alpha(D)$$

by a natural bijective map $S_n^{(n)} \rightarrow S_{n-1}$, while, since the map $\pi \mapsto \tilde{\pi}$ is bijective on each $S_{2n}^{(k)}$, the terms corresponding to $k \in [n-1]$ in the second sum are equal to

$$\begin{aligned} & \sum_{\pi \in S_n^{(k)}} \left(\frac{\alpha}{2}\right)^{\nu(\pi)} (Q_{u_n(\pi)}(A) + Q_{u_n(\tilde{\pi})}(A)) \prod_{c \in C(\pi) \setminus \{u_n(\pi)\}} P_c(A) \\ &= \sum_{\pi \in S_n^{(k)}} \left(\frac{\alpha}{2}\right)^{\nu(\pi)} (A_{2k,2n}P_{u'_n(\pi)}(B^{(2k)}) + A_{2k-1,2n}P_{u'_n(\pi)}(B^{(2k-1)})) \prod_{c \in C(\pi) \setminus \{u_n(\pi)\}} P_c(A) \\ &= \sum_{\pi' \in S_{n-1}} \left(\frac{\alpha}{2}\right)^{\nu(\pi')} (A_{2k,2n}P_{\pi'}(B^{(2k)}) + A_{2k-1,2n}P_{\pi'}(B^{(2k-1)})) \\ &= A_{2k,2n} \widetilde{\text{hf}}_\alpha(B^{(2k)}) + A_{2k-1,2n} \widetilde{\text{hf}}_\alpha(B^{(2k-1)}). \end{aligned}$$

Here the first equality follows by (2.4), and the second equality follows from the bijection $S_n^{(k)} \ni \pi \mapsto \pi' = u'_n(\pi) \prod_{c \in C(\pi) \setminus \{u_n(\pi)\}} c \in S_{n-1}$. Hence (2.3) follows, and we end the proof of Proposition 3.

3 Proof of Theorem 1

Let m_1, \dots, m_n and x be $d \times d$ matrices. Given a cycle $c = (c_r \rightarrow c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_r)$ on $[n]$, we define

$$R_c(x; m_1, \dots, m_n) = \text{tr} (xm_{c_1}xm_{c_2} \cdots xm_{c_r}).$$

More generally, for a permutation $\pi \in S_n$, we define

$$R_\pi(x; m_1, \dots, m_n) = \prod_{c \in C(\pi)} R_c(x; m_1, \dots, m_n).$$

For example, if $n = 6$ and $\pi = (1 \rightarrow 5 \rightarrow 3 \rightarrow 1)(2 \rightarrow 6 \rightarrow 2)(4 \rightarrow 4)$, then

$$R_\pi(x; m_1, m_2, m_3, m_4, m_5, m_6) = \text{tr}(xm_1xm_5xm_3)\text{tr}(xm_2xm_6)\text{tr}(xm_4).$$

The following proposition, given in [GLM2], is our starting point for the proof of Theorem 1. Let d, β, σ be as in Introduction.

Proposition 4. *Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and let $s_1, \dots, s_n \in \text{Sym}(d)$. Then*

$$\mathbb{E}[\text{tr}(Ws_1)\text{tr}(Ws_2)\cdots\text{tr}(Ws_n)] = \sum_{\pi \in S_n} \beta^{\nu(\pi)} R_\pi(\sigma; s_1, \dots, s_n).$$

Proof. See Proposition 1 in [GLM2]. See also Theorem 1 in [LM1]. \square

Theorem 1 is a consequence of Proposition 4 and Proposition 3. For $1 \leq a, b \leq d$, denote by $E_{ab} = E_{ab}^{(d)}$ the matrix unit of size d , whose (i, j) -entry is $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$. We apply Proposition 4 with $s_j = (E_{k_{2j-1}k_{2j}} + E_{k_{2j}k_{2j-1}})/2$ ($1 \leq j \leq n$). Since W is symmetric, we have $\text{tr}(Ws_j) = (W_{k_{2j-1}k_{2j}} + W_{k_{2j}k_{2j-1}})/2 = W_{k_{2j-1}k_{2j}}$, and therefore it follows from Proposition 4 that

$$\begin{aligned} & \mathbb{E}[W_{k_1k_2}W_{k_3k_4}\cdots W_{k_{2n-1}k_{2n}}] \\ &= 2^{-n} \sum_{\pi \in S_n} \beta^{\nu(\pi)} R_\pi(\sigma; E_{k_1k_2} + E_{k_2k_1}, \dots, E_{k_{2n-1}k_{2n}} + E_{k_{2n}k_{2n-1}}). \end{aligned}$$

From Proposition 3, in order to prove Theorem 1, it is sufficient to show

$$(3.1) \quad R_\pi(\sigma; E_{k_1k_2} + E_{k_2k_1}, \dots, E_{k_{2n-1}k_{2n}} + E_{k_{2n}k_{2n-1}}) = P_\pi((\sigma_{k_pk_q})_{p,q \in [2n]})$$

for any permutation $\pi \in S_n$.

To show (3.1), let $A = (A_{pq})_{p,q \in [2n]}$ be a symmetric matrix and let $c = (c_r \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r)$ be a cycle. Equation (3.1) follows from

$$(3.2) \quad \text{tr}(A(E_{2c_1-1,2c_1} + E_{2c_1,2c_1-1}) \cdots A(E_{2c_r-1,2c_r} + E_{2c_r-1,2c_r})) = P_c(A),$$

with $A = (\sigma_{k_pk_q})_{p,q \in [2n]}$. Here the $E_{ab} = E_{ab}^{(2n)}$ are $2n \times 2n$ matrix units. However we may show (3.2) as follows:

$$\begin{aligned} & \text{tr}(A(E_{2c_1-1,2c_1} + E_{2c_1,2c_1-1}) \cdots A(E_{2c_r-1,2c_r} + E_{2c_r-1,2c_r})) \\ &= \sum_{j_1, j_2, \dots, j_{2r}=1}^{2n} A_{j_{2r}j_1}(E_{2c_1-1,2c_1} + E_{2c_1,2c_1-1})_{j_1j_2} A_{j_2j_3} \cdots A_{j_{2r-2}j_{2r-1}}(E_{2c_r-1,2c_r} + E_{2c_r,2c_r-1})_{j_{2r-1}j_{2r}} \\ &= \sum_{j_1, \dots, j_{2r}} A_{j_{2r}j_1} A_{j_2j_3} \cdots A_{j_{2r-2}j_{2r-1}}. \end{aligned}$$

Here the last sum is over $(j_{2k-1}, j_{2k}) \in \{(2c_k - 1, 2c_k), (2c_k, 2c_k - 1)\}$ ($k = 1, 2, \dots, r$). Hence we obtain (3.2) and therefore (3.1). It ends the proof of Theorem 1.

4 Orthogonal Weingarten functions

We review the theory of the Weingarten function for orthogonal groups; see [CM, Mat2] for details. Claims in Subsections 4.1–4.4 are also seen in [Mac, VII-2].

4.1 Hyperoctahedral groups and perfect matchings

Let H_n be the subgroup in S_{2n} generated by transpositions $(2k-1 \rightarrow 2k \rightarrow 2k-1)$ ($1 \leq k \leq n$) and by double transpositions $(2i-1 \rightarrow 2j-1 \rightarrow 2i-1) \cdot (2i \rightarrow 2j \rightarrow 2i)$ ($1 \leq i < j \leq n$). The group H_n is called the *hypercubic group*. Note that $|H_n| = 2^n n!$.

We embed the set $\mathcal{M}(2n)$ into S_{2n} via the mapping

$$\mathcal{M}(2n) \ni \mathbf{m} \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n \\ \mathbf{m}(1) & \mathbf{m}(2) & \mathbf{m}(3) & \mathbf{m}(4) & \cdots & \mathbf{m}(2n) \end{pmatrix} \in S_{2n}$$

where $(\mathbf{m}(1), \dots, \mathbf{m}(2n))$ is the unique sequence satisfying

$$\begin{aligned} \mathbf{m} &= \{\{\mathbf{m}(1), \mathbf{m}(2)\}, \dots, \{\mathbf{m}(2n-1), \mathbf{m}(2n)\}\}, \\ \mathbf{m}(2k-1) &< \mathbf{m}(2k) \quad (1 \leq k \leq n), \quad \text{and} \quad 1 = \mathbf{m}(1) < \mathbf{m}(3) < \cdots < \mathbf{m}(2n-1). \end{aligned}$$

The $\mathbf{m} \in \mathcal{M}(2n)$ are representatives of the cosets gH_n of H_n in S_{2n} :

$$(4.1) \quad S_{2n} = \bigsqcup_{\mathbf{m} \in \mathcal{M}(2n)} \mathbf{m}H_n.$$

4.2 Coset-types

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of nonnegative integers such that $|\lambda| := \sum_{i \geq 1} \lambda_i$ is finite. If $|\lambda| = n$, we call λ a *partition of n* and write $\lambda \vdash n$. Define the length $\ell(\lambda)$ of λ by the number of nonzero λ_i .

Given $g \in S_{2n}$, we attach a graph $G(g)$ with vertices $1, 2, \dots, 2n$ and with the edge set

$$\{\{2k-1, 2k\} \mid k \in [n]\} \sqcup \{\{g(2k-1), g(2k)\} \mid k \in [n]\}.$$

Each connected component of $G(g)$ has even vertices. Let $2\lambda_1, 2\lambda_2, \dots, 2\lambda_l$ be numbers of vertices of components. We may suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. Then the sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a partition of n . We call the λ the *coset-type* of $g \in S_{2n}$.

For example, the coset-type of $(\frac{1}{7} \frac{2}{1} \frac{3}{6} \frac{4}{3} \frac{5}{2} \frac{6}{8} \frac{7}{4} \frac{8}{5})$ in S_8 is $(2, 2)$.

In general, given $g, g' \in S_{2n}$, their coset-types coincide if and only if $H_n g H_n = H_n g' H_n$. Hence we have the double coset decomposition of H_n in S_{2n} :

$$(4.2) \quad S_{2n} = \bigsqcup_{\rho \vdash n} H_\rho, \quad \text{where } H_\rho = \{g \in S_{2n} \mid \text{the coset-type of } g \text{ is } \rho\}.$$

Note $H_{(1^n)} = H_n$ and $|H_\rho| = (2^n n!)^2 / (2^{\ell(\rho)} z_\rho)$. Here

$$(4.3) \quad z_\rho = \prod_{r \geq 1} r^{m_r(\rho)} m_r(\rho)!$$

with multiplicities $m_r(\rho) = |\{i \geq 1 \mid \rho_i = r\}|$ of r in ρ .

For $g \in S_{2n}$, denote by $\kappa(g)$ the number of connected components of $G(g)$. Equivalently, $\kappa(g)$ is the length of the coset-type of g . Under the embedding $\mathcal{M}(2n) \subset S_{2n}$, we may define $G(\mathfrak{m})$ and $\kappa(\mathfrak{m})$ for each $\mathfrak{m} \in \mathcal{M}(2n)$. They are compatible with their definitions in Subsection 1.2.

4.3 Zonal spherical functions

For two functions f_1, f_2 on S_{2n} , their convolution $f_1 * f_2$ is defined by

$$(f_1 * f_2)(g) = \sum_{g' \in S_{2n}} f_1(g(g')^{-1}) f_2(g') \quad (g \in S_{2n}).$$

Let \mathcal{H}_n be the set of all complex-valued H_n -biinvariant functions on S_{2n} :

$$\mathcal{H}_n = \{f : S_{2n} \rightarrow \mathbb{C} \mid f(\zeta g) = f(g\zeta) = f(g) \ (g \in S_{2n}, \zeta \in H_n)\}.$$

It is known that this is a commutative algebra under convolution, with unit $\mathbf{1}_{\mathcal{H}_n}$ given by

$$(4.4) \quad \mathbf{1}_{\mathcal{H}_n}(g) = \begin{cases} (2^n n!)^{-1} & \text{if } g \in H_n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (S_{2n}, H_n) is a *Gelfand pair* in the sense of [Mac, VII.1]. The algebra \mathcal{H}_n is called the *Hecke algebra* associated with the Gelfand pair (S_{2n}, H_n) .

For each $\lambda \vdash n$ we define the *zonal spherical function* ω^λ by

$$\omega^\lambda(g) = \frac{1}{2^n n!} \sum_{\zeta \in H_n} \chi^{2\lambda}(g\zeta) \quad (g \in S_{2n}).$$

Here $\chi^{2\lambda}$ is the irreducible character of S_{2n} associated with $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$. The ω^λ ($\lambda \vdash n$) form a basis of \mathcal{H}_n and have the property

$$(4.5) \quad \omega^\lambda * \omega^\mu = \delta_{\lambda\mu} \frac{(2n)!}{f^{2\lambda}} \omega^\lambda \quad \text{for all } \lambda, \mu \vdash n.$$

Here $f^{2\lambda}$ is the value of $\chi^{2\lambda}$ at the identity of S_{2n} , or equivalently the dimension of the irreducible representation of character $\chi^{2\lambda}$. We denote by ω_ρ^λ the value of ω^λ at the double coset H_ρ . Note $\omega_{(1^n)}^\lambda = 1$ for all $\lambda \vdash n$.

4.4 Zonal polynomials

We now need the theory of symmetric functions. Let Λ be the algebra of symmetric functions in infinitely-many variables x_1, x_2, \dots and with coefficients in \mathbb{Q} . Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n . We denote by p_λ the *power-sum symmetric function*:

$$p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i} \quad \text{and} \quad p_k(x_1, x_2, \dots) = x_1^k + x_2^k + \dots.$$

Let Z_λ be the *zonal polynomial* (or zonal symmetric function):

$$(4.6) \quad Z_\lambda = 2^n n! \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \omega_\rho^\lambda p_\rho.$$

Here z_ρ is the quantity defined in (4.3). Alternatively, for $\rho \vdash n$,

$$(4.7) \quad p_\rho = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega_\rho^\lambda Z_\lambda.$$

Recall that Λ is the algebra generated by $\{p_r \mid r \geq 1\}$ and that the p_r are algebraically independent. Let z be a complex number and let $\phi_z : \Lambda \rightarrow \mathbb{C}$ be the algebra homomorphism defined by $\phi_z(p_r) = z$ for all $r \geq 1$. Then we have the *specializations*

$$(4.8) \quad \phi_z(p_\rho) = z^{\ell(\rho)} \quad \text{and} \quad \phi_z(Z_\lambda) = C_\lambda(z) := \prod_{(i,j) \in \lambda} (z + 2j - i - 1)$$

where the product $\prod_{(i,j) \in \lambda}$ stands for $\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i}$, which is over all boxes of the Young diagram of λ . It follows by (4.6) and (4.7) that

$$(4.9) \quad C_\lambda(z) = 2^n n! \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \omega_\rho^\lambda z^{\ell(\rho)} \quad \text{and} \quad z^{\ell(\rho)} = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega_\rho^\lambda C_\lambda(z).$$

4.5 Weingarten functions

Let z be a complex number such that $C_\lambda(z) \neq 0$ for all $\lambda \vdash n$. We define a function $\text{Wg}^O(\cdot; z)$ in \mathcal{H}_n by

$$(4.10) \quad \text{Wg}^O(g; z) = \frac{1}{(2n-1)!!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C_\lambda(z)} \omega^\lambda(g) \quad (g \in S_{2n}).$$

We call it the *orthogonal Weingarten function* (or *Weingarten function for orthogonal groups*).

The function $g \mapsto \text{Wg}^O(g; z)$ is constant at each double coset H_ρ ($\rho \vdash n$). We denote by (the same symbol) $\text{Wg}^O(\rho; z)$ its value at H_ρ .

Example 1.

$$\begin{aligned} \text{Wg}^O((1); z) &= \frac{1}{z}. \\ \text{Wg}^O((2); z) &= \frac{-1}{z(z+2)(z-1)}. \quad \text{Wg}^O((1^2); z) = \frac{z+1}{z(z+2)(z-1)}. \end{aligned}$$

The list of $\text{Wg}^O(\rho; z)$ for $|\rho| \leq 6$ is seen in [CM].

Define the function $G^O(\cdot; z)$ in \mathcal{H}_n by

$$G^O(g; z) = z^{\kappa(g)} \quad (g \in S_{2n}).$$

The following lemma is a key in our proof of Theorem 2.

Lemma 5 ([CM]).

$$G^O(\cdot; z) * \text{Wg}^O(\cdot; z) = (2^n n!)^2 \mathbf{1}_{\mathcal{H}_n}.$$

Here $\mathbf{1}_{\mathcal{H}_n}$ is defined in (4.4).

Proof. Recall that if ρ is the coset-type of g , then $\kappa(g) = \ell(\rho)$. From the second formula in (4.9), we have

$$(4.11) \quad G^O(\cdot; z) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} C_\lambda(z) \omega^\lambda,$$

so that

$$G^O(\cdot; z) * \text{Wg}^O(\cdot; z) = \frac{(2^n n!)^2}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega^\lambda$$

by (4.10) and (4.5).

On the other hand, since $\lim_{t \in \mathbb{R}, t \rightarrow +\infty} t^{-n} C_\lambda(t) = 1$, using the second formula in (4.9) again, we may see that

$$\frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega^\lambda(g) = \lim_{t \rightarrow +\infty} t^{-n} \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} C_\lambda(t) \omega^\lambda(g) = \lim_{t \rightarrow +\infty} t^{-(n-\kappa(g))},$$

which is equal to 1 if $g \in H_n$, or to zero otherwise. Hence we have

$$\mathbf{1}_{\mathcal{H}_n} = \frac{1}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega^\lambda.$$

This finishes the proof. \square

4.6 Weingarten calculus for orthogonal groups

The content in this subsection will not be used in the latter sections. We here review how the Weingarten function Wg^O appears in the theory of random orthogonal matrices.

Let $O(N)$ be the compact Lie group of $N \times N$ real orthogonal matrices. The group $O(N)$ is equipped with the *Haar probability measure* dO such that $d(U_1 O U_2) = dO$ for fixed $U_1, U_2 \in O(N)$ and that $\int_{O(N)} dO = 1$.

Let $O = (O_{ij})_{i,j \in [N]}$ be a Haar-distributed orthogonal matrix. Consider a general moment

$$\mathbb{E}[O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_k j_k}] \quad (i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_k \in [N]).$$

From the biinvariant property for the Haar measure, we can see immediately that $\mathbb{E}[O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_k j_k}] = 0$ if k is odd.

Proposition 6 ([CM, CS]). *Let $i_1, \dots, i_{2n}, j_1, \dots, j_{2n}$ be indices in $[N]$. Assume that $N \geq n$ and let $O = (O_{ij})_{i,j \in [N]}$ be a Haar-distributed orthogonal matrix. Then we have*

$$\mathbb{E}[O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_{2n} j_{2n}}] = \sum_{\mathbf{m}, \mathbf{n} \in \mathcal{M}(2n)} \text{Wg}^O(\mathbf{m}^{-1} \mathbf{n}; N) \left(\prod_{\{p,q\} \in \mathbf{m}} \delta_{i_p, i_q} \right) \left(\prod_{\{p,q\} \in \mathbf{n}} \delta_{j_p, j_q} \right).$$

Here each $\mathbf{m} \in \mathcal{M}(2n)$ is regarded as a permutation in S_{2n} .

In particular, using Example 1, we have

$$\mathbb{E}[O_{1,j_1} O_{1,j_2} O_{2,j_3} O_{2,j_4}] = \frac{1}{N(N+2)(N-1)} ((N+1)\delta_{j_1 j_2} \delta_{j_3 j_4} - \delta_{j_1 j_3} \delta_{j_2 j_4} - \delta_{j_1 j_4} \delta_{j_2 j_3})$$

for $N \geq 2$ and $j_1, j_2, j_3, j_4 \in [N]$.

Remark 3. Proposition 6 was first proved in [CS] with a function Wg^O , which was implicitly defined via the equation of Lemma 5. The explicit expression (4.10) was first given in [CM]. Zinn–Justin [Z] (see also [Mat2]) gave another expression, involving Jucys–Murphy elements.

Remark 4. If $\ell(\lambda) > N$ then $C_\lambda(N) = 0$, and therefore the definition (4.10) does not make sense unless $N \geq n$. For $z = N \in \{1, 2, \dots, n-1\}$ we extend the definition of the Weingarten function by

$$\text{Wg}^O(g; N) = \frac{1}{(2n-1)!!} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq N}} \frac{f^{2\lambda}}{C_\lambda(N)} \omega^\lambda(g) \quad (g \in S_{2n}).$$

Then $\text{Wg}^O(g; N)$ does make sense for all $g \in S_{2n}$, and Proposition 6 holds true without any condition for N ; see [CM] for details.

5 Proof of Theorem 2

Let d, β, σ be as in Introduction. We also use symbols defined in Section 4. Our starting point for the proof of Theorem 2 is the following lemma.

Lemma 7. Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and let $s_1, \dots, s_n \in \text{Sym}(d)$. Put $\gamma = \beta - \frac{d+1}{2}$ and suppose $\gamma > 0$. Then

$$\text{tr}(\sigma^{-1} s_1) \text{tr}(\sigma^{-1} s_2) \cdots \text{tr}(\sigma^{-1} s_n) = (-1)^n \sum_{\pi \in S_n} (-\gamma)^{\nu(\pi)} \mathbb{E}[R_\pi(W^{-1}; s_1, \dots, s_n)],$$

where $R_\pi(\cdot; \dots)$ is defined in Section 3.

Proof. We can obtain the proof in the same way to [GLM1, Theorem 3]. Therefore we omit it here. (The assumption $\gamma = \beta - \frac{d+1}{2} > 0$ implies that the real Wishart distribution $\mathfrak{W}_{d, \beta, \sigma}$ has the density $f(w; d, \beta, \sigma)$ given by (1.1), and that $f(w; d, \beta, \sigma)$ vanishes on the boundary of Ω . Therefore, we can apply Stokes' formula for f ; see pages 298–299 in [GLM1].) \square

Lemma 8. Let W and γ be as in Lemma 7. Given indices k_1, k_2, \dots, k_{2n} from $\{1, \dots, d\}$, we have

$$(5.1) \quad \sigma^{k_1 k_2} \sigma^{k_3 k_4} \cdots \sigma^{k_{2n-1} k_{2n}} = (-1)^n 2^{-n} \sum_{\mathfrak{m} \in \mathcal{M}(2n)} (-2\gamma)^{\kappa(\mathfrak{m})} \mathbb{E} \left[\prod_{\{p, q\} \in \mathfrak{m}} W^{k_p k_q} \right].$$

Proof. By using Lemma 7, one can prove it in the same way to the proof of Theorem 1. Indeed, applying Lemma 7 with $s_j = (E_{k_{2j-1}, k_{2j}} + E_{k_{2j}, k_{2j-1}})/2$ ($1 \leq j \leq n$), and using (3.1) and Proposition 3, we see that

$$\begin{aligned}
& \sigma^{k_1 k_2} \sigma^{k_3 k_4} \dots \sigma^{k_{2n-1} k_{2n}} \\
&= (-1)^n 2^{-n} \sum_{\pi \in S_n} (-\gamma)^{\nu(\pi)} \mathbb{E}[R_\pi(W^{-1}; E_{k_1 k_2} + E_{k_2 k_1}, \dots, E_{k_{2n-1} k_{2n}} + E_{k_{2n} k_{2n-1}})] \\
&= (-1)^n 2^{-n} \sum_{\pi \in S_n} (-\gamma)^{\nu(\pi)} \mathbb{E}\left[P_\pi\left((W^{k_p k_q})_{p, q \in [2n]}\right)\right] \\
&= (-1)^n 2^{-n} \mathbb{E}\left[\text{hf}_{-2\gamma}(W^{k_p k_q})_{p, q \in [2n]}\right].
\end{aligned}$$

□

Suppose $\gamma > n - 1$. Then $\text{Wg}^O(g; -2\gamma)$ ($g \in S_{2n}$) can be defined (see Subsection 4.5). Set

$$(5.2) \quad \widetilde{\text{Wg}}(g; \gamma) = (-1)^n 2^n \text{Wg}^O(g; -2\gamma) = \frac{2^n n!}{(2n)!} (-1)^n 2^n \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C_\lambda(-2\gamma)} \omega^\lambda(g) \quad (g \in S_{2n}).$$

We finally prove Theorem 2. Recall that the functions $g \mapsto \kappa(g)$ and $g \mapsto \text{Wg}(g; z)$ are H_n -biinvariant. We can rewrite (5.1) in the form

$$\sigma^{k_1 k_2} \sigma^{k_3 k_4} \dots \sigma^{k_{2n-1} k_{2n}} = (-1)^n 2^{-n} (2^n n!)^{-1} \sum_{g \in S_{2n}} (-2\gamma)^{\kappa(g)} \mathbb{E}\left[W^{k_{g(1)} k_{g(2)}} \dots W^{k_{g(2n-1)} k_{g(2n)}}\right]$$

by the coset decomposition (4.1). Therefore, the right hand side of (1.4) is equal to

$$\begin{aligned}
& (-1)^n 2^n (2^n n!)^{-1} \sum_{g' \in S_{2n}} \text{Wg}^O(g'; -2\gamma) \sigma^{k_{g'(1)} k_{g'(2)}} \dots \sigma^{k_{g'(2n-1)} k_{g'(2n)}} \\
&= (2^n n!)^{-2} \sum_{g, g' \in S_{2n}} (-2\gamma)^{\kappa(g)} \text{Wg}^O(g'; -2\gamma) \mathbb{E}\left[W^{k_{g'g(1)} k_{g'g(2)}} \dots W^{k_{g'g(2n-1)} k_{g'g(2n)}}\right] \\
&= (2^n n!)^{-2} \sum_{g, g'' \in S_{2n}} (-2\gamma)^{\kappa(g)} \text{Wg}^O(g'' g^{-1}; -2\gamma) \mathbb{E}\left[W^{k_{g''(1)} k_{g''(2)}} \dots W^{k_{g''(2n-1)} k_{g''(2n)}}\right]
\end{aligned}$$

by letting $g'' = g'g$. Since Lemma 5 implies

$$\sum_{g \in S_{2n}} z^{\kappa(g)} \text{Wg}^O(g'' g^{-1}; z) = \begin{cases} 2^n n! & \text{if } g'' \in H_n, \\ 0 & \text{otherwise,} \end{cases}$$

the last equation equals

$$(2^n n!)^{-1} \sum_{g'' \in H_n} \mathbb{E}\left[W^{k_{g''(1)} k_{g''(2)}} \dots W^{k_{g''(2n-1)} k_{g''(2n)}}\right] = \mathbb{E}[W^{k_1 k_2} W^{k_3 k_4} \dots W^{k_{2n-1} k_{2n}}].$$

Hence we have proved Theorem 2.

Remark 5. Theorem 2 holds true for any positive real number γ such that $C_\lambda(-2\gamma) \neq 0$ for all $\lambda \vdash n$.

Remark 6. The complex-Wishart version of Theorem 2 is obtained by Graczyk et al. [GLM1]. They employ a class function on S_n defined by

$$\text{Wg}^U(\pi; -q) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{(i,j) \in \lambda} (-q + j - i)} \chi^\lambda(\pi) \quad (\pi \in S_n),$$

where $q > n - 1$ is a parameter in [GLM1], corresponding to our γ . The function $\text{Wg}^U(\pi; N)$ coincides with the Weingarten function for the unitary group $U(N)$, studied in [C] (see also [MN]).

6 Applications

In this section, we give applications of Theorem 1 and Theorem 2.

6.1 Mixed moments of traces

Recall the symbol $R_\pi(x; m_1, \dots, m_n)$ defined in Section 3, where x is a $d \times d$ symmetric matrix, m_1, \dots, m_n are $d \times d$ complex matrices, and $\pi \in S_n$. For example,

$$\begin{aligned} R_{(1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1)}(x; m_1, m_2, m_3, m_4) &= \text{tr}(xm_1xm_3xm_2xm_4), \\ R_{(1 \rightarrow 4 \rightarrow 5 \rightarrow 1)(2 \rightarrow 7 \rightarrow 2)(6 \rightarrow 6)}(x; m_1, m_2, \dots, m_7) &= \text{tr}(xm_1xm_4xm_5)\text{tr}(xm_2xm_7)\text{tr}(xm_6). \end{aligned}$$

Thus $R_\pi(x; m_1, \dots, m_n)$ is a product of traces of the form $\text{tr}(xm_{i_1}xm_{i_2} \cdots xm_{i_k})$. Our purpose in this section is to compute moments of the forms

$$\mathbb{E}[R_\pi(W; m_1, \dots, m_n)] \quad \text{and} \quad \mathbb{E}[R_\pi(W^{-1}; m_1, \dots, m_n)]$$

where $W \sim W_d(\beta, \sigma; \mathbb{R})$ as usual.

First we observe a simple example.

Example 2. We compute $\mathbb{E}[\text{tr}(Wm_1Wm_2)]$. Expanding the trace, we have

$$\mathbb{E}[\text{tr}(Wm_1Wm_2)] = \sum_{k_1, k_2, k_3, k_4} (m_1)_{k_2 k_3} (m_2)_{k_4 k_1} \mathbb{E}[W_{k_1 k_2} W_{k_3 k_4}].$$

From Theorem 1 or (1.3), it is equal to

$$\begin{aligned} & \sum_{k_1, k_2, k_3, k_4} (m_1)_{k_2 k_3} (m_2)_{k_4 k_1} \left(\beta^2 \sigma_{k_1 k_2} \sigma_{k_3 k_4} + \frac{\beta}{2} \sigma_{k_1 k_3} \sigma_{k_2 k_4} + \frac{\beta}{2} \sigma_{k_1 k_4} \sigma_{k_2 k_3} \right) \\ &= \beta^2 \text{tr}(\sigma m_1 \sigma m_2) + \frac{\beta}{2} \text{tr}(\sigma m_1^t \sigma m_2) + \frac{\beta}{2} \text{tr}(\sigma m_1) \text{tr}(\sigma m_2), \end{aligned}$$

where m^t is the transpose of m . In other words,

$$\mathbb{E}[R_{(1 \rightarrow 2 \rightarrow 1)}(W; m_1, m_2)] = \beta^2 R_{(1 \rightarrow 2 \rightarrow 1)}(\sigma; m_1, m_2) + \frac{\beta}{2} R_{(1 \rightarrow 2 \rightarrow 1)}(\sigma; m_1^t, m_2) + \frac{\beta}{2} R_{(1 \rightarrow 1)(2 \rightarrow 2)}(\sigma; m_1, m_2).$$

This example indicates that we should deal with not only m_1, \dots, m_n but also with their transposes m_1^t, \dots, m_n^t .

Given a matrix $m = (m_{ij})$ and a signature $\epsilon \in \{-1, +1\}$, we put

$$m^\epsilon = \begin{cases} m & \text{if } \epsilon = +1, \\ m^t & \text{if } \epsilon = -1. \end{cases}$$

Let m_1, \dots, m_n be $d \times d$ complex matrices and let $x = (x_{i,j})$ be a $d \times d$ real symmetric matrix. Given a permutation $g \in S_{2n}$, we define $T_g(x; m_1, \dots, m_n)$ by

$$T_g(x; m_1, \dots, m_n) = \sum_{j_1, \dots, j_{2n}=1}^d (m_1)_{j_1, j_2} (m_2)_{j_3, j_4} \cdots (m_n)_{j_{2n-1}, j_{2n}} x_{j_{g(1)}, j_{g(2)}} x_{j_{g(3)}, j_{g(4)}} \cdots x_{j_{g(2n-1)}, j_{g(2n)}}.$$

In our situation, the symbol T_g is more useful than R_π .

Given $\pi \in S_n$, we denote by $\tilde{\pi}$ the permutation in S_{2n} given by $\tilde{\pi}(2j-1) = 2\pi(j) - 1$ and $\tilde{\pi}(2j) = 2j$ for $j = 1, 2, \dots, n$. Denote by ζ_i the transposition $(2i-1 \rightarrow 2i \rightarrow 2i-1)$.

Lemma 9. For $\pi \in S_n$ and $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ we have

$$R_\pi(x; m_1^{\epsilon_1}, \dots, m_n^{\epsilon_n}) = T_g(x; m_1, \dots, m_n) \quad \text{with } g = \left(\prod_{i: \epsilon_i = -1} \zeta_i \right) \cdot \tilde{\pi}.$$

Proof. First we will show

$$(6.1) \quad R_\pi(x; m_1, \dots, m_n) = T_{\tilde{\pi}}(x; m_1, \dots, m_n).$$

Take a cycle $c = (c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_r \rightarrow c_1)$ in π . Then we see that

$$\begin{aligned} & \sum_{j_{2c_1-1}, j_{2c_1}, \dots, j_{2c_r-1}, j_{2c_r}} \prod_{k=1}^r (m_{c_k})_{j_{2c_k-1}, j_{2c_k}} x_{j_{\tilde{\pi}(2c_k-1)}, j_{\tilde{\pi}(2c_k)}} \\ &= \sum_{j_{2c_1-1}, j_{2c_1}, \dots, j_{2c_r-1}, j_{2c_r}} \prod_{k=1}^r (m_{c_k})_{j_{2c_k-1}, j_{2c_k}} x_{j_{2\pi(c_k)-1}, j_{2c_k}} \\ &= \sum_{j_{2c_1-1}, j_{2c_1}, \dots, j_{2c_r-1}, j_{2c_r}} (m_{c_1})_{j_{2c_1-1}, j_{2c_1}} x_{j_{2c_1}, j_{2c_2-1}} (m_{c_2})_{j_{2c_2-1}, j_{2c_2}} x_{j_{2c_2}, j_{2c_3-1}} \cdots (m_{c_r})_{j_{2c_r-1}, j_{2c_r}} x_{j_{2c_r}, j_{2c_1-1}} \\ &= \text{tr}(m_{c_1} x m_{c_2} x \cdots m_{c_r} x) = R_c(x; m_1, \dots, m_n). \end{aligned}$$

We obtain (6.1) by taking the product over all cycles in π .

Next we will show

$$(6.2) \quad R_\pi(x; m_1, \dots, m_i^t, \dots, m_n) = T_{\zeta_i \tilde{\pi}}(x; m_1, \dots, m_n).$$

We have

$$T_{\zeta_i \tilde{\pi}}(x; m_1, \dots, m_n) = \sum_{j_1, \dots, j_{2n}} \prod_{k=1}^n (m_k)_{j_{2k-1}, j_{2k}} x_{j_{\zeta_i \tilde{\pi}(2k-1)}, j_{\zeta_i \tilde{\pi}(2n)}}.$$

Letting $j'_k = j_{\zeta_i(k)}$ for all $k = 1, 2, \dots, 2n$, it is equal to

$$\begin{aligned} & \sum_{j'_1, \dots, j'_{2n}} \prod_{k=1}^n (m_k)_{j'_{\zeta_i(2k-1)}, j'_{\zeta_i(2k)}} x_{j'_{\tilde{\pi}(2k-1)}, j'_{\tilde{\pi}(2k)}} \\ &= \sum_{j'_1, \dots, j'_{2n}} (m_i^t)_{j'_{2i-1}, j'_{2i}} x_{j'_{\tilde{\pi}(2i-1)}, j'_{\tilde{\pi}(2i)}} \prod_{k \neq i} (m_k)_{j'_{2k-1}, j'_{2k}} x_{j'_{\tilde{\pi}(2k-1)}, j'_{\tilde{\pi}(2k)}} \\ &= T_{\tilde{\pi}}(x; m_1, \dots, m_i^t, \dots, m_n). \end{aligned}$$

Therefore, (6.2) follows by (6.1). Now the result can be obtained from (6.1) and (6.2). \square

Example 3. Consider

$$\text{tr}(xm_1xm_4^txm_5^txm_2)\text{tr}(xm_3xm_7^t)\text{tr}(xm_6),$$

which is equal to $R_{\pi}(x; m_1^{\epsilon_1}, \dots, m_7^{\epsilon_7})$ with

$$\begin{aligned} \pi &= (1 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 1)(3 \rightarrow 7 \rightarrow 3)(6 \rightarrow 6) \in S_7, \\ (\epsilon_1, \dots, \epsilon_7) &= (+1, +1, +1, -1, -1, +1, -1). \end{aligned}$$

It coincides with $T_g(x; m_1, \dots, m_7)$, where $g = \zeta_4 \zeta_5 \zeta_7 \tilde{\pi}$, i.e.,

$$g = (7 \rightarrow 8 \rightarrow 7)(9 \rightarrow 10 \rightarrow 9)(13 \rightarrow 14 \rightarrow 13)(1 \rightarrow 7 \rightarrow 9 \rightarrow 3 \rightarrow 1)(5 \rightarrow 13 \rightarrow 5)(11 \rightarrow 11).$$

Lemma 10. The function $S_{2n} \ni g \mapsto T_g(x; m_1, \dots, m_n)$ is right H_n -invariant:

$$T_{g\zeta}(x; m_1, \dots, m_n) = T_g(x; m_1, \dots, m_n) \quad \text{for all } \zeta \in H_n \text{ and } g \in S_{2n}.$$

Proof. It is enough to check for $\zeta = (2i-1 \rightarrow 2i \rightarrow 2i-1)$ and $(2i-1 \rightarrow 2j-1 \rightarrow 2i-1)(2i \rightarrow 2j \rightarrow 2i)$ because H_n is generated by them. However, it is clear. \square

The moment of the form $\mathbb{E}[R_{\pi}(W^{\pm 1}; m_1^{\epsilon_1}, \dots, m_n^{\epsilon_n})]$ may be given by $\mathbb{E}[T_g(W^{\pm 1}; m_1, \dots, m_n)]$ with some $g \in S_{2n}$. Hence we now compute the moments $\mathbb{E}[T_g(W^{\pm 1}; m_1, \dots, m_n)]$. First of all, we note that the formulas in Theorem 1 and Theorem 2 can be expressed in the forms

$$(6.3) \quad \mathbb{E}[W_{k_1 k_2} \cdots W_{k_{2n-1}, k_{2n}}] = 2^{-n} (2^n n!)^{-1} \sum_{g \in S_{2n}} (2\beta)^{\kappa(g)} \sigma_{k_{g(1)}, k_{g(2)}} \cdots \sigma_{k_{g(2n-1)}, k_{g(2n)}},$$

$$(6.4) \quad \mathbb{E}[W^{k_1 k_2} \cdots W^{k_{2n-1}, k_{2n}}] = (2^n n!)^{-1} \sum_{g \in S_{2n}} \widetilde{Wg}(g; \gamma) \sigma^{k_{g(1)}, k_{g(2)}} \cdots \sigma^{k_{g(2n-1)}, k_{g(2n)}}.$$

Theorem 3. Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and let γ be as in Theorem 2. Let m_1, \dots, m_n be $d \times d$ matrices and let $g \in S_{2n}$. Then

$$\begin{aligned}\mathbb{E}[T_g(W; m_1, \dots, m_n)] &= 2^{-n} \sum_{\mathbf{n} \in \mathcal{M}(2n)} (2\beta)^{\kappa(g^{-1}\mathbf{n})} T_{\mathbf{n}}(\sigma; m_1, \dots, m_n), \\ \mathbb{E}[T_g(W^{-1}; m_1, \dots, m_n)] &= \sum_{\mathbf{n} \in \mathcal{M}(2n)} \widetilde{Wg}(g^{-1}\mathbf{n}; \gamma) T_{\mathbf{n}}(\sigma^{-1}; m_1, \dots, m_n).\end{aligned}$$

Proof. Using (6.3) (or Theorem 1),

$$\begin{aligned}& \mathbb{E}[T_g(W; m_1, \dots, m_n)] \\ &= \sum_{j_1, \dots, j_{2n}} \left(\prod_{k=1}^n (m_k)_{j_{2k-1}, j_{2k}} \right) \mathbb{E}[W_{j_{g(1)}, j_{g(2)}} \cdots W_{j_{g(2n-1)}, j_{g(2n)}}] \\ &= \sum_{j_1, \dots, j_{2n}} \left(\prod_{k=1}^n (m_k)_{j_{2k-1}, j_{2k}} \right) 2^{-n} (2^n n!)^{-1} \sum_{g' \in S_{2n}} (2\beta)^{\kappa(g')} \sigma_{j_{gg'(1)}, j_{gg'(2)}} \cdots \sigma_{j_{gg'(2n-1)}, j_{gg'(2n)}}\end{aligned}$$

and, letting $h = gg'$,

$$\begin{aligned}&= 2^{-n} (2^n n!)^{-1} \sum_{h \in S_{2n}} (2\beta)^{\kappa(g^{-1}h)} \sum_{j_1, \dots, j_{2n}} \prod_{k=1}^n (m_k)_{j_{2k-1}, j_{2k}} \sigma_{j_{h(2k-1)}, j_{h(2k)}} \\ &= 2^{-n} (2^n n!)^{-1} \sum_{h \in S_{2n}} (2\beta)^{\kappa(g^{-1}h)} T_h(\sigma; m_1, \dots, m_n) \\ &= 2^{-n} \sum_{\mathbf{n} \in \mathcal{M}(2n)} (2\beta)^{\kappa(g^{-1}\mathbf{n})} T_{\mathbf{n}}(\sigma; m_1, \dots, m_n).\end{aligned}$$

Here the last equality follows from Lemma 10 and (4.1). Thus the first formula has been proved. The same applies to the second formula. \square

It follows from Lemma 9 and Theorem 3 that, for $\pi \in S_n$ and $(\epsilon_1, \dots, \epsilon_n) \in \{-1, +1\}^n$,

$$(6.5) \quad \mathbb{E}[R_\pi(W; m_1^{\epsilon_1}, \dots, m_n^{\epsilon_n})] = 2^{-n} \sum_{\mathbf{n} \in \mathcal{M}(2n)} (2\beta)^{\kappa(g^{-1}\mathbf{n})} T_{\mathbf{n}}(\sigma; m_1, \dots, m_n),$$

$$(6.6) \quad \mathbb{E}[R_\pi(W; m_1^{\epsilon_1}, \dots, m_n^{\epsilon_n})] = \sum_{\mathbf{n} \in \mathcal{M}(2n)} \widetilde{Wg}(g^{-1}\mathbf{n}; \gamma) T_{\mathbf{n}}(\sigma^{-1}; m_1, \dots, m_n),$$

where g is as in Lemma 9. We remark that (6.5) is equivalent to [GLM2, Corollary 14].

6.2 Averages of invariant polynomials

Given a partition λ of n , we define two functions \mathbf{Z}_λ and \mathbf{p}_λ on $\Omega = \text{Sym}^+(d)$ by

$$\mathbf{Z}_\lambda(x) = Z_\lambda(a_1, a_2, \dots, a_d, 0, 0, \dots) \quad \text{and} \quad \mathbf{p}_\lambda(x) = p_\lambda(a_1, a_2, \dots, a_d, 0, 0, \dots),$$

where a_1, \dots, a_d are eigenvalues of $x \in \Omega$, and Z_λ, p_λ are symmetric functions defined in Subsection 4.4. In particular, we have

$$\mathbf{p}_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} \text{tr}(x^{\lambda_i}) = \prod_{r \geq 1} (\text{tr}(x^r))^{m_r(\lambda)},$$

where $m_r(\lambda)$ is the multiplicity of r in λ . From (4.6) and (4.7), we have

$$(6.7) \quad \mathbf{Z}_\lambda = 2^n n! \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \omega_\rho^\lambda \mathbf{p}_\rho \quad \text{and} \quad \mathbf{p}_\rho = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} f^{2\lambda} \omega_\rho^\lambda \mathbf{Z}_\lambda.$$

Recall $C_\lambda(z) = \prod_{(i,j) \in \lambda} (z + 2j - i - 1)$. The following theorem, derived from Theorem 1 and Theorem 2, is exactly the real case of Proposition 5 and 6 in [LM1].

Theorem 4. *Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and let γ be as in Theorem 2. For a partition λ of n ,*

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_\lambda(W)] &= 2^{-n} C_\lambda(2\beta) \mathbf{Z}_\lambda(\sigma). \\ \mathbb{E}[\mathbf{Z}_\lambda(W^{-1})] &= (-1)^n 2^n C_\lambda(-2\gamma)^{-1} \mathbf{Z}_\lambda(\sigma^{-1}). \end{aligned}$$

Proof. First of all, we note that

$$\mathbf{p}_\rho(x) = T_g(x; \underbrace{I_d, \dots, I_d}_n)$$

for a permutation g in S_{2n} of coset-type ρ and for a matrix x in Ω . Indeed, since the function $S_{2n} \ni g \mapsto T_g(x; I_d, \dots, I_d)$ is H_n -biinvariant, the image depends only on the coset-type. If π is a permutation in S_n of cycle-type ρ , then $\tilde{\pi}$ is of coset-type ρ , and therefore $T_g(x; I_d, \dots, I_d) = T_{\tilde{\pi}}(x; I_d, \dots, I_d) = R_\pi(x; I_d, \dots, I_d) = \mathbf{p}_\rho(x)$ by Lemma 9.

From the first formula in (6.7) and the double decomposition (4.2), we have

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_\lambda(W)] &= 2^n n! \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \omega_\rho^\lambda \mathbb{E}[\mathbf{p}_\rho(W)] \\ &= 2^n n! \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \frac{1}{|H_\rho|} \sum_{g \in H_\rho} \omega^\lambda(g) \mathbb{E}[T_g(W; I_d, \dots, I_d)] \\ &= (2^n n!)^{-1} \sum_{g \in S_{2n}} \omega^\lambda(g) \mathbb{E}[T_g(W; I_d, \dots, I_d)]. \end{aligned}$$

It follows from Theorem 3 that

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_\lambda(W)] &= (2^n n!)^{-2} \sum_{g \in S_{2n}} \omega^\lambda(g) 2^{-n} \sum_{g' \in S_{2n}} (2\beta)^{\kappa(g^{-1}g')} T_{g'}(\sigma; I_d, \dots, I_d) \\ &= (2^n n!)^{-2} 2^{-n} \sum_{g' \in S_{2n}} \left((\omega^\lambda * G^O(\cdot; 2\beta))(g') \right) T_{g'}(\sigma; I_d, \dots, I_d). \end{aligned}$$

Since $\omega^\lambda * G^O(\cdot; z) = 2^n n! C_\lambda(z) \omega^\lambda$ by (4.11) and (4.5), we have

$$\mathbb{E}[\mathbf{Z}_\lambda(W)] = (2^n n!)^{-1} 2^{-n} C_\lambda(2\beta) \sum_{g' \in S_{2n}} \omega^\lambda(g') T_{g'}(\sigma; I_d, \dots, I_d).$$

Since

$$\sum_{g' \in S_{2n}} \omega^\lambda(g') T_{g'}(\sigma; I_d, \dots, I_d) = \sum_{\rho \vdash n} |H_\rho| \omega_\rho^\lambda \mathbf{p}_\rho(\sigma) = \sum_{\rho \vdash n} \frac{(2^n n!)^2}{2^{\ell(\rho)} z_\rho} \omega_\rho^\lambda \mathbf{p}_\rho(\sigma) = 2^n n! \mathbf{Z}_\lambda(\sigma)$$

by the first formula in (6.7), our first result follows. The proof of our second result is similar. \square

The following is equivalent to the real case of [LM1, Theorem 2].

Corollary 5. *Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and let γ be as in Theorem 2. For a partition μ of n ,*

$$\begin{aligned} \mathbb{E}[\mathbf{p}_\mu(W)] &= \frac{(2^n n!)^2}{(2n)!} \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \left(2^{-n} \sum_{\lambda \vdash n} C_\lambda(2\beta) f^{2\lambda} \omega_\mu^\lambda \omega_\rho^\lambda \right) \mathbf{p}_\rho(\sigma), \\ \mathbb{E}[\mathbf{p}_\mu(W^{-1})] &= \frac{(2^n n!)^2}{(2n)!} \sum_{\rho \vdash n} 2^{-\ell(\rho)} z_\rho^{-1} \left((-1)^n 2^n \sum_{\lambda \vdash n} C_\lambda(-2\gamma)^{-1} f^{2\lambda} \omega_\mu^\lambda \omega_\rho^\lambda \right) \mathbf{p}_\rho(\sigma^{-1}). \end{aligned}$$

Proof. They follow from Theorem 4 and (6.7). \square

Corollary 6. *Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and let γ be as in Theorem 2. Then*

$$\begin{aligned} \mathbb{E}[(\text{tr } W)^n] &= \sum_{\rho \vdash n} \frac{n!}{z_\rho} \beta^{\ell(\rho)} \mathbf{p}_\rho(\sigma), \\ \mathbb{E}[(\text{tr } W^{-1})^n] &= \sum_{\rho \vdash n} 2^{n-\ell(\rho)} \frac{n!}{z_\rho} \widetilde{\text{Wg}}(\rho; \gamma) \mathbf{p}_\rho(\sigma^{-1}). \end{aligned}$$

Proof. The first result follows by letting $\mu = (1^n)$ in Corollary 5 and by using the second formula in (4.9). The second one also follows by (4.10). \square

6.3 Examples for low degrees

If we apply our theorems (Theorem 1 – Corollary 6), we can compute various Wishart moments for low degrees $n = 1, 2, 3, 4$ easily; see Appendix.

Acknowledgements

I would like to thank Piotr Graczyk for getting me interested in Wishart distributions on May 2009, and thank Hideyuki Ishi, who organized the meeting where I met P. Graczyk. I also thank Yasuhide Numata for his talk on noncentral Wishart distributions in March 2010.

References

- [C] B. Collins, Moments and cumulants of polynomial random variables on unitary groups, the Itzykson–Zuber integral, and free probability, *Int. Math. Res. Not.* (2003), no. 17, 953–982.
- [CM] B. Collins and S. Matsumoto, On some properties of orthogonal Weingarten functions, *J. Math. Phys.* **50** (2009), 113516, 14 pp.
- [CS] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic group, *Comm. Math. Phys.* **264** (2006), no. 3, 773–795.
- [GLM1] P. Graczyk, G. Letac, and H. Massam, The complex Wishart distribution and the symmetric group, *Ann. Statist.* **31** (2003), no. 1, 287–309.
- [GLM2] P. Graczyk, G. Letac, and H. Massam, The hyperoctahedral groups, symmetric group representations and the moments of the real Wishart distribution, *J. Theoret. Probab.* **18** (2005), no. 1, 1–42.
- [KN1] S. Kuriki and Y. Numata, Graph presentations for moments of noncentral Wishart distributions and their applications, *Ann. Inst. Statist. Math.* **62** (2010), no. 4, 645–672.
- [KN2] S. Kuriki and Y. Numata, On formulas for moments of the Wishart distributions as weighted generating functions of matchings, *DMTCS Proceedings, 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*, pp. 953–964. on line journal, <http://math.sfsu.edu/fpsac/pdfpapers/dmAN0172.pdf>.
- [LM1] G. Letac and H. Massam, All invariant moments of the Wishart distribution, *Scand. J. Statist.* **31** (2004), no. 2, 295–318.
- [LM2] G. Letac and H. Massam, The noncentral Wishart as an exponential family and its moments, *J. Multivariate Analysis* **99** (2008), 1393–1417.
- [LR] I-Li Lu and D. St. P. Richards, MacMahon’s master theorem, representation theory, and moments of Wishart distributions, *Adv. in Appl. Math.* **27** (2001), no. 2-3, 531–547.
- [Mac] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford University Press, Oxford, 1995.
- [Mat1] S. Matsumoto, α -Pfaffians, pfaffian point process and shifted Schur measure, *Linear Alg. Appl.* **403** (2005), 369–398.
- [Mat2] S. Matsumoto, Jucys-Murphy elements, orthogonal matrix integrals, and Jack measures, preprint, arXiv:1001.2345v1, 35 pp.
- [MN] S. Matsumoto and J. Novak, Jucys-Murphy elements and unitary matrix integrals, preprint, arXiv:0905.1992v2, 44 pp.

- [Mu] R. J. Muirhead, Aspects of multivariate statistical theory, John Wiley & Sons, Inc., 1982.
- [Sh] T. Shirai, Remarks on the positivity of α -determinants, Kyushu J. Math. **61** (2007), 169–189.
- [Ve] D. Vere-Jones, A generalization of permanents and determinants, Linear Alg. Appl. **111** (1988), 119–124.
- [Vo] D. von Rosen, Moments for the inverted Wishart distribution, Scand. J. Statist. **15** (1988), no. 2, 97–109.
- [Z] P. Zinn-Justin, Jucys–Murphy elements and Weingarten matrices, Lett. Math. Phys. **91** (2010), 119–127.

SHO MATSUMOTO

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN.

E-mail: sho-matsumoto@math.nagoya-u.ac.jp

A Appendix: Examples for low degrees

We give explicit examples of our theorems obtained in the present papers. Let $W \sim W_d(\beta, \sigma; \mathbb{R})$ and set $\gamma = \beta - \frac{d+1}{2}$ as usual. Let m_1, m_2, \dots be $d \times d$ matrices.

Degree 1

Suppose $\gamma > 0$. It follows from Theorem 1 and Theorem 2 that

$$(A.1) \quad \mathbb{E}[W_{ij}] = \beta \sigma_{ij} \quad \text{and} \quad \mathbb{E}[W^{ij}] = \frac{1}{\gamma} \sigma^{ij}$$

for $1 \leq i, j \leq d$. It is immediate to see that

$$(A.2) \quad \mathbb{E}[W] = \beta \sigma, \quad \mathbb{E}[W^{-1}] = \gamma^{-1} \sigma^{-1},$$

$$(A.3) \quad \mathbb{E}[\text{tr}(W m_1)] = \beta \text{tr}(\sigma m_1), \quad \mathbb{E}[\text{tr}(W^{-1} m_1)] = \gamma^{-1} \text{tr}(\sigma^{-1} m_1).$$

Degree 2

Suppose $\gamma > 0$ but $\gamma \neq 1$ (see Remark 5). From (5.2) and Example 1,

$$\begin{aligned} \widetilde{\text{Wg}}(\{\{1, 2\}, \{3, 4\}\}; \gamma) &= \frac{2\gamma - 1}{\gamma(\gamma - 1)(2\gamma + 1)}, \\ \widetilde{\text{Wg}}(\{\{1, 3\}, \{2, 4\}\}; \gamma) &= \widetilde{\text{Wg}}(\{\{1, 4\}, \{2, 3\}\}; \gamma) = \frac{1}{\gamma(\gamma - 1)(2\gamma + 1)}. \end{aligned}$$

It follows from Theorem 1 and Theorem 2 that

$$(A.4) \quad \mathbb{E}[W_{k_1 k_2} W_{k_3 k_4}] = \beta^2 \sigma_{k_1 k_2} \sigma_{k_3 k_4} + \frac{\beta}{2} (\sigma_{k_1 k_3} \sigma_{k_2 k_4} + \sigma_{k_1 k_4} \sigma_{k_2 k_3}),$$

$$(A.5) \quad \mathbb{E}[W^{k_1 k_2} W^{k_3 k_4}] = \frac{1}{\gamma(\gamma - 1)(2\gamma + 1)} \left[(2\gamma - 1) \sigma^{k_1 k_2} \sigma^{k_3 k_4} + \sigma^{k_1 k_3} \sigma^{k_2 k_4} + \sigma^{k_1 k_4} \sigma^{k_2 k_3} \right],$$

for $(k_1, k_2, k_3, k_4) \in [d]^4$.

The average for the (i, j) -entry of W^2 is

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^d W_{ik} W_{kj} \right] &= \beta^2 \sum_{k=1}^d \sigma_{ik} \sigma_{kj} + \frac{\beta}{2} \sum_{k=1}^d (\sigma_{ik} \sigma_{kj} + \sigma_{ij} \sigma_{kk}) \\ &= \left(\beta^2 + \frac{\beta}{2} \right) (\sigma^2)_{ij} + \frac{\beta}{2} (\text{tr} \sigma) \sigma_{ij}, \end{aligned}$$

and the average for the (i, j) -entry of W^{-2} is

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^d W^{ik} W^{kj} \right] &= \frac{1}{\gamma(\gamma - 1)(2\gamma + 1)} \left[(2\gamma - 1) \sum_{k=1}^d \sigma^{ik} \sigma^{kj} + \sum_{k=1}^d (\sigma^{ik} \sigma^{kj} + \sigma^{ij} \sigma^{kk}) \right] \\ &= \frac{1}{\gamma(\gamma - 1)(2\gamma + 1)} (2\gamma (\sigma^{-2})_{ij} + \text{tr}(\sigma^{-1}) \sigma^{ij}). \end{aligned}$$

Therefore

$$(A.6) \quad \mathbb{E}[W^2] = \left(\beta^2 + \frac{\beta}{2} \right) \sigma^2 + \frac{\beta}{2} (\text{tr } \sigma) \sigma,$$

$$(A.7) \quad \mathbb{E}[W^{-2}] = \frac{1}{\gamma(\gamma-1)(2\gamma+1)} (2\gamma\sigma^{-2} + \text{tr } (\sigma^{-1})\sigma).$$

As we saw in Example 2,

$$(A.8) \quad \mathbb{E}[\text{tr } (Wm_1Wm_2)] = \beta^2 \text{tr } (\sigma m_1 \sigma m_2) + \frac{\beta}{2} \text{tr } (\sigma m_1^t \sigma m_2) + \frac{\beta}{2} \text{tr } (\sigma m_1) \text{tr } (\sigma m_2),$$

and in a similar way we have

$$(A.9) \quad \mathbb{E}[\text{tr } (W^{-1}m_1W^{-1}m_2)] = \frac{1}{\gamma(\gamma-1)(2\gamma+1)} \left[(2\gamma-1) \text{tr } (\sigma^{-1}m_1\sigma^{-1}m_2) \right. \\ \left. + \text{tr } (\sigma^{-1}m_1^t\sigma^{-1}m_2) + \text{tr } (\sigma^{-1}m_1) \text{tr } (\sigma^{-1}m_2) \right].$$

Moreover

$$(A.10) \quad \mathbb{E}[\text{tr } (Wm_1) \text{tr } (Wm_2)] = \beta^2 \text{tr } (\sigma m_1) \text{tr } (\sigma m_2) + \frac{\beta}{2} \text{tr } (\sigma m_1 \sigma m_2) + \frac{\beta}{2} \text{tr } (\sigma m_1^t \sigma m_2),$$

$$(A.11) \quad \mathbb{E}[\text{tr } (W^{-1}m_1) \text{tr } (W^{-1}m_2)] = \frac{1}{\gamma(\gamma-1)(2\gamma+1)} \left[(2\gamma-1) \text{tr } (\sigma^{-1}m_1) \text{tr } (\sigma^{-1}m_2) \right. \\ \left. + \text{tr } (\sigma^{-1}m_1\sigma^{-1}m_2) + \text{tr } (\sigma^{-1}m_1^t\sigma^{-1}m_2) \right].$$

Degree 3

Suppose $\gamma > 0$ but $\gamma \neq 1, 2$. From (5.2) and a list in [CM] (see also [CS]), the $\widetilde{Wg}(\rho; \gamma)$ ($\rho \vdash 3$) are given by

$$\widetilde{Wg}((3); \gamma) = \frac{1}{u_3(\gamma)}, \quad \widetilde{Wg}((2, 1); \gamma) = \frac{\gamma-1}{u_3(\gamma)}, \quad \widetilde{Wg}((1^3); \gamma) = \frac{2\gamma^2 - 3\gamma - 1}{u_3(\gamma)},$$

where

$$u_3(\gamma) = \gamma(\gamma-1)(\gamma-2)(\gamma+1)(2\gamma+1).$$

It follows from Theorem 1 and Theorem 2 that

$$(A.12) \quad \mathbb{E}[W_{k_1k_2}W_{k_3k_4}W_{k_5k_6}] \\ = \beta^3 \sigma_{k_1k_2} \sigma_{k_3k_4} \sigma_{k_5k_6} + \frac{\beta^2}{2} (\sigma_{k_1k_3} \sigma_{k_2k_4} \sigma_{k_5k_6} + \sigma_{k_1k_4} \sigma_{k_2k_3} \sigma_{k_5k_6} + \sigma_{k_1k_5} \sigma_{k_2k_6} \sigma_{k_3k_4} \\ + \sigma_{k_1k_6} \sigma_{k_2k_5} \sigma_{k_3k_4} + \sigma_{k_1k_2} \sigma_{k_3k_5} \sigma_{k_4k_6} + \sigma_{k_1k_2} \sigma_{k_3k_6} \sigma_{k_4k_5}) \\ + \frac{\beta}{4} (\sigma_{k_1k_4} \sigma_{k_2k_5} \sigma_{k_3k_6} + \sigma_{k_1k_3} \sigma_{k_2k_5} \sigma_{k_4k_6} + \sigma_{k_1k_4} \sigma_{k_2k_6} \sigma_{k_3k_5} + \sigma_{k_1k_3} \sigma_{k_2k_6} \sigma_{k_4k_5} \\ + \sigma_{k_1k_6} \sigma_{k_2k_3} \sigma_{k_4k_5} + \sigma_{k_1k_5} \sigma_{k_2k_3} \sigma_{k_4k_6} + \sigma_{k_1k_6} \sigma_{k_2k_4} \sigma_{k_3k_5} + \sigma_{k_1k_5} \sigma_{k_2k_4} \sigma_{k_3k_6})$$

and

$$\begin{aligned}
 (A.13) \quad & \mathbb{E}[W^{k_1 k_2} W^{k_3 k_4} W^{k_5 k_6}] \\
 &= u_3(\gamma)^{-1} \left[(2\gamma^2 - 3\gamma - 1) \sigma^{k_1 k_2} \sigma^{k_3 k_4} \sigma^{k_5 k_6} \right. \\
 &\quad + (\gamma - 1) (\sigma^{k_1 k_3} \sigma^{k_2 k_4} \sigma^{k_5 k_6} + \sigma^{k_1 k_4} \sigma^{k_2 k_3} \sigma^{k_5 k_6} + \sigma^{k_1 k_5} \sigma^{k_2 k_6} \sigma^{k_3 k_4} \\
 &\quad + \sigma^{k_1 k_6} \sigma^{k_2 k_5} \sigma^{k_3 k_4} + \sigma^{k_1 k_2} \sigma^{k_3 k_5} \sigma^{k_4 k_6} + \sigma^{k_1 k_2} \sigma^{k_3 k_6} \sigma^{k_4 k_5}) \\
 &\quad + (\sigma^{k_1 k_4} \sigma^{k_2 k_5} \sigma^{k_3 k_6} + \sigma^{k_1 k_3} \sigma^{k_2 k_5} \sigma^{k_4 k_6} + \sigma^{k_1 k_4} \sigma^{k_2 k_6} \sigma^{k_3 k_5} + \sigma^{k_1 k_3} \sigma^{k_2 k_6} \sigma^{k_4 k_5} \\
 &\quad \left. + \sigma^{k_1 k_6} \sigma^{k_2 k_3} \sigma^{k_4 k_5} + \sigma^{k_1 k_5} \sigma^{k_2 k_3} \sigma^{k_4 k_6} + \sigma^{k_1 k_6} \sigma^{k_2 k_4} \sigma^{k_3 k_5} + \sigma^{k_1 k_5} \sigma^{k_2 k_4} \sigma^{k_3 k_6}) \right].
 \end{aligned}$$

From Corollary 5 we have

$$(A.14) \quad \mathbb{E}[\mathbf{p}_\mu(W)] = \frac{16}{5} \left(\frac{1}{6} A(\mu, (3)) \mathbf{p}_{(3)}(\sigma) + \frac{1}{8} A(\mu, (2, 1)) \mathbf{p}_{(2,1)}(\sigma) + \frac{1}{48} A(\mu, (1^3)) \mathbf{p}_{(1^3)}(\sigma) \right),$$

$$(A.15) \quad \mathbb{E}[\mathbf{p}_\mu(W^{-1})] = \frac{16}{5} \left(\frac{1}{6} B(\mu, (3)) \mathbf{p}_{(3)}(\sigma^{-1}) + \frac{1}{8} B(\mu, (2, 1)) \mathbf{p}_{(2,1)}(\sigma^{-1}) + \frac{1}{48} B(\mu, (1^3)) \mathbf{p}_{(1^3)}(\sigma^{-1}) \right),$$

for each $\mu \vdash 3$, where

$$A(\mu, \rho) = \frac{1}{8} \sum_{\lambda \vdash 3} C_\lambda(2\beta) f^{2\lambda} \omega_\mu^\lambda \omega_\rho^\lambda \quad \text{and} \quad B(\mu, \rho) = -8 \sum_{\lambda \vdash 3} C_\lambda(-2\gamma)^{-1} f^{2\lambda} \omega_\mu^\lambda \omega_\rho^\lambda.$$

We compute the matrices $A = (A(\mu, \rho))_{\mu, \rho \vdash 3}$ and $B = (B(\mu, \rho))_{\mu, \rho \vdash 3}$. Here indices of rows and columns of the matrices are labeled by (3) , $(2, 1)$, (1^3) in order. By using results in [Mac, VII.2], we have

$$Z := (\omega_\mu^\lambda)_{\lambda, \mu \vdash 3} = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{1}{4} & \frac{1}{6} & 1 \\ \frac{1}{4} & -\frac{1}{2} & 1 \end{pmatrix}.$$

Since $f^{2\lambda}$ coincides with the number of standard Young tableaux of shape 2λ (see, e.g., [Sa]²), we may have

$$f^{2(3)} = f^{(6)} = 1, \quad f^{2(2,1)} = f^{(4,2)} = 9, \quad \text{and} \quad f^{2(1^3)} = f^{(2^3)} = 5.$$

From the definition of $C_\lambda(z)$, it is immediate to see

$$C_{(3)}(z) = z(z+2)(z+4), \quad C_{(2,1)}(z) = z(z+2)(z-1), \quad \text{and} \quad C_{(1^3)}(z) = z(z-1)(z-2).$$

Now, letting $F := \text{diag}(f^{2(3)}, f^{2(2,1)}, f^{2(1^3)})$ and $C(z) := \text{diag}(C_{(3)}(z), C_{(2,1)}(z), C_{(1^3)}(z))$, we can calculate

$$A = \frac{1}{8} Z^t \cdot F \cdot C(2\beta) \cdot Z = \begin{pmatrix} \frac{15}{16}\beta(2\beta^2 + 3\beta + 2) & \frac{15}{8}\beta(2\beta + 1) & \frac{15}{4}\beta \\ \frac{15}{8}\beta(2\beta + 1) & \frac{5}{4}\beta(2\beta^2 + \beta + 2) & \frac{15}{2}\beta^2 \\ \frac{15}{4}\beta & \frac{15}{2}\beta^2 & 15\beta^3 \end{pmatrix},$$

²[Sa] B. E. Sagan, The symmetric group. Representations, combinatorial algorithms, and symmetric functions, second ed., Graduate Texts in Mathematics, **203**. Springer-Verlag, New York, 2001.

and

$$B = -8Z^t \cdot F \cdot C(-2\gamma)^{-1} \cdot Z = \frac{1}{u_3(\gamma)} \begin{pmatrix} \frac{15}{4}\gamma^2 & \frac{15}{2}\gamma & 15 \\ \frac{15}{2}\gamma & 5(\gamma^2 - \gamma + 1) & 15(\gamma - 1) \\ 15 & 15(\gamma - 1) & 15(2\gamma^2 - 3\gamma - 1) \end{pmatrix}.$$

Hence

$$(A.16) \quad \mathbb{E}[\mathbf{p}_{(3)}(W)] = \frac{1}{2}\beta(2\beta^2 + 3\beta + 2)\mathbf{p}_{(3)}(\sigma) + \frac{3}{4}\beta(2\beta + 1)\mathbf{p}_{(2,1)}(\sigma) + \frac{1}{4}\beta\mathbf{p}_{(1^3)}(\sigma),$$

$$(A.17) \quad \mathbb{E}[\mathbf{p}_{(2,1)}(W)] = \beta(2\beta + 1)\mathbf{p}_{(3)}(\sigma) + \frac{1}{2}\beta(2\beta^2 + \beta + 2)\mathbf{p}_{(2,1)}(\sigma) + \frac{1}{2}\beta^2\mathbf{p}_{(1^3)}(\sigma),$$

$$(A.18) \quad \mathbb{E}[\mathbf{p}_{(1^3)}(W)] = 2\beta\mathbf{p}_{(3)}(\sigma) + 3\beta^2\mathbf{p}_{(2,1)}(\sigma) + \beta^3\mathbf{p}_{(1^3)}(\sigma),$$

and

$$(A.19) \quad \mathbb{E}[\mathbf{p}_{(3)}(W^{-1})] = \frac{2\gamma^2\mathbf{p}_{(3)}(\sigma^{-1}) + 3\gamma\mathbf{p}_{(2,1)}(\sigma^{-1}) + \mathbf{p}_{(1^3)}(\sigma^{-1})}{\gamma(\gamma - 1)(\gamma - 2)(\gamma + 1)(2\gamma + 1)},$$

$$(A.20) \quad \mathbb{E}[\mathbf{p}_{(2,1)}(W^{-1})] = \frac{4\gamma\mathbf{p}_{(3)}(\sigma^{-1}) + 2(\gamma^2 - \gamma + 1)\mathbf{p}_{(2,1)}(\sigma^{-1}) + (\gamma - 1)\mathbf{p}_{(1^3)}(\sigma^{-1})}{\gamma(\gamma - 1)(\gamma - 2)(\gamma + 1)(2\gamma + 1)},$$

$$(A.21) \quad \mathbb{E}[\mathbf{p}_{(1^3)}(W^{-1})] = \frac{8\mathbf{p}_{(3)}(\sigma^{-1}) + 6(\gamma - 1)\mathbf{p}_{(2,1)}(\sigma^{-1}) + (2\gamma^2 - 3\gamma - 1)\mathbf{p}_{(1^3)}(\sigma^{-1})}{\gamma(\gamma - 1)(\gamma - 2)(\gamma + 1)(2\gamma + 1)}.$$

We remark that those formulas for $\mathbb{E}[\mathbf{p}_\mu(W)]$ ($\mu \vdash 3$) are seen in [LM1, equation (37)].

Degree 4 and higher degrees

First we note that, when $n = 4$, the sums in Theorem 1, 2 and 3 are over $|\mathcal{M}(8)| = 7 \cdot 5 \cdot 3 \cdot 1 = 105$ terms.

Consider Corollary 5 for any degree n . As we did in the degree 3 case, we can apply it to any degree n . The $f^{2\lambda}$ may be computed by the well-known hook formula, see e.g. [Sa, Theorem 3.10.2], and the $C_\lambda(z)$ may be done easily by the definition (4.8). The ω^λ are the most complicated among quantities appearing in Corollary 5 but we can know their explicit values from the table of zonal polynomials in [PJ]³.

In closing, we give the explicit expressions of Corollary 6 for $n = 4$. Its first formula is given

$$(A.22) \quad \mathbb{E}[(\text{tr } W)^4] = 6\beta\mathbf{p}_{(4)}(\sigma) + 8\beta^2\mathbf{p}_{(3,1)}(\sigma) + 3\beta^2\mathbf{p}_{(2^2)}(\sigma) + 6\beta^3\mathbf{p}_{(2,1^2)}(\sigma) + \beta^4\mathbf{p}_{(1^4)}(\sigma).$$

Suppose $\gamma > 0$ but $\gamma \neq \frac{1}{2}, 1, 2, 3$. Put

$$u_4(\gamma) = \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)(2\gamma - 1)(\gamma + 1)(2\gamma + 1)(2\gamma + 3),$$

³ [PJ] A. M. Parkhurst and A. T. James, Zonal polynomials of order 1 through 12, Selected Tables in Mathematical Statistics (1974), vol. 2, 199–388.

which is non-zero. From (5.2) and a list in [CM] (see also [CS]), we have the explicit values

$$\begin{aligned}\widetilde{\text{Wg}}((4); \gamma) &= \frac{5\gamma - 3}{u_4(\gamma)}, & \widetilde{\text{Wg}}((3, 1); \gamma) &= \frac{4\gamma(\gamma - 2)}{u_4(\gamma)}, \\ \widetilde{\text{Wg}}((2^2); \gamma) &= \frac{2\gamma^2 - 5\gamma + 9}{u_4(\gamma)}, & \widetilde{\text{Wg}}((2, 1^2); \gamma) &= \frac{4\gamma^3 - 12\gamma^2 + 3\gamma + 3}{u_4(\gamma)}, \\ \widetilde{\text{Wg}}((1^4); \gamma) &= \frac{(\gamma + 1)(2\gamma - 3)(4\gamma^2 - 12\gamma + 1)}{u_4(\gamma)}.\end{aligned}$$

Hence the second formula of Corollary 6 at $n = 4$ is given

$$\begin{aligned}(\text{A.23}) \quad u_4(\gamma) \cdot \mathbb{E}[(\text{tr } W^{-1})^4] &= 48(5\gamma - 3)\mathbf{p}_{(4)}(\sigma^{-1}) + 128\gamma(\gamma - 2)\mathbf{p}_{(3,1)}(\sigma^{-1}) \\ &\quad + 12(2\gamma^2 - 5\gamma + 9)\mathbf{p}_{(2^2)}(\sigma^{-1}) \\ &\quad + 12(4\gamma^3 - 12\gamma^2 + 3\gamma + 3)\mathbf{p}_{(2,1^2)}(\sigma^{-1}) \\ &\quad + (\gamma + 1)(2\gamma - 3)(4\gamma^2 - 12\gamma + 1)\mathbf{p}_{(1^4)}(\sigma^{-1}).\end{aligned}$$