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# A Study for the Moment of Wishart Distribution

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**Abstract**

The skewness of a matrix quadratic form $XX'$ is obtained using the expectation of stochastic matrix and applying the properties of commutation matrices, where $X \sim \mathrm{N}_{p,n}(0, \Sigma, I_n)$.

**Keywords:** Commutation matrix, Moment, Wishart distribution

## 1 Introduction

Let $p \times n$ stochastic matrix $X$ be distributed as $\mathrm{N}_{p,n}(0, \Sigma, \Phi)$, where $\Sigma : p \times p$ and $\Phi : n \times n$ are positive semidefinite, and $\mathrm{E}(X) = 0$. Then, the variance-covariance matrix of the vectorization of $X$ is $\mathrm{Cov}(\mathrm{vec}X, \mathrm{vec}X) = \Phi \otimes \Sigma$. For $\Phi = I_n$, $XX'$ is said to have Wishart distribution with scale matrix $\Sigma$ and degrees of freedom parameter $n$, where $I_n$ is an $n \times n$ identity matrix. Von Rosen (1988) has obtained moments of arbitrary order of matrix $X$, and using these has calculated the second order moment of quadratic form $XAX'$, where $A : n \times n$ is an arbitrary non-stochastic matrix. Neudecker and Wansbeek (1987) have also obtained the second order moment of $XAX'$ by calculating the expectation of $YAY'CYBY'$, where $Y = X + M$ is the matrix of means, and $A$, $B$ and $C$ are $n \times n$ arbitrary non-stochastic matrices. Tracy and Sultan (1993) obtained the third order moment of $XAX'$ using the sixth moment of matrix $X$. Kang and Kim (1996a) derived the vectorization of the general moment of $XAX'$. Also Kang and Kim (1996b) derived the vectorization of the general moment of non-central Wishart distribution, using the vectorization of the general moment of $XAX'$. We obtain the skewness of $XX'$ using the third moment of $XX'$.

Commutation matrices play a major role here. These and some useful results are presented in Section 2. The skewness of matrix quadratic form $XX'$ is obtained in Section 3.

# 2 Some Useful Results

**Definition 2.1** *Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of $A$ and $B$ is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

**Definition 2.2** *Let $A$ be an $m \times n$ matrix and $a_i$ the $i$th column of $A$ ; then* vec$A$ *is the $mn$ column vector*

$$\text{vec}A = [a_1', a_2', \cdots, a_n']'.$$

**Definition 2.3** *Commutation matrix $I_{m,n}$ is an $mn \times mn$ matrix containing $mn$ blocks of order $m \times n$ such that $(ij)$th blcok has a 1 in its $(ji)$th position and zeroes elsewhere. One has*

$$I_{m,n} = \sum_{i=1}^{n} \sum_{j=1}^{m} (H_{i,j} \otimes H_{i,j}')$$

*where $H_{i,j}$ is an $n \times m$ matrix with a 1 in its $(ij)$th position and zeroes elsewhere, and can be written as $H_{i,j} = e_i e_j'$, where $e_i$ is the $i$th unit column vector of order $n$.*

For an $m \times m$ identity matrix $I_m$ and $n \times n$ identity matrix $I_n$,

$$I_m \otimes I_n = I_{mn}, \tag{1}$$

$$I_{m,n}^{-1} = I_{m,n}' = I_{n,m}, \tag{2}$$

and

$$\begin{aligned}&(I_n \otimes I_{n,n} \otimes I_n)\,(I_n \otimes I_n \otimes I_{n,n})\,(I_n \otimes I_{n,n} \otimes I_n)\\ &\qquad = (I_n \otimes I_n \otimes I_{n,n})\,(I_n \otimes I_{n,n} \otimes I_n)\,(I_n \otimes I_n \otimes I_{n,n})\,.\end{aligned}$$

For an $m \times n$ matrix $A$, $I_{n,m}\mathrm{vec}A = \mathrm{vec}A'$,

$$(A \otimes B)' = A' \otimes B'.$$

For $A$, $B$, $C$, and $D$ conformable matrices,

$$(A \otimes B) \otimes (C \otimes D) = (A \otimes B) + (A \otimes D) + (B \otimes C) + (B \otimes D)\,.$$

For $A$ and $B$ conformable matrices,

$$\mathrm{vec}'A'\mathrm{vec}B = \mathrm{tr}\,(AB)\,.$$

For an $m \times n$ matrix $A$ and a $p \times q$ matrix $B$,

$$\mathrm{vec}\,(A \otimes B) = (I_n \otimes I_{m,q} \otimes I_p)\,(\mathrm{vec}A \otimes \mathrm{vec}B)\,.$$

For $A$, $B$, and $C$ conformable matrices,

$$\mathrm{vec}(ACB) = (B' \otimes A)\,\mathrm{vec}C.$$

For $A$, $B$, $C$, and $D$ conformable matrices,

$$(AB) \otimes (CD) = (A \otimes C)\,(B \otimes D)\,. \tag{3}$$

For an $m \times n$ matrix $A$ and a $p \times q$ matrix $B$,

$$I_{m,p}\,(A \otimes B) = (B \otimes A)\,I_{n,q}. \tag{4}$$

For an $m \times n$ matrix $A$ and a $p \times q$ matrix $B$, and an $r \times s$ matrix $C$,

$$A \otimes B \otimes C = I_{r,mp}(C \otimes A \otimes B)I_{qm,s} = I_{pr,s}(B \otimes C \otimes A)I_{n,sq}. \tag{5}$$

For an $m \times n$ matrix $A$ and a $p \times q$ matrix $B$, and an $r \times s$ matrix $C$,

$$\begin{aligned}&\mathrm{vec}(A \otimes B \otimes C)\\ &\quad = (I_m \otimes I_{m,qs} \otimes I_{pr})\,(I_{mnq} \otimes I_{p,s} \otimes I_r)\,(\mathrm{vec}A \otimes \mathrm{vec}B \otimes \mathrm{vec}C)\,.\end{aligned}$$

Balestra (1976) and Neudecker and Wansbeek (1983) discussed higher order commutation matrices, for which

$$I_{ab,c} = (I_{a,c} \otimes I_b)\,(I_a \otimes I_{b,c}) = (I_{b,c} \otimes I_a)\,(I_b \otimes I_{a,c})\,, \tag{6}$$

$$I_{a,bc} = (I_b \otimes I_{a,c})\,(I_{a,b} \otimes I_c) = (I_c \otimes I_{a,b})\,(I_{a,c} \otimes I_b)\,, \tag{7}$$

$$I_{n,n^2} I_{n,n^2} = I_{n^2,n} \quad \text{and} \quad I_{n^2,n} I_{n^2,n} = I_{n,n^2}. \tag{8}$$

When $A = I_n = \Phi$, $\mathrm{E}(XX') = n\Sigma$ where $\mathrm{E}(XX')$ is the first moment of $XX'$ according to Magnus and Neudecker (1979).

Von Resen (1988) obtained the second moment of $XX'$,

$$\mathrm{E}\left(\otimes^2 (XX')\right) = n(\mathrm{vec}\Sigma \mathrm{vec}'\Sigma) + n^2 \left(\otimes^2 \Sigma\right) + n I_{p,p} \left(\otimes^2 \Sigma\right).$$

# 3 Skewness of Wishart Distribution

**Notation**

$$\begin{aligned} &P(k,l;m) = I_{p^k,p^l} \otimes I_{p^m}, \\ &P(k;l,m) = I_{p^k} \otimes I_{p^l,p^m}, \\ &V = (\mathrm{vec}\Sigma \mathrm{vec}'\Sigma) \otimes \Sigma, \\ &\text{and} \quad S_X = XX'. \end{aligned}$$

Tracy and Sultan(1993) gave the third moment of matrix quadratic form $XX'$, in square matrix form,

$$\begin{aligned} &\mathrm{E}\left(\otimes^3 S_X\right) \\ &= n^3 \left(\otimes^3 \Sigma\right) + n^2 V + n^2 Q \left(\otimes^3 \Sigma\right) + nQPVP + nPQ\left(\otimes^3 \Sigma\right) + n^2 QPVPQ \\ &\quad + nQPV + n^2 QP\left(\otimes^3 \Sigma\right) + nVP + nVPQ + n^2 P\left(\otimes^3 \Sigma\right) + nPV + nPVPQ \\ &\quad + n^2 PVP + n^2 QPQ\left(\otimes^3 \Sigma\right). \end{aligned}$$

In fact, $XX'$ is distributed as Wishart distribution.

**Lemma 3.1** *Let $P = P(1;1,1)$, $Q = P(1,1;1)$. Then*
*(a) $P^2 = I_{p^3} \quad Q^2 = I_{p^3}$.*
*(b) $P^{-1} = P, \quad Q^{-1} = Q$.*
*(c) $PQPQ = QP, \quad QPQP = PQ$.*

*Proof.* (a) Using (1) and (2), we get $Q^{-1} = Q$.

$$P^2 = (I_p \otimes I_{p,p})(I_p \otimes I_{p,p}) = I_p \otimes I_{p^2} = I_{p^3}.$$

Similarly, we get $Q^2 = I_{p^3}$.
(b) They are trivial because of (a) in lemma 3.1.
(c) Using (6), (7), and (8), we can derive

$$\begin{aligned} PQPQ &= (I_p \otimes I_{p,p})(I_{p,p} \otimes I_p)(I_p \otimes I_{p,p})(I_{p,p} \otimes I_p) \\ &= I_{p,p^2} I_{p,p^2} = I_{p^2,p} = QP \end{aligned}$$

and

$$QPQP = (I_{p,p} \otimes I_p)(I_p \otimes I_{p,p})(I_{p,p} \otimes I_p)(I_p \otimes I_{p,p}) = I_{p^2,p} I_{p^2,p} = I_{p,p^2} = PQ.$$

**Lemma 3.2** $QPQ\left(\otimes^3\Sigma\right) = P\left(\otimes^3\Sigma\right)QP$

*Proof.* Let $L = QPQ\left(\otimes^3\Sigma\right)$. Then

$$PLPQ = P\left(QPQ\left(\otimes^3\Sigma\right)\right)PQ = QP\left(\otimes^3\Sigma\right)PQ = \otimes^3\Sigma$$

by (5) and (c) in lemma 3.1.
Using $P = P^{-1}$ and $QP = (PQ)^{-1}$, $L = P\left(\otimes^3\Sigma\right)QP$.

**Lemma 3.3** $PQ\left(\otimes^3\Sigma\right)QP = QP\left(\otimes^3\Sigma\right)PQ = \otimes^3\Sigma$

*Proof.* They are trivial using (4) and (5).

**Lemma 3.4** $QPQ\left(\otimes^3\Sigma\right)PQ = P\left(\otimes^3\Sigma\right)$.

*Proof.* Let $N = QPQ\left(\otimes^3\Sigma\right)PQ$. Then

$$PN = PQPQ\left(\otimes^3\Sigma\right)PQ = QP\left(\otimes^3\Sigma\right)PQ = \otimes^3\Sigma$$

by (5) and (c) in lemma 3.1.
Using $P = P^{-1}$, $N = P^{-1}\left(\otimes^3\Sigma\right) = P\left(\otimes^3\Sigma\right)$.

**Theorem 3.5** *Let a $p \times n$ random matrix $X$ be distributed as* $\mathrm{N}_{p,n}\left(0, \Sigma, \Phi\right)$. *Then* $\mathrm{E}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right) = n^2V + n^3\left(\otimes^3\Sigma\right) + n^2Q\left(\otimes^3\Sigma\right)$.

*Proof.*
Since $\mathrm{E}S_X = n\Sigma$ and $\mathrm{E}\left(\otimes^2 S_X\right) = n(\mathrm{vec}\Sigma\mathrm{vec}'\Sigma) + n^2\left(\otimes^2\Sigma\right) + nI_{p,p}\left(\otimes^2\Sigma\right)$,

$$\begin{aligned}
\mathrm{E}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right) &= \mathrm{E}\left(\otimes^2 S_X\right) \otimes \mathrm{E}S_X \\
&= \left[n\mathrm{vec}\Sigma\mathrm{vec}'\Sigma + n^2\left(\otimes^2\Sigma\right) + nI_{p,p}\left(\otimes^2\Sigma\right)\right] \otimes n\Sigma \\
&= n^2\left[(\mathrm{vec}\Sigma\mathrm{vec}'\Sigma) \otimes \Sigma\right] + n^3\left(\otimes^3\Sigma\right) + n^2\left[I_{p,p}\left(\otimes^2\Sigma\right) \otimes I_p\Sigma\right] \\
&= n^2V + n^3\left(\otimes^3\Sigma\right) + n^2Q\left(\otimes^3\Sigma\right)
\end{aligned}$$

using (4).

**Theorem 3.6** *Let a $p \times n$ random matrix $X$ be distributed as* $\mathrm{N}_{p,n}\left(0, \Sigma, \Phi\right)$. *Then* $\mathrm{E}\left(S_X \otimes \mathrm{E}S_X \otimes S_X\right) = n^2PQVQP + n^3\left(\otimes^3\Sigma\right) + n^2P\left(\otimes^3\Sigma\right)QP$.

*Proof.* Using (5), lemma 3.2, and lemma 3.3,

$$\begin{aligned}
\mathrm{E}\left(S_X \otimes \mathrm{E}S_X \otimes S_X\right) &= \mathrm{E}\left[I_{p,p^2}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right)I_{p^2,p}\right] \\
&= I_{p,p^2}\mathrm{E}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right)I_{p^2,p} = PQ\mathrm{E}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right)QP \\
&= PQ\left[n^2V + n^3\left(\otimes^3\Sigma\right) + n^2Q\left(\otimes^3\Sigma\right)\right]QP \\
&= n^2PQVQP + n^3PQ\left(\otimes^3\Sigma\right)QP + n^2P\left(\otimes^3\Sigma\right)QP \\
&= n^2PQVQP + n^3\left(\otimes^3\Sigma\right) + n^2P\left(\otimes^3\Sigma\right)QP.
\end{aligned}$$

**Theorem 3.7** *Let a $p \times n$ random matrix $X$ be distributed as* $\mathrm{N}_{p,n}\left(0, \Sigma, \Phi\right)$. *Then* $\mathrm{E}\left(\mathrm{E}S_X \otimes S_X \otimes S_X\right) = n^2QPVPQ + n^3\left(\otimes^3\Sigma\right) + n^2P\left(\otimes^3\Sigma\right)$.

*Proof*. Using (5), lemma 3.3, and lemma 3.4,

$$\begin{aligned}
&\mathrm{E}\left(S_X \otimes \mathrm{E}S_X \otimes S_X\right) = \mathrm{E}\left[I_{p^2,p}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right) I_{p,p^2}\right] \\
&\quad = QP\mathrm{E}\left(S_X \otimes S_X \otimes \mathrm{E}S_X\right) PQ \\
&\quad = QP\left[n^2V + n^3\left(\otimes^3\Sigma\right) + n^2Q\left(\otimes^3\Sigma\right)\right] PQ \\
&\quad = n^2QPVPQ + n^3QP\left(\otimes^3\Sigma\right) PQ + n^2QPQ\left(\otimes^3\Sigma\right) PQ \\
&\quad = n^2QPVPQ + n^3\left(\otimes^3\Sigma\right) + n^2P\left(\otimes^3\Sigma\right).
\end{aligned}$$

Here is the key result of this paper.

**Theorem 3.8** $\mathrm{E}\left[\otimes^3\left(S_X - \mathrm{E}S_X\right)\right]$

$$\begin{aligned}
&= n\left[PV + VP + QPV + VPQ + PVPQ + QPVP + PQ\left(\otimes^3\Sigma\right)\right] \\
&\quad + n^2\left[PVP - PQVQP + QP\left(\otimes^3\Sigma\right)\right] - 2n^3\left(\otimes^3\Sigma\right).
\end{aligned}$$

*Proof*.

$$\begin{aligned}
&\mathrm{E}\left[\otimes^3\left(S_X - \mathrm{E}S_X\right)\right] \\
&= \mathrm{E}\left[\left(\otimes^3 S_X\right) + \left(\mathrm{E}S_X \otimes S_X \otimes \mathrm{E}S_X\right) + \left(\mathrm{E}S_X \otimes \mathrm{E}S_X \otimes S_X\right) + \left(S_X \otimes \mathrm{E}S_X \otimes \mathrm{E}S_X\right)\right] \\
&-\mathrm{E}\left[\left(\mathrm{E}S_X \otimes S_X \otimes S_X\right) + \left(S_X \otimes S_X \otimes \mathrm{E}S_X\right) + \left(S_X \otimes \mathrm{E}S_X \otimes S_X\right) + \left(\otimes^3\mathrm{E}S_X\right)\right] \\
&= \mathrm{E}\left(\otimes^3 S_X\right) - \mathrm{E}(S_X \otimes S_X \otimes \mathrm{E}S_X) - \mathrm{E}(S_X \otimes \mathrm{E}S_X \otimes S_X) \\
&-\mathrm{E}(\mathrm{E}S_X \otimes S_X \otimes S_X) + \left(\otimes^3\mathrm{E}S_X\right).
\end{aligned}$$

Using theorem 3.6, theorem 3.7, and theorem 3.8,

$$\begin{aligned}
&\mathrm{E}\left[\otimes^3\left(S_X - \mathrm{E}S_X\right)\right] \\
&\quad = n\left[PV + VP + QPV + VPQ + PVPQ + QPVP + PQ\left(\otimes^3\Sigma\right)\right] \\
&\quad + n^2\left[PVP - PQVQP + QP\left(\otimes^3\Sigma\right)\right] - 2n^3\left(\otimes^3\Sigma\right).
\end{aligned}$$

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