Problem 1

a) The only system that is not in SS form is system 2

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -cx_2 - g\left(1 - \left(\frac{x_d}{x_1}\right)^k\right) \end{cases}$$

b) 1) $\dot{x}_1 = 0 \Rightarrow au_1 = b\sqrt{x_1} \Rightarrow x_1 = \left(\frac{au_1}{b}\right)^2$

$\dot{x}_2 = 0 \Rightarrow u_1(u_2 - x_2) + c(u_3 - x_2) = 0$

$\Rightarrow u_1u_2 + cu_3 = u_1x_2 + cx_2$

$\Rightarrow x_2 = \dfrac{u_1u_2 + cu_3}{u_1 + c}$

$$\Rightarrow x^* = \begin{bmatrix} \left(\frac{au_1}{b}\right)^2 \\ \frac{u_1u_2 + cu_3}{u_1 c} \end{bmatrix}, \quad u^* = u$$

2) $\dot{x}_1 = 0 \Rightarrow x_2 = 0$

$\dot{x}_2 = 0 \Rightarrow 1 = \left(x_d/x_1\right)^k \Rightarrow x_1 = x_d$

$$\Rightarrow x^* = \begin{bmatrix} x_d \\ 0 \end{bmatrix}, \quad u^* = u$$

b) 3) $\dot{x} = 0 \Rightarrow [x\ y] = [0\ 0]$

or $\ln(\sqrt{x^2+y^2})\,y = x$ , $\sqrt{x^2+y^2} \neq 1$

In polar coordinates we get

$x = r \cdot \cos(\alpha)$

$y = r \cdot \sin(\alpha)$

$\Rightarrow \ln(r) \cdot \sin(\alpha) \cdot r = \cos(\alpha) \cdot r$ , $r \neq 1$

$$\Rightarrow \ln(r) = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)}$$

$\dot{y} = 0$

$$\Rightarrow \ln(r) = \frac{\sin(\alpha)}{-\cos(\alpha)} = -\tan(\alpha) \quad , \quad r \neq 1$$

finally

$$\Rightarrow \frac{1}{\tan(\alpha)} = -\tan(\alpha) \Rightarrow 1 = -\tan^2(\alpha)$$

This is clearly not possible, so the only solution is

$[x\ y] = [0\ 0]$

c) I used Python and the sympy library to solve this task. See the attachment for source code.

1)
$$A = \begin{bmatrix} -\frac{b^2}{2au_1} & 0 \\ 0 & -\frac{b^2}{au_1^2}(u_1+c) \end{bmatrix}$$

$$B = \begin{bmatrix} a & 0 & 0 \\ \frac{b^2c(u_2-u_3)}{au_1^2(c+u_1)} & \frac{b^2}{au_1} & \frac{b^2c}{au_1^2} \end{bmatrix}$$

2)
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{gk}{x_d} & -c \end{bmatrix}, \quad B = 0$$

3) $r = \sqrt{x^2+y^2}$

$$\frac{\partial \dot{x}}{\partial x} = \frac{x^2}{r^2 \cdot \ln^2(r)} - \frac{1}{\ln(r)}$$

$$\lim_{x,y \to [0,0]} \frac{\partial \dot{x}}{\partial x} = \frac{0}{-\infty} - \frac{1}{-\infty} = 0$$

$$\frac{\partial \dot{x}}{\partial y} = \frac{xy}{r^2 \ln^2(r)} + 1$$

$$\lim \frac{\partial \dot{x}}{\partial y} = 1$$

$$\frac{\partial \dot{y}}{\partial x} = \frac{xy}{r^2 \ln^2(r)} - 1$$

$$\lim \frac{\partial \dot{y}}{\partial x} = -1$$

$$\frac{\partial \dot{y}}{\partial y} = \frac{y^2}{r^2 \cdot \ln^2(r)} - \frac{1}{\ln(r)}$$

$$\lim \frac{\partial \dot{y}}{\partial y} = 0$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

d) 1) A is diagonal with negative entrys, so the ~~system~~ linearized system is asymptotically stable.

2) $\lambda_1 = -\frac{c}{2} - \frac{\sqrt{c^2 x_d - 4gk}}{2\sqrt{x_d}}$

$\lambda_2 = -\frac{c}{2} + \frac{\sqrt{c^2 x_d - 4gk}}{2\sqrt{x_d}}$

The linearized system is stable if and only if

$c \cdot \sqrt{x_d} \geq \sqrt{c^2 x_d - 4gk}$

$\Rightarrow \quad c^2 x_d \geq c^2 x_d - 4gk$

$\Rightarrow \quad 0 \geq -4gk$

which is always true, as $g, k > 0$
Hence the linearized system is asymtotically stable.

3) The eigenvalues are $\pm i$, hence the ~~syst~~ linearized system is only stable.

Problem 2

a) The first simulation goes toward $\infty$ when $t$ goes toward 1. It throws an error: "unable to meet integration tolerances..."

The second simulation is 0 for all $t$.

b) The first ODE is given by $\dot{x} = f(x) = x^2$. $f(x)$ is not globally Lipshitz, so no global solution exists. A local solution can be found ~~by~~ using separation of variables.

$$\frac{dx}{dt} = x^2$$

$$\Rightarrow \int_{x(t_0)}^{x(t)} \frac{1}{x^2} dx = \int_{t_0}^{t} dt$$

$$\Rightarrow \left[ -\frac{1}{x} \right]_{x=x(t_0)}^{x(t)} = \left[ t \right]_{t=t_0}^{t=t}$$

$$\Rightarrow \frac{1}{x(t_0)} - \frac{1}{x(t)} = t - t_0$$

$$\Rightarrow x(t) = \frac{1}{-(t-t_0) + \frac{1}{x(t_0)}} = \frac{x(t_0)}{1 - x(t_0)(t-t_0)}$$

with $t_0 = 0$ and $x(t_0) = 1$ we get

$$x(t) = \frac{1}{1-t}$$

It is clear to see $x(t)$ goes to $\infty$ when $t$ goes to $1$. ode45 is an adaptive solver and due to numerical precision, it is not capable of solving the system when $t$ goes to $1$, and it is obviously incapable of solving the system for $t > 1$

The second ODE is given by $\dot{x} = f(x) = \sqrt{|x|}$.
$f'(x) = 1/(2\sqrt{|x|}) \cdot \text{sign}(x)$, and is well defined for $x \neq 0$. Thus, the ODE has a unique solution if and only if $x$ never is $0$.

$$\frac{dx}{dt} = \sqrt{|x|} \cdot \operatorname{sign}(x)$$

$$\int_{x_0}^{x(t)} \frac{1}{\sqrt{|x|} \cdot \operatorname{sign}(x)} \, dx = t - t_0$$

$$\left[ 2\sqrt{|x|} \cdot \operatorname{sign}(x) \right]_{x_0}^{x(t)} = t - t_0$$

$$\sqrt{|x(t)|} \cdot \operatorname{sign}(x(t)) = \frac{1}{2} t - t_0 + \sqrt{|x_0|} \cdot \operatorname{sign}(x_0)$$

and $x > 0 \Leftrightarrow f(x) > 0$, $f(0) = 0$

As $f(x)$ is monotone, $\operatorname{sign}(x(t)) = \operatorname{sign}(x_0)$

$$\Rightarrow \sqrt{|x(t)|} = \frac{t - t_0}{2} \cdot \operatorname{sign}(x_0) + \sqrt{|x_0|}$$

$$\Rightarrow x(t) = \left( \frac{t - t_0}{2} \cdot \operatorname{sign}(x_0) + \sqrt{|x_0|} \right)^2 \cdot \operatorname{sign}(x_0)$$

With $t_0 = 0$, $x_0 = 0$ we get $x(t) = \frac{1}{4} t^2$

As $t_0 = 0$ any function $x(t) = C \cdot \frac{1}{4} \cdot t^2$, $C \in \mathbb{R}$ is valid. ode45 finds the trivial solution $x(t) = 0$.

## Problem 3

a) The system can be modeled as

$$\dot{H} = (b-d)H - b_d H^2 - iHZ$$

$$\dot{I} = -(a+d)I + iHZ$$

$$\dot{Z} = aI + rD - nHZ$$

$$\dot{D} = d(H+I) - rD + nHZ$$

$$H, I, Z, D \geq 0 \quad , \quad a, b_d, d, r, n > 0$$

b) The new system can be modeled as

$$\dot{H} = \text{——} \, '' \, \text{——}$$

$$\dot{I} = -(a + d + q_i)I + iHZ$$

$$\dot{Z} = aI + rD - nHZ - q_z Z$$

$$\dot{D} = d(H+I) - rD + nHZ + d_q Q$$

$$\dot{Q} = q_i I + q_z Z - d_q Q$$