Student Information

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Answer 1

a)

First we should calculate the **probability mass function** for all values of **X**:

$$
\begin{aligned}
P_X(0) = P\{X = 0\} = \sum_y P_{(X,Y)}(0, y) = P(0, 0) + P(0, 2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} \\
P_X(1) = P\{X = 1\} = \sum_y P_{(X,Y)}(1, y) = P(1, 0) + P(1, 2) = \frac{4}{12} + \frac{2}{12} = \frac{6}{12} \\
P_X(2) = P\{X = 2\} = \sum_y P_{(X,Y)}(2, y) = P(2, 0) + P(2, 2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12}
\end{aligned}
\tag{1}
$$

Calculation of **expected value** of random variable **X**:

$$
\begin{aligned}
\mu = E(X) &= \sum_x xP(x) \\
&= 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) \\
&= 0 \cdot \frac{3}{12} + 1 \cdot \frac{6}{12} + 2 \cdot \frac{3}{12} = \frac{6}{12} + \frac{6}{12} = 1
\end{aligned}
\tag{2}
$$

Calculation of **variance** of random variable **X**:

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}(x) &= \sum_x (x - \mu)^2 P(x) \\
&= (0 - 1)^2 \cdot P(0) + (1 - 1)^2 \cdot P(1) + (2 - 1)^2 \cdot P(2) \\
&= 1 \cdot \frac{3}{12} + 1 \cdot \frac{3}{12} = \frac{3}{12} + \frac{3}{12} = \frac{6}{12} = \frac{1}{2}
\end{aligned}
\tag{3}
$$

b)

Let $Z = X + Y$. So, **minimum** value of Z is $0 + 0 = 0$ and **maximum** value of Z is $2 + 2 = 4$. Next,

$$
\begin{aligned}
P_Z(0) &= P\{Z=0\} \\
&= P\{X=0, Y=0\} = \frac{1}{12} \\
P_Z(1) &= P\{Z=1\} \\
&= P\{X=1, Y=0\} = \frac{4}{12} \\
P_Z(2) &= P\{Z=2\} \\
&= P\{X=0, Y=2\} + P\{X=2, Y=0\} = \frac{2}{12} + \frac{1}{12} = \frac{3}{12} \\
P_Z(3) &= P\{Z=3\} \\
&= P\{X=1, Y=2\} = \frac{2}{12} \\
P_Z(4) &= P\{Z=4\} \\
&= P\{X=2, Y=2\} = \frac{2}{12}
\end{aligned} \tag{4}
$$

c)

First we should calculate the **probability mass function** for all values of **Y**:

$$
\begin{aligned}
P_Y(0) = P\{Y=0\} = \sum_x P_{(X,Y)}(x,0) = P(0,0) + P(1,0) + P(2,0) = \frac{1}{12} + \frac{4}{12} + \frac{1}{12} = \frac{6}{12} \\
P_Y(2) = P\{Y=2\} = \sum_x P_{(X,Y)}(x,2) = P(0,2) + P(1,2) + P(2,2) = \frac{2}{12} + \frac{2}{12} + \frac{2}{12} = \frac{6}{12}
\end{aligned} \tag{5}
$$

After that we should calculate the **expected value** of random variable Y:

$$
\begin{aligned}
\mu = E(Y) = \sum_y yP(Y) \\
= 0 \cdot P(0) + 2 \cdot P(2) = 0 \cdot \frac{6}{12} + 2 \cdot \frac{6}{12} = \frac{12}{12} = 1.
\end{aligned} \tag{6}
$$

The formula of **covariance** of random variables X and Y is:

$$
\text{Cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) \tag{7}
$$

$\mathbf{E}(X) = 1.$ (obtained from 1.(a))
$\mathbf{E}(Y) = 1.$

We should calculate $\mathbf{E}(XY)$:

$$\begin{aligned}
\mathbf{E}(XY) &= \sum_x \sum_y xyP(x,y) \\
&= (0)(0)P(0,0) + (0)(2)P(0,2) + (1)(0)P(1,0) + (1)(2)P(1,2) + (2)(0)P(2,0) + (2)(2)P(2,2) \\
&= (1)(2)P(1,2) + (2)(2)P(2,2) \\
&= 1 \cdot 2 \cdot \frac{2}{12} + 2 \cdot 2 \cdot \frac{2}{12} \\
&= \frac{4}{12} + \frac{8}{12} = \frac{12}{12} = 1.
\end{aligned} \tag{8}$$

Calculation of **covariance** of random variables X and Y:

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) \\
&= 1 - 1 \cdot 1 = 0
\end{aligned} \tag{9}$$

d)

The formula of **covariance** is:

$$\mathrm{Cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) \tag{10}$$

For **independent** X and Y, *(from (3.5) on page 49 on textbook)*

$$\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) \tag{11}$$

So,

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) \\
&= \mathbf{E}(X)\mathbf{E}(Y) - \mathbf{E}(X)\mathbf{E}(Y) = 0.
\end{aligned} \tag{12}$$

e)

If random variables X and Y are independent:

$$P_{(X,Y)}(x,y) = P_X(x)P_Y(y) \tag{13}$$

We should try for all values of random variables X and Y:

$$P_X(0)P_Y(0) = \frac{3}{12} \cdot \frac{6}{12} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \neq P_{(X,Y)}(x,y) = \frac{1}{12} \tag{14}$$

So, the random variables X and Y are **not independent**.

Answer 2

a)

We can solve this question with **Binomial distribution**. Binomial probability mass function is:

$$P(x) = P\{X = x\} = \binom{n}{x} p^x q^{n-x} \tag{15}$$

We need to find probability of $P\{X \geq 3\}$, where X is the number of pens out of 12, which are broken. X has binomial distribution with parameters $n = 12$, $p = 0.2$ and $q = 0.8$. We can obtain $P\{X \geq 3\}$ in this way:

$$P\{X \geq 3\} = 1 - F(2) \tag{16}$$

The value of $F(2) = 0.558$, obtained from *Table A2* from textbook. Let's calculate $F(2)$ by ourselves:

$$\begin{aligned}
F(2) &= \sum_{x=0}^{2} P(x) \\
&= \sum_{x=0}^{2} \binom{n}{x} p^x q^{n-x} \\
&= \binom{12}{0}(0.2)^0(0.8)^{12} + \binom{12}{1}(0.2)^1(0.8)^{11} + \binom{12}{2}(0.2)^2(0.8)^{10} \\
&= 1 \cdot 1 \cdot 0.0687 + 12 \cdot 0.2 \cdot 0.0858 + 66 \cdot 0.04 \cdot 0.1073 \\
&= 0.0687 + 0.2059 + 0.2832 \\
&= 0.5578 \approx 0.558
\end{aligned} \tag{17}$$

After that we can put this value in equation (15):

$$P\{X \geq 3\} = 1 - F(2) = 1 - 0.558 = 0.442 \tag{18}$$

b)

We can solve this question with **Negative Binomial distribution**. Negative Binomial probability function is:

$$P(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}. \tag{19}$$

Where p is probability of success *(being broken)*. We need to find probability of $P(5)$ with parameters $k = 2$ and $p = 0.2$. So,

$$\begin{aligned}
P(5) &= \binom{4}{1}(0.2)^2(0.8)^3 \\
&= 4 \cdot 0.04 \cdot 0.512 \\
&= 0.08192 \approx 0.082
\end{aligned} \tag{20}$$

c)

In this question we can use property of **Negative Binomial distribution**. *(3.12) on page 63 from textbook*

$$E(X) = \frac{k}{p} = \frac{4}{0.2} = 20 \tag{21}$$

Answer 3

a)

In this question:

$$\mu = E(X) = 4 \tag{22}$$

We can calculate λ:

$$\lambda = \frac{1}{\mu} = \frac{1}{4} = \frac{1}{4} \tag{23}$$

Now we should calculate $P(X \geq 2)$:

$$P(X \geq 2) = 1 - F(2) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x} = e^{-\frac{1}{4} \cdot 2} = e^{-\frac{1}{2}} = 0.6065 \tag{24}$$

b)

We can solve this question with **Poisson distribution**.
First we should calculate the value of λ. If the average number of calls in 4 hours is 1. So, the average number of calls in 10 hours is 2.5. So,

$$\lambda = 2.5 \tag{25}$$

Now we can find the answer

$$\mathbf{P}\{X \leq 3\} = F_X(3) = \sum_{x=0}^{3} e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=0}^{3} e^{-2.5} \frac{(2.5)^x}{x!} = 0.758 \tag{26}$$

The value of $F_X(3)$ obtained from *Table A3 on textbook.*

c)

Exponential distribution has **memoryless property**:

$$\mathbf{P}\{T > t + x | T > t\} = \mathbf{P}\{T > x\} \tag{27}$$

That means we should just consider 6 hours part. After that we can solve this question with **Poisson distribution**. That's why first we should find the value of λ. If the average number of calls in 4 hours is 1. So, the average number of calls in 6 hours is 1.5. So,

$$\lambda = 1.5 \tag{28}$$

We should divide this question into 4 parts.
1) In 10 hours period, Bob didn't get any call. We should calculate the probability of getting at most 3 calls in 6 hours period:

$$\mathbf{P}\{X \leq 3\} = F_X(3) = \sum_{x=0}^{3} e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=0}^{3} e^{-1.5}\frac{(1.5)^x}{x!} = 0.934 \tag{29}$$

2) In 10 hours period Bob got 1 call. We should calculate the probability of getting at most 2 calls in 6 hours period:

$$\mathbf{P}\{X \leq 2\} = F_X(2) = \sum_{x=0}^{2} e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=0}^{2} e^{-1.5}\frac{(1.5)^x}{x!} = 0.809 \tag{30}$$

3) In 10 hours period Bob got 2 calls. We should calculate the probability of getting at most 1 call in 6 hours period:

$$\mathbf{P}\{X \leq 1\} = F_X(1) = \sum_{x=0}^{1} e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=0}^{1} e^{-1.5}\frac{(1.5)^x}{x!} = 0.558 \tag{31}$$

4) In 10 hours period Bob got 3 calls. We should calculate the probability of not getting a call in 6 hours period:

$$\mathbf{P}\{X \leq 0\} = F_X(0) = e^{-1.5} = 0.223 \tag{32}$$

All values obtained from *Table A3. Poisson distribution from textbook.*