Student Information

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Answer 1

a)

$E_1 = \{$ picking a green ball from box X $\}$

$$P\{E_1\} = \frac{2}{6} = \frac{1}{3}$$

b)

$R = \{$ Picking red ball $\}$
$R_1 = \{$ Picking red ball from box X when box X is chosen $\}$
$R_2 = \{$ Picking red ball from box Y when box Y is chosen $\}$
$R_3 = \{$ Choosing box X $\}$
$R_4 = \{$ Choosing box Y $\}$

$$P\{R\} = P\{R_1\}P\{R_3\} + P\{R_2\}P\{R_4\} = \frac{1}{3} \cdot \frac{4}{10} + \frac{1}{5} \cdot \frac{6}{10} = \frac{38}{150} = \frac{19}{75}$$

c)

$B = \{$ Picking blue ball $\}$
$Y_1 = \{$ Choosing box Y $\}$
$\overline{Y_1} = \{$ Not choosing box Y $\}$

$$P\{Y_1|B\} = \frac{P\{Y_1 \cap B\}}{P\{B\}} = \frac{P\{Y_1 \cap B\}}{P\{B|Y_1\}P\{Y_1\} + P\{B|\overline{Y_1}\}P\{\overline{Y_1}\}} = \frac{\frac{4}{10} \cdot \frac{6}{10}}{\frac{4}{10} \cdot \frac{6}{10} + \frac{1}{3} \cdot \frac{4}{10}} = \frac{0.24}{\frac{112}{300}} =$$
$$= \frac{24}{100} \cdot \frac{300}{112} = \frac{9}{14}$$

Answer 2

a)

The statement, "A and B are **mutually exclusive** if and only if $\overline{A}$ and $\overline{B}$ are exhaustive" implies that "If A and B are mutually exclusive, so they are exhaustive." and "If $\overline{A}$ and $\overline{B}$ are exhaustive, so they are mutually exclusive."

Let's assume $\overline{A}$ and $\overline{B}$ are exhaustive. So, $\overline{A} \cup \overline{B} = \Omega$.
If we take the complement of the both sides of this equation, the equation stays same.

$$\overline{\overline{A} \cup \overline{B}} = \overline{\Omega}.$$

According to the De Morgan's laws

$$\overline{\overline{A} \cup \overline{B}} = A \cap B$$

The complement of Ω is $\emptyset$.

$$\begin{gathered} P\{\Omega\} = 1 \\ P\{\overline{\Omega}\} = 1 - P\{\Omega\} = 0 = P\{\emptyset\}. \\ \overline{\Omega} = \emptyset. \end{gathered}$$

So,

$$A \cap B = \emptyset$$

Now let's assume A and B are mutually exclusive: $A \cap B = \emptyset$.
Take complement of both sides:

$$\overline{A \cap B} = \overline{\emptyset}$$

Again according to De Morgan's laws:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

The complement of $\emptyset$ is Ω.

$$\begin{gathered} P\{\emptyset\} = 0 \\ P\{\overline{\emptyset}\} = 1 - P\{\emptyset\} = 1 - 0 = 1 = P\{\Omega\} \\ \overline{\emptyset} = \Omega \end{gathered}$$

So,

$$\overline{A} \cup \overline{B} = \Omega$$

These results implies that:

$$A \cap B = \emptyset \Longleftrightarrow \overline{A} \cup \overline{B} = \Omega$$

So, the statement, "A and B are **mutually exclusive** if and only if $\overline{A}$ and $\overline{B}$ are exhaustive." has been proven.

b)

The statement, "A, B and C are **mutually exclusive** if and only if $\overline{A}$, $\overline{B}$ and $\overline{C}$ are exhaustive" implies that "If A, B and C are mutually exclusive, so they are exhaustive." and "If $\overline{A}$, $\overline{B}$ and $\overline{C}$ are exhaustive, so they are mutually exclusive."

Let's assume $\overline{A}$, $\overline{B}$ and $\overline{C}$ are exhaustive: $\overline{A} \cup \overline{B} \cup \overline{C} = \Omega$
If A, B and C sets are mutually exclusive, so the following equations should hold for those sets:

$$A \cap B = \emptyset$$
$$A \cap C = \emptyset$$
$$B \cap C = \emptyset$$

Let's try these for the following sets A_1 ,B_1 and C_1:

$$A_1 = \{2, 3\},\ B_1 = \{4, 5, 6\} \text{ and } C_1 = \{4\}$$

Our Ω is $\{2, 3, 4, 5, 6\}$.

$$\overline{A_1} \cup \overline{B_1} \cup \overline{C_1} = \{2, 3, 4, 5, 6\}$$

So, $\overline{A_1}$, $\overline{B_1}$ and $\overline{C_1}$ are exhaustive.

$$A_1 \cap B_1 = \emptyset$$
$$A_1 \cap C_1 = \{4\}$$
$$B_1 \cap C_1 = \{4\}$$

So, A_1, B_1 and C_1 are not mutually exclusive.
As a result, the statement, "A, B and C are **mutually exclusive** if and only if $\overline{A}$, $\overline{B}$ and $\overline{C}$ are exhaustive." has been disproven.

Answer 3

a)

$E_1 = \{$ having exactly 2 successful dice $\}$

$$P\{E_1\} = \binom{5}{2} \cdot (\tfrac{1}{3})^2 \cdot (\tfrac{2}{3})^3 = 10 \cdot \frac{1}{9} \cdot \frac{8}{27} = \frac{80}{243}$$

b)

$E = \{$ having at least 2 successful dice $\}$
$E_2 = \{$ having exactly 2 successful dice $\}$
$E_3 = \{$ having exactly 3 successful dice $\}$
$E_4 = \{$ having exactly 4 successful dice $\}$

$E_5 = \{$ having exactly 5 successful dice $\}$

$P\{E\} = P\{E_2\}+P\{E_3\}+P\{E_4\}+P\{E_5\} = \binom{5}{2}(\frac{1}{3})^2(\frac{2}{3})^3+\binom{5}{3}(\frac{1}{3})^3(\frac{2}{3})^2+\binom{5}{4}(\frac{1}{3})^4(\frac{2}{3})^1+\binom{5}{5}(\frac{1}{3})^5(\frac{2}{3})^0 =$
$10 \cdot \frac{1}{9} \cdot \frac{8}{27} + 10 \cdot \frac{1}{27} \cdot \frac{4}{9} + 5 \cdot \frac{1}{81} \cdot \frac{2}{3} + \frac{1}{243} = \frac{80}{243} + \frac{40}{243} + \frac{10}{243} + \frac{1}{243} = \frac{131}{243}$

Answer 4

a)

$P_{(A,C)}(1,0) = P\{A=1, C=0\} = \Sigma_b P_{A,B,C}(1,b,0) = P\{A=1, B=0, C=0\} + P\{A=1, B=1, C=0\} = 0.06 + 0.09 = 0.15$

b)

$P_B(1) = P\{B=1\} = \Sigma_a \Sigma_c P_{(A,B,C)}(a,1,c) = P\{A=0, B=1, C=0\} + P\{A=0, B=1, C=1\} + P\{A=1, B=1, C=0\} + P\{A=1, B=1, C=1\} = 0.21 + 0.02 + 0.09 + 0.08 = 0.4$

c)

$P_A(1) = P\{A=1\} = \Sigma_b \Sigma_c P_{(A,B,C)}(1,b,c) = P\{A=1, B=0, C=0\} + P\{A=1, B=0, C=1\} + P\{A=1, B=1, C=0\} + P\{A=1, B=1, C=1\} = 0.06 + 0.32 + 0.09 + 0.08 = 0.55$

$P_B(1) = P\{B=1\} = \Sigma_a \Sigma_c P_{(A,B,C)}(a,1,c) = P\{A=0, B=1, C=0\} + P\{A=0, B=1, C=1\} + P\{A=1, B=1, C=0\} + P\{A=1, B=1, C=1\} = 0.21 + 0.02 + 0.09 + 0.08 = 0.4$

$P_{A,B}(1,1) = P\{A=1, B=1\} = \Sigma_c P_{A,B,C}(1,1,c) = P\{A=1, B=1, C=0\} + P\{A=1, B=1, C=1\} = 0.09 + 0.08 = 0.17$

If A and B are independent,

$$P\{A=1, B=1\} = P\{A=1\} \cdot P\{B=1\}.$$

But, $P\{A=1, B=1\} = 0.17 \neq P\{A=1\}P\{B=1\} = 0.4 \cdot 0.55 = 0.22$. So A and B are not independent.

d)

If A and B are conditionally independent, so then the formula

$$P\{A, B|C=1\} = P\{A|C=1\} \cdot P\{B|C=1\}$$

should hold.

First let's calculate
$P_C(1) = P\{C = 1\} = \Sigma_a \Sigma_b P_{(A,B,C)}(a, b, 1) = P\{A = 0, B = 0, C = 1\} + P\{A = 0, B = 1, C = 1\} + P\{A = 1, B = 0, C = 1\} + P\{A = 1, B = 1, C = 1\} = 0.08 + 0.02 + 0.32 + 0.08 = 0.5$

$P_{A,B}(0, 0|C = 1) = P\{A = 0, B = 0|C = 1\} = \dfrac{P\{A, B, C\}}{P\{C\}} = \dfrac{0.08}{0.5} = 0.16$
$P\{A = 0|C = 1\}P\{B = 0|C = 1\} = \dfrac{P\{A, C\}}{P\{C\}} \cdot \dfrac{P\{B, C\}}{P\{C\}} = \dfrac{0.1}{0.5} \cdot \dfrac{0.4}{0.5} = 0.2 \cdot 0.8 = 0.16$

$$P_{A,B}(0, 0|C = 1) = P\{A = 0|C = 1\}P\{B = 0|C = 1\}$$

$P_{A,B}(0, 1|C = 1) = P\{A = 0, B = 1|C = 1\} = \dfrac{P\{A, B, C\}}{P\{C\}} = \dfrac{0.02}{0.5} = 0.04$
$P\{A = 0, C = 1\}P\{B = 1, C = 1\} = \dfrac{P\{A, C\}}{P\{C\}} \cdot \dfrac{P\{B, C\}}{P\{C\}} = \dfrac{0.1}{0.5} \cdot \dfrac{0.1}{0.5} = 0.2 \cdot 0.2 = 0.04$

$$P_{A,B}(0, 1|C = 1) = P\{A = 0|C = 1\}P\{B = 1|C = 1\}$$

$P_{A,B}(1, 0|C = 1) = P\{A = 1, B = 0|C = 1\} = \dfrac{P\{A, B, C\}}{P\{C\}} = \dfrac{0.32}{0.5} = 0.64$
$P\{A = 1, C = 1\}P\{B = 0, C = 1\} = \dfrac{P\{A, C\}}{P\{C\}} \cdot \dfrac{P\{B, C\}}{P\{C\}} = \dfrac{0.4}{0.5} \cdot \dfrac{0.4}{0.5} = 0.8 \cdot 0.8 = 0.64$

$$P_{A,B}(1, 0|C = 1) = P\{A = 1|C = 1\}P\{B = 0|C = 1\}$$

$P_{A,B}(1, 1|C = 1) = P\{A = 1, B = 1|C = 1\} = \dfrac{P\{A, B, C\}}{P\{C\}} = \dfrac{0.08}{0.5} = 0.16$
$P\{A = 1, C = 1\}P\{B = 1, C = 1\} = \dfrac{P\{A, C\}}{P\{C\}} \cdot \dfrac{P\{B, C\}}{P\{C\}} = \dfrac{0.4}{0.5} \cdot \dfrac{0.1}{0.5} = 0.8 \cdot 0.2 = 0.16$

$$P_{A,B}(1, 1|C = 1) = P\{A = 1|C = 1\}P\{B = 1|C = 1\}$$

The equation holds for all the values of A and B when $C = 1$. So A and B are conditionally independent for the $C = 1$.