CENG 384 - Signals and Systems for Computer Engineers
Spring 2020
Written Assignment 2

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1. (a) i. The given system is:

$$y[n] = \Sigma_{k=1}^{\infty} x[n-k].$$

Let $x_1[n]$ be an arbitrary input to this system, and let

$$y_1[n] = \Sigma_{k=1}^{\infty} x_1[n-k].$$

Then consider a second input obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n-n_0].$$

The output corresponding to $x_2[n]$ is:

$$y_2[n] = \Sigma_{k=1}^{\infty} x_2[n-k] = \Sigma_{k=1}^{\infty} x_1[n-n_0-k].$$

Now calculate the $y_1[n]$ with time shift n_0

$$y_1[n-n_0] = \Sigma_{k=1}^{\infty} x[n-n_0-k].$$

We see that $y_2[n] = y_1[n-n_0]$. So, the given system is **time invariant**.

ii. The given system is:

$$y[n] = \Sigma_{k=1}^{\infty} x[n-k].$$

For determining the system is linear or not, we consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \rightarrow y_1[n] = \Sigma_{k=1}^{\infty} x_1[n-k]$$
$$x_2[n] \rightarrow y_2[n] = \Sigma_{k=1}^{\infty} x_2[n-k]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

a and b are arbitrary scalars. The output of input $x_3[n]$

$$\begin{aligned} y_3[n] &= \Sigma_{k=1}^{\infty} x_3[n-k] \\ &= \Sigma_{k=1}^{\infty} ax_1[n-k] + bx_2[n-k] \\ &= \Sigma_{k=1}^{\infty} ax_1[n-k] + \Sigma_{k=1}^{\infty} bx_2[n-k] \\ &= a\Sigma_{k=1}^{\infty} x_1[n-k] + b\Sigma_{k=1}^{\infty} x_2[n-k] \\ &= ay_1[n] + by_2[n] \end{aligned} \tag{1}$$

So, the given system is **linear**.

iii. $y[n] = \sum_{k=1}^{\infty} x[n-k]$ is **not memoryless**, because $y[n]$ depends on values of $x[\cdot]$ before time instant n.

iv. The given system is:

$$y[n] = \sum_{k=1}^{\infty} x[n-k] = x[n-1] + x[n-2] + x[n-3] + ...$$

The given system is **causal** because the value of $y[\cdot]$ at any instant n depends on only the previous values of $x[\cdot]$.

v. The given system is:

$$x[n] \longrightarrow y[n] = \Sigma_{k=1}^{\infty} x[n-k]$$

The inverse of the given system is:

$$y[n] \longrightarrow x[n] = y[n] - y[n-1]$$

This system is **invertible**.

vi. A stable system produces a bounded output for any given bounded input. However if we try the $u[n]$ as input which is bounded, the output of the system is not bounded for that input. So, system is **not stable**.

(b) i. The given system is:

$$y(t) = tx(2t+3)$$

Let $x_1(2t+3)$ be an arbitrary input to this system, and let

$$y_1(t) = tx_1(2t+3)$$

Let $x_2(t)$ is an input obtained by shifting $x_1(2t+3)$ in time:

$$x_2(t) = x_1(2(t-t_0)+3)$$

The output corresponding to $x_2(t)$ is:

$$y_2(t) = tx_2(t) = tx_1(2(t-t_0)+3).$$

But,

$$y_1(t-t_0) = (t-t_0)x_1(2(t-t_0)+3).$$

As a result $y_2(t) \neq y(t-t_0)$. So the system is **not time invariant**.

ii. $y(t) = tx(2t+3)$ is **not causal** because $y(\cdot)$ is depend on future values of $x(\cdot)$.

iii. $y(t) = tx(2t+3)$ is **not memoryless** because $y(\cdot)$ is not just depend on values of $x(\cdot)$ at that time t.

iv. For determining the system is linear or not, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \longrightarrow y_1(t) = tx_1(2t+3)$$
$$x_2(t) \longrightarrow y_2(t) = tx_2(2t+3)$$

Let $x_3(t)$ be an linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

a and b are arbitrary scalars. We will provide $x_3(t)$ as an input to that system and output will be

$$\begin{aligned} y_3(t) &= tx_3(2t+3) \\ &= t(ax_1(2t+3) + bx_2(2t+3)) \\ &= atx_1(2t+3) + btx_2(2t+3) \\ &= ay_1(t) + by_2(t) \end{aligned} \tag{2}$$

So the system is **linear**.

v. The system $y(t) = tx(2t+3)$ is **not invertible**, because $y(t) = 0$, for $t = 0$ independent from the value of $x(t)$. So, we can not determine the value of $x(t)$ from the output of the system.

vi. For system $y(t) = tx(2t+3)$, a constant input $x(2t+3) = 1$ yields $y(t) = t$, which is unbounded. So the system is **unstable**.

2. (a) $x(t) - 5y(t) = y'(t)$
$y'(t) + 5y(t) = x(t)$

(b) We can find $y(t)$ by calculating and adding *particular* solution and *homogeneous* solution:

$$y(t) = y_h(t) + y_p(t)$$

For calculating homogeneous solution part $y_h(t)$, we need the calculation of:

$$y'(t) + 5y(t) = 0$$

For that calculation We assume

$$y_h(t) = Ke^{\lambda t}$$

and plug it into differential equation. So,

$$\begin{aligned} y'(t) + 5y(t) &= x(t) \\ K\lambda e^{\lambda t} + 5Ke^{\lambda t} &= 0 \\ Ke^{\lambda t}(\lambda + 5) &= 0 \\ \lambda &= -5 \end{aligned} \tag{3}$$

Homogeneous solution is:

$$y_h(t) = K_1 e^{-5t} \quad \text{for } t > 0$$

For particular solution part y_p, given input $x(t)$

$$y_p(t) = (ae^{-t} + be^{-3t})u(t)$$

So,

$$\begin{aligned} y'(t) + 5y(t) =& x(t) \\ (-ae^{-t} - 3be^{-3t} + 5ae^{-t} + 5be^{-3t})u(t) =& (e^{-t} + e^{-3t})u(t) \\ 4a =& 1 \\ 2b =& 1 \end{aligned} \tag{4}$$

We obtain,

$$a = \frac{1}{4} \text{ and } b = \frac{1}{2}$$

For getting $y(t)$, we need to add $y_h(t)$ and $y_p(t)$:

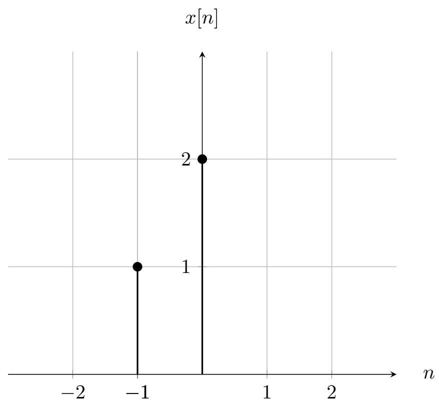
$$\begin{aligned} y(t) =& y_h(t) + y_p(t) \\ =& K_1 e^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t} \quad \text{for } t > 0 \end{aligned} \tag{5}$$

System is initially at rest. So,

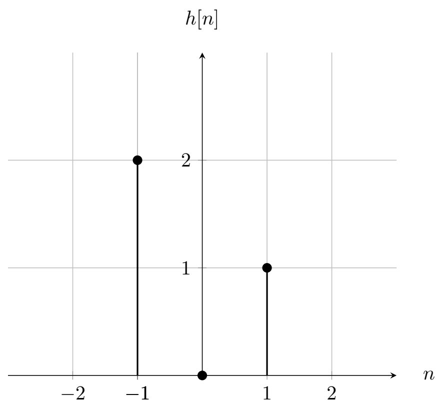
$$\begin{aligned} y(0) &= 0. \\ K_1 + \frac{1}{4} + \frac{1}{2} &= 0 \\ K_1 &= \frac{-3}{4} \\ y(t) &= (\frac{-3}{4}e^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t})u(t) \end{aligned} \tag{6}$$

3. (a)

$x[n] = 2\delta[n] + \delta[n+1]$

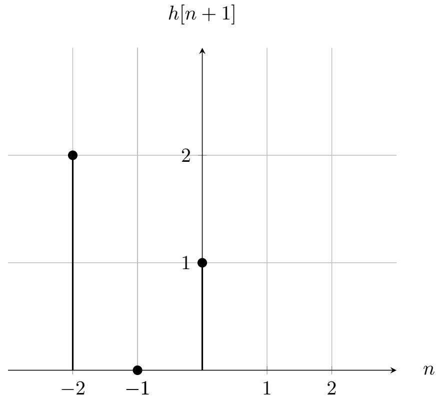


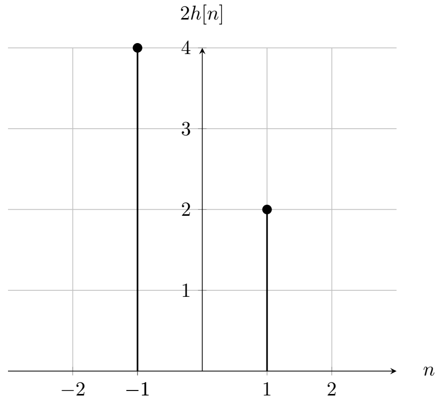
$h[n] = \delta[n-1] + 2\delta[n+1]$

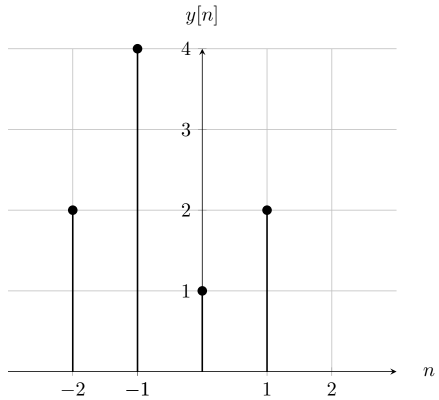


$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$y[n] = x[-1]h[n+1] + x[0]h[n]$, since for all $n \neq -1, 0$ $x[n] = 0$
$y[n] = h[n+1] + 2h[n]$







(b) We provide a solution for $y(t)$ below:

$$
\begin{aligned}
x(t) &= u(t-1) + u(t+1) \\
h(t) &= e^{-t} sin(t) u(t) \\
y(t) &= \frac{dx(t)}{dt} * h(t) \\
&= \frac{d(u(t-1) + u(t+1))}{dt} * h(t) \\
&= \left(\frac{du(t-1)}{dt} + \frac{du(t+1)}{dt} \right) * h(t) \\
&= (\delta(t-1) + \delta(t+1)) * h(t) \\
&= (\delta(t-1) * h(t)) + (\delta(t+1) * h(t)) \\
&= h(t-1) + h(t+1) \\
&= e^{-(t-1)} sin(t-1) u(t-1) + e^{-(t+1)} sin(t+1) u(t+1)
\end{aligned} \tag{7}
$$

4. (a) $y(t) = x(t) * h(t)$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{+\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau \\
&= \int_{0}^{+\infty} e^{-\tau}e^{-2(t-\tau)}u(t-\tau)d\tau \\
&= \int_{0}^{t} e^{-\tau}e^{-2(t-\tau)}d\tau \\
&= \int_{0}^{t} e^{-\tau}e^{-2t}e^{2\tau}d\tau \\
&= e^{-2t}\int_{0}^{t} e^{-\tau}e^{2\tau}d\tau \\
&= e^{-2t}\int_{0}^{t} e^{\tau}d\tau \\
&= e^{-2t}\left(e^{\tau}\right)|_0^t \\
&= e^{-2t}\left(e^{t}-e^{0}\right) \\
&= e^{-2t}\left(e^{t}-1\right) \\
&= e^{-t}-e^{-2t}
\end{aligned}
\tag{8}
$$

Thus, the convolution of $x(t)$ with $h(t)$:

$$
\begin{aligned}
&y(t) = x(t) * h(t) \\
&y(t) = 0, \quad \text{for } t < 0 \\
&y(t) = \left(e^{-t}-e^{-2t}\right)u(t)
\end{aligned}
\tag{9}
$$

(b) For this question, we need to consider three cases for the values of t:

$$
y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \int_0^t e^{3(t-\tau)}d\tau & \text{if } 0 \le t \le 1 \\ \int_0^1 e^{3(t-\tau)}d\tau & \text{if } 1 < t \end{cases}
\tag{10}
$$

After evaluation of integrals:

$$
y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{3}\left(e^{3t}-1\right) & \text{if } 0 \le t \le 1 \\ -\frac{1}{3}\left(e^{3(t-1)}-e^{3t}\right) & \text{if } 1 < t \end{cases}
\tag{11}
$$

5. (a) $2y[n+2] - 3y[n+1] + y[n] = 0$

The characteristic polynomial of the system is:

$$
\begin{aligned}
D(z) = 2z^2 - 3z + 1 &= 0 \\
(2z-1)(z-1) &= 0 \\
z_1 = \frac{1}{2} \quad \text{and } z_2 &= 1
\end{aligned}
\tag{12}
$$

$y[n] = c_1\left(\frac{1}{2}\right)^n + c_2$

Calculation of $c1$ and $c2$:

$$
\begin{aligned}
y[0] &= c_1 + c_2 = 1 \\
y[1] &= \frac{c_1}{2} + c_2 = 0 \\
c_1 &= 2 \quad \text{and } c_2 = -1
\end{aligned}
\tag{13}
$$

Overall solution is:

$$
y[n] = 2\left(\frac{1}{2}\right)^n - 1
\tag{14}
$$

(b) We assume $y(t) = Ke^{\lambda t}$ and derivatives of $y(t)$:

$$
\begin{aligned}
y'(t) &= K\lambda e^{\lambda t} \\
y''(t) &= K\lambda^2 e^{\lambda t} \\
y^{(3)}(t) &= K\lambda^3 e^{\lambda t}
\end{aligned}
\tag{15}
$$

Put these values into the equation:

$$
\begin{aligned}
K\lambda^3 e^{\lambda t} - 3K\lambda^2 e^{\lambda t} + 4K\lambda e^{\lambda t} - 2Ke^{\lambda t} = 0 \\
Ke^{\lambda t}(\lambda^3 - 3\lambda^2 + 4\lambda - 2) = 0 \\
(\lambda - 1)(\lambda^2 - 2\lambda + 2) = 0 \\
\lambda_1 = 1, \lambda_2 = 1 - j, \lambda_3 = 1 + j
\end{aligned}
\tag{16}
$$

Solution of y(t) and corresponding derivatives are the following:

$$
\begin{aligned}
y(t) &= K_1 e^t + K_2 e^t \sin(t) + K_3 e^t \cos(t) \\
y'(t) &= K_1 e^t + K_2 e^t \sin(t) + K_2 e^t \cos(t) + K_3 e^t \cos(t) - K_3 e^t \sin(t) \\
y''(t) &= K_1 e^t + 2K_2 e^t \cos(t) - 2K_3 e^t \sin(t)
\end{aligned}
\tag{17}
$$

To find constant coefficients, we try initial conditions:

$$
\begin{aligned}
y(0) = K_1 + K_3 = 3 \\
y'(0) = K_1 + K_2 + K_3 = 1 \\
y''(0) = K_1 + 2K_2 = 2 \\
K_1 = 6, K_2 = -2, K_3 = -3
\end{aligned}
\tag{18}
$$

Overall solution is:

$$
y(t) = 6e^t - 2e^t \sin(t) - 3e^t \cos(t) \tag{19}
$$

6. (a)

$$
w[n] - \frac{1}{2}w[n-1] = x[n] \tag{20}
$$

We can express $w[n]$ as following:

$$
w[n] = x[n] + \frac{1}{2}w[n-1] \tag{21}
$$

We need an initial condition. So,

$$
\begin{aligned}
&x[n] = A\delta[n] \\
&w[0] = x[0] + \frac{1}{2}w[-1] = A, \\
&w[1] = x[1] + \frac{1}{2}w[0] = \frac{1}{2}A, \\
&w[2] = x[2] + \frac{1}{2}w[1] = \left(\frac{1}{2}\right)^2 A \\
&. \\
&. \\
&. \\
&w[n] = x[n] + \frac{1}{2}w[n-1] = \left(\frac{1}{2}\right)^n A.
\end{aligned}
\tag{22}
$$

Setting A = 1, the impulse response for the system is considered as:

$$
h_0[n] = \left(\frac{1}{2}\right)^n u[n] \tag{23}
$$

(b) Overall impulse response of the system is the following:

$$
\begin{aligned}
h[n] &= h_0[n] * h_0[n] \\
&= \Sigma_{k=-\infty}^{\infty} h_0[k] h_0[n-k] \\
&= \Sigma_{k=0}^{\infty} h_0[k] h_0[n-k] \\
&= \Sigma_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\
&= \Sigma_{k=0}^{n} \left(\frac{1}{2}\right)^n \\
&= (n+1)\left(\frac{1}{2}\right)^n
\end{aligned}
\tag{24}
$$

(c)

$$
\begin{aligned}
x[n] &= w[n] - \frac{1}{2}w[n-1] \\
w[n] &= y[n] - \frac{1}{2}y[n-1] \\
w[n-1] &= y[n-1] - \frac{1}{2}y[n-2] \\
x[n] &= y[n] - \frac{1}{2}y[n-1] - \frac{1}{2}\left(y[n-1] - \frac{1}{2}y[n-2]\right) \\
x[n] &= y[n] - \frac{1}{2}y[n-1] - \frac{1}{2}y[n-1] + \frac{1}{4}y[n-2] \\
x[n] &= y[n] - y[n-1] + \frac{1}{4}y[n-2]
\end{aligned}
\tag{25}
$$