Envariabelanalys 2018-02-15 #15

Från sist: $\int e^{ax}\sin bx\,dx = \frac{1}{1+\frac{b^2}{a^2}}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right)+C$ med partiell integration 2 ggr

alternativt: $\underbrace{\int e^{(a+bi)x}dx}_{\substack{\text{im-del: } e^{ax}\sin bx \\ \text{re: } \cos}} = \frac{1}{a+bi}e^{(a+bi)x}+C = \frac{e^{ax}}{a^2+b^2}\underbrace{(a-bi)(\cos bx + i\sin bx)}_{\substack{\text{im-del: } a\sin bx - b\cos bx \\ \text{re } \quad a\cos bx + b\sin bx}}+C$

Idag först om integration av rationella funktioner

ex. $\int \frac{x^2+8x+9}{x^2+3x+2}dx = \int\left(1+\frac{5x+7}{(x+1)(x+2)}\right)dx = \int\left(1+\underbrace{\frac{A}{x+1}+\frac{B}{x+2}}_{\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}}\right)dx = x+2\ln|x+1| + 3\ln|x+2|+C$

kan vara olika i $]-\infty,-2[$, $]-2,-1[$, $]-1,\infty[$

$A,B?$: $(A+B)x+(2A+B) = 5x+7$

$$\begin{cases}A+B=5\\2A+B=7\end{cases} \Leftrightarrow \begin{cases}A=2\\B=3\end{cases}$$

ex $\int \frac{x^4-x^2+4x}{\underbrace{x^3-x^2-x+1}_{(x-1)(x^2-1)}}dx =$

$$\begin{array}{l} \quad\ x+1 \\ \hline x^4 - x^2 + 4x \ \big|\ x^3-x^2-x+1 \\ -(x^4-x^3-x^2+x) \\ \hline \quad x^3+3x \\ \ -(x^3-x^2-x+1) \\ \hline \quad x^2+4x-1 \end{array}$$

$= \int\left(x+1+\frac{x^2+4x-1}{(x-1)^2(x+1)}\right)dx =$

$= \int\left(x+1+\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\right)dx = \frac{x^2}{2}+x+A\ln|x-1| - \frac{B}{x-1}+C\ln|x+1|+K$

$A,B,C?$ $A(x-1)(x+1)+B(x+1)+C(x-1)^2 = x^2+4x-1$

identifiera koefficienter:
$$\begin{array}{l} x^2: \\ x: \\ 1: \end{array}\begin{cases}A+C=1\\B-2C=4\\-A+B+C=-1\end{cases} \quad \left(\begin{array}{ccc|c}1&0&1&1\\0&1&-2&4\\-1&1&1&-1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&1&1\\0&1&-2&4\\0&1&2&0\end{array}\right) \sim$$

$$\sim\left(\begin{array}{ccc|c}1&0&1&1\\0&1&-2&4\\0&0&4&-4\end{array}\right)\begin{array}{l}\\ \\ \frac{1}{4}\ \ 2\ \ -1\end{array} \sim \left(\begin{array}{ccc|c}1&0&0&2\\0&1&0&2\\0&0&1&-1\end{array}\right) \quad \begin{array}{l}A=B=2\\C=-1\end{array}$$

Sats (om partialbråk): Om $q(x) = (x-\alpha_1)^{m_1}\cdots(x-\alpha_k)^{m_k}(x^2+a_1+b_1)^{n_1}\cdots$ och grad p < grad q så finns $A..,B..,C..$ sådana att

$$\frac{p(x)}{q(x)} = \frac{A_{11}}{(x-\alpha_1)}+\frac{A_{12}}{(x-\alpha_1)^2}+\ldots+\frac{A_{1m_1}}{(x-\alpha_1)^{m_1}}+\ldots+\frac{B_{11}+C_{11}}{x^2+a_1x+b}+\frac{B_{12}+C_{12}}{(x^2+a_1x+b_1)^2}+\ldots+\frac{B_{1n_1}+C_{1n_1}}{(x^2+a_1x+b_1)^{n_1}}+\ldots$$

ex. $\displaystyle\int \frac{3x^2-3x+5}{x^3-x^2+4x-4}\,dx = \int \frac{3x^2-3x+5}{(x-1)(x^2+4)} = \int \left(\frac{Ax+B}{x^2+4} + \frac{C}{x-1}\right)dx$

$$\frac{5x+7}{(x+1)(x+2)} = \frac{2}{(x+1)} + \frac{3}{x+2}$$

A, B, C ? $\underbrace{(Ax+B)(x-1)}_{Ax^2+(-A+B)x-B} + C(x^2+4) = 3x^2-3x+5$, id. koeff.

$$\begin{array}{l} x^2: \\ x: \\ 1: \end{array} \begin{cases} A + C \\ -A + B \\ -B + 4C = 5 \end{cases} \Leftrightarrow \begin{cases} A = 2 \\ B = -1 \\ C = 1 \end{cases}$$

$$= \int \left(\frac{2x-1}{x^2+4} + \frac{1}{x-1}\right)dx = \ln(x^2+4) - \int \frac{dx}{\underbrace{x^2+4}_{4\left(\left(\frac{x}{2}\right)^2+1\right)}} + \ln|x-1| \left\{ \begin{matrix} t = \frac{x}{2} \\ dx = 2dt \end{matrix} \right\} =$$

$$= \ln(x^2+4) - \frac{1}{2}\arctan\frac{x}{2} + \ln|x-1| + C$$

"Handpåläggningsmetoden": $^{1)}$ $\displaystyle\frac{5x+7}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

A: "multiplicera med $x+1$": $\displaystyle\frac{5x+7}{x+2} = A + \frac{B(x+1)}{x+2}$

$^{2)}$ sätt $x=-1$ $\displaystyle\frac{-5+7}{-1+2} = A + 0,\ A = 2$

p.s.s: $\displaystyle B = \frac{5(-2)+7}{(-2+1)} = 3$

Fungerar inte alltid:

$$\frac{x^2+4x-1}{(x-1)^2(x+1)} = \overset{A}{\frac{2}{x-1}} + \overset{B}{\frac{2}{(x-1)^2}} - \overset{C}{\frac{1}{(x+1)}}$$

hpl: $\displaystyle C = \frac{1-4-1}{(-2)^2} = -1$

$\displaystyle B = \frac{1+4-1}{1+1} = 2$

A, t.ex. sätt $x=0$

Integraler som dyker upp vid partialbråksuppdelning (efter variabelbyte):

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{dx}{x^k} = \frac{x^{1-k}}{1-k} + C, \quad k = 2,3,\ldots$$

$$\int \frac{dx}{x^2+1} = \arctan x + C$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C$$

$$\int \frac{x}{(x^2+1)^k}\,dx = \frac{1}{2}\frac{(x^2+1)^{1-k}}{1-k} + C$$

$$\int \frac{dx}{(x^2+1)^k} \ ? \quad k = 2,3,\ldots$$

rekursion: $\displaystyle I_n = \int \frac{dx}{(1+x^2)^n} \quad n \geq 1$

om $n \geq 2$: $\displaystyle I_{n-1} = \int \underset{f}{1} \cdot \underset{g}{\frac{dx}{(1+x^2)^{n-1}}} \overset{\text{P.I.}}{=} \frac{x}{(1+x^2)^{n-1}} - \int x(-n+1)\frac{2x}{(1+x^2)^n}\,dx =$

$$= \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{x^2}{(1+x^2)^n}\,dx = \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\left(I_{n-1} - \underline{I_n}\right)$$

lös ut

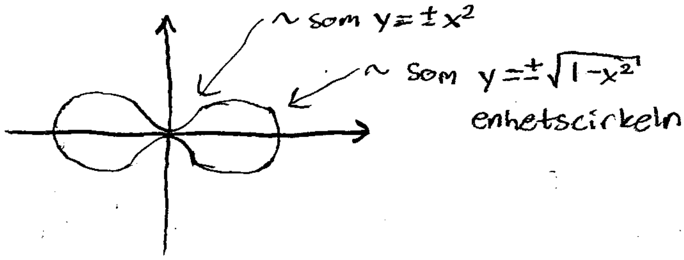
$$\begin{cases} I_n = \dfrac{1}{2(n-1)}\dfrac{x}{(1+x^2)^{n-1}} + \dfrac{2n-3}{2n-2} I_{n-1} \\ I_1 = \arctan x + C \end{cases}$$

ex. $I_3 = \int \frac{dx}{(1+x^2)^3} = \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{4} I_2 = \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{4}\left(\frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2} I_1\right) =$

$= \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{8}\frac{x}{1+x^2} + \frac{3}{8}\arctan x + C$

ex. Vad är arean av det begränsade området inom kurvan $y^2 = x^4 - x^6$

$y^2 = x^4 - x^6 \iff y = \pm x^2\sqrt{1-x^2}$



Sökta arean: $2\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx =$

$= 4\int_0^1 x^2\sqrt{1-x^2}\,dx \left\{\begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array} \quad \begin{array}{c|c} x & t \\ \hline 1 & \pi/2 \\ 0 & 0 \end{array}\right\} =$

$= 4\int_0^{\pi/2} \sin^2 t \underbrace{\sqrt{1-\sin^2 t}}_{= \cos t\ (\text{då } 0 \le t \le \frac{\pi}{2})} \cos t\,dt = \int_0^{\pi/2} \sin^2 2t\,dt =$

$= \frac{1}{2}\int_0^{\pi/2} (1 - \cos 4t)\,dt = \frac{1}{2}\left[t - \frac{\sin 4t}{4}\right]_0^{\pi/2} = \underline{\underline{\frac{\pi}{4}}}$

$\cos 2t = \cos^2 t - \sin^2 t =$

$= 1 - 2\sin^2 t$

$\sin^2 t = \frac{1 - \cos 2t}{2}$

En annan "standardsubstitution" för integranden en rationell funktion av x och $\sqrt{x+b}$ (eller $\sqrt[n]{\frac{ax+b}{cx+d}}$),

tag $t = \sqrt{x+b}$ (eller $t = \sqrt[n]{\frac{ax+b}{cx+d}}$)

ex $\int_1^3 \frac{dx}{\sqrt{x}(x+1)} = \left\{\begin{array}{l} t = \sqrt{x},\ x = t^2 \\ dx = 2t\,dt \end{array} ; \begin{array}{c|c} x & t \\ \hline 3 & \sqrt{3} \\ 1 & 1 \end{array}\right\} = \int_1^{\sqrt{3}} \frac{2t\,dt}{t(t^2+1)} = 2\left[\arctan t\right]_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \underline{\underline{\frac{\pi}{6}}}$

sist $\int \frac{dx}{2+\sin x} = \ldots = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2} + 1}{\sqrt{3}} + C$

$t = \tan\frac{x}{2}$ $\quad \int_0^{2\pi} \frac{dx}{2+\sin x} \neq 0$, men $\left[\frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right]_0^{2\pi} = 0$

$= \left[\quad\right]_0^{\pi-} + \left[\quad\right]_{\pi+}^{2\pi}$