Envariabelanalys 2018-02-19 #16

Först ett par exempel på integration av rationella funktioner

ex. $\displaystyle\int_0^1 \frac{x}{x^2+5x+6}\,dx = \int_0^1 \frac{x}{(x+2)(x+3)}\,dx = \int_0^1 \left(\frac{A}{(x+2)}+\frac{B}{(x+3)}\right)dx =$

A, B ?

$A(x+3)+B(x+2)=x$

$\begin{matrix} x: \\ 1: \end{matrix}\begin{cases} A+B=1 \\ 3A+2B=0 \end{cases} \quad \begin{cases} A=-2 \\ B=3 \end{cases}$

1) täljarens grad < nämnarens? Ja

2) dela upp i partialbråk

$= \left[3\ln|x+3| - 2\ln|x+2|\right]_0^1 = \overset{6\ln 2}{3\ln 4} - 2\ln 3 - (3\ln 3 - 2\ln 2) =$

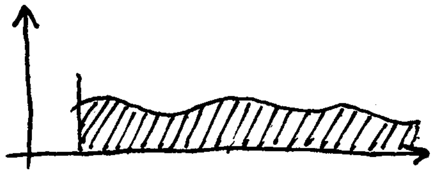
$= 8\ln 2 - 5\ln 3$

Handpåläggning (genväg)

ex. $\displaystyle\int \frac{dx}{x^2+5x+6} = \int \frac{dx}{(x+2)(x+3)} = \int \left(\frac{1}{x+2} + \frac{-1}{x+3}\right)dx = \ln|x+2| - \ln|x+3| + C$

Dagens: Generaliserade integraler, 6.5

$\displaystyle\int_a^\infty f(x)\,dx$



$\displaystyle\sum_{i=1} f(c_i)\,\Delta x_i$

arean över pos. x-axeln

Kan inte göras med riemannsummor, utan som ett gränsvärde.

Def: $\displaystyle\int_a^\infty f(x)\,dx$ betyder $\displaystyle\lim_{R\to\infty} \int_a^R f(x)\,dx$ om det existerar. Då kallas integralen konvergent, annars divergent.

ex. $\displaystyle\int_0^\infty x^3 e^{-x^2}\,dx$? $\displaystyle\int_0^R x^3 e^{-x^2}\,dx = \left\{\begin{matrix} t=x^2 \\ dt=2x\,dx \end{matrix} \;\begin{array}{c|c} x & t \\ R & R^2 \\ 0 & 0 \end{array}\right\} = \int_0^{R^2} \underset{g}{t}\,\underset{f}{e^{-t}} \cdot \tfrac{1}{2}\,dt =$ partiell integration

$= \left[\tfrac{1}{2}\underset{F}{(-e^{-t})}\,t\right]_0^{R^2} - \displaystyle\int_0^{R^2} \tfrac{1}{2}\underset{F}{(-e^{-t})}\cdot \underset{g'}{1}\,dt = \left[-\tfrac{1}{2}te^{-t} - \tfrac{1}{2}e^{-t}\right]_0^{R^2} = -\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} - (0-\tfrac{1}{2}) \xrightarrow[R\to\infty]{} \tfrac{1}{2},$

så $\displaystyle\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2}$

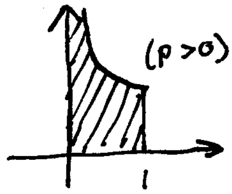
ex. enl. ovan $\displaystyle\int_0^\infty \frac{x}{x^2+5x+6}\,dx = \ldots = \lim_{R\to\infty} \Big[\underbrace{3\ln|x+3| - 2\ln|x+2|}_{2\ln\frac{x+3}{x+2} + \ln(x+3)}\Big]_0^R \to \infty$ så integralen är divergent

$\underset{\to 0}{2\ln\frac{x+3}{x+2}} + \underset{\to\infty}{\ln(x+3)}$

$\displaystyle\int_0^\infty \underbrace{\frac{1}{x^2+5x+6}}_{\sim\frac{1}{x^2},\text{ stora } x}\,dx = \ldots = \lim_{R\to\infty} \Big[\underbrace{\ln(x+2) - \ln(x+3)}_{\ln\frac{x+2}{x+3}}\Big]_0^R = 0 - \ln\frac{2}{3} = \ln\frac{3}{2}$, så konvergent integral

ex (sats) $R>1$ $\displaystyle\int_1^R \frac{dx}{x^p} = \begin{cases} [\ln x]_1^R, & p=1 \\ \left[\frac{x^{1-p}}{1-p}\right]_1^R, & p\neq 1 \end{cases}$ så konvergent, $=\frac{1}{p-1}$ om $p>1$
divergent om $p\leq 1$

och (generaliserad i en punkt):



$$\int_0^1 \frac{dx}{x^p} \overset{\text{def}}{\to} \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{dx}{x^p}$$

$$\int_\varepsilon^1 \frac{dx}{x^p} = \begin{cases} [\ln x]_\varepsilon^1, & p=1 \\ \left[\frac{x^{1-p}}{1-p}\right]_\varepsilon^1, & p\neq 1 \end{cases}$$ så $\displaystyle\int_0^1 \frac{dx}{x^p}$ är konvergent då $p<1$
divergent då $p\geq 1$

ex $$\int_0^1 \frac{dx}{\sqrt{x}(x+1)} = \left\{ \begin{array}{l} t=\sqrt{x} \\ dt=\frac{dx}{2\sqrt{x}} \end{array}, \begin{array}{c|c} x & t \\ 1 & 1 \\ 0 & 0 \end{array} \right\} = \int_0^1 \frac{2\,dt}{t^2+1} = 2[\arctan t]_0^1 = 2\left(\frac{\pi}{4}-0\right) = \underline{\underline{\frac{\pi}{2}}}$$

generaliserad integral

Sats: Om $0\leq f(x)\leq g(x)$ för alla x i integrationsintervallet (eller för alla x nära singularitetspunkten) så

$$\int g(x)\,dx \text{ konv.} \Rightarrow \int f(x)\,dx \text{ konv.}$$

ekvivalent: $\int f(x)\,dx$ divergent $\Rightarrow \int g(x)$ divergent

ex. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ är konvergent, ty $x^3+1>x^3>0$ då $x>1$, så $\sqrt{x^3+1}>\sqrt{x^3}>0$,
så $\frac{1}{\sqrt{x^3}} > \frac{1}{\sqrt{x^3+1}} > 0$ och $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ konvergent $\left(\frac{3}{2}>1\right)$

ex. $\displaystyle\int_1^2 \frac{dx}{\sqrt{x^2-1}}$, konv/div? Jo $\underset{>1}{\sqrt{x^2-1}} = \sqrt{x-1}\underbrace{\sqrt{x+1}}_{>1} > \sqrt{x-1} > 0$ då $x>1$

så $\frac{1}{\sqrt{x-1}} > \frac{1}{\sqrt{x^2-1}} > 0$

$\left(\displaystyle\int_0^1 \frac{dt}{t^{1/2}}\right)\int_1^2 \frac{dx}{\sqrt{x-1}}$ konvergent, så den givna $\int$ också konvergent.

$$\left(\int_1^2 \frac{dx}{\sqrt{x^2-1}} = \left[\ln\left|x+\sqrt{x^2-1}\right|\right]_1^2 = \ln(2+\sqrt{3})-\ln(2+\sqrt{3})\right)$$

Varning: $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$ är divergent, trots att $\displaystyle\int_{-R}^{R}\frac{x}{x^2+1}\,dx = 0$ för alla R

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{x}{1+x^2}\,dx + \int_0^\infty \frac{x}{1+x^2}\,dx \qquad \int_0^\infty \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\ln(x^2+1)\right]_0^\infty \text{ divergent}$$

kräver vi

Divergenta

ex. $\int_0^\infty \frac{\ln x}{x^2+1}\,dx$?

generaliserad i båda gränsvärdena

$\int_0^1 \frac{\ln x}{x^2+1}\,dx$ konvergent, ty $0 \le -\frac{\ln x}{x^2+1} \le \ln x$ i $]0,1[$

och $\int_\varepsilon^1 -\ln x\,dx \overset{PI}{=} -[x\ln x]_\varepsilon^1 + \int_\varepsilon^1 \frac{x}{x}\,dx =$

$$= -[x\ln x - x]_\varepsilon^1 = -(0-1-(\underbrace{\varepsilon\ln\varepsilon}_{\to 0} - \underbrace{\varepsilon}_{\to 0})) \to 1$$

$$\int_1^\infty \frac{\ln x}{x^2+1}\,dx = \left\{ \begin{array}{l} x=\frac{1}{t} \\ dx=-\frac{dt}{t^2} \end{array} ; \begin{array}{c|c} x & t \\ \hline \infty & 0+ \\ 1 & 1 \end{array} \right\} = \int_1^{0+} \frac{\ln(\frac{1}{t})}{\frac{1}{t^2}+1}\cdot\frac{-1}{t^2}\,dt = -\int_0^1 \frac{\ln t}{1+t^2}\,dt \text{ så } \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0\text{, konv}$$

Ex Vad är $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=0/1}^{n} k\arctan\frac{k}{n} =$

$$= \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \frac{k}{n}\arctan\frac{k}{n}\cdot\frac{1}{n}}_{\text{Riemannsumma}} = \int_0^1 \underset{f}{x}\,\underset{g}{\arctan x}\,dx = \Big[\underset{F}{\frac{x^2}{2}}\,\underset{g}{\arctan x}\Big]_0^1 - \int_0^1 \underset{F}{\frac{\overset{x^2+1-1}{x^2}}{2}}\cdot\underset{g'}{\frac{1}{x^2+1}}\,dx =$$

$$= \Big[\frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x\Big]_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\cdot\frac{\pi}{4} = \underline{\frac{\pi}{4} - \frac{1}{2}} \qquad \left(\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}\right)$$

Ex $I_n = \int_0^\infty \underset{g}{t^n}\,\underset{f}{e^{-t}}\,dt \overset{PI}{=} \Big[\underset{F}{-e^{-t}}\cdot\underset{g}{t^n}\Big]_0^\infty - \int_0^\infty (-e^{-t})\cdot n\cdot t^{n-1}\,dt = 0 + n\cdot I_{n-1}$, $I_0 = \int_0^\infty e^{-t}\,dt = 1$

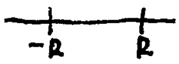
$I_n \begin{cases} n I_{n-1} & n=1,2,\dots \\ 1 & n=0 \end{cases}$, så $I_n = n!$ för alla $n = 0,1,2,\dots$

Vi kan definiera $\alpha! = \underbrace{\int_0^\infty t^\alpha e^{-t}\,dt}_{=\,\Gamma(\alpha+1)}$ då $\alpha > -1$ t.ex. $\frac{1}{2}! = \frac{\sqrt{\pi}}{2}$, se flervariabelanalys

PI ger $\alpha! = (\alpha-1)!$ om $\alpha > 0$

Så vi kan definiera $(\alpha-1)! = \frac{1}{\alpha}\cdot\alpha!$ som ger $\alpha!$ för alla $\alpha \ne -1,-2,\dots$

Volymen av en "boll" i n dimensioner? (radie R)

n		"volym"
1		$2R$
2		πR^2
3		$\frac{4\pi}{3}R^3$
4		$\frac{\pi^2}{2}R^4$

Jo, $\frac{\pi^{n/2}}{\frac{n}{2}!}R^n$