Envariabelanalys 2018-02-13 #14

Om integraler med primitiva funktioner (antiderivator)

Minns • Om $F'(x)=f(x)$ i $]a,b[$, $F(x)$ kont. i $[a,b]$, så är $\int_a^b f(x)dx = F(b)-F(a) = \left[F(x)\right]_a^b$

• Om $f(x)$ är kontinuerlig i ett intervall kring $x=a$ så är $\frac{d}{dx}\underbrace{\int_a^x f(t)dt}_{F(x)} = f(x)$

Ex. $\int_3^{16} \underbrace{\sqrt[3]{33-2x}}_{(33-2x)^{\frac{1}{3}}}dx = \left[-\frac{3}{8}(33-2x)^{4/3}\right]_3^{16} = -\frac{3}{8}\cdot 1^{4/3} - \left(-\frac{3}{8}\cdot 27^{4/3}\right) = -\frac{3}{8}+\frac{3}{8}\cdot 81 = 30$ $D(33-2x)^{\frac{4}{3}} = \frac{4}{3}(33-2x)^{\frac{1}{3}}(-2) = -\frac{8}{3}\sqrt[3]{33-2x}$

$D\ln|x| = \frac{1}{x}$

ex. $\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x}dx = \left[\ln|\cos x + \sin x|\right]_0^{\pi/4} = \ln\sqrt{2} - \ln 1 = \underline{\frac{1}{2}\ln 2}$ $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$

ex. $\int_0^{\pi/4} \frac{dx}{1+\cos 2x} = \frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^2 x} = \frac{1}{2}\left[\tan x\right]_0^{\pi/4} = \frac{1}{2}(1-0) = \frac{1}{2}$ $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

<u>men</u> $\int_0^{\pi} \frac{dx}{1+\cos 2x}\ ? \neq \frac{1}{2}\left[\tan x\right]_0^{\pi} = 0$ $\tan x$ är inte en primitiv funktion i hela $(0,\pi)$!

ex. $\frac{d}{dx}\int_x^{x^2} e^{-t^2}dt = \frac{d}{dx}\left(\int_0^{x^2}e^{-t^2}dt - \int_0^x e^{-t^2}dt\right) = \underbrace{e^{-(x^2)^2}}_{\text{y.d.}}\cdot\underbrace{2x}_{\text{i.d}} - e^{-x^2}$

Variabelsubstitution 5.6 "kedjeregeln baklänges"

• Om $f(t)$ är kontinuerlig, $g(x)$ deriverbar m. $g'(x)$ integrerbar:

$\int f(g(x))g'(x)dx = \int f(t)dt\Big|_{t=g(x)}$ $\frac{d}{dx}F(g(x)) = f(g(x))\cdot g'(x)$

$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt$ skriver oftast "$t=g(x)$, så $dt = g'(x)dx$, $\begin{array}{c|c} x & t \\ \hline b & g(b) \\ a & g(a)\end{array}$"

ex. $\int \frac{\sin(\ln x)}{x}dx \begin{cases} t=\ln x \\ dt = \frac{1}{x}dx\end{cases} \int \sin t\,dt = -\cos t + C = -\cos(\ln x)+C$

ex. $\int_0^{\pi/2} \sin x\cdot\underbrace{\sin 2x}_{2\sin x\cos x}dx = 2\int_0^{\pi/2}\sin^2 x\cos x\,dx \left\{\begin{array}{l} t=\sin x \\ dt=\cos x\,dx\end{array}\ \begin{array}{c|c} x & t \\ \hline \pi/2 & 1 \\ 0 & 0\end{array}\right\} 2\int_0^1 t^2dt = \left[\frac{2t^3}{3}\right]_0^1 = \frac{2}{3}$

$\left[\text{alt. } \sin x\cdot\sin 2x = \frac{1}{2}(\cos(2x-x) - \cos(2x+x))\ldots\right]$

Vanliga substitutioner: $\int f(\cos x)\sin x\,dx = -\int f(t)dt\Big|_{t=\cos x}$

$\int f(\sin x)\cos x\,dx = \int f(t)dt\Big|_{t=\sin x}$

$\int f(\tan x)\,dx \begin{cases} t=\tan x \\ dt=(1+\tan^2 x)dx \quad dx=\frac{dt}{1+t^2}\end{cases} \int\frac{f(t)}{1+t^2}dt\Big|_{t=\tan x}$ i intervall där $\tan x$ är snäll

Integranden kan behöva omformas

$\cos^2 x = \frac{1}{1+\tan^2 x}$, $\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x}$, $\cos x\sin x = \frac{\tan x}{1+\tan^2 x}$

ex $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx = \int \frac{\tan x}{\tan x+1}\,dx \left\{\begin{matrix} t=\tan x \\ \dots \end{matrix}\right\} \int \frac{t}{(1+t)(1+t^2)}\,dt \Big|_{t=\tan x}$ i intervall där $\tan x$ def och $\tan x \neq -1$

kan vi snart lösa!

(enklare $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx = \int \frac{\frac{1}{2}(\sin x+\cos x)+\frac{1}{2}(\sin x-\cos x)}{\sin x+\cos x}\,dx = \frac{x}{2} - \frac{1}{2}\ln|\sin x+\cos x| + C$)

med $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

ex $\displaystyle\int \frac{dx}{2+\sin x}$ "Fungerar alltid": $t=\tan\frac{x}{2}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$

$dt = \frac{1}{2}(1+\tan^2\frac{x}{2})\,dx$, $dx = \frac{2}{1+t^2}\,dt$

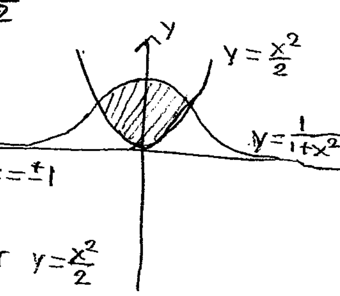
$$\int \frac{1}{2+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int\frac{dt}{t^2+t+1} = \int\frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{4}{3}\int\frac{dt}{1+\left(\frac{2(t+\frac{1}{2})}{\sqrt{3}}\right)^2} \left\{\begin{matrix} u=\frac{2(t+\frac{1}{2})}{\sqrt{3}} \\ du=\frac{2}{\sqrt{3}}\,dt \end{matrix}\right\}$$

$$= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{du}{1+u^2} = \frac{2}{\sqrt{3}}\arctan u + C = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}} + C \quad (!)$$

Ex. Finn arean av det begränsade området mellan $y=\frac{1}{1+x^2}$ och $y=\frac{x^2}{2}$

Skärningspunkterna mellan graferna:

$$\frac{1}{1+x^2} = \frac{x^2}{2} \Leftrightarrow (x^2)^2 + x^2 - 2 = 0 \Leftrightarrow x^2 = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = \begin{cases} 1 \\ (-2) \end{cases}, \text{ så (reella) } x=\pm 1$$

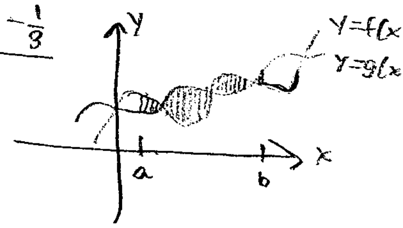


Det gäller området mellan $x=-1$ och $x=1$, under $y=\frac{1}{1+x^2}$ och över $y=\frac{x^2}{2}$

$$\int_{-1}^{1}\left(\frac{1}{1+x^2}-\frac{x^2}{2}\right)dx = 2\int_0^1\left(\frac{1}{1+x^2}-\frac{x^2}{2}\right)dx = 2\left[\arctan x - \frac{x^3}{6}\right]_0^1 = 2\left(\frac{\pi}{4}-\frac{1}{6}\right) = \frac{\pi}{2}-\frac{1}{3}$$

Allmänt: arean mellan $y=f(x)$, $y=g(x)$, $x=a$, $x=b$:

$$\int_a^b |f(x)-g(x)|\,dx$$



Partiell integration, 6.1 "derivatan av en produkt baklänges"

- $F(x)$ primitiv funktion till $f(x)$, $g(x)$ deriverbar ($f(x)g(x)$, $F(x)g'(x)$ integrerbara),

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

ty $(Fg)' = F'g + Fg' = fg + Fg'$

(I boken: $\displaystyle\int U\underbrace{dV}_{V'dx} = UV - \int V\overbrace{dU}^{U'dx}$)

ex $$\int \underset{f}{x}\,\underset{g}{\ln x}\,dx = \underset{F}{\frac{x^2}{2}}\cdot\underset{g}{\ln x} - \int \underset{F}{\frac{x^2}{2}}\cdot\underset{g'}{\frac{1}{x}}\,dx = \frac{x^2}{2}\ln x - \int\frac{x}{2}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$$\int \ln x\,dx = \int \underset{f}{1}\cdot\underset{g}{\ln x}\,dx = \underset{F}{x}\cdot\underset{g}{\ln x} - \int \underset{F}{x}\cdot\underset{g'}{\frac{1}{x}}\,dx = x\cdot\ln x - x + C$$

$$\int \ln(x^2+1)\,dx = \int \underset{f}{1}\cdot\underset{g}{\ln(x^2+1)}\,dx = \underset{F}{x}\cdot\underset{g}{\ln(x^2+1)} - \int \underset{F}{x}\cdot\underset{g'}{\frac{2x}{x^2+1}} = x\cdot\ln(x^2+1) - 2\int\frac{x^2+1-1}{x^2+1}\,dx =$$

$$= x\ln(x^2+1) - 2x + 2\arctan x + C$$

ex $\int \sin\sqrt{x}\,dx \begin{cases} t=\sqrt{x} \\ dt=\frac{1}{2\sqrt{x}}dx\ ,\ dx=2t\,dt \end{cases} \int \underset{f}{\sin t}\cdot \underset{g}{2t}\,dt = \underset{F}{-\cos t}\cdot \underset{g}{2t} - \int \underset{F}{(-\cos t)}\cdot \underset{g'}{2}\,dt = -2t\cos t + 2\sin t + C =$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

ex $\int \underset{f}{e^{ax}}\,\underset{g}{\sin bx}\,dx = \underset{F}{\frac{1}{a}e^{ax}}\,\underset{g}{\sin bx} - \int \underset{F}{\frac{1}{a}e^{ax}}\,\underset{g'}{b\cos bx}\,dx =$

$a,b \neq 0$

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int \underset{f_1}{e^{ax}}\,\underset{g_1}{\cos bx}\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left(\underset{F_1}{\frac{1}{a}e^{ax}}\,\underset{g_1}{\cos bx} - \int \underset{F_1}{\frac{1}{a}e^{ax}}\,\underset{g_1'}{b(-\sin bx)}\,dx\right) =$

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$

$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx + C$ ger $\int e^{ax}\sin bx\,dx = \frac{1}{1+\frac{b^2}{a^2}}\cdot(\ldots)$