SF1626 2018-09-06 #6

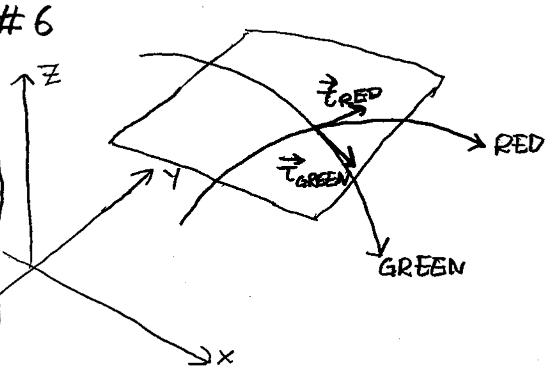
RED CURVE: $\gamma_1(t)\vec{i} + \gamma_2(t)\vec{j} + \gamma_3(t)\vec{k}$ $= f(\gamma_1(t), \gamma_2(t))$

LIES ON THE GRAPH OF $f(x,y)$

GREEN CURVE: $\vec{\tau}_{GREEN}(t)$

$(a, b, f(a,b))$ $\quad a\vec{i} + y\vec{j} + f(a,y)\vec{k}$

$$\vec{\tau}_{RED}(t) = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \gamma_3'(t) \end{pmatrix} \qquad \tau_{GREEN}(y) = \begin{pmatrix} 0 \\ 1 \\ f_2(a,y) \end{pmatrix}$$



General tangent plane: $z = f_1(a,b)(x-a) + f_2(a,b)(y-b) + f(a,b)$

Normal line: $a\vec{i} + b\vec{j} + f(a,b)\vec{k} + t(n_1\vec{i} + n_2\vec{j} + n_3\vec{k})$

Higher-order derivatives

$$f_{11}(x,y) = \frac{\partial}{\partial x} f_1(x,y) \qquad f_{12}(x,y) = \frac{\partial}{\partial y} f_1(x,y)$$

$$f_{21}(x,y) = \frac{\partial}{\partial x} f_2(x,y) \quad \ldots \quad f_{321} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} f\right)\right) \left(= f_{zyx}\right)$$

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b) \text{ if } f_x, f_y, f_{xy} \text{ and } f_{yx} \text{ are all continous at } (a,b)}$$

Similarly $f_{xyx} = f_{xxy} = f_{yxx}$ if partial derivatives are continous

Laplace and wave equation

Laplace equation for $(x,y) \to \mathbb{R}$: $u_{xx} + u_{yy} = 0$
If fullfilled then u is called harmonic

Wave equation for $(x,y,t) \to \mathbb{R}$: $u_{tt} = u_{xx} + u_{yy}$

$\Delta f = 0$

Δ = $f_{xx} + f_{yy}$ — Laplace operator

Ex. $f(x,y) = e^{kx}\cos(ky)$ Show that f solves the laplace equation.

$$f_x = kf \Rightarrow f_{xx} = k^2 f \qquad f_y = -ke^{kx}\sin(ky) \Rightarrow f_{yy} = -k^2 e^{kx}\cos(ky) = -k^2 f$$

$$\Rightarrow f_{xx} + f_{yy} = 0$$

Linear approximations

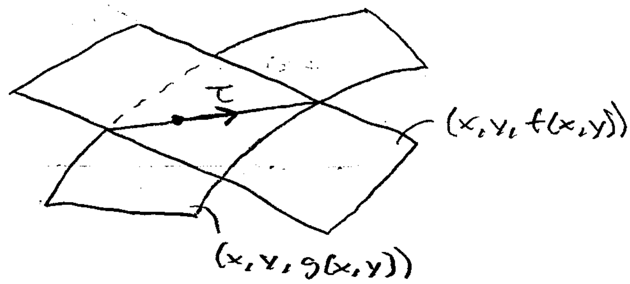
Tangent plane $L(x,y)$ of f at (a,b) as a function is called the linearization of f at a,b)

$$L(x,y): z \quad z = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$$

Similar things to taylor-polynomials exist for multiple variables

Given two functions $f(x,y)$, $g(x,y)$. Find the tangent to the intersection

$\vec{n_f} = \nabla(z - f(x,y))$ normal to f
$\vec{n_g} = \nabla(z - g(x,y))$ normal to g



$\tau \,//\, \underbrace{\vec{n_f} \times \vec{n_g}}$

orthogonal to both normal vectors

$L(x,y)$ is an affine function. $L(x,y) - f(a,b)$ is a linear function of x and y

$L(a+h, b+y) \approx f(a,b)$: in general, L is a very good approximation of f near (a,b).

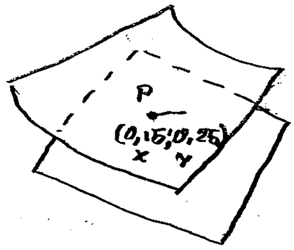
Later, if f can be approximated "well enough" by L, we say that f is differentiable

ex. $f(x,y) = \sqrt{2x^2 + e^{2y}}$ find partial derivatives:

Find the tangent plane to f at the point $(0{,}15;0{,}25)$
$\qquad\qquad\qquad\qquad\qquad\qquad x \qquad y$

$$f_x = \frac{2x}{f(x,y)}$$

$$f_y = \frac{e^{2y}}{f(x,y)}$$



tangent plane:
$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

The plane is a good approximation to the curve near the point P.

Def

Def $\underline{n}$ 5 Let (x,y) be the linearization of f at (a,b)
We say that $f(x,y)$ is differentiable at the point (a,b) if:

$$\lim_{(h,k)\to(0,0)} \left(\frac{L(x+h, y+k) - f(a,b)}{\sqrt{h^2+k^2}} \right) = 0$$

This means that the difference between $L(x+h, y+k)$ and $f(a,b)$ approaches zero "faster" than $\sqrt{h^2+k^2}$ aproaches zero, as $(h,k) \to (0,0)$

Mean Value theorem

If $f_1(x,y)$ and $f_2(x,y)$ are constants in a neighborhood of the point (a,b) and if $|h|$ and $|k|$ are sufficiently small, then there exist θ_1 and θ_2 between 0 and 1 such that

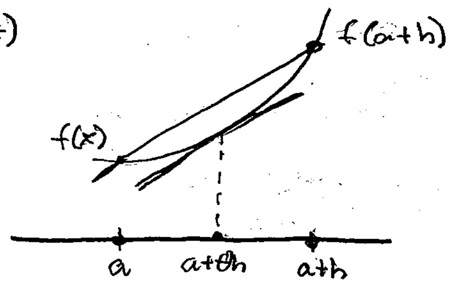
$$f(a+h, b+k) - f(a,b) = h \cdot f_x(a+\theta_1 h, b+k) + k f_y(a, b+\theta_2 k)$$

see image (*) (next page)

Theorem 4

If $f_1(x,y)$ and $f_2(x,y)$ are continous in a neighborhood of (a,b), then f is differentiable at (a,b)

(*)



Differentials

The differential of $f(x,y)$ is defined as $df : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$df \equiv df(x,y,dx,dy) := f_x(x,y)dx + f_y(x,y)dy$$

Compare df to the linearization of f around (x,y)

$$L(x+dx, y+dy) := f(x,y) + f_1(x,y)dx + f_2(x,y)dy$$ ← Almost like Taylor-polynomials (SF1625)

It approximates the change of f

General formulas for $f: \mathbb{R}^n \to \mathbb{R}^m$

$y_1 = f(x_1, x_2, \dots, x_n)$ (1)
$y_2 = f(x_1, x_2, \dots, x_n)$ (2)

$y_m = f(x_1, x_2, \dots, x_n)$ (m)

The Jacobian matrix of f

$$Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right) \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & \ddots & \vdots \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$$dy := Df(x)dx$$

Chain rule

$g: \mathbb{R}^m \to \mathbb{R}^k$

$$D(g \circ f)(x) = Dg(f(x))\, Df(x)$$