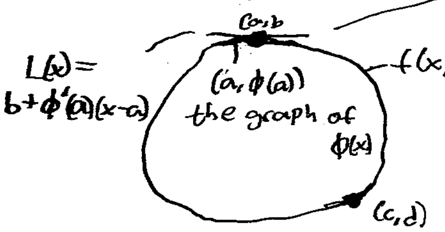
$f(x,y)=0$



if $f_y(a,b) \neq 0$
$\Rightarrow y=\phi(x),\quad b=\phi(a)$

$$\phi'(a) = -\frac{f_1(a,b)}{f_2(a,b)}$$

- $\dfrac{\partial \phi_i}{\partial x_j} = ?$

(Last lecture)

$$F_{(k)}(x_1,\dots,x_j,\dots,x_m,\ \phi_1(x_1,\dots,x_m),\dots,\phi_i(x_1,\dots,x_m),\dots)=0$$

$$\frac{d}{dx_j}F_{(k)} = \frac{\partial F_{(k)}}{\partial x_j} + \sum_{\ell=1}^{n}\frac{\partial F_{(k)}}{\partial y_\ell}\frac{\partial \phi_\ell}{\partial x_j} = 0,\quad k=1,2,\dots,n$$

$$\left(\frac{\partial F_{(k)}}{\partial y_\ell}\right)_{j,\ell} D\phi = -\left(\frac{\partial F_{(k)}}{\partial x_j}\right)_{k,j}$$

Extreme values

We work with: $U \subset \mathbb{R}^n \to \mathbb{R}$ Important:

- Def^{n} (a,b) is a critical point of f if $\nabla f(a,b)=0$
- f attains an extreme value (max or min) at a critical point, singular point or a boundary point
- The Hessian matrix of a function
- How to use the Hessian matrix to classify the critical point of f.
- The definition of a saddle point of f.

All directional derivatives must have a positive/negative derivative (same sign) at a point for that point to be a minimum/maximum

Examples of different extremums in highlights on GitHub.

Classifying interior critical points by the Hessian matrix of f

$f(x,y)$ $\vec{a} \in \mathbb{R}^2$ $f(\vec{a})$ $\vec{a}+t\vec{h}$ — a line in the domain
$f(\vec{a}+t\vec{h})$

The function $g(t) := f(\vec{a}+t\vec{h})$ is a single variable function that reveals how the value of f changes along the curve $\vec{a}+t\vec{h}$

- a is an interior critical point of f:

$$g''(t) = (\vec{h}\cdot\nabla)f(\vec{a}+t\vec{h})h_ih_j = \vec{h}^T\mathcal{H}(\vec{a}+t\vec{h})\vec{h} = h_1f_1(\vec{a}+t\vec{h}) + h_2f_2(\vec{a}+t\vec{h}) + \dots$$

- Classify the interior critical points by g'':

$$g''(t) = (\vec{h}\cdot\nabla)^2 f = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(\vec{a}+t\vec{h})h_ih_j = \vec{h}^T\mathcal{H}(\vec{a}+t\vec{h})\vec{h}$$

$f : \mathbb{R}^n \to \mathbb{R}$

$\vec{a} \in \mathbb{R}^n$, $\vec{h} \in \mathbb{R}^n$ } column vectors

the line: $\vec{a} + t\vec{h}$

The restriction of f along the curve

$g(t) = f(\vec{a} + t\vec{h})$

$g''(0) = \vec{h}^T H \vec{h}$

The Hessian matrix of f

$$H(\vec{x}) := \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}$$

- For every $\vec{x}$, $H(\vec{x})$ is a square matrix (symmetric if f has continous partial derivatives up to at least second order)
- f has a local minimum at $\vec{a}$ if $g''(0) > 0$ for every possible $\vec{h}$; i.e if H is positive definite
- A matrix is positive (negative) definite if all it's eigenvalues are positive (negative)

Ex $f(x,y,z) = x^2y + y^2z + z^2 - 2x$

CRITICAL PT:

$$\nabla f(x,y,z) = \vec{0} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} 2xy - 2 \\ x^2 + 2yz \\ y^2 + 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow P = (1, 1, -\frac{1}{2})$

$$H(x,y,z) = \begin{pmatrix} \underset{f_{xx}}{2y} & \underset{f_{xy}}{2y} & \underset{f_{xz}}{0} \\ \underset{f_{yx}}{2x} & \underset{f_{yy}}{2z} & \underset{f_{yz}}{2y} \\ \underset{f_{zx}}{0} & \underset{f_{zy}}{2y} & \underset{f_{zz}}{2} \end{pmatrix} \quad H(P) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

Eigen values $= \begin{pmatrix} -2{,}701\ldots \\ 2 \\ 3{,}701\ldots \end{pmatrix}$

Ex $f(x,y) := 1 - x$

- Interior critical point? NO

$\nabla f = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

- Singular point? NO
- Boundary point?

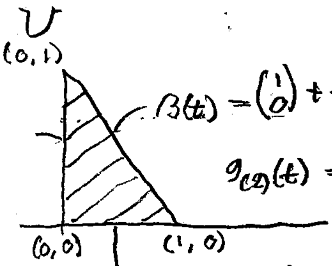
Boundary: $(\cos(t), \sin(t))$

$g(t) = f(\cos(t), \sin(t))$

$g'(t) = \underset{-1}{f_1}(-\sin(t)) + \cancel{f_2 \cdot (\cos(t))}$

$\sin(t) = 0$, $t = 0$ (loc min) or π (loc max)

$g''(t) = \cos(t)$ $\quad \cos(0) = 1 > 0$



$\beta(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$g_{(2)}(t) = f(\beta) = f(1-t, 0+t)$

$\alpha(t) = (t, 0)$ $\quad 0 \le t \le 1$

$g_{(1)}(t)\ f(t, \alpha(t))$