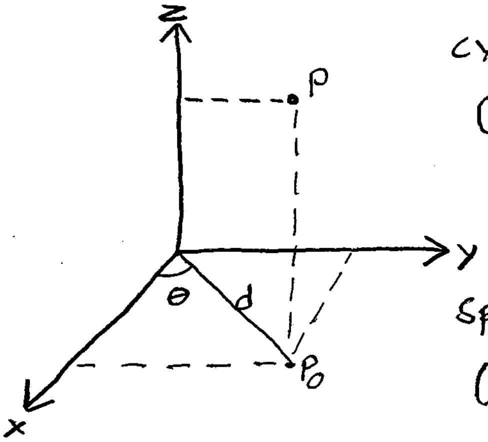
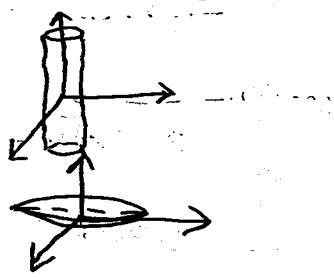
SF1626 2018-08-28 #2

Back to the cylindrical and spherical coordinates.



cylindrical coordinates:

$(x, y, z) \to (d, \theta, z)$

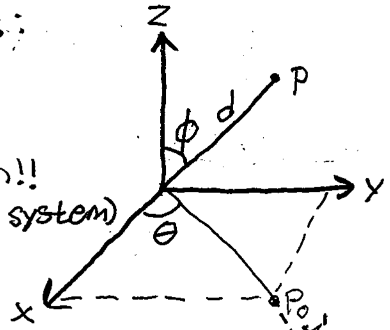


Spherical coordinates:

$(x, y, z) \to (d, \phi, \theta)$

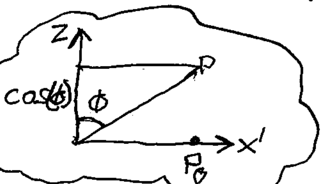
⇑ new position!! (right handed system)

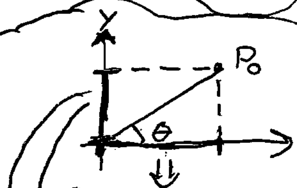
$d \geq 0$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \pi$



Problem: Given a point in spherical coordinates (d, ϕ, θ), what are the cartesian coordinates for that point?

$$x = f_1(\theta, d, \phi) = \overline{OP_0}\cos\theta = d\sin(\phi)\cos(\theta)$$
$$y = f_2(\theta, d, \phi) = \overline{OP_0}\sin\theta = d\sin(\phi)\sin(\theta)$$
$$z = f_3(\theta, d, \phi) = f_3(d, \phi) = d\cos(\phi)$$





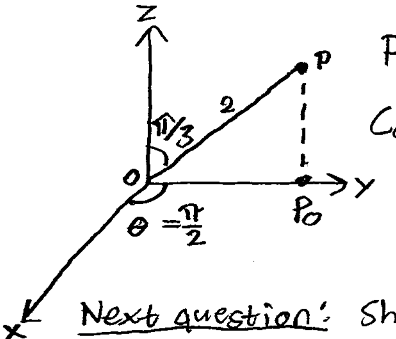
$\overline{OP_0}\cos(\theta) = d\sin(\phi)\cos(\theta)$

$\overline{OP_0}\sin(\theta) = d\sin(\phi)\sin(\theta)$

Question: We consider a system with spherical coordinates and a point P with coordinates $P = (2, \pi/3, \pi/2)$

Find the cartesian coordinates of P.

$$x = 2\sin(\pi/3)\cos(\pi/2) = 2 \cdot \quad \cdot 0 = 0$$
$$y = 2\sin(\pi/3)\sin(\pi/2) = 2 \cdot \tfrac{\sqrt{3}}{2} \cdot 1 = \sqrt{3}$$
$$z = 2\cos(\pi/3) = 2 \cdot \tfrac{1}{2} = 1$$



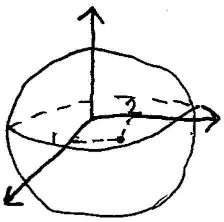
P lies on the plane $x = 0$

Cartesian coordinates of P are $(0, 2\sin(\pi/3), 2\cos(\pi/3)) = (0, \sqrt{3}, 1)$

Next question: Show that the surface $p = 4\sin(\phi)\sin(\theta) + 2\sin(\phi)\cos(\theta)$ corresponds to $\boxed{(x-1)^2 + (y-2)^2 + z^2 = 5}$

↓

sphere with center $(1, 2, 0)$ and radius $\sqrt{5}$



(Not answered during lecture)

Problem: Given (x,y,z), how do we find the corresponding (d,ϕ,θ)?

$$d=\sqrt{x^2+y^2+z^2} \quad \text{(length of segment } \overline{OP}\text{)}$$

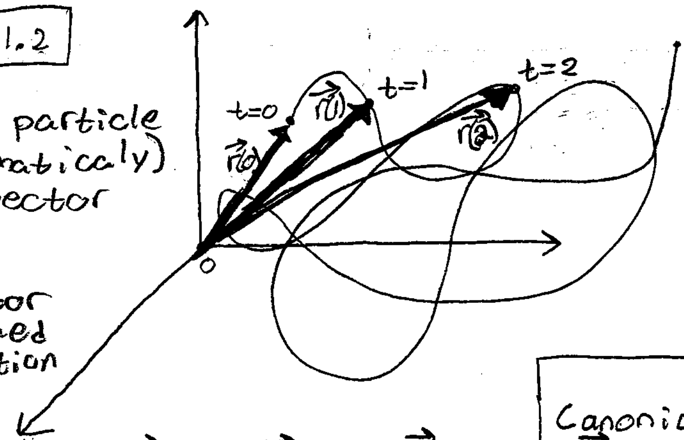
$$\cos(\phi)=\frac{z}{d}=\frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \phi=\arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\tan(\theta)=\frac{y}{x} \Rightarrow \theta=\arctan\left(\frac{y}{x}\right)$$

$$\Rightarrow \begin{cases} d=\sqrt{x^2+y^2+z^2} \\ \phi=\arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \theta=\arctan(y/x) \end{cases}$$

Sections 11.1, 11.2

Position of the particle can be (mathematicaly) described by a vector



$f(t)=\vec{r}(t)=$

time ↑ vector valued function

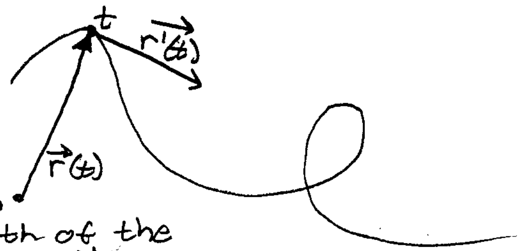
$=(r_1(t), r_2(t), r_3(t))=r_1(t)\vec{i}+r_2(t)\vec{j}+r_3(t)\vec{k}$

Canonical base of $\mathbb{R}^3$:
$\vec{i}=(1,0,0)$
$\vec{j}=(0,1,0)$
$\vec{k}=(0,0,1)$

Given the vector-valued function $\vec{r}(t)=(x(t), y(t), z(t))$

velocity: $\frac{d}{dt}\vec{r}(t)=(x'(t), y'(t), z'(t))=\vec{r}'(t)$

tangent vector to the curve at the time t.

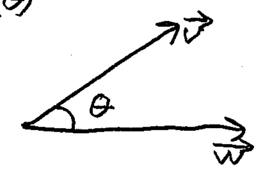


magnitude of the veloöty: $|\vec{r}'(t)|$ =: length of the vector $\vec{r}'(t)$

$$=\sqrt{(x')^2(t)+(y')^2(t)+(z')^2(t)}$$

Accelleration: $\vec{r}''(t)=(x''(t), y''(t), z''(t))$

Given two vectors $\vec{v}=(v_1,v_2,v_3)$, $\vec{w}=(w_1,w_2,w_3)$, then the DOT-PRODUCT (OR SCALAR PRODUCT) is defined as: $\boxed{\vec{v}\cdot\vec{w}:=v_1w_1+v_2w_2+v_3w_3}=|\vec{v}||\vec{w}|\cos(\theta)$



$$\vec{v}\cdot\vec{v}=v_1^2+v_2^2+v_3^2=|\vec{v}|^2$$

$$\Rightarrow |\vec{v}|=\sqrt{\vec{v}\cdot\vec{v}}$$

$\vec{v}(t) = (v_1(t), v_2(t), v_3(t))$, $\vec{w}(t) = (w_1(t), w_2(t), w_3(t))$

$$\frac{d}{dt}\vec{v}(t)\cdot\vec{w}(t) = \frac{d}{dt}\left[v_1(t)w_1(t) + v_2(t)w_2(t) + v_3(t)w_3(t)\right] = v_1'(t)w_1(t) + v_1(t)w_1'(t) + v_2'(t)w_2(t) + v_2(t)v_2'(t) + v_3'(t)w_3(t) + v_3(t)w_3'(t)$$

$$= \vec{v}'(t)\cdot\vec{w}(t) + \vec{v}(t)\,\vec{w}'(t)$$

$\Rightarrow$ ① $\frac{d}{dt}\vec{v}\cdot\vec{w} = \vec{v}'\cdot\vec{w} + \vec{v}\cdot\vec{w}'$

$$\frac{d}{dt}|\vec{v}(t)| = \frac{d}{dt}\sqrt{v_1^2(t) + v_2^2(t) + v_3^2(t)}$$

$$= \frac{1}{2\sqrt{v_1^2(t) + v_2^2(t) + v_3^2(t)}}\cdot\frac{d}{dt}\left(v_1^2(t) + v_2^2(t) + v_3^2(t)\right)$$

$$= \frac{1}{\cancel{2}\underbrace{\sqrt{v_1^2(t) + v_2^2(t) + v_3^2(t)}}_{|\vec{v}|}}\left(\cancel{2}v_1(t)v_1'(t) + \cancel{2}v_2(t)v_2'(t) + \cancel{2}v_3(t)v_3'(t)\right) = \frac{1}{|\vec{v}|}\vec{v}\cdot\vec{v}'$$

$\Rightarrow$ ② $\frac{d}{dt}|\vec{v}| = \frac{\vec{v}\cdot\vec{v}'}{|\vec{v}|}$