SF1626 2018-08-30 #3 1

CROSS PRODUCT (Section 11.2)

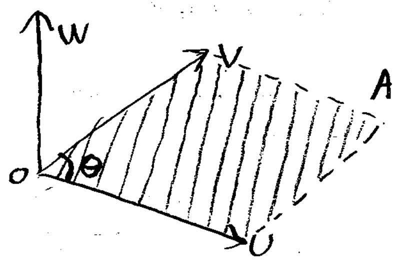
Given two vectors $U, V \in \mathbb{R}^3$

Definition: $W := U \times V$ is a vector in $\mathbb{R}^3$ with the following properties:

① $W \perp U \Rightarrow (U \times V) \cdot U = 0$

Remark: The fact that two vectors $\vec{V}_1$ and $\vec{V}_2$ are $\perp$ means:
① angle between $\vec{V}_1$ and $\vec{V}_2$ is $\frac{\pi}{2}$
② $\vec{V}_1 \cdot \vec{V}_2 = 0$

② $W \perp V \Rightarrow (U \times V) \cdot V = 0$



$A = |U \times V| = |W|$

③ $\{W, U, V\}$ form a right-hand triad $\Rightarrow (U \times V) = -(V \times U)$

④ $|W| = |U||V| \sin\theta$

⑤ if $U // V \Rightarrow U \times V = 0$

$$\boxed{\begin{aligned} i &= (1,0,0) \\ j &= (0,1,0) \\ k &= (0,0,1) \end{aligned}}$$

$$V_1 = (x_1, y_1, z_1),\ V_2 = (x_2, y_2, z_2) \Rightarrow V_1 \times V_2 = \det \begin{pmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} =$$

$$= i \det \begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \end{pmatrix} - j \det \begin{pmatrix} x_1 & z_1 \\ x_2 & z_2 \end{pmatrix} + k \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} =$$

$$= i(y_1 z_2 - z_1 y_2) - j(x_1 z_2 - z_1 x_2) + (x_1 y_2 - y_1 x_2) = \boxed{(y_1 z_2 - z_1 y_2,\ z_1 x_2 - x_1 z_2,\ x_1 y_2 - y_1 x_2)}$$

Exercise: Assume we have a particle moving in the space. At the time $t=0$ the particle is at the position $i + 3j = (1,3,0)$

The particle moves following the law:

$$\begin{cases} \frac{d}{dt} \vec{r}(t) = \vec{r}(t) \times 2\vec{i} \\ \vec{r}(0) = \vec{i} + 3j \end{cases} \qquad \vec{r}(t) = (x(t), y(t), z(t))$$

Find $\vec{r}(t)$.

$$\vec{r}(t) \times \vec{i} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y(t) \cdot 0 - z(t) \cdot 0 \\ -x(t) \cdot 0 + z(t) \cdot 1 \\ x(t) \cdot 0 - y(t) \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ z(t) \\ -y(t) \end{pmatrix}$$

$$\frac{d}{dt} \vec{r}(t) = \frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ z(t) \\ -y(t) \end{pmatrix} \Rightarrow \begin{cases} \frac{d}{dt} x(t) = 0 \\ \frac{d}{dt} y(t) = 2z(t) \\ \frac{d}{dt} z(t) = -2y(t) \end{cases}$$

initial values
$x(0) = 1$
$y(0) = 3$
$z(0) = 0$

$\Rightarrow \begin{cases} x(t) = \text{const since } x(0) = 1 \\ \quad \Rightarrow x(t) = 1 \text{ for all } t \geq 0. \\ \quad \Downarrow \\ \text{particle will move on the plane } x = 1 \end{cases}$

Solve: $\begin{cases} y' = 2z \\ z' = -2y \end{cases}$ $\Rightarrow$ derive with respect to the time the second equation: $z'' = -2y' \underset{\uparrow}{=} -2(2z) = -4z$

substitute 1st equation.

Solve: $\boxed{z'' + 4z = 0}$ $\rightarrow$ characteristic polynomial $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$$\Rightarrow \boxed{z(t) = c_1 \cos(2t) + c_2 \sin(2t)}$$

Try to solve the equation for $y(t)$: $y'(t) = 2(c_1 \cos(2t) + c_2 \sin(2t))$ $\Rightarrow$
$= 2c_1 \cos(2t) + 2c_2 \sin(2t)$

$\Rightarrow y(t) = c_1 \sin(2t) - c_2 \cos(2t)$

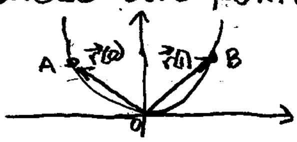
Last step: Compute c_1 and c_2 using $y(0) = 3$, $z(0) = 0$

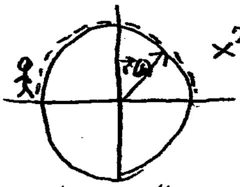
$$\begin{aligned} z(0) &= c_1 \cos(0) + c_2 \sin(0) = c_1 \Rightarrow c_1 = 0 \\ y(0) &= c_1 \sin(0) - c_2 \cos(0) = -c_2 \Rightarrow c_2 = -3 \end{aligned} \Rightarrow \begin{cases} y(t) = 3\cos(2t) \\ z(t) = -3\sin(2t) \end{cases}$$

$$\vec{r}(t) = \begin{pmatrix} 1 \\ 3\cos(2t) \\ -3\sin(2t) \end{pmatrix}$$

Given a curve and two points on this curve, how can we compute the distance between these two points?

$\boxed{\text{PARAMETRIZATION}}$





$x^2 + y^2 = 1$

Find a parameter t that allows you to describe the curve in the following form:

$t \rightarrow (x(t), y(t))$ in $\mathbb{R}^2$ or $t \rightarrow (x(t), y(t), z(t))$ in $\mathbb{R}^3$

Example #1: parabola $y = x^2$. Write $y = x^2$ in parametrized form.

$$t \rightarrow (x(t), y(t))$$

try #1: $t = x \Rightarrow t \mapsto (t, t^2)$ $t \in (-\infty, \infty)$

$\vec{r}(t) = (t, t^2)$ for $-\infty < t < +\infty$ describes the parabola $y = x^2$

try #2: $t = 5x$, $\vec{r}(t) = \left(\frac{t}{5}, \left(\frac{t}{5}\right)^2\right)$

Example #2: $x^2 + y^2 = 1$, parametrize

try #1: $x = t \Rightarrow t \rightarrow (t, \pm\sqrt{1 - t^2})$ NOT a function, not working!

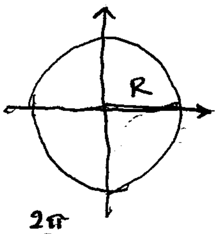
$\vdots$

try #5: $t = \arctan\left(\frac{y}{x}\right)$ $t \rightarrow (\cos(t), \sin(t))$ $0 \leq t \leq 2\pi$

Given a curve $t \to \vec{r}(t) = (x(t), y(t))$
$= (x(t), y(t), z(t))$
and two points on the curve $\vec{r}(t_1), \vec{r}(t_2)$

__Def:__ The length of the curve between $\vec{r}(t_1)$ and $\vec{r}(t_2)$ is:

$$\int_{t_1}^{t_2} |\vec{r}\,'(t)|\,dt$$



$\vec{r}(t) = (R\cos(t), R\sin(t))$, $0 \le t \le 2\pi$
$\vec{r}\,'(t) = (-R\sin(t), R\cos(t))$

length of the circle: $\displaystyle\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}\,dt$

$$= \int_0^{2\pi} \sqrt{R^2(\sin^2(t) + \cos^2(t))}\,dt = \int_0^{2\pi} R\,dt = \boxed{2\pi R}$$

Curve is the intersection between: $\begin{cases} x + y = 1, & y = 1 - x \\ z = x^2 + y^2 & z = x^2 + (1-x)^2 \end{cases}$

Find the parametrization: $t \to (x(t), y(t), z(t))$

__try #1:__ $t = x$ $\quad t \to (t, 1-t, t^2 + (1-t)^2)$

__#2:__ $t = y$ $\quad t \to (1-t, t, t^2 + (1-t)^2)$

__#3:__ $t = z$ $\quad t \to (\times, \times, t)$ Not working!

$\begin{cases} x + 2y + 2z = 4 \\ x^2 + 4y^2 = 4 \end{cases}$ $\quad t \to \left(2\cos(t), \sin(t), \frac{1}{2}(4 - 2\cos(t) - 2\sin(t))\right)$

$4\cos^2(t) + 4\sin^2(t) = 4$

$\boxed{\text{Section 11.3}}$