SF1626 2018-09-24 #13

Vector fields and Streamlines

A vector field in $\mathbb{R}^2$ $\quad \vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$$

So at every point in the domain, $\vec{F}$ is a vector.
Examples: Weather reports, gravitation

<u>Def^n</u> A curve whose tangents are prescribed by $\vec{F}$ is called a streamline of $\vec{F}$.

- Streamline = integral curve = flow line = trajectory
- Imagine a particle whose trajectory is described by $\vec{\gamma}(t)$.

The trajectory is a streamline of $\vec{F}$ if the differential equation

$$\vec{\gamma}'(t) = \vec{F} \circ \vec{\gamma}(t)$$
$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t)),$$
$$\vec{F} \circ \vec{\gamma}(t) = \vec{F}(\gamma_1(t), \gamma_2(t)) \equiv \vec{F}(\vec{\gamma}(t))$$

is satisfied.

In the book, the streamline equation of $\vec{F}$ is described with an additional scaling factor, $\lambda(t)$:

$$\vec{\gamma}'(t) = \lambda(t)\vec{F} \circ \vec{\gamma}(t)$$

The differential equation can also be written in the form

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)}$$

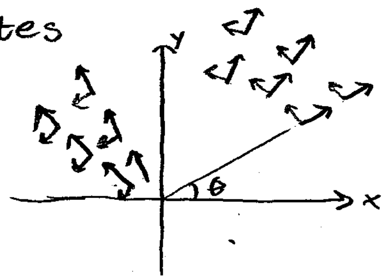
$$\vec{\gamma}(t) = (x(t), y(t), z(t)),\ \vec{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$$

Vector fields in polar coordinates

$$\vec{F}(r,\theta) = F_1(r,\theta)\vec{r} + F_2(r,\theta)\vec{\theta}$$

$$\vec{r} := \cos(\theta)\vec{i} + \sin(\theta)\vec{j}$$
$$\vec{\theta} := -\sin(\theta)\vec{i} + \cos(\theta)\vec{j}$$

?



Conservative vector fields

<u>Def^n</u> $\vec{F}(x,y,z) = \nabla\phi(x,y,z), \quad (x,y,z) \in D$
- $\vec{F}$ is called a conservative vector field
- ϕ is a potential for $\vec{F}$

Differential form: $d\phi = F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$

Necessary conditions for a conservative vector field in 2D and 3D

$$\vec{F}(x,y,z) = \nabla \phi(x,y,z) = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$$

$$\phi_{xy} = \phi_{yx}, \quad \phi_{xz} = \phi_{zx}, \quad \phi_{yz} = \phi_{zy}$$

$$\Updownarrow \qquad\qquad (✡)$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

(F is a conservative vector field) $\Rightarrow$ (✡) — (✡) are necessary conditions for this statement

$\Leftarrow$ (□) — (□) is a sufficient condition...

Ex $\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, tangent of $\vec{\gamma}$: $\frac{d\vec{\gamma}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$

$$\vec{F}(x,y) = \begin{pmatrix} -\Omega y \\ \Omega x \end{pmatrix}$$

Field line conditions:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \lambda(t) \begin{pmatrix} -\Omega y \\ \Omega x \end{pmatrix} \qquad \frac{\frac{dx}{dt}}{-\cancel{\Omega} y} = \lambda(t) = \frac{\frac{dy}{dt}}{\cancel{\Omega} x}$$

$$x\frac{dx}{dt} = -y\frac{dy}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow x^2+y^2 = \text{const}$$

Ex $$\vec{F}(x,y,z) = -km \frac{(x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}}$$

D = denominator

$$\frac{dx}{\frac{\cancel{-km}(x-x_0)}{\cancel{D}}} = \frac{dy}{\frac{\cancel{-km}(y-y_0)}{\cancel{D}}} = \frac{dz}{\frac{\cancel{-km}(z-z_0)}{\cancel{D}}}$$

($?$ $\nabla\phi = F$)

Ex $$\phi(x,y,z) = \frac{km}{|r-r_0|} = \frac{km}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = km \cdot ((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{-\frac{1}{2}}$$

$$\phi_x = km\left(-\tfrac{1}{2}\right)\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{-\frac{3}{2}} \cdot 2(x-x_0) = \frac{-km(x-x_0)}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{3/2}}$$

$r_0 = (x_0, y_0, z_0)$ $\quad r = (x,y,z)$ $\quad |r-r_0| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$