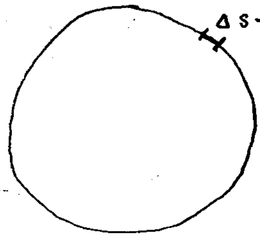
SF1626 2018-09-25 #14

Line integrals

$$\int_C f(x,y,z)\,ds = \int_a^b f(\gamma(t))\left|\frac{d\gamma}{dt}\right|dt \quad (*)$$ (Book uses $\vec{r}$ instead of γ)

Integrate f along the curve C, parametrized as $\gamma(t)$ for $a \le t \le b$

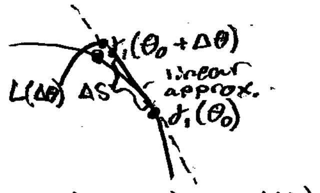
$\boxed{\text{– What is } ds \text{, the arc length element? – How is } ds \text{ related to } dt\text{?}}$



$\gamma_1(\theta) := (\cos 4\pi\theta)\vec{i} + (\sin 4\pi\theta)\vec{j}, \quad 0 \le \theta < \frac{1}{2}$

$\gamma_2(\theta) := (\cos(-\theta))\vec{i} + (\sin(-\theta))\vec{j}, \quad 0 \le \theta \le 2\pi$

$\gamma_3(\theta) := \cos\theta\,\vec{i} + \sin\theta\,\vec{j}, \quad 0 \le \theta \le 2\pi$



$L(\theta) := \gamma_1(\theta_0) + \theta\,\gamma_1'(\theta_0)$

$\Delta s = |L(0) - L(\Delta\theta)| =$

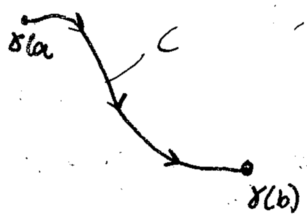
$= \Delta\theta \cdot |\gamma'(\theta_0)|$

$(\theta = t)$

$I \quad \int_C f(x,y)\,\underline{\underline{ds}}$ integrate f along the curve C with the arc length element

- C is parametrized by $\gamma(t)$, $a \le t \le b$

$$I \cong \sum_{j=0}^{N-1} f(\gamma(t_j))\overbrace{|\gamma'(t_j)|\,\Delta t}^{\Delta s}$$



$\Delta s \approx |\gamma(t_{j+1}) - \gamma(t_j)|$

A special case: $f \equiv 1$, $\int_C 1\,ds$ = the length of the curve C.

$$* \begin{cases} \sum_{j=0}^{N-1} f(\gamma(t_j))\,\Delta t \to \int_a^b f(\gamma(t))\,dt & \underline{\text{NOT what we want}} \\ \sum_{j=0}^{N-1} f(\gamma(t_j))\,\underline{\underline{v_j}}\,\Delta t \to \int_C f(\gamma(t))\,ds \end{cases}$$

Line integrals of projection of vector fields along a curve

$$\left.\begin{aligned} &F(x,y) = (F_1(x,y),\ F_2(x,y)) \\ &\int_C F(x,y)\,ds = \left(\int_C F_1(x,y)\,ds,\ \int_C F_2(x,y)\,ds\right) \end{aligned}\right\}$$ NOT what we are doing

$\int_C \vec{F}\cdot d\vec{r}$ (work done by the force F along the curve γ

- The vector field $\vec{F}(x,y,z)$
- The projection of $\vec{F}$ onto the curve $\gamma(t)$

$$F_\tau(t) := \vec{F}(\gamma(t)) \cdot \frac{\gamma'(t)}{|\gamma'(t)|}$$

(no work is done perpendicular to the direction of motion)

Total work done:

$$\int_C F_\tau(t)\,ds = \int_C F_\gamma(t)|\gamma'(t)|dt$$

Following is equivalent:

$$\int_C \vec{F}\cdot d\vec{r} := \int_C \vec{F}(\gamma(t))\cdot \frac{\gamma'(t)}{|\gamma'(t)|}|\gamma'(t)|dt = \int_C \vec{F}(\gamma(t))\,\gamma'(t)dt$$

In particular, if the curve C is a closed curve,

$\oint_C \vec{F}\cdot d\vec{r}$ denotes the circulation of $\vec{F}$ around C

Very important theorem:

D is an open connected domain. $\vec{F}$ is a smooth vector field on D. The following statements are equivalent:

(a) $\vec{F}$ is conservative in D

(b) $\oint_C \vec{F}\cdot d\vec{r} = 0$ for every piecewise smooth curve C in D

(c) Given two points P_0 and P_1 in D. For any piecewise smooth curves C_1 and C_2 which start at P_0 and end at P_1.

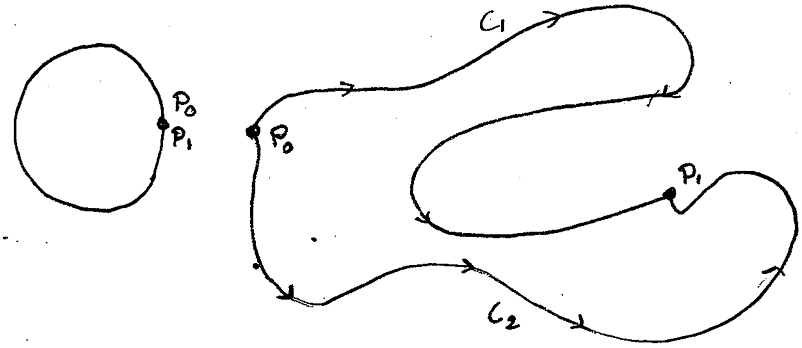
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

- $\vec{F}$ is conservative $\Rightarrow$ for some potential function ϕ $\vec{F} = \nabla\phi$
- Chain rule:

$$F(\gamma(t))\gamma'(t) = \nabla\phi(\gamma(t))\,\gamma'(t) = \frac{d}{dt}\phi(\gamma(t))$$

So $$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot\gamma' dt = \int_0^1 \frac{d}{dt}\phi(\gamma(t))dt = \phi(\gamma(1)) - \phi(\gamma(0))$$

Here the curve C is parametrized by $\gamma(t)$ for $0 \le t \le 1$



$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

Simply connected domains

D is connected but not simply connected

"C cannot shrink without exiting the domain"