Föreläsning 11

Improper integrals

– With a change of variables we might be able to solve

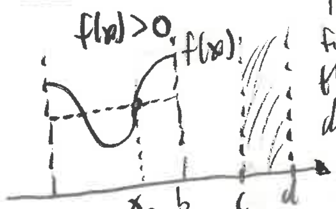
- $\int_0^1 \frac{1}{x^2}\,dx$ integrand is not defined in the domain. $\frac{1}{x^2}$ blows up close to 0, "Singular integral"
- $\int_1^\infty \frac{1}{x^2}\,dx$ the domain is infinite.

Redefine as:

$$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$$

Mean value Theorem

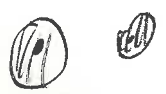
? water tank



The value for f at the point in domain a,b is its average value over the domain (a,b)

$$\int_a^b f(x)\,dx + \int_c^d f(x)\,dx$$

The domain has to be closed, bounded & connected

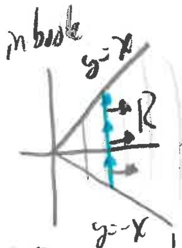


Not connected!

I can also define average or mean value of f over D.

Example 1 in book

$$\iint_R e^{-x^2}\,\frac{dA}{dxdy}$$



Improper since R is not unbounded

R is an x-simple domain

$$\int_{x=0}^{\infty}\left(\int_{-x}^{x} e^{-x^2}\,dy\right)dx =$$

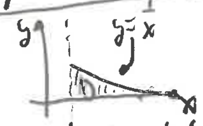
$$= \int_{x=0}^{\infty} e^{-x^2}\left(\int_{-x}^{x} 1\,dy\right)dx = \int_0^{\infty} e^{-x^2}\,[y]_{-x}^{x}\,dx =$$

$$= \int_0^{\infty} 2x e^{-x^2}\,dx = \lim_{L\to\infty}\int_0^L \left(\frac{d}{dx} e^{-x^2}\right)dx =$$

$$= \lim_{L\to\infty}\left(\underbrace{e^{-L^2}}_{0} - \underbrace{e^{-0^2}}_{1}\right) = 1$$

Or $z = x^2$, $dz = 2x\,dx$

$$\iint_D \frac{1}{xey}\,dA$$



Improper since D is unbounded

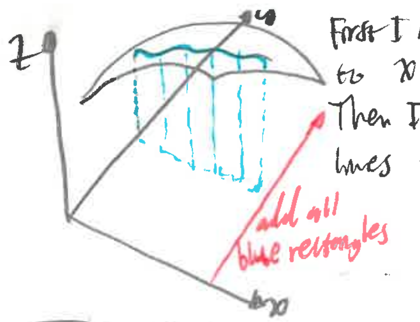
D is x & y simple.

x-simple:

$$\int_{x=1}^{\infty}\left(\int_{y=0}^{1/x} \frac{1}{xey}\,dy\right)dx$$

y-simple:

$$\int_{y=0}^{1}\int_{x=1}^{1/y} \frac{1}{xey}\,dx\,dy$$



First I integrate f with respect to x for every y value. Then I sum all of the blue lines together.

$F(x) = \int f(x)\,dx$

Double integrals in polar coordinates

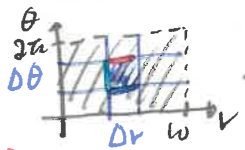
$x(r,\theta) = r\cos\theta \quad y(r,\theta) = r\sin\theta$

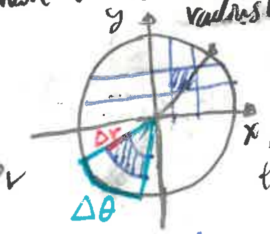
$(x_0,y_0)\; r(x,y) = \sqrt{x^2+y^2}$ radius

$\theta(x,y) = \arcsin\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

We have to be careful about range of θ radius to

$$\iint_D \overbrace{f(x^2+y^2)}^{f(r^2)}\,\overbrace{dxdy}^{?\,drd\theta} =$$





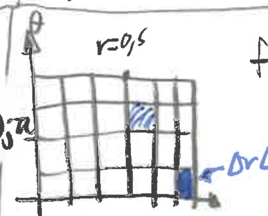
For $0 \le \theta < 2\pi$ the area of the rectangle is not the same as for the circle

$$= \iint_D f(r^2)\,J(r,\theta)\,drd\theta\,?$$

Area of [shaded sector] $= r \cdot \Delta r \cdot \Delta\theta$

$dxdy = r\,drd\theta$, $drd\theta = \frac{1}{\sqrt{x^2+y^2}}\,dxdy$

$$\iint_D f(r,\theta)\,dxdy = \iint_D f(r,\theta)\,r\,drd\theta$$



$j\Delta\theta = \theta_j\,2\pi$

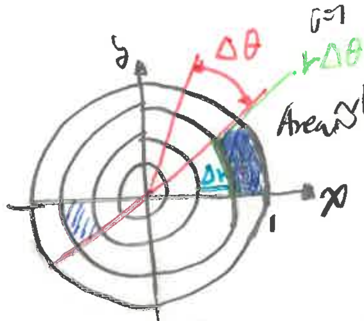
$j = 0,1,\dots,M$

$r_i = i\Delta r$

$i = 0,1,\dots,N$

$\Delta r = \frac{1}{N} \quad \Delta\theta = \frac{2\pi}{M}$

$f(r,\theta)$ Riemann Sum $= \sum_{j=1}^{M}\sum_{i=1}^{N} f(r_i,\theta_j)\,\Delta r\Delta\theta$



$r\Delta\theta$ length of the arc

Area $\approx \boxed{r\Delta\theta} \cdot \Delta r$ Approximation of the [shaded] area.

for small squares, [square] we get an area in the circle: almost a rectangle i.e. linear transformation

$$\iint_{\substack{0\le r\le 1\\ 0\le\theta\le 2\pi}} f(r,\theta)\,drd\theta \approx \sum_j\sum_i f(r_i,\theta_j)\,\Delta r\Delta\theta = \iint_D \tilde{f}(x,y)\,J(x,y)\,dxdy =$$

$$= \sum_{k=1}\sum_{h=1} \tilde{f}(x_h,y_h)\,\frac{1}{r_{hk}}\,\Delta x\Delta y$$

$J_{xy} = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \quad J_{r\theta} = \left|\frac{\partial(r,\theta)}{\partial(x,y)}\right|$

$J_{r\theta} = \frac{1}{J_{xy}}$