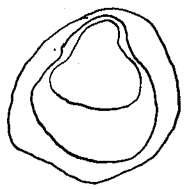
SF1626 2018-09-03 #4

Two ways to visualize a function $f(x,y)$ of two variables

1. Graph of the function $(x, y, f(x,y))$
2. Level sets of the function. "level set" = "contours" = "isosurface"

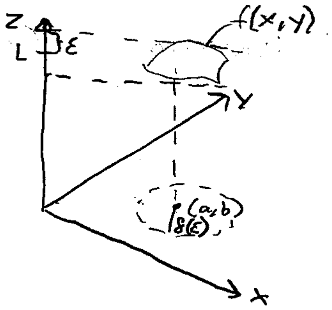


12.2 Limits and continuity

Def$^{\underline{n}}$ 2 $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if (i) & (ii).

(i) every neighborhood of (a,b) contains points in the domain of f, which are different from (a,b).

(ii) For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta(\varepsilon)$ ✡



$f(x,y)$ may approach different values depending on from which direction a point is approached.

Partial derivatives

The partial derivative $f_1(a,b)$ is the rate of change of $f(x,y)$ at $x=a$, while $y=b$. i.e. it's the rate of change of f along the line $(x, y=b)$

$$f_1(a,b) := \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \qquad f_2(a,b) := \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Example $f(x,y) = x^2 \sin(y)$

$$f_1(x,y) \equiv f_x(x,y) \equiv \frac{\partial f}{\partial x} \equiv \frac{\partial}{\partial x} f(x,y) = 2x\sin(y)$$

$$f_2(x,y) \equiv f_y(x,y) \equiv \frac{\partial f}{\partial y} \equiv \frac{\partial}{\partial y} f(x,y) = x^2\cos(y)$$

Example $f(x,y) = e^{xy}\cos(x+y)$

$$\frac{\partial}{\partial x} e^{xy}\cos(x+y) = \underbrace{\left(\frac{\partial}{\partial x} e^{xy}\right)}_{= e^{xy}\cdot\left(\frac{\partial}{\partial x} xy\right)} \cos(x+y) + e^{xy}\underbrace{\left(\frac{\partial}{\partial x}\cos(x+y)\right)}_{= -\sin(x+y)\cdot\left(\frac{\partial}{\partial x}(x+y)\right)} = y e^{xy}\cos(x+y) - e^{xy}\sin(x+y)$$

$$(\Delta)\ \boxed{u_x + u_y = 0} \qquad \text{e.g. } \cos(x-y) =: f(x,y)$$

Show that $f(x,y)$ satisfies (Δ)

$$f_x(x,y) = \frac{\partial}{\partial x}\cos(x-y) = \boxed{\cancel{-\sin(x-y)}}$$

$$\overset{-1}{\shortparallel}$$

$$+)\ f_y(x,y) = \frac{\partial}{\partial x}\cos(x-y) = -\sin(x-y)\left(\frac{\partial}{\partial y}(x-y)\right) = \boxed{\cancel{\sin(x-y)}}$$

$$f_x + f_y = 0$$

Gradients

The gradient of f is a vector in $\mathbb{R}^2$, denoted and defined as

$$\nabla(f(x,y)) = f_1(x,y)\,i + f_2(x,y)\,j \qquad i = \begin{pmatrix}1\\0\end{pmatrix} \quad j = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\nabla f:\ (x,y) \in \mathbb{R}^2 \longrightarrow \begin{pmatrix}f_1(x,y)\\f_2(x,y)\end{pmatrix} \equiv f_1(x,y)\,i + f_2(x,y)\,j$$

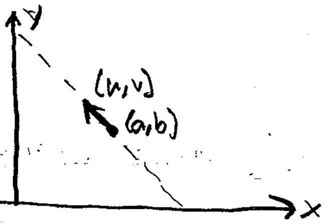
Theorem The gradient of f (when it is not a zero vector) is perpendicular/orthogonal to the level curve of f.

Maximal rate of increase/decrease is $\pm|\nabla f(x,y)|$

Directional Derivatives Let $u = ui + vj$ be a unit vector, i.e. $u^2 + v^2 = 1$

Def^{n} The directional derivative of f at (a,b) is the rate of change of f along the line/ray in the direction u which passes through (a,b)

$\vec{u}\begin{pmatrix}u\\v\end{pmatrix}$



$$t \to \begin{pmatrix}u\cdot t + a\\ v\cdot t + b\end{pmatrix} \qquad \nabla f = \begin{pmatrix}f_1\\f_2\end{pmatrix} \quad \vec{u} = \begin{pmatrix}u\\v\end{pmatrix}$$

$$f(t) = f(u\cdot t + a,\ v\cdot t + b)$$

$$\frac{d}{dt}f(t) = f_1 \cdot \underbrace{\frac{d}{dt}(u\cdot t + a)}_{u} + f_2 \cdot \underbrace{\frac{d}{dt}(v\cdot t + b)}_{v} = \vec{u}\cdot\nabla f$$