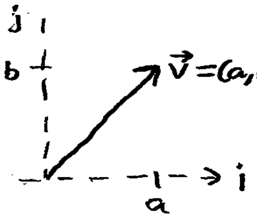
SF1626 2018-09-04 #5

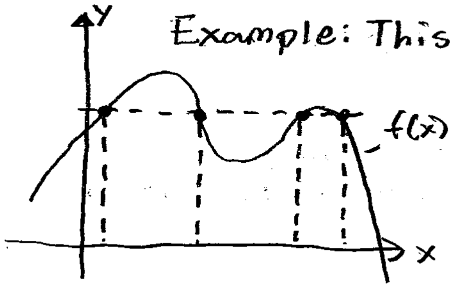
Highlights will be avaliable to download at github.com/yhrtsai



$\vec{v}=(a,b)$ or $\binom{a}{b}$ $\quad |\vec{v}| = \sqrt{a^2+b^2}$

$\vec{v}$ is a unit vector if $|\vec{v}|=1$ ($|\vec{v}|=0$ zero vector)

Example: This level set of $f(x)$ consists of four points.



Chain rule $\quad \frac{\partial}{\partial r} f(x(r,\theta), y(r,\theta)) = f_1 \frac{\partial x}{\partial r} + f_2 \frac{\partial y}{\partial r}$

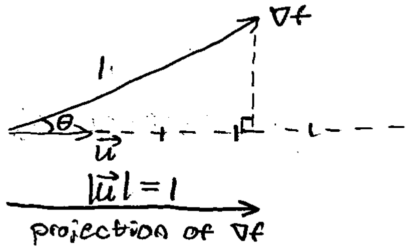
$$(f_r \; f_\theta) = (f_x \; f_y)\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$ See slides and examples in book

Directional Derivatives $\quad \vec{u}=(u,v)$ is the direction vector $|\vec{u}|=1$

$$D_{\vec{u}}f(a,b) := \frac{d}{dt} f(a+tu, b+tv) = f_1(a,b)u + f_2(a,b)v = \nabla f(a,b) \cdot \vec{u}$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta$$

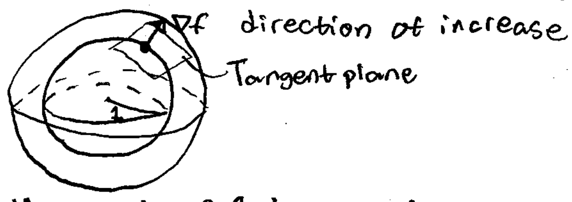
Maximal value of $\cos\theta$ is 1 when
$\theta = 0 \pm 2\pi$ ($\vec{u}$ and ∇f in same direction)
Minimal value of $\cos\theta$ is -1 when
$\theta = \pi \pm 2\pi$
$\cos\theta = 0$ when $\vec{u} \perp \nabla f$

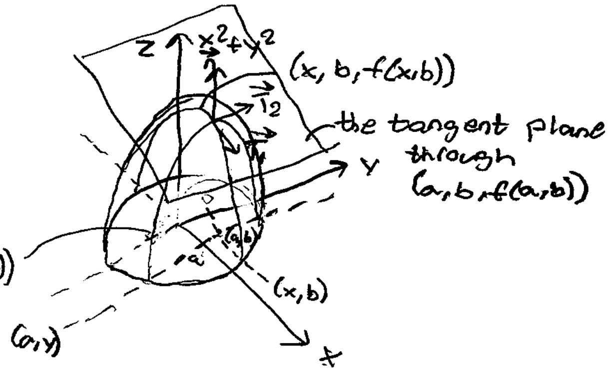


Tangent planes and normal lines (of the graph of f)

Example $\quad f(x,y,z) = x^2+y^2+z^2 \qquad f: \mathbb{R}^3 \to \mathbb{R}, \; (x,y,z) \to f(x,y,z)$

1-level set of f is $x^2+y^2+z^2=1 \quad$ 1.44 level set of f is $x^2+y^2+z^2=1.44=1.2^2$





Cut the graph of f by the plane $y=b$ / $x=a$

The curve at the intersection can be written parametrically as $x \mapsto (x,b,f(x,b))$ / $y \mapsto (a,y,f(a,y))$

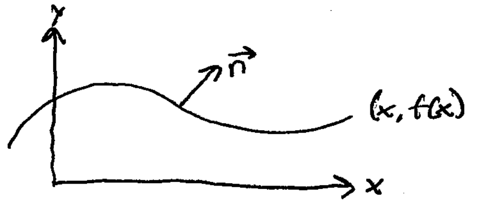
The tangent vectors:

$T_1 := \vec{i} + f_1(a,b)\vec{k}, \quad T_2 := \vec{j} + f_2(a,b)\vec{k}$

Normal vector $\vec{n} := T_2 \times T_1$

Getting the normal vectors from the gradient

First find the normal vector of the graph of a function of one variable $f(x)$

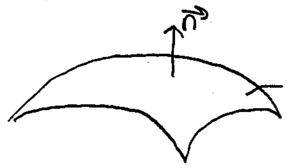


- Define a new function of two variables by $g(x, y) = y - f(x)$
- The 0-level set of g is the graph of f.
- Since $\nabla g(x,y) = (-f'(x), 1)$ is perpendicular to the level sets of g $\nabla g(a, f(a))$ is perpendicular to the graph of f at $(a, f(a))$
- A unit normal vector to the graph of f as a function of x is then

$$\vec{n}(x) = \frac{1}{\sqrt{f'(x)^2 + 1}} (f'(x)\vec{i} - \vec{j})$$

- This generates to functions of two variables very easily.

$f(x, y)$



$g(x, y, z) := z - f(x-y)$

$$\nabla g = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix} \parallel \vec{n}$$

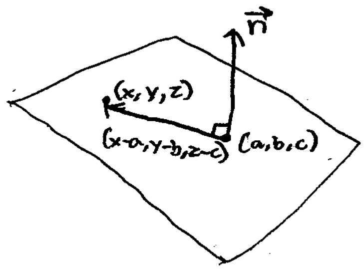
Equation for the tangent plane

Given the normal vector $\vec{n} = (n_1, n_2, n_3)$ of the plane passing through (a, b, c) we can write a equation for the plane as

$$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$$

In our particular case: $\vec{n} = (-f_1(a,b), -f_2(a,b), 1)$, $c = f(a,b) \Rightarrow$

$\Rightarrow f_1(a,b)(x-a) + f_2(a,b)(y-b) + f(a,b)$



$(x-a, y-b, z-c) \cdot \vec{n} = 0$