SF1626 2018-09-27 #15

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \int_C \vec{F}\cdot d\vec{r}, \quad \vec{F} = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$$

Ex. Evaluate $I = \oint_C (e^x \sin y + 3y)dx + (e^x \cos y + 2x - 2y)dy$ counterclockwise around the ellipse $4x^2 + y^2 = 4$

$$I = \oint_C \vec{F}\cdot d\vec{r}, \quad \vec{F} = (e^x \sin y + 3y, e^x \cos y + 2x - 2y)$$

Parametrize: $y = \pm 2\sqrt{1-x^2}$ (or $x(\theta) = \cos\theta$, $y(\theta) = 2\sin\theta$)

$$\oint_C \vec{F}\cdot d\vec{r} = \underbrace{\int_{C_1} \vec{F}\cdot d\vec{r}}_{\downarrow} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\int_{-1}^{1} \vec{F}\cdot \frac{\gamma'(x)}{|\gamma'(x)|}|\gamma'(x)|\,dt \qquad \gamma(x) = (x, 2\sqrt{1-x^2})$$

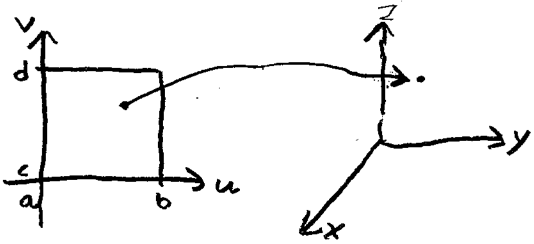
$$\gamma'(x) = \left(1, \frac{-2x}{\sqrt{1-x^2}}\right)$$

Surfaces and surface integrals

Parametric surfaces

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

Graph of a function $f(x,y)$ is a surface parametrized by x, y



Surface integrals

$$\int_\Sigma f(x,y,z)dS = \int\int_{D_{uv}} f(\vec{r}(u,v))\left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right| du\,dv$$

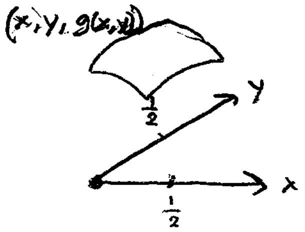
$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i \to \int_\Sigma f(x,y,z)dS$$

"Here you see ΔS. This means nothing to you. Pretend you're a compiler or something."
- Richard

Normal vectors of the function

$$\begin{cases} \dfrac{\partial \bar{r}}{\partial u} = \text{tangent vector 1} \\ \dfrac{\partial \bar{r}}{\partial v} = \text{tangent vector 2} \\ \text{normal vector } \bar{n} = \dfrac{\partial \bar{r}}{\partial u} \times \dfrac{\partial r}{\partial v} \end{cases}$$

ex. The surface is the graph of $g(x,y) := \sqrt{1-(x^2+y^2)}$



$(x,y) \to x\hat{\imath} + y\hat{\jmath} + \sqrt{1-(x^2+y^2)}\,\hat{k}$

$0 \leq x \leq \frac{1}{2}, \quad 0 \leq y \leq \frac{1}{2}$

$f(x,y,z) := 1$

$$\iint_{\Sigma} 1\, dS = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 1 \cdot \overbrace{\left|\frac{\partial \bar{r}}{\partial x} \times \frac{\partial \bar{r}}{\partial y}\right|}^{\text{Jacobian}} dx\,dy$$

↖ length of vector

$$\frac{d\bar{r}}{dx} = \begin{pmatrix} 1 \\ 0 \\ \frac{-x}{\sqrt{1-(x^2+y^2)}} \end{pmatrix} \leftarrow g_x \qquad \frac{d\bar{r}}{dy} = \begin{pmatrix} 0 \\ 1 \\ \frac{-y}{\sqrt{1-(x^2+y^2)}} \end{pmatrix} \leftarrow g_y$$

$$\frac{\partial \bar{r}}{\partial x} \times \frac{\partial \bar{r}}{\partial y} = \frac{x}{\sqrt{1-(x^2+y^2)}}\hat{\imath} + \frac{y}{\sqrt{1-(x^2+y^2)}}\hat{\jmath} + 1\hat{k}$$

$$\iint_{\Sigma} f(x,y,z)\, dS = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f(x,y,\underbrace{\sqrt{1-(x^2+y^2)}}_{g(x,y)}) \sqrt{1^2 + \underbrace{\frac{x^2}{1-(x^2+y^2)}}_{g_x^2} + \underbrace{\frac{y^2}{1-(x^2+y^2)}}_{g_y^2}}\, dx\,dy = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\, dx\,dy$$

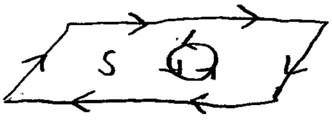
What you should know:

- Orientation of a surface and its bounding (if it has boundaries)
- Flux of a vector field across an oriented surface

ex (Hemisphere has boundaries
Donut has no boundaries)
(torus)

A surface can be orientable or non-orientable

(möbius strip)



When you go along the edge, S has to always be on the same side of you (left or right)