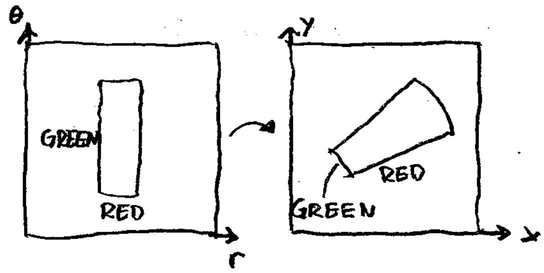
SF1626 2018-09-20 #12

$u,v \to x,y \qquad x(u,v) \quad y(u,v)$



$dx(u,v,du,dv) = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$

$dy(u,v,du,dv) = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$

$\Delta A^* = \Delta u \, \Delta v \to dA^* = dA$

$$\left.\begin{aligned} \overrightarrow{RED} &= dx(u,v,du,0)\vec{i} + dy(u,v,du,0)\vec{j} \\ \overrightarrow{GREEN} &= dx(u,v,0,dv)\vec{i} + dy(u,v,0,dv)\vec{j} \end{aligned}\right\} \text{Right side}$$

The area of the parallelogram:

$$dA = |\overrightarrow{RED} \times \overrightarrow{GREEN}| = \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{Jacobian determinant}} du\,dv$$

For integration, linear approximations of a function is "enough"

$$\vec{f}(u,v) := (x(u,v), y(u,v))$$

Linear approximation of $\vec{f}$ at (u_i, v_j):

$$\vec{f}(u,v) \approx \vec{f}(u_i, v_j) + (u-u_i,\, v-v_j) D\vec{f}(u_i, v_j)$$

Change of variables formula for double integrals

$$\begin{aligned} x(r,\theta) &= r\cos\theta \\ y(r,\theta) &= r\sin\theta \end{aligned} \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r$$

Using chain rule, we have: $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}} = \frac{1}{r}$

Triple integrals $\qquad \iiint_D f(x,y,z)\,dV \qquad$ "Think of f as a density"

Change of variable formula: $(u,v,w) \to (x,y,z)$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D} f(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dz$$

In practice we write the integrals into a sequence of one dimensional integrals
For example

$$\iiint_D f(x,y,z)\,dV = \int_{D_x} \int_{D_y(x)} \int_{D_z(x,y)} f(x,y,z)\,dz\,dy\,dx$$

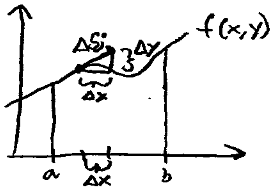
Or via suitable change of variables $(u,v,w) \to (x,y,z)$

$$\iiint_D f(x,y,z)\,dV = \iiint_{D_{uvw}} f(x(u,v,w), y(u,v,w), z(u,v,w))\, J(u,v,w)\,du\,dv\,dw$$

The graph of $f(x,y)$ is a surface parametrized by (x,y)
$(x, y, f(x,y)) \in S$

Unit normal vector of S

$$g(x,y,z) = z - f(x,y) \Rightarrow n(x,y) = \frac{-f_1(x,y)\vec{i} - f_2(x,y)\vec{j} + \vec{k}}{\sqrt{f_1(x,y)^2 + f_2(x,y)^2 + 1}}$$

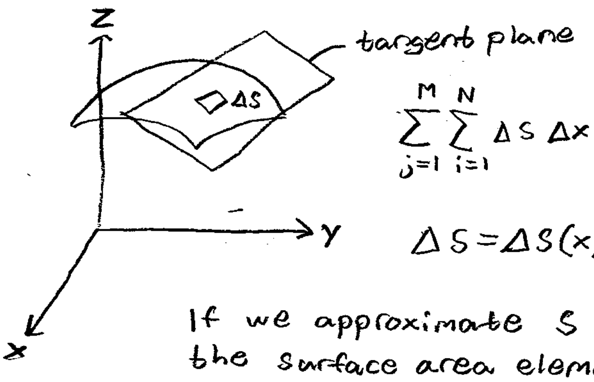


$$\Delta y = f'(x)\Delta x = df(x, \Delta x)$$

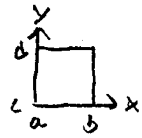
$$\Delta S = \sqrt{\Delta x^2 + (f'(x)\Delta x)^2} = \sqrt{1 + (f'(x))^2} \cdot \Delta x$$

2D

$$\int_a^b ?\,dx = \text{length of the graph of } f \approx \sum_{j=1}^{N} \Delta S(x_j, \Delta x)$$



$$\sum_{j=1}^{M}\sum_{i=1}^{N} \Delta S\, \Delta x \Delta y \approx \text{area of the graph of } f \text{ over}$$



$$\Delta S = \Delta S(x, y, \Delta x, \Delta y)$$

If we approximate S by $(x, y, f(x,y))$ by the tangent plane, then the surface area element of the tangent plane (which is the same as that of S, denoted by dS.

$dA = dx\,dy = n \cdot k\, dS$ $\qquad dS = \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}\, dA$

Ex: Let B be the rectangular box $\quad B := \begin{cases} 0 \le x \le a \\ 0 \le y \le b \\ 0 \le z \le c \end{cases}$

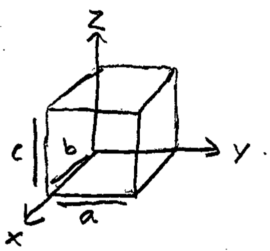
$$I = \iiint_B (xy^2 + z^3)\,dV = \int_0^c \int_0^b \underbrace{\int_0^a (xy^2 + z^3)\,dx}\,dy\,dz$$

$$\left[\tfrac{1}{2}x^2y^2 + z^3x\right]_0^a = \tfrac{1}{2}a^2y^2 + az^3$$

$$\int_0^b \left(\tfrac{1}{2}a^2y^2 + az^3\right)dy = \left[\tfrac{1}{6}a^2y^3 + ayz^3\right]_0^b = \tfrac{1}{6}a^2b^3 + abz^3$$

$$\int_0^c \left(\tfrac{1}{6}a^2b^3 + abz^3\right)dz = \left[\tfrac{1}{6}a^2b^3z + \tfrac{1}{4}abz^4\right]_0^c = \tfrac{1}{6}a^2b^3c + \tfrac{1}{4}abc^4$$

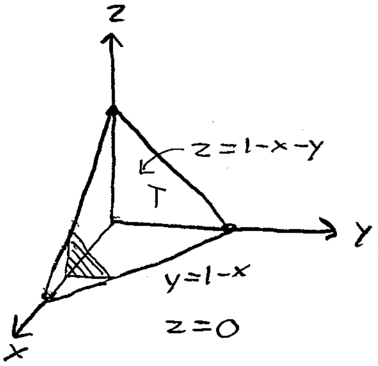
B is a cube, symmetric. You can just as well do $\int_0^a\int_0^b\int_0^c$ or $\int_0^b\int_0^a\int_0^c$ instead



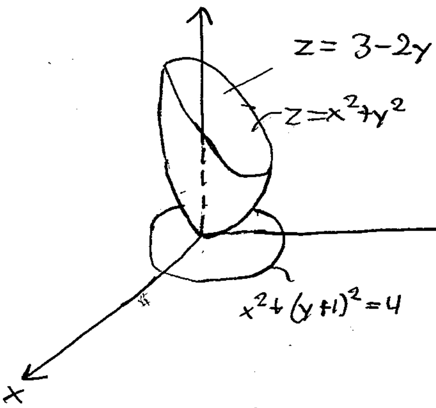
Ex: T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

calculate $I = \iiint_T y\,dV =$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\,dz\,dy\,dx$$

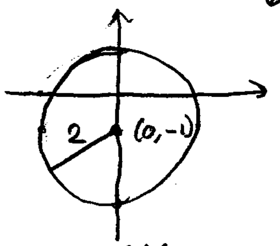


Ex: Find the volume of the region R lying below the plane $z=3-2y$ and above the paraboloid $z=x^2+y^2$



$$\left.\begin{aligned} x^2+y^2&=3-2y \\ x^2+y^2+2y-3&=0 \\ x^2+(y+1)^2-4&=0 \end{aligned}\right\} \text{intersection}$$

this is a circle with $r=2$



$$\iiint_R 1\,dV = \iint_D \left(\int 1\,dz\right)dx\,dy = \iint_D \left((3-2y-x^2-y^2)\,z\right)dx\,dy$$

Now we change variables to polar coordinates! But with center $(0,-1)$

$$\begin{cases} x = r\cos(\theta) \\ y+1 = r\sin(\theta) \end{cases}$$

Then you solve it.