SF1626 2018-09-13 #9

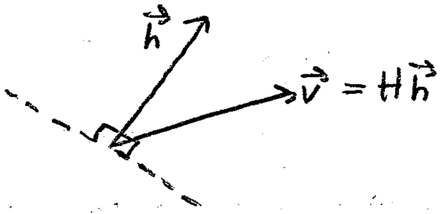
$g(t) := f(\vec{a}, t\vec{h})$

$\frac{d}{dt} g(t) = Df(\vec{a} + t\vec{h})\vec{h} = \nabla f(\vec{a} + t\vec{h}) \cdot \vec{h} = 0$ when $t = 0$

$$g''(t) = \vec{h} \cdot \boxed{\text{Hessian}}\, \vec{h} \Big|_{t=0} = \begin{cases} > 0, & \text{local min} \\ < 0, & \text{local max} \\ \text{else it is neither} \end{cases}$$

$\forall \vec{h} \in \mathbb{R}^{\#\text{variables}}$

Positive/negative definite



Find critical points via Lagrange function: $L(x, y, \lambda) := f(x, y) + \lambda g(x, y)$

Solve for x, y and λ if needed

example 2: $f(x, y) = x^2 + y^2$
$g(x, y) = 17x^2 + 12xy + 8y^2 - 100$

$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$

Find the critical points of L:

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + \lambda(34x + 12y) = 0 \Rightarrow x = -\lambda(17x + 6y) \\ \frac{\partial L}{\partial y} = 2y + \lambda(12x + 16y) = 0 \Rightarrow y = -\lambda(6x + 8y) \\ \frac{\partial L}{\partial \lambda} = g(x, y) = 0 \Rightarrow 17x^2 + 12xy + 8y^2 - 100 = 0 \end{cases}$$

$$\frac{x}{17x + 6y} = -\lambda = \frac{y}{6x + 8y}$$

$$\Updownarrow$$

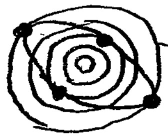
$6x^2 + 8xy = 17xy + 6y^2 \Leftrightarrow$
$6x^2 = 9xy + 6y^2 \Leftrightarrow$
(*) $2x^2 = 3xy + 2y^2 \Leftrightarrow$
$-8x^2 + 12xy + 8y^2 = 0 \Leftrightarrow$
$25x^2 - 100 = 0$
$\Rightarrow x^2 = 4$
$x = \pm 2$

Critical points of L: $x = 2$, $x = -2$

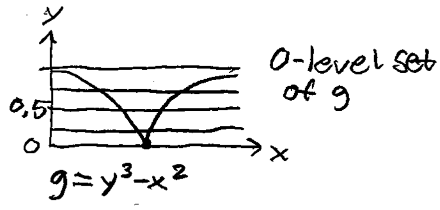
plug in x in (*)

$8 = 32y + 2y^2 \Rightarrow y^2 + 3y - 4 = 0 \Rightarrow y = \frac{-3 \pm \sqrt{9 + 16}}{2} = \frac{-3 \pm 5}{2} = -4 \text{ or } 1$

do the same for $x = -2$ to get 2 more points where a critical point could be.



yes, there are 4 points where L has critical points.



We can't find solutions using Lagrange in this situation because the function is not continous in that one point

(Singular point where $\nabla g = \vec{0}$)

New example

Maximize $f(x,y,z)$ subject to $g(x,y,z)=0$
$h(x,y,z)=0$

Look at points where $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

To solve, look at the critical points of

$$L(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z)$$

Example 3

$f(x,y) = y$
$g(x,y) = y^3 - x^2$

$L(x,y,\lambda) = y + \lambda(y^3 - x^2)$

$\frac{\partial L}{\partial x} = -2\lambda x = 0 \Rightarrow x = 0$ or $\cancel{\lambda = 0}$

$\frac{\partial L}{\partial y} = 1 + 3\lambda y^2 = 0 \Rightarrow$ contradiction $x = 0 \Rightarrow y = 0$