Green's theorem

$$\iint_D \text{curl}\,\bar{F}\cdot\hat{k}\,dA = \oint_C \bar{F}\cdot d\bar{r}$$

Forelasning 17

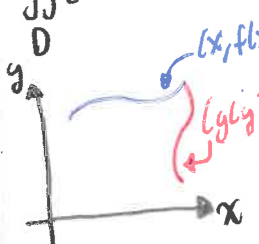
Divergence - How things spread or converge
Curl - How things rotate

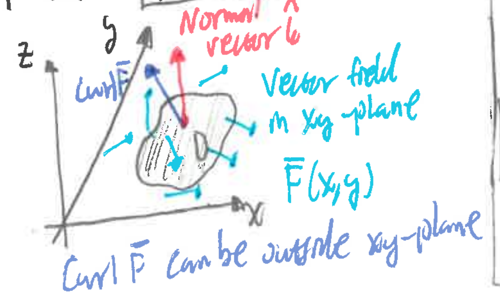
The curl of a conservative field is zero.
Example: Gravitational field. Things do not rotate in it.

Green's theorem 2 dimensional version of FTC
Regular domain D, x & y simple. C is the boundary of the domain

$\iint_D \text{curl}\,\bar{F}\cdot\hat{k}\,dA$ should be thought of as adding all the numbers $\text{curl}\,\bar{F}\cdot\hat{k}$ for every point in domain D

$$\iint_D \text{curl}\,\bar{F}\cdot\hat{k}\,dA = \oint_C \bar{F}\cdot d\bar{r}$$





"independent of z"
$\bar{F}(x,y,z) = \bar{F}(x,y)$.

$$\text{curl}\,\bar{F}(x,y,z) = \begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\times\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix} = \left\{\begin{matrix}F_3=0 \text{ since the field lines are on the plane} & \partial/\partial z \text{ is also } 0\end{matrix}\right\} = \begin{bmatrix}0\\0\\ \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\end{bmatrix}.$$

$\bar{F}\cdot\hat{k}$ then gives $\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$ so we can rewrite Green's theorem as:

$$\iint_D\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dA = \oint_C F_1(x,y)\,dx + F_2(x,y)\,dy$$

*Example 1 in book

$\bar{F}(x,y,z) = 0\hat{i} + x\hat{j} + 0\hat{k}$

$\bar{r}(t) = \begin{bmatrix}x(t)\\y(t)\\0\end{bmatrix}$ $\quad \bar{r}'(t) = \begin{bmatrix}x'(t)\\y'(t)\\0\end{bmatrix}$

Green's theorem:

$$\iint_D \text{curl}\,\bar{F}\cdot\hat{k}\,dA = \oint_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(x(t),y(t),0)\cdot\bar{r}'(t)\,dt = \int_a^b\begin{bmatrix}0\\x\\0\end{bmatrix}\cdot\begin{bmatrix}x'(t)\\y'(t)\\0\end{bmatrix}dt = \int_a^b x\,y'(t)\,dt = \int_a^b x\frac{dy}{dt}dt = \int_a^b x\,dy =$$

$$= \iint_D 1\,dA. \quad \left\{\text{curl}\,\bar{F}\cdot\hat{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} = 1\right\}$$

Example 2 in book

Find area of ellips: $\bar{r}(t) = \begin{bmatrix}3(\cos t+\sin t)\\2(\sin t-\cos t)\\0\end{bmatrix}$

$\bar{r}'(t) = \begin{bmatrix}3(-\sin t+\cos t)\\2(\cos t+\sin t)\\0\end{bmatrix}$

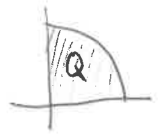
I want a vector field $\bar{F}$ s.t. $\text{curl}\,\bar{F}\cdot\hat{k}=1$. Then I can find the area. We choose $\bar{F} = \begin{bmatrix}0\\x\\0\end{bmatrix}$ like in example 1*:

$$\oint_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi}\begin{bmatrix}0\\3\cos t+3\sin t\\0\end{bmatrix}\cdot\begin{bmatrix}3\cos t-3\sin t\\2\cos t+2\sin t\\0\end{bmatrix}dt =$$

$$\int_0^{2\pi}(6\cos^2 t + 6\cos t\sin t + 6\cos t\sin t + 6\sin^2 t)\,dt = \int_0^{2\pi}(6+12\cos t\sin t)\,dt = \int_0^{2\pi}(6+6\sin 2t)\,dt =$$

$$= \Big[6t - 3\cos 2t\Big]_0^{2\pi} = 12\pi - 3 + 3 = 12\pi$$

Example 3 in the book



Use Green's theorem to simplify:

$$I = \oint_C (x-y^3)\,dx + (y^3+x^3)\,dy$$

$$I = \iint_Q \frac{\partial(y^3+x^3)}{\partial x} + \frac{\partial(x-y^3)}{\partial y}\,dA = 3\iint_Q (x^2+y^2)\,dA = \left\{\begin{matrix}x=r\cos\theta\\ y=r\sin\theta\\ dA = r\,dr\,d\theta\end{matrix}\right\} =$$

$$= 3\int_0^{\pi/2}d\theta\int_0^a r^3\,dr = \frac{3}{8}\pi a^4$$

Regular domains:

y-simple R_1:
Blue line: $f(g(y),y)$
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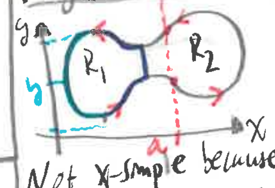
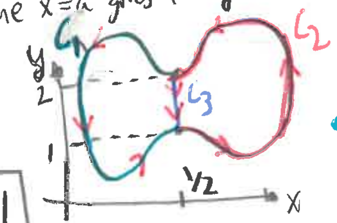
Split up the

$R = R_1 \cup R_2$
- R is not x-simple
- R_1 & R_2 are both x/y-simple



Not x-simple because one x=a gives two y



$$\oint_{\partial R}\bar{F}\cdot d\bar{r} = \oint_{\partial R_1}\bar{F}\cdot d\bar{r} + \oint_{\partial R_2}\bar{F}\cdot d\bar{r}$$

- $\oint_{\partial R_1}\bar{F}\cdot d\bar{r} = \int_{C_1}\bar{F}\cdot d\bar{r} + \boxed{\int_{C_3}\bar{F}\cdot d\bar{r}}$ opposite orientation
- $\oint_{\partial R_2}\bar{F}\cdot d\bar{r} = \int_{C_2}\bar{F}\cdot d\bar{r} + \boxed{\int_{C_3}\bar{F}\cdot d\bar{r}}$

$\partial R = C_1 \cup C_2$
$\partial R_1 = C_1 \cup C_3$
$\partial R_2 = C_2 \cup C_3$

C_3 can be parametrised in two ways:

- $\gamma_1(t) = \begin{bmatrix}1/2\\t\end{bmatrix}$ t: 1→2 so $\gamma_1'(t) = \begin{bmatrix}0\\1\end{bmatrix}$
- $\gamma_2(t) = \begin{bmatrix}1/2\\(1-t)+2\end{bmatrix}$ t: 1→2 so $\gamma_2'(t) = \begin{bmatrix}0\\-1\end{bmatrix}$

Personlig kommentar:
$\gamma_2(t) = \begin{bmatrix}1/2\\t\end{bmatrix}$ t: 2→1 då blir integralen negativ så det stämmer också.