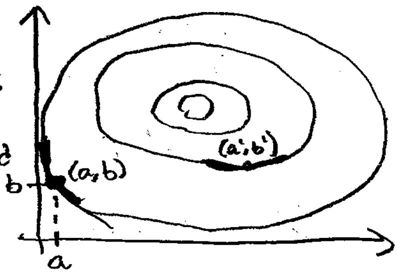
SF1626 2018-09-10 #7

Ch 12.8 Implicit functions

Can a chosen level curve be regarded as the graph of a function of x or a function of y ?

Pick the zero level set as an example.
Consider a neighborhood of (a,b). Call this neigborhood U.
Suppose the level curve can be regarded as a graph of a function $y=g(x)$ in U, then



$f(x,g(x))=0 \Leftrightarrow \frac{d}{dx}f(x,g(x)) = f_1(x,g(x))+f_2(x,g(x))g'(x)=0$

$g'(x)$ has to exist.

If it does: $g' = -\frac{f_1}{f_2}$. So f_2 cannot be zero in U.

- Knowing g' lets us construct a linear approximation to the solution of $(x,y)=0$ by $(x, g(x))$

The other view is this:
$f(x,y)=0$ consists of one equation and two variables
So in general, one of the variables is not "free", i.e. one of the variables is dependent on the other

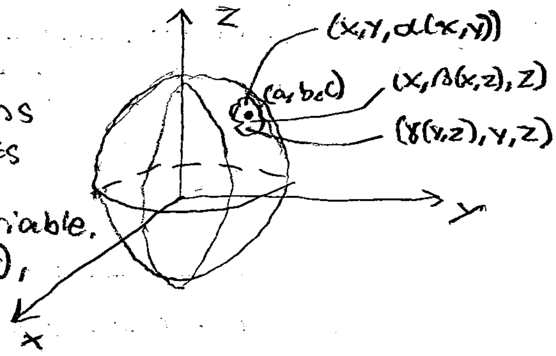
- The level set of $f(x,y,z)$ is in general a two dimentional surface.
- If this surface can be regarded as a graph of the function $g(x,z)$, then

$f(x,g(x,z),z)=0 \Rightarrow \frac{\partial f}{\partial x} = f_1+f_2 g_x = 0, \ \frac{\partial f}{\partial z} = f_2 g_z + f_3 = 0$

- g_x and g_z have to exist. If they do $g_x = -f_1/f_2 \quad g_y = -f_3/f_2 \quad (f_2 \neq 0)$

$f(x,y,z)=0, \quad g(x,y,z)=0$

- The points that satisfy these two equations lie on the intersection of the zero level sets of f and g, which is generally a curve.
- A curve is parametrized by only one variable.
- So we ask: in a neighborhood of (a,b,c), can this curve be parametrized by the x variable (or y or z)?



- Assume that we can use x to parametrize the curve: $x \to (x, y(x), z(x))$
- Two equations and three unknowns. So two of the unknowns may be dependent on the third.

$$\begin{pmatrix} f_2 & f_3 \\ g_2 & g_3 \end{pmatrix} \begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} = -\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$$

Jacobian determinants

In the previous example, in order to define the implicit function, the matrix

$$\begin{pmatrix} f_2 & f_3 \\ g_2 & g_3 \end{pmatrix}$$

has to be invertable.
It is invertible if the "Jacobian" is non-zero:

$$\frac{\partial(f,g)}{\partial(y,z)} := \det\begin{pmatrix} f_2 & f_3 \\ g_2 & g_3 \end{pmatrix} \neq 0 \quad \left(f_y g_z - g_y f_z = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z} = \frac{\partial g \partial f}{\partial y \partial z} = \frac{\partial(g,f)}{\partial(y,z)} \right)$$

The implicit function theorem

A system of n equations and $n+m$ variables:

$$F_{(i)}(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = 0$$
$$\llcorner i = 1, 2, \dots, n$$

$m+n$ unknowns with n equations. In general, there can be m independent variables.

Let U be a neighborhood of the point $P_0 = (a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n)$

Suppose that

- In U, $F_{(j)}$ has continous first partial derivatives with respective to each of the variables.
- At P_0, if $\dfrac{\partial(F_{(1)}, \dots, F_{(n)})}{\partial(y_1, y_2, \dots, y_n)} \neq 0$

Then $\exists \phi_1(x_1, \dots, x_m), \dots, \phi_n(x_1, \dots, x_m)$ such that

- $\phi_k(a_1, a_2, \dots, a_m) = b_k$, $\llcorner k = 1, 2, \dots, n$
- $F_{(j)}(x_1, \dots, x_m, \llcorner \phi_1(x_1, \dots, x_m), \dots, \phi_n(x_1, \dots, x_m)) = 0$ in U
- $\dfrac{\partial \phi_i}{\partial x_j} = ?$ (Chain rule)

$$\frac{\partial \phi_i}{\partial x_j} = -\frac{\dfrac{\partial(F_{(1)}, \dots, F_{(n)})}{\partial(y_1, \dots, x_j, \dots, y_n)}}{\dfrac{\partial(F_{(1)}, \dots, F_{(n)})}{\partial(y_1, \dots, y_i, \dots, y_n)}}$$

Ex. $n = 1$, $m = 1$ $\left.\begin{matrix}(1 \text{ eq'n}) \\ 2 \text{ variables})\end{matrix}\right\}$ $F_{(1)}(x, y) = 0$

$$\boxed{f(x, \phi(x)) = 0}$$

"there exists"

$$\text{if } \frac{\partial(f)}{\partial \phi} = f_2 \neq 0 \Rightarrow \exists \phi(x)$$

$$\frac{\partial}{dx} f(x, \phi(x)) = 0$$

$$= f_1(x, \phi(x)) \underset{\substack{\| \\ \frac{dx}{dx}}}{1} + f_2(x, \phi(x)) \frac{d\phi}{\partial x}(x) = 0 \Rightarrow \frac{d\phi}{dx} = -\frac{f_1(x, \phi(x))}{f_2(x, \phi(x))}$$

Ex 7)
$$\begin{cases} f_{(1)}: & xy^2 + xzu + yv^2 = 3 \\ f_{(2)}: & x^3yz + 2xv - u^2v^3 = 3 \end{cases}$$

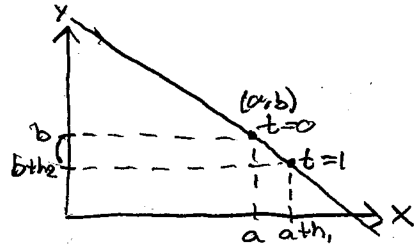
Show that this system of equations can be solved for $u = u(x, y, z)$, $v = v(x, y, z)$ near $P_0 = (x, y, z, u, v) = (1, 1, 1, 1, 1)$

$x \to x_1$ $\quad u \to y_1$
$y \to x_2$ $\quad v \to y_2$
$z \to x_3$

$$\frac{\partial(f_{(1)}, f_{(2)})}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial f_{(1)}}{\partial u} & \frac{\partial f_{(1)}}{\partial v} \\ \frac{\partial f_{(2)}}{\partial u} & \frac{\partial f_{(2)}}{\partial v} \end{pmatrix} = \det\begin{pmatrix} xy & 2yv \\ -2v^2u & 2x-2u^2v \end{pmatrix}\Bigg|_{(x,y,z,u,v)=(1,1,1,1,1)} = \det\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} = 4 \neq 0$$

Ch 12.9 Taylor's formula

We try to approximate the values of $f(x,y)$ near (a,b), restricted along the curve $t \to (a+th_1)\vec{i} + (b+th_2)\vec{j}$:



$$\begin{aligned} F(1) &= F(0) + F'(0) + \tfrac{1}{2}F''(0) + \dots && (1) \\ f(a+h_1, b+h_2) &= f(a,b) + (h_1 f_1(a,b) + h_2 f_2(a,b)) + \dots && (2) \end{aligned}$$

$$F''(t) = \frac{d}{dt} g(a+th_1, b+th_2) = (\vec{h}\cdot\nabla)g = (\vec{h}\cdot\nabla)^2 f$$