SF1626 2018-10-11 #20

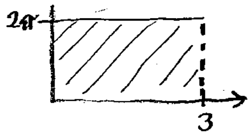
Exam problem (2018-06-04 #4)

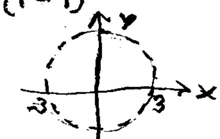
$f(x,y) = (\sqrt{x^2+y^2} - 1)^2 \qquad D := \{(x,y) \mid x^2+y^2 < 9\}$

a) Find all local min and local max of f on D
- critical points in D $\rightarrow$ solve for $\nabla f(x,y) = \vec{0}$
- boundary points $\rightarrow$ consider points on boundary of D.
- singular points $\rightarrow$ consider where $f(x,y)$ is NOT differentiable

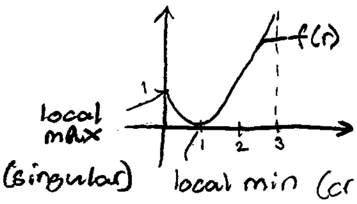
Polar coordinates: $r^2 = x^2+y^2 \qquad f(x,y) = f(r,\theta) = (r-1)^2$

$D_{(x,y)}$ is described by $D_{(r,\theta)}$:





We won't need to consider boundary points since the boundary is not included.



$$\nabla f = \begin{pmatrix} 2(\sqrt{x^2+y^2}-1) \cdot \frac{1}{2} \frac{2x}{(x^2+y^2-1)^{3/2}} \\ (\sqrt{x^2+y^2}-1) \cdot \frac{2y}{(x^2+y^2)^{3/2}} \end{pmatrix} = \vec{0}$$

$f_x = 0 \Rightarrow x = 0$ or $\sqrt{x^2+y^2} = 1$

$f_y = 0 \Rightarrow y = 0$ or $\sqrt{x^2+y^2} = 1$

if $x = 0$, $f_y(0,y) = (|y|-1)\frac{2y}{|y|^3} = 0 \Rightarrow |y| = 1$ $\quad (0, \pm 1)$ on the unit circle

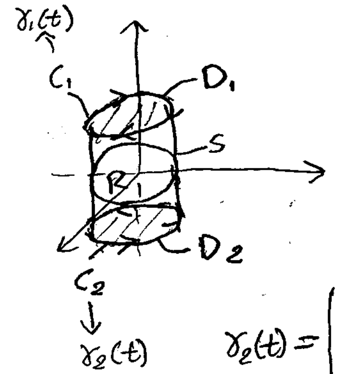
Exam problem (2018-06-04 #5)

Find $\iint_S \text{curl}\,\bar{F} \cdot \hat{N}\, dS$ where $\bar{F} = (F_1, F_2, F_3)$ is a smooth vector field with the property

$\bar{F} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \\ F_3(x,y,z) \end{pmatrix} \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial z} = 0$ and S is given by $S = \{(x,y,z) : x^2+y^2=1, -1<z<1\}$

Stoke's theorem:

$$\iint_S \text{curl}\,\bar{F} \cdot \hat{N}\, dS = \int_{\partial S} \bar{F} \cdot d\bar{r} = \int_{C_1} \bar{F} \cdot d\bar{r} + \int_{C_2} \bar{F} \cdot d\bar{r}$$



$$\oint_{C_2} \bar{F} \cdot d\bar{r} = \int_0^{2\pi} \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \\ F_3(x,y,z) \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$$

$$\oint_{C_1} \bar{F} \cdot d\bar{r} = \int_{2\pi}^{0} \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \\ F_3(x,y,z) \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = -\oint_{C_2} \bar{F} \cdot d\bar{r}$$

$\gamma_2(t) = \begin{pmatrix} \cos t \\ \sin t \\ -1 \end{pmatrix}$

$\gamma_2'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}$

This gives

$$\int_{C_1} \bar{F} \cdot d\bar{r} + \int_{C_2} \bar{F} \cdot d\bar{r} = 0 \quad \Rightarrow \iint_S \text{curl}\,\bar{F} \cdot \hat{N}\, dS = 0$$

An other solution using divergence theorem on next page

Divergence theorem

G vector field in R

$$\iint_{\partial R} G \cdot \hat{N}\, dS = \iiint_R \operatorname{div} G \, dV$$

$$\Rightarrow \quad \iint_S \operatorname{curl} \bar{F} \cdot \hat{N}\, dS = \iint_{S \cup D_1 \cup D_2} \underbrace{\operatorname{curl} F}_{G} \cdot \hat{N}\, dS = \iiint_R \operatorname{div}(\operatorname{curl} \bar{F})\, dV = 0$$