TDT4171 - Assignment 2

Task 1

$Fish_t$: Fish nearby
$Bird_t$: Birds nearby

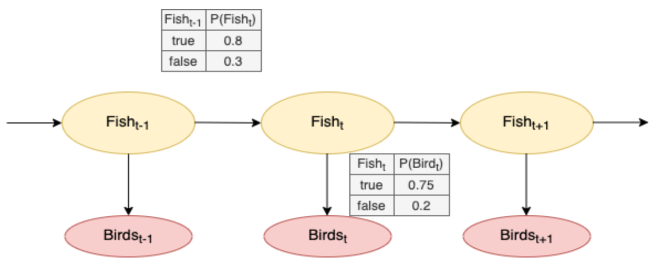
$e_1 = \{birds\ nearby\}$
$e_2 = \{birds\ nearby\}$
$e_3 = \{\ no\ birds\ neabry\}$
$e_4 = \{\ birds\ neaby\}$
$e_5 = \{\ no\ birds\ neaby\}$
$e_6 = \{\ birds\ neaby\}$

$$P(Fish_t) = \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix}$$

$$\boldsymbol{T} = P(Fish_t | Fish_{t-1}) = \begin{pmatrix} 0.80 & 0.20 \\ 0.30 & 0.70 \end{pmatrix}$$

$$\boldsymbol{B} = P(Bird_t | Fish_t) = \begin{pmatrix} 0.75 & 0.25 \\ 0.20 & 0.80 \end{pmatrix}$$

Problem a)



Problem b)

$P(X_t | e_{1:t}), for\ t = 1, 2, 3, 4, 5, 6.$

This is filtering, which shows the posterior distribution over the most recent state, given all evidence up to t = 5. Uses only past data in estimation. Solved by ***filtering.py***.

Day	True	False
1	0.8209	0.1791
2	0.902	0.098
3	0.4852	0.5148
4	0.8165	0.1835
5	0.4313	0.5687
6	0.7997	0.2003

Problem c)

$P(X_t|e_{1:4}), for\ t = 7, ..., 30.$

This is prediction, which shows the posterior distribution over the future states for t = 7, …, 30. This is done using the evidence up to data, i.e. for t = 1, …, 5. Solved by ***prediction.py***.

Day	True	False
7	0.6999	0.3001
8	0.6499	0.3501
9	0.625	0.375
10	0.6125	0.3875
11	0.6062	0.3938
12	0.6031	0.3969
13	0.6016	0.3984
14	0.6008	0.3992
15	0.6004	0.3996
16	0.6002	0.3998
17	0.6001	0.3999
18	0.6	0.4
19	0.6	0.4
20	0.6	0.4
21	0.6	0.4
22	0.6	0.4
23	0.6	0.4
24	0.6	0.4
25	0.6	0.4
26	0.6	0.4
27	0.6	0.4
28	0.6	0.4
29	0.6	0.4
30	0.6	0.4

Problem d)

$P(X_t|e_{1:6}), for\ t = 0, 1, 2, 3, 4, 5.$

This is smoothing. This shows the distribution of the past states for t = 1,.. 5, given evidence up present time, .i.e. t = 5. Uses both past and present data in estimation. Therefore more evidence is incorporated, and a better estimation of the state at the time is available. Solved by ***smoothing.py***.

Day	True	False
0	0.7482	0.2518
1	0.9137	0.0863
2	0.9052	0.0948
3	0.6847	0.3153
4	0.8211	0.1789
5	0.6332	0.3668

Problem e)

$$\underset{x_1,\ldots,x_{t-1}}{\arg\max} P(x_1, \ldots x_{t-1}, X_t | e_{1:t})\, , for\ t = 1, \ldots , 6$$

This is the most likely sequence, which is the sequence of states that is most likely to have generated the observations. I.e. given our observations, what is the most likely sequence of hidden states that led to the observations. Solved by ***mls.py***.

```
Day     True    False
-----------------------
1       0.375   0.1
2       0.225   0.015
3       0.045   0.036
4       0.027   0.005
5       0.0054  0.0043
6       0.0032  0.0006

Most likely sequence
------------------------
Day 1:  True
Day 2:  True
Day 3:  True
Day 4:  True
Day 5:  True
Day 6:  True
```

Task 2

$e_1 = \{animal\ tracks, food\ gone\}$
$e_2 = \{\ no\ animal\ tracks, food\ gone\}$
$e_3 = \{\ no\ animal\ tracks, food\ not\ gone\}$
$e_4 = \{\ animal\ tracks, food\ not\ gone\}$

$$P(Animal_t) = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

$$P(Animal_t \mid Animal_{t-1}) = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$$P(AnimalTracks_t \mid Animal_t) = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

$$P(FoodGone_t \mid Animal_t) = \begin{pmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{pmatrix}$$

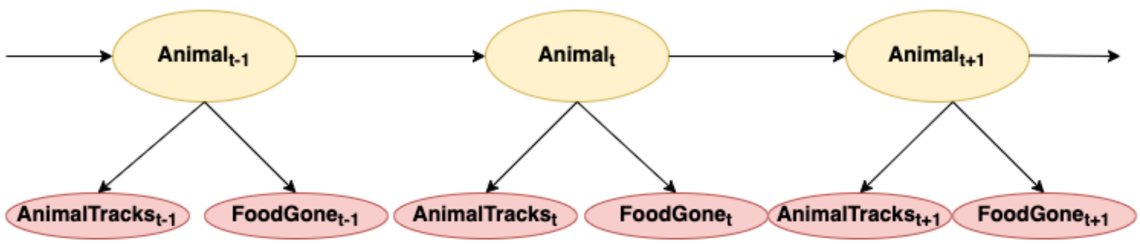
Problem a)

$Animal_{t-1}$	$P(Animal_t)$
true	0.8
false	0.3

$Animal_t$	$P(AnimalTracks_t)$
true	0.7
false	0.3

$Animal_t$	$P(FoodGone_t)$
true	0.3
false	0.1

$Animal_t$	$P(AnimalTracks_t, FoodGone_t)$
true	0.24
false	0.14



$$AnimalTracks_t \perp\!\!\!\perp FoodGone_t \mid \{Animal_t\}$$

$$\rightarrow P(AnimalTracks_t, FoodGone_t \mid Animal_t) =$$

$$= P(AnimalTracks_t \mid Animal_t) * P(FoodGone_t \mid Animal_t)$$

$$= \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \cdot \begin{pmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.24 & 0.76 \\ 0.14 & 0.86 \end{pmatrix}$$

$$T = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

Problem b)

$P(X_t|e_{1:t}), for\ t = 1, 2, 3, 4.$

Observations on matrix form:

$$O_1 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.3 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.21 & 0 \\ 0 & 0.02 \end{pmatrix}$$

$$O_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.8 \end{pmatrix} \cdot \begin{pmatrix} 0.3 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.08 \end{pmatrix}$$

$$O_3 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.8 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0 \\ 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.21 & 0 \\ 0 & 0.72 \end{pmatrix}$$

$$O_4 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0 \\ 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.49 & 0 \\ 0 & 0.18 \end{pmatrix}$$

This is filtering → Using the forward algorithm.

$$(f_{0:0})^T = \begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$$

$$(f_{0:1})^T = c_1^{-1} \begin{pmatrix} 0.21 & 0 \\ 0 & 0.02 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix} = c_1^{-1} \begin{pmatrix} 0.1365 \\ 0.0070 \end{pmatrix} = \begin{pmatrix} 0.9512 \\ 0.0488 \end{pmatrix}$$

$$(f_{0:2})^T = c_1^{-1} \begin{pmatrix} 0.09 & 0 \\ 0 & 0.08 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.9512 \\ 0.0488 \end{pmatrix} = c_1^{-1} \begin{pmatrix} 0.0698 \\ 0.0180 \end{pmatrix} = \begin{pmatrix} 0.7954 \\ 0.2046 \end{pmatrix}$$

$$(f_{0:3})^T = c_1^{-1} \begin{pmatrix} 0.21 & 0 \\ 0 & 0.72 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.7954 \\ 0.2046 \end{pmatrix} = c_1^{-1} \begin{pmatrix} 0.1465 \\ 0.2176 \end{pmatrix} = \begin{pmatrix} 0.4024 \\ 0.5976 \end{pmatrix}$$

$$(f_{0:4})^T = c_1^{-1} \begin{pmatrix} 0.49 & 0 \\ 0 & 0.18 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.4024 \\ 0.5976 \end{pmatrix} = c_1^{-1} \begin{pmatrix} 0.2456 \\ 0.0897 \end{pmatrix} = \begin{pmatrix} 0.7323 \\ 0.2677 \end{pmatrix}$$

Problem c)

$P(X_t|e_{1:4}), for\ t = 5,6,7,8.$

From problem c) we have that

$$(f_{0:4})^T = \begin{pmatrix} 0.7323 \\ 0.2677 \end{pmatrix}$$

$$t_5 \rightarrow \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.7323 \\ 0.2677 \end{pmatrix} = \begin{pmatrix} 0.6661 \\ 0.3339 \end{pmatrix}$$

$$t_6 \rightarrow \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6661 \\ 0.3339 \end{pmatrix} = \begin{pmatrix} 0.6331 \\ 0.3669 \end{pmatrix}$$

$$t_7 \rightarrow \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6331 \\ 0.3669 \end{pmatrix} = \begin{pmatrix} 0.6165 \\ 0.3835 \end{pmatrix}$$

$$t_8 \rightarrow \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6165 \\ 0.3835 \end{pmatrix} = \begin{pmatrix} 0.6083 \\ 0.3917 \end{pmatrix}$$

Problem d)

$\lim_{t \rightarrow \infty} P(X_t|e_{1:4}) = \langle 0.6, 0.4 \rangle$.

Stationary transition matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

$0.8 \cdot \theta_1 + 0.3 \cdot \theta_2 = \theta_1$ (1)

$0.2 \cdot \theta_1 + 0.7 \cdot \theta_2 = \theta_2$ (2)

$\theta_1 + \theta_2 = 1$ (3)

(3) $\rightarrow \theta_1 = 1 - \theta_2$

(1) $\rightarrow \theta_1 = 1.5 \cdot \theta_2$

$\rightarrow (\theta_1, \theta_2) = (0.6, 0.4)$

$\rightarrow Converges\ to$ $\langle 0.6, 0.4 \rangle$.

Problem e)

$P(X_t|e_{1:4}), for\ t = 0, 1, 2, 3.$

Observations on matrix form:

$$O_1 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.3 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.21 & 0 \\ 0 & 0.02 \end{pmatrix}$$

$$O_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.8 \end{pmatrix} \cdot \begin{pmatrix} 0.3 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.08 \end{pmatrix}$$

$$O_3 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.8 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0 \\ 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.21 & 0 \\ 0 & 0.72 \end{pmatrix}$$

$$O_4 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0 \\ 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.49 & 0 \\ 0 & 0.18 \end{pmatrix}$$

This is smoothing → Using the results from forward algorithm

For backward phase we start with:

$$(b_{4:4})^T = \begin{pmatrix} 1.0000 \\ 1.0000 \end{pmatrix}$$

$$(b_{3:4})^T = \alpha \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.49 & 0 \\ 0 & 0.18 \end{pmatrix} \cdot \begin{pmatrix} 1.0000 \\ 1.0000 \end{pmatrix} = \alpha \begin{pmatrix} 0.4460 \\ 0.2240 \end{pmatrix} = \begin{pmatrix} 0.6657 \\ 0.3343 \end{pmatrix}$$

$$(b_{2:4})^T = \alpha \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.21 & 0 \\ 0 & 0.72 \end{pmatrix} \cdot \begin{pmatrix} 0.6657 \\ 0.3343 \end{pmatrix} = \alpha \begin{pmatrix} 0.1840 \\ 0.1965 \end{pmatrix} = \begin{pmatrix} 0.4837 \\ 0.5163 \end{pmatrix}$$

$$(b_{1:4})^T = \alpha \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.09 & 0 \\ 0 & 0.08 \end{pmatrix} \cdot \begin{pmatrix} 0.4837 \\ 0.5263 \end{pmatrix} = \alpha \begin{pmatrix} 0.0472 \\ 0.0376 \end{pmatrix} = \begin{pmatrix} 0.5566 \\ 0.4434 \end{pmatrix}$$

$$(b_{0:4})^T = \alpha \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.21 & 0 \\ 0 & 0.02 \end{pmatrix} \cdot \begin{pmatrix} 0.5566 \\ 0.4434 \end{pmatrix} = \alpha \begin{pmatrix} 0.0961 \\ 0.0296 \end{pmatrix} = \begin{pmatrix} 0.7647 \\ 0.2353 \end{pmatrix}$$

Finally, the smoothed probabilities are calculated using the forward and backward probabilities:

$$(\gamma_0)^T = \alpha \begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix} \circ \begin{pmatrix} 0.7647 \\ 0.2353 \end{pmatrix} = \alpha \begin{pmatrix} 0.5353 \\ 0.0706 \end{pmatrix} = \begin{pmatrix} 0.8835 \\ 0.1165 \end{pmatrix}$$

$$(\gamma_1)^T = \alpha \begin{pmatrix} 0.9512 \\ 0.0488 \end{pmatrix} \circ \begin{pmatrix} 0.5566 \\ 0.4434 \end{pmatrix} = \alpha \begin{pmatrix} 0.5294 \\ 0.0216 \end{pmatrix} = \begin{pmatrix} 0.9607 \\ 0.0393 \end{pmatrix}$$

$$(\gamma_2)^T = \alpha \begin{pmatrix} 0.7954 \\ 0.2046 \end{pmatrix} \circ \begin{pmatrix} 0.4837 \\ 0.5163 \end{pmatrix} = \alpha \begin{pmatrix} 0.3847 \\ 0.1056 \end{pmatrix} = \begin{pmatrix} 0.7846 \\ 0.2154 \end{pmatrix}$$

$$(\gamma_3)^T = \alpha \begin{pmatrix} 0.4024 \\ 0.5976 \end{pmatrix} \circ \begin{pmatrix} 0.6657 \\ 0.3343 \end{pmatrix} = \alpha \begin{pmatrix} 0.2678 \\ 0.1998 \end{pmatrix} = \begin{pmatrix} 0.5727 \\ 0.4273 \end{pmatrix}$$

$$(\gamma_4)^T = \alpha \begin{pmatrix} 0.7323 \\ 0.2677 \end{pmatrix} \circ \begin{pmatrix} 1.0000 \\ 1.0000 \end{pmatrix} = \alpha \begin{pmatrix} 0.7323 \\ 0.2677 \end{pmatrix} = \begin{pmatrix} 0.7323 \\ 0.2677 \end{pmatrix}$$