Theory Of Estimations

Estimation theory is a branch of statistics which deals with estimating the value of parameters based on measured or empirical data that has a random component.

Parameters described physical setting in such a way that the value of the parameters affects the distribution of measured data. an estimater attempt to aproximate the unknown parameter using the measurments.

eg: It is decide to estimate the propotion of a popⁿ of voters who will vote for a particular canidate that propotion is the unobservable parameter the estimate is based on small random sample of voters.

Types of Estimation

The theory of estimation is divided into two groups

i) Point estimation

ii) Interval estimation.

I) Point estimation

In Statistics, point estimation involves the use of sample data to calculate the single value (known us statistics) which is to serve us a best guess for an unknown popⁿ parameter. i e, the yield single valued result although this includes the posibility of single vector valued results and results can be expressed us a single function. a good estimator is one which is us closed to the true value of the parameter us possible.

ii) Interval Estimation

In statistics interval estimation is the use of sample data to calculate an interval of possible values of an unknown popⁿ parameter. This is in contrast to an point estimator.

where the results would be a range of possible values.

The most prevelent forms of interval estimation are i) confidence intervals ii) credible intervals, iii) tolerence intervals production intervals, . . . etc

Properties of estimators

The most important desirable properties of good estimator are

i) unbiasedness
ii) consistency
iii) efficiency
iv) sufficiency

i) unbiasedness

A statistic $t = E(x_1, x_2 \ldots x_n)$ a function of sample observations $x_1, x_2 \ldots x_n$ is said to be an unbiased estimate of the corresponding popn parameter θ if

$E(t) = \theta$ ie, if the mean value of the sampling distribution of the statistic is equal to the parameter.

eg: The sample mean $\bar{x}$ is an unbiased estimate of popⁿ mean μ ie, $E(\bar{x}) = \mu$

D S.T the sample mean is an unbiased estimate of popⁿ mean

→ Let $\bar{x}$ is the sample mean

n: size of samples

μ: popⁿ mean

$E(\bar{x}) = \mu$

$$\bar{x} = \frac{x_1 + x_2 \cdots + x_n}{n}$$

$$E(\bar{x}) = 1/n \left[E(x_1 + x_2 \cdots x_n)\right]$$

$$= 1/n \left[\mu + \mu + \cdots\right]$$

$$= \mu$$

2 $X_1, X_2 \cdots X_n$ is a random sample from a popⁿ following poisson distribution with parameter λ. Suggest any three unbiased estimation of λ

→ X_i is a r.s from a poisson popⁿ with parameter λ

$E(x_i) = \lambda$ $i = 1, 2 \cdots n$

$$t_1 = X_1, \quad t_2 = \frac{X_1 + X_2}{2}, \quad t_n = \frac{X_1 + X_2 + \cdots X_n}{n}$$

are unbiased estimates of λ

$\therefore E(t_1) = E(x_1) = \lambda$

$E(t_2) = \frac{E(x_1+x_2)}{2} = \frac{2\lambda}{\lambda} = \lambda$

$E(t_n) = \lambda$.

$\therefore t_1\ t_2 \ldots t_n$ are unbiased estimates of λ

3) If $x_1, x_2 \ldots x_n$ is r.s from a normal popⁿ with mean μ & variance 1. S.T $t = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is an unbiased estimate of μ^2+1

→ Sample is from normal popⁿ with mean μ & variance 1 $i = 1, \ldots n$

$V(x_i) = E(x_i^2) - [E(x_i)]^2$

$1 = E(x_i^2) - \mu^2$

$E(x_i^2) = 1 + \mu^2$

$E(t) = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\right]$

$= \frac{1}{n}\sum_{i=1}^{n} E(x_i^2)$

$= \frac{1}{n}\, n(1+\mu^2)$

$= \mu^2 + 1$ //

4 If T is an unbiased estimator for θ S.T. T^2 is a biased estimator for θ^2

$\Rightarrow$ Since T is an unbiased estimator for θ

$$E(T) = \theta$$

$$V(T) = E(x_i^2) - [E(x_i)]^2$$

$$= E(T^2) - \theta^2$$

$$E(T^2) = V(T) + \theta^2$$

$V(T) > 0$

$\therefore E(T^2)$ is not equal to θ^2 $\therefore T^2$ is biased estimate of θ^2

Note:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \quad \text{---(1)}$$ is not an unbiased estimate of pop^n variance.

however

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \quad \text{---(2)}$$ is an unbiased estimate of σ^2

ie, $E(s^2) \neq \sigma^2$ & $E(S^2) = \sigma^2$

Remarks

I From eqn (1) & (2) we have $ns^2 = (n-1)S^2$

$$S^2 = \frac{n}{(n-1)} s^2$$

II) If the sample size is large $(n-1)$ can be aproximate by n

$$S^2 = \frac{n}{n-1} s^2 \simeq s^2$$

thus for large samples the sample variance gives an unbiased estimate of pop^n variance σ^2 hence σ^2 is not known, then for large samples its estimate is provided by S^2

III If $E(t) \neq \theta$ then the statistic t is ~~use~~ said to be biased estimate of θ.

IV. $E(t) = b + \theta$

then b is called amount of bias in the estimate.

if $b > 0$ ie, $E(t) > \theta$

then t is called +vly biased

$b < 0$ ie, $E(t) < \theta$ then t is called -vly biased.

ii) S.T $2\bar{x}$ is an unbiased estimate of θ

$f(x,\theta) = 1/\theta \qquad 0 \leq x \leq \theta$

$$2\bar{x} = \frac{2.(x_1 + x_2 + \cdots x_n)}{n}$$

$$E(2\bar{x}) = \frac{2\sum_{i=1}^{n} E(x_i)}{n}$$

$$E(x_i) = \int_0^\theta 1/\theta \cdot x_i \cdot dx_i$$

$$\frac{1}{\theta}\left[\frac{x^2}{2}\right]_0^{\theta}$$

$$= \frac{1}{\theta} \cdot \frac{\theta^2}{2} = \frac{\theta}{2}$$

$$E(2\bar{x}) = \frac{2}{\cancel{n}} \cdot \frac{\cancel{n}\theta}{\cancel{2}}$$

$$= \underline{\underline{\theta}}$$

11) Consistency

An estimator $t_n = t(x_1, x_2, \dots x_n)$ based on a random sample of size n is said to be consistent estimator of θ, ~~$\theta \in$~~ $\theta \in \Theta$ the parameter space.

If t_n converges to θ in probability

ie, $t_n \xrightarrow{P} \theta$ as $n \to \infty$

iv If $E(t_n) = \theta$ or $E(t_n) \to \theta$ and $V(t_n) \to 0$ as $n \to \infty$, t_n is a consistant estimate of θ this may be proved by tchebychelebs inequality

① S.T the mean of sample taken from a normal popⁿ are Consistant estimate of popⁿ mean.

Let the popⁿ be $N(\mu, \sigma^2)$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

$$= \frac{1}{n}\left[E(x_1) + E(x_2) + \cdots + E(x_n)\right]$$

$$= \frac{1}{n}\left[\mu + \mu + \cdots + \mu\right]$$

$$= \frac{1}{\cancel{n}}\left[\cancel{n}\mu\right] = \underline{\underline{\mu}}$$

$$V(\bar{x}) = V\left(\frac{1}{n}\sum x_i\right)$$

$$= V\left(\frac{1}{n}\left[x_1 + x_2 + \cdots + x_n\right]\right)$$

$$= \frac{1}{n^2} V(x_1 + x_2 + \cdots + x_n)$$

$$= \frac{\cancel{n}\sigma^2}{n^{\cancel{2}}} = \frac{\sigma^2}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$\therefore$ $\bar{x}$ is Consistant estimate of μ

② P.T $t_n = \frac{n\bar{x}}{n+1}$ is a consistant estimate of λ where $\bar{x}$ is the mean of the sample size n. taken from a Poisson popⁿ.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$E(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\left[E(x_1) + E(x_2) + \cdots + E(x_n)\right]$$

$$= \frac{1}{n}\left[\lambda + \lambda + \cdots + \lambda\right] = \frac{1}{\cancel{n}}\cancel{n}\lambda = \underline{\underline{\lambda}}$$

$$\therefore E(t_n) = E\left(\frac{n\bar{x}}{n+1}\right) = \frac{n\, E(\bar{x})}{n+1}$$

$$= \frac{n\lambda}{n+1}$$

$E(t_n) \to \lambda$ as $n \to \infty$

$$V(\bar{x}) = \frac{1}{n^2} \sum V(x_i)$$

$$= \frac{1}{n^2}\left[V(x_1) + V(x_2) + \cdots + V(x_n)\right]$$

$$= \frac{1}{n^{\cancel{2}}} \cancel{n}\lambda = \frac{\lambda}{n}$$

$$V(t_n) = V\left(\frac{n\bar{x}}{n+1}\right)$$

$$= \frac{n^2\, V(\bar{x})}{(n+1)^2} = \frac{n^{\cancel{2}}}{(n+1)^2} \cdot \frac{\lambda}{\cancel{n}} = \frac{n\lambda}{(n+1)^2} \to 0 \text{ as } n \to \infty$$

$\therefore$ t_n is consistent estimate of λ.

Efficiency

If the two consistent estimators t_1 & t_2 of a certain parameter θ we have $V(t_1) < V(t_2)$ $\forall n$ then t_1 is more efficient than t_2 $\forall$ sample size

If t_1 is most efficient estimator with variance V_1 & t_2 is any other estimator with variance V_2 then efficiency $\epsilon(t_2)$ is defind as $\epsilon = V_1/V_2$. ϵ can not exceed unity.

So we may say that variance of an estimate decreases its efficiency increases.

$V(t_1)/V(t_2)$ is called relative efficiency of t_2 with respect to t_1.

① $X_1 \ldots X_n$ is a random sample from normal popⁿ $n(\mu, \sigma)$, $t_1 = X_1$, $t_2 = \frac{X_1 + X_2}{2}$

$\ldots$ $t_n = \frac{X_1 + X_2 + \cdots X_n}{n}$ are

proposed as estimates of μ. examine whether the estimates are unbiased. and compare their efficiency.

$\Rightarrow$

$E(t_1) = E(x_1) = \mu$

$E(t_2) = E\left(\frac{x_1 + x_2}{2}\right) = \mu$

$E(t_n) = \mu$

all the estimates are unbiased

$V(t_1) = V(x_1) = \sigma^2$

$V(t_2) = V\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{4} V(x_1 + V(x_2))$

$= \frac{2\sigma^2}{4} = \sigma^2/2$

$V(t_n) = n\sigma^2/n^2 = \sigma^2/n$

now $\sigma^2 > \sigma^2/2 > \sigma^2/3 > \ldots > \sigma^2/n$

$\therefore t_n$ is more efficient than others.

2 x_1, x_2, x_3 are three independed observations from a popⁿ with mean μ & variance σ^2 if $t_1 = x_1 + x_2 - x_3$ and $t_2 = 2x_1 + 3x_2 - 4x_3$. compare the efficiencies of t_1 & t_2

$\Rightarrow$ $$E(t) = E(x_1) + E(x_2) - E(x_3)$$

$$= \mu + \mu - \mu$$

$$= \mu$$

$$E(t_2) = 2E(x_1) + 3E(x_2) - 4E(x_3)$$

$$2\mu + 3\mu - 4\mu$$

$$= \mu$$

t_1 & t_2 are unbiased estimate of μ

$$V(t_1) = V(x_1) + V(x_2) + V(x_3)$$

$$= \sigma^2 + \sigma^2 + \sigma^2$$

$$= 3\sigma^2$$

$$V(t_2) = \cancel{2V(x} \; 2V(x_1) + 3V(x_2) + 4V(x_3)$$

$$= 4\sigma^2 + 9\sigma^2 + 16\sigma^2$$

$$= 29\sigma^2$$

$\therefore V(t_1) < V(t_2)$

$\therefore t_1$ is more efficient than t_2

relative efficiency of t_1 w.r.t t_2

$$E = \frac{V(t_1)}{V(t_2)}$$

$$= 3\sigma^2 / 29\sigma^2 = \underline{\underline{3/29}}$$

HW 4 X_1 X_2 X_3 are random sample of size three from a popⁿ with mean value μ & variance σ^2. t_1 t_2 t_3 are the estimators used to estimate the mean value μ where $t_1 = \frac{1}{3}[\lambda x_1 + x_2 + x_3]$

$$t_2 = \frac{x_1 + x_2 + x_3}{3}$$

i) are t_1 & t_2 unbiased estimators

ii) find the value of λ s.t. t_1 is unbiased estimator of μ

iii) With this value of λ is t_1 is a consistent estimator.

iv) Which is the best estimator.

(5) / iv) Sufficiency

An estimator is said to be sufficient for a parameter if it contains all the information in the sample regarding the parameter.

Neyman's condition for sufficiency

Let $x_1, x_2 \ldots x_n$ be a sample from popⁿ with ~~the~~ pdf $f(x, \theta)$ the joint pdf of sample [Likelihood of the sample] is $L(x_1, x_2, \ldots, x_n, \theta)$
$= f(x_1, \theta)\, f(x_2, \theta) \ldots f(x_n, \theta)$

t is a sufficient estimate of θ iff it is possible to write $L(x_1, x_2, \ldots, x_n, \theta) = L_1(t, \theta) \cdot L_2(x_1, x_2, \ldots x_n)$

Where $L_1(t, \theta)$ is a function of t & θ alone. & L_2 is independed of θ.

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1) $f(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad x = 0, 1 \ldots$

Let $x_1, x_2 \ldots x_n$ be the sample

then $L(x_1, x_2 \ldots x_n), \lambda) =$

$$\frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{x_1!, x_2! \ldots x_n!}$$

$$= \frac{e^{-n\lambda} \cdot \lambda^{n\bar{x}}}{x_1!, x_2! \ldots x_n!} = e^{-n\lambda} \lambda^{n\bar{x}} \left[\frac{1}{x_1!\, x_2! \ldots x_n!} \right]$$

by the factorisation theorem, the sample mean $\bar{x}$ is a sufficient estimator for λ

① S.T $\bar{x}$ is a sufficient estimate of p when samples of size n are taken from a binomial popⁿ with parameters n & p

$$f(x) = \binom{N}{x} p^{x} q^{N-x}$$

Let $x_1, x_2 \ldots x_n$ be the samples then

$L(x_1, x_2, \ldots x_n, p) =$

$$\binom{N}{x_1}\binom{N}{x_2}\cdots\binom{N}{x_n} \cdot p^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n}(N-x_i)}$$

$$= \binom{N}{x_1}\binom{N}{x_2}\cdots\binom{N}{x_n} \cdot p^{n\bar{x}} (1-p)^{nN-n\bar{x}}$$

$$L_1 = \binom{N}{x_1}\binom{N}{x_2}\cdots\binom{N}{x_n}$$

$$L_2 = p^{n\bar{x}} (1-p)^{nN-n\bar{x}}$$

$\therefore L = L_1 \cdot L_2$

L_1 is independed of p & L_2 is a function of $\bar{x}$ & p only so $\bar{x}$ is a sufficient

Methods of Estimation

I Methods of maximum likelihood estimation (MLE)

The maximum likelihood estimate is an ~~ex~~ estimation techniques in statistics to estimate non-random parameters. A maximum likelihood estimate is a maximiser of log likelihood function. It is distributed as a Guassian random variable.

Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a popⁿ with density function $f(x, \theta)$. Then the likelihood function of this sample values $x_1, x_2 \ldots x_n$ usually denoted by ~~LOL~~ $L = L(\theta)$ is their joined density function is given by.

$$L = f(x_1, \theta) \cdot f(x_2, \theta) \cdot \ldots \cdot f(x_n, \theta).$$

$$= \prod_{i=1}^{n} f(x_i, \theta)$$

L gives the relative likelihood

that the random variables assume a particular set of values $x_1, x_2 \ldots x_n$. L becomes a function of the variable θ, parameter

The principle of maximum likelihood consist in finding an estimated for the unknown parameter $\theta = (\theta_1, \theta_2 \cdots \theta_k)$, which maximises the likelihood function $L(\theta)$ for variations in parameter ie, we wish to find

$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_k)$ so that

$L(\hat{\theta}) > L(\theta) \;\forall\; \theta \in \Theta$

$L(\hat{\theta}) = \sup L(\theta) \;\forall\; \theta \in \Theta$

$\hat{\theta}$ is usually called MLE. Thus $\hat{\theta}$ is the solution if any of $\frac{\partial L}{\partial \theta} = 0$ and

$$\left.\begin{aligned} &\frac{\partial L}{\partial \theta} = 0 \text{ and} \\ &\frac{\partial^2 L}{\partial \theta^2} < 0 \end{aligned}\right\} \text{ — (1)}$$

since $L > 0$ and $\log L$ is a non

decreasing function of L; L and log L attain their extreme values at the same values of $\hat{\theta}$. The equation one can be re written as

$$\frac{1}{L}\frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial \log L}{\partial \theta} = 0 \quad \text{——— (2)}$$

If θ is vector valued parameter then $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_k)$ is given by the solution of simultaneous equ

$$\frac{\partial}{\partial \theta_i} \log L = \frac{\partial}{\partial \theta_i} \log L(\theta_1, \theta_2 \ldots \theta_k)$$

$$i = 1, 2 \ldots k \quad \text{——— (3)}$$

Equations (2) and (3) are usually reffered to as likelihood equations for estimating the parameters

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Properties of MLE

i) All MLEs are asymptotically unbiased [approximately unbiased for large number]

ii) All MLEs are asymptotically normally distributed

iii) All MLEs are consistant

iv) All MLEs are most efficient

v) All MLEs are sufficient, if sufficient estimates exist

vi) All MLEs are achieved the Cramer Rao lower bound for consistant estimators

vii) Invariance property

If $\hat{\theta}$ is the MLE of θ, $h(\theta)$ is some function of θ. Then $h(\hat{\theta})$ is the MLE of $h(\theta)$.

Find M.L.E of λ based on a sample taken from Pop^n p.d.f $f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2 \dots$

$$L(x_1, x_2 \dots x_n, \lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \cdot \frac{e^{-\lambda}\lambda^{x_3}}{x_3!} \cdots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-n\lambda}\lambda^{n\bar{x}}}{x_1!\, x_2! \cdots x_n!}$$

$\because x_1 + x_2 + \dots + x_n = n\bar{x}$

$\Rightarrow \cancel{n}\left[\frac{x_1 + x_2 + \dots + x_n}{\cancel{n}}\right] = x_1 + x_2 + \dots + x_n$

$$\log L = \log\left[\frac{e^{-n\lambda}\lambda^{n\bar{x}}}{x_1!\, x_2! \cdots x_n!}\right]$$

$\because \log(a/b) = \log a - \log b$

$$= \log\left(e^{-n\lambda}\lambda^{n\bar{x}}\right) - \log(x_1! + x_2! + \dots + x_n!)$$

$$= \log e^{-n\lambda} + \log \lambda^{n\bar{x}} - \log(x_1! + x_2! + \dots + x_n!)$$

$$= -n\lambda \log e + n\bar{x}\log\lambda - \log(x_1! + x_2! + \dots + x_n!)$$

$$= -n\lambda + n\bar{x}\log\lambda - [\log x_1! + \log x_2! + \dots + \log x_n!]$$

$$\frac{\partial \log L}{\partial \lambda} = -n + \frac{n\bar{x}}{\lambda} = 0$$

$$= \frac{-n\lambda + n\bar{x}}{\lambda} = 0$$

$$= -n\lambda + n\bar{x} = 0$$

$$= n\bar{x} = n\lambda$$

$$\therefore \lambda = \bar{x}$$

Methods of Moments

Let $f(x : \theta_1, \theta_2 \dots \theta_k)$ be the density function of the parent popⁿ with k parameters $\theta_1, \theta_2 \dots \theta_k$ if μ_r' denote the r^{th} moment about origin. then

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x; \theta_1, \theta_2 \dots \theta_k)\, dx \quad \text{--- (1)}$$

$$r = 1, 2 \dots k$$

in general $\mu_1', \mu_2' \dots \mu_k'$ will be the function of parameter $\theta_1, \theta_2 \dots \theta_k$

Let X_i, $i = 1, 2 \dots n$ be a random Sample of size n from the given popⁿ. the method of moments consist in Solving the k eqns (1) for $\theta_1, \theta_2 \dots \theta_k$ in terms of $\mu_1', \mu_2' \dots \mu_k'$ and then replacing this moments μ_r' $r = 1, 2 \dots k$ by the Sample moments.

3) Obtain the moment estimators of parameters a, b in rectangular distribution.

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

$$\mu_1' = \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$\frac{1}{b-a} \int_a^b \frac{x^2}{2}\, dx$$

$$= \frac{1}{b-a}\left[\frac{b^2 - a^2}{2}\right] = \frac{1}{\cancel{b-a}}\left[\frac{(b+a)\cancel{(b-a)}}{2}\right]$$

$$\frac{b+a}{2}$$

$$\mu_2' = \int_a^b \frac{1}{b-a} \cdot x^2$$

$$= \frac{1}{b-a}\left[\frac{b^3 - a^3}{3}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{\cancel{(b-a)}\,(b^2+ba+a^2)}{3}\right]$$

$$= \frac{b^2+ba+a^2}{3}$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{b^2+ba+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ba+a^2}{3} - \frac{b^2+2ba+a^2}{4}$$

$$= \frac{4b^2+4ba+4a^2 - 3b^2 + 6ba + 3a^2}{12}$$

$$\frac{b^2-2ba+a^2}{12} = \frac{(a-b)^2}{12}$$

$$\mu_1' = \frac{b+a}{2}, \quad \mu_2' = \frac{b^2+ba+a^2}{3}, \quad \mu_2 = \frac{(a-b)^2}{12}$$

$\therefore 12\mu_2 = (a-b)^2$

$a-b = \sqrt{12\mu_2}$ —(1)

$2\mu_1' = b+a$ —(2)

—(1) + —(2)

$$-b+a = \sqrt{12\mu_2} +$$
$$b+a = 2\mu_1'$$

$$2a = \sqrt{12\mu_2} + 2\mu_1'$$

so $a = \dfrac{\sqrt{12\mu_2} + 2\mu_1'}{2}$

$$a = \frac{\cancel{2}(\sqrt{3\mu_2} + \mu_1')}{\cancel{2}} = \mu_1' + \sqrt{3\mu_2}$$

$\therefore$ From —(2)

$2\mu_1' = b+a \Rightarrow 2\mu_1' - a = b$

$= 2\mu_1' - [\mu_1' + \sqrt{3\mu_2}] = b$

$= 2\mu_1' - \mu_1' - \sqrt{3\mu_2}$

$b = \mu_1' - \sqrt{3\mu_2}$

The method of minimum variance

Let $f(x, \theta)$ be the pdf of pop^n with one parameter θ and x_i's are random sample. Let $L(x_1, x_2 \ldots x_n, \theta)$ be the likelyhood function. If $\frac{\partial \log L}{\partial \theta}$ can be put in the form $k(t-\theta)$ where k is either constant or function of θ and t is a function of the observation fully, t is the minimum variance unbiased estimate of θ (minimum variance unbiased estimates) (MVUE).

2) examine whether their exist a minimum Variance for the parameter λ of the Poisson population

$$f(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$L(x_1, x_2, \ldots, \lambda) = \frac{e^{-n\lambda} \lambda^{n\bar{x}}}{x_1! \, x_2! \ldots x_n!}$$

$$e^{-n\lambda} \cdot \lambda^{n\bar{x}} \left[\frac{1}{x_1! \, x_2! \ldots x_n!} \right]$$

$$\text{Log } L = \frac{-n\lambda + n\bar{x}\log\lambda}{\log x_1! + \log x_2! + \dots + \log x_n!}$$

$$\Rightarrow \frac{\partial \log L}{\partial \lambda} = -n + \frac{n\bar{x}}{\lambda}$$

$$= \frac{-n\lambda + n\bar{x}}{\lambda}$$

$$= \frac{n}{\lambda}(\bar{x} - \lambda)$$

$\therefore \bar{x}$ is the minimum variance estimator and it's variance is λ/n

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Cramer - Rao Inequality

If t is an unbiased estimator of $\gamma(\theta)$ a function of parameter θ then

$$\operatorname{Var}(t) \geq \frac{\left[\frac{d}{d\theta}\gamma(\theta)\right]^2}{E\left[\frac{\partial}{\partial\theta}\log L\right]^2} = \frac{[\gamma'(\theta)]^2}{I(\theta)}$$

Where $I(\theta)$ is called by R.A fisher on the amount of information on θ supplied by the samples and its resiprocal $\frac{1}{I(\theta)}$ as the information limit to the variance of estimator $t = t(x_1 + x_2 + \cdots x_n)$

Interval estimation of mean

1) confidence interval for mean when σ is known

Let $\bar{x}$ be the mean of the sample of size n taken from the normal population $N(\mu, \sigma)$. We know that $t = \left(\frac{\bar{x} - \mu}{\sigma}\right)\sqrt{n}$ follows standard normal distribution $N(0, 1)$.

From the table of standard normal distribution we can find $t_{\alpha/2}$ s. $P(|t| \leq t_{\alpha/2}) = \alpha$ where α is the confidence coefficient.

$\therefore (-t_{\alpha/2} \le t \le t_{\alpha/2}) = 1-\alpha$

$$\left(-t_{\alpha/2} \le \left(\frac{\bar{x}-\mu}{\sigma}\right)\sqrt{n} \le t_{\alpha/2}\right) = 1-\alpha$$

$$\left(-\sigma/\sqrt{n}\, t_{\alpha/2} \le (\bar{x}-\mu) \le \sigma/\sqrt{n}\, t_{\alpha/2}\right) = 1-\alpha$$

$$\left(-\sigma/\sqrt{n}\, t_{\alpha/2} - \bar{x} \le -\mu \le \sigma/\sqrt{n}\, t_{\alpha/2} - \bar{x}\right) = 1-\alpha$$

multiply by -1

$$P\left(\sigma/\sqrt{n}\, t_{\alpha/2} + \bar{x} \ge \mu \ge -\sigma/\sqrt{n}\, t_{\alpha/2} + \bar{x}\right) = 1-\alpha$$

$$P\left(\bar{x} - \sigma/\sqrt{n}\, t_{\alpha/2} \le \mu \le \bar{x} + \sigma/\sqrt{n}\, t_{\alpha/2}\right) = 1-\alpha$$

i.e, $[\bar{x} - \sigma/\sqrt{n}\, t_{\alpha/2},\ \bar{x} + \sigma/\sqrt{n}\, t_{\alpha/2}]$ It is the confidence interval

for 95 % confidence interval

$t_{\alpha/2} = 1.96$

confidence interval is

$[\bar{x} - \sigma/\sqrt{n}\, 1.96,\ \bar{x} + \sigma/\sqrt{n}\, 1.96]$

For 99% confidence interval

$t_{\alpha/2} = 2.58$

Then confidence interval is

$[\bar{x} - 2.58\, \sigma/\sqrt{n},\ \bar{x} + 2.58\, \sigma/\sqrt{n}]$

90% = 1.65

1. The mean of the sample of size 20 from normal population $N(\mu, 8)$ was found to be 81.2. Find 90% confidence interval for μ.

Ans $n = 20$, $\sigma = 8$, $\bar{x} = 81.2$

$\alpha = 90\% = 0.9$

0.45 0.45

$P(|t| \leq t_{\alpha/2}) = 1-\alpha$

$P(|t| \leq 1.65) = 0.9$

$$\left[\bar{x} - \sigma/\sqrt{n}\, t_{\alpha/2},\ \bar{x} + \sigma/\sqrt{n}\, t_{\alpha/2}\right]$$

$$\left[81.2 - 8/\sqrt{20}\ 1.65,\ 81.2 + 8/\sqrt{20}\ 1.65\right]$$

$$= [78.25,\ 84.15]$$

2. Find the least sample size required if the length of the 95% confidence interval for mean of the normal population with S.D 5 should be less than 6.

Ans $\alpha = 95\%$, $\sigma = 5$

$$P\left(|t| \le t_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(|t| \le 1.96\right) = 0.95$$

$$\left[\bar{x} - \frac{\sigma}{\sqrt{n}} t_{\frac{\alpha}{2}} , \bar{x} + \frac{\sigma}{\sqrt{n}} t_{\frac{\alpha}{2}}\right]$$

$$\left[\bar{x} - \frac{5}{\sqrt{n}} \times 1.96 , \bar{x} + \frac{5}{\sqrt{n}} \times 1.96\right]$$

Length of the interval =

$$\left(\bar{x} + \frac{5}{\sqrt{n}} \cdot 1.96\right) - \left(\bar{x} - \frac{5}{\sqrt{n}} 1.96\right)$$

$$= 2 \times \frac{5}{\sqrt{n}} \times 1.96$$

the width is to be less than 6

$$2 \times \frac{5}{\sqrt{n}} \times 1.96 \le 6$$

$$\frac{2 \times 5 \times 1.96}{6} \le \sqrt{n}$$

$$n = 10.6\cancel{7}$$

$$\approx 11$$

Confidence interval for μ when σ is unknown.

$$t = \frac{(x-\mu)\sqrt{n-1}}{s} \quad \text{follows}$$

~~$t = \bar{x} - \mu$~~ students t dis. with $(n-1)$ degree of freedom from the table of Students 't' the value of $t_{\frac{\alpha}{2}}$ $\therefore$

$$P\left(|t| \leq t_{\frac{\alpha}{2}}\right) = 1-\alpha$$

where α is the confidence interval

$$\therefore \; P(-t_{\alpha/2} \leq t \leq t_{\alpha/2}) = 1-\alpha$$

$$P\left(-t_{\alpha/2} \leq \frac{(\bar{x}-\mu)\sqrt{n-1}}{s} \leq t_{\alpha/2}\right) = 1-\alpha$$

$$P\left(-\frac{s}{\sqrt{n-1}} t_{\alpha/2} \leq \bar{x}-\mu \leq \frac{s}{\sqrt{n-1}} t_{\alpha/2}\right) = 1-\alpha$$

$$P\left(-\frac{s}{\sqrt{n-1}} t_{\alpha/2} - \bar{x} \leq -\mu \leq \frac{s}{\sqrt{n-1}} t_{\alpha/2} - \bar{x}\right) = 1-\alpha$$

multiplying with -1

$$P\left(\frac{s}{\sqrt{n-1}} t_{\alpha/2} + \bar{x} \geq \mu \geq -\frac{s}{\sqrt{n-1}} t_{\alpha/2} + \bar{x}\right) = 1-\alpha$$

$$P\left(\bar{x} - \frac{s}{\sqrt{n-1}} t_{\alpha/2} \leq \mu \leq \bar{x} + \frac{s}{\sqrt{n-1}} t_{\alpha/2}\right) = 1-\alpha$$

$$\left[\bar{x} - \frac{s}{\sqrt{n-1}}\,t_{\frac{\alpha}{2}},\ \bar{x} + \frac{s}{\sqrt{n-1}}\,t_{\frac{\alpha}{2}}\right]$$

is the required confidence interval.

① A random Sample of size 17 from a normal popⁿ is find to have $\bar{x} = 4.7$ & $s^2 = 5.76$. find 90% of confidence interval for mean of the popⁿ.

$$P\left(|t| \le t_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$1 - 0.9 = 0.1$

$$P\left(|t| \le 1.75\right) = 0.90$$

$$= \left[\bar{x} - \frac{s}{\sqrt{n-1}}\,t_{\frac{\alpha}{2}},\ \bar{x} + \frac{s}{\sqrt{n-1}}\,t_{\frac{\alpha}{2}}\right]$$

$$\left[4.7 - \frac{2.4 \times 1.75}{\sqrt{16}},\ 4.7 + \frac{2.4 \times 1.75}{\sqrt{16}}\right)$$

$$= (3.65, 5.75)$$

Confidence interval for variance

Let s^2 is the Variance of sample size n taken from a normal popn $N(\mu, \sigma)$ we know that $u = \frac{ns^2}{\sigma^2}$ follows χ^2 distribution with $(n-1)$ degree of freedom.

Let u_1, u_2 be the two numbers s.t. $P\left(u_1 \le \frac{ns^2}{\sigma^2} \le u_2\right) = 1-\alpha$

$$P\left(\frac{1}{u_1} \ge \frac{\sigma^2}{ns^2} \ge \frac{1}{u_2}\right) = 1-\alpha$$

$$P\left(\frac{ns^2}{u_1} \ge \sigma^2 \ge \frac{ns^2}{u_2}\right) = 1-\alpha$$

$$P\left(\frac{ns^2}{u_2} \le \sigma^2 \le \frac{ns^2}{u_1}\right) = 1-\alpha$$

$P\left(\frac{ns^2}{u_2}, \frac{ns^2}{u_1}\right)$ is the required confident Coifficient.

Confidence interval for Propotion of a binomial popⁿ

Consider a sample of size n from a binomial popⁿ with parameters n & p. Where n is assumed to be known. Let $\bar{p}$ be the sample propotion. We know that

$$t = \frac{\bar{p} - p}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}}$$ follows also to follow $N(0,1)$ from the table we can find out $t_{\alpha/2}$ so

$$P(|t| \le t_{\alpha/2}) = 1-\alpha$$

$$P(-t_{\alpha/2} \le t \le t_{\alpha/2}) = 1-\alpha$$

$$P\left(-t_{\alpha/2} \le \frac{\bar{p}-p}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}} \le t_{\alpha/2}\right) = 1-\alpha$$

$$P\left(-t_{\alpha/2} \cdot \sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \le \bar{p} - p \le t_{\alpha/2} \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}\right) = 1-\alpha$$

$$P\left(-t_{\alpha/2} \sqrt{\frac{\bar{p}(1-\bar{p})}{n}} - \bar{p} \le -p \le t_{\alpha/2} \sqrt{\frac{\bar{p}(1-\bar{p})}{n}} - \bar{p}\right) = 1-\alpha$$

multiply with -1

$$\Rightarrow P\left(t_{\alpha/2}\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}+\bar{P} \geq P \geq -t_{\alpha/2}\cdot\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}+\bar{P}\right)$$

$$=1-\alpha$$

$$P\left(-t_{\alpha/2}\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}+\bar{P} \leq P \leq t_{\alpha/2}\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}+P\right)$$

$$=1-\alpha$$

$\therefore$ the confidence interval is

$$\left[\bar{P}-t_{\alpha/2}\cdot\sqrt{\frac{\bar{P}(1-\bar{P})}{n}},\ P+t_{\alpha/2}\cdot\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}\right]$$

1) In a sample of 20 ~~person~~ Persons from a town it was seen ~~that~~ ~~for~~ four are suffering from T.B. Find a 95% confidence interval for the propotion of T.B patients in the town.

$n = 20$

$\bar{P} = \frac{4}{20} = 0.2$

$\alpha = 0.95$

$$t = \frac{\bar{P}-P}{\sqrt{\frac{\bar{P}(1-\bar{P})}{n}}} = \frac{0.2-P}{\sqrt{\frac{0.2\times0.8}{20}}}$$

$$P\left(|t| \le t_{\alpha/2}\right) = 0.95$$

$$t_{\alpha/2} = 1.96$$

$$\left[\bar{p} - t_{\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})}{n}},\ \bar{p} + t_{\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}\right]$$

$$\left[0.2 - 1.96 \times \sqrt{\frac{0.2 \times 0.8}{20}},\ 0.2 + 1.96\sqrt{\frac{0.2 \times 0.8}{20}}\right)$$

$$(0.025,\ 0.375)$$