Module 2

Linear Programming Problems

Mathematical formulation of a LPP

Programming problems in general deal with determining optimal allocation of limited resources to meet given objectives. The resources may be in the form of men, materials, machines etc. The objective is to optimize the total profit or total cost, using the limited resources, subject to the various restrictions.

Definition

Linear programming may be defined as a method of determining an optimum programme of independent activities in view of available resources. The objective in a linear programming problem is to maximize profit or minimize cost, subject to a number of limitations known as constraints. For this an objective function is constructed which represents total profit or total cost. The constraints are expressed in the form of equations or inequalities. Both the objective function and the constraints are linear relationship between the variables. The solution of a linear programming problem shows how much should be produced or sold or purchased which will optimize the objective function and satisfy the constraints.

Uses of LPP

LPP technique is used to achieve the best allocation of available resources. Available resources may be man-hours, money, machine-hours, raw materials etc.

Examples

1. A production manager wants to allocate the available machine-time, labour and raw materials to the activities of producing the different products. The manager would like to determine the number units of the products to be produced so as to maximize the profit.
2. A manufacturer wants to develop a production schedule and an inventory policy that will minimize the total production and inventory costs.
3. A marketing manager wants to allocate fixed advertising budget among alternative advertising media scuh as radio, television, newspapers and magazines. He wants to determine the media scheduled that maximizes the advertising effectiveness.

Applications of LP in industry and management (Uses of LP in management)

LP is extensively used to solve variety of industrial and management problems. Some examples are given below.

1. **Product mix**. This problem deals with determining the quantum of different products to be manufactured knowing the marginal contribution of each product and the amount of available resources used by each product. The objective is to maximize the total contribution subject to constraints formed by available resources.
2. **Product smoothing**. This is a problem in which the manufacturer has to determine the best plan for producing a product with a fluctuating demand.
3. **Media selection**. The problem is to select the advertising mix that will maximize the advertising effectiveness subject to the constraints like the total advertising budget, usage rate of various media etc.
4. **Travelling salesman problem.** The problem is to find the shortest route for a salesman starting from a given city, visiting each of the specified cities and then returning to the original point of departure.

5. **Capital investment.** The problem is to find the allocation which maximizes the total return when a fixed amount of capital is allocates to a number of activities.
6. **Transportation problem.** Using transportation technique of LP we can determine the distribution system that will minimize total shipping cost from several warehouses to various market locations.
7. **Assignment problems.** The problem of assigning the given number of personnel to different jobs can be solved with the help of assignment model. The objective is to minimize the total time taken or total cost.
8. **Blending problem.** These problems are likely to arise when a product can be made from a variety of available raw materials of various composition and prices. The problem is to find the number of units of each raw material to be blended to make one unit of product.
9. **Communication industry.** LP methods are used in solving problems involving facilities for transmission, switching, relaying etc.
10. **Rail road industry.** LP technique can be used to minimize the total crew and engine expenses subject to restrictions on hiring and paying the trainmen, the scheduling of ship capacities of rail road etc.
11. **Staffing problem.** LP method can be used to minimize the total number of employees in restaurant, hospital, police station etc. meeting the staff need at all hours.

Advantages of LP

1. It provides an insight and perspective into problem environment related with a multidimensional phenomenon. This generally results in clear picture of the true problem.
2. It makes a scientific and mathematical analysis of the problem situations. It also considers all possible aspects and remedies associated with the problem.
3. It gives an opportunity to the decision maker to formulate his strategies consistent with the constraints and the objectives.
4. It deals with changing situations. Once a plan is arrived through the LP, it can be revaluated for changing conditions.
5. By using LP the decision maker makes sure that he is considering the best solution.

Limitations of LP

1. It treats all relationships as linear. This assumption may not hold good in many real life situations.
2. Usually the decision variables would have physical significance only if they have integer values. But sometimes we get fractional valued answers in LP.
3. All parameters in the LP model are assumed to be known constants, but in reality they are frequently neither known nor constants.
4. Many problems are complex since the number of variables and constraints are quite a large number.

Basic assumptions

The LPP are solved on the basis of the following assumptions.

1. **Proportionality.** There must be proportionality between objectives and constraints.
2. **Additivity.** Sum of the resources used by different activities must be equal to the total quantity of resources.
3. **Divisibility.** The solution need not be in whole numbers.
4. **Certainty.** Coefficients in the objective function and constraints are completely known and do not change during the period under study.
5. **Finiteness.** Activities and constraints are of finite number.

6. **Optimality.** The solution to a problem is to be optimum (maximum or minimum).

Requirements for employing LP technique

1. **Definition of the objective.** A well defined objective is the first basic requirement. The objective may be (i) production at least cost (ii) obtaining maximum profit (iii) the best distribution of the productive factors within a fixed period of time.
2. **Quantitative measurement of the elements of problem.** It is essential that each element of a problem is capable of being expressed numerically.
3. **Alternatives.** There must be alternative courses of action to choose.
4. **Constraints.** There must be limitations of resources. These give rise to constraints.
5. **Non-negative restrictions.** Decision variables must assume only non-negative values.
6. **Linearity.** Both the objective function and constraints must be expressed in terms of linear equations or inequalities.
7. **Finiteness.** There must be finite number of activities and constraints. Otherwise optimal solution cannot be obtained.

Characteristics of LPP

1. **Objective function.** The objective function is of the form z = $c_1x_1 + c_2x_2 + \ . \ . \ . + \ c_nx_n$, where $c_1, c_2, \ldots, c_n$ are constants and $x_1, x_2, \ldots, x_n$ are decision variables.
2. **Linear constraints.** There are a set of restrictions imposed on the variables appearing in the objective function. These restrictions are due to limitations of resources. Constraints are of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq$ b (or $\leq$ b) where $a_1, a_2, \ldots, a_n$ and b are constants. There may be more than one constraint.
3. **Feasible solution.** A feasible solution to a LPP is a set of values of the decision variables $x_1, x_2, \ldots, x_n$ which satisfy the constraints.
4. **Optimal solution.** There can be one or more feasible solutions to a LPP. A feasible solution which optimizes the objective function is known as optimal solution.

Formation of Mathematical model to a LPP

Formation of suitable mathematical model to explain the given situation is the starting point of linear programming. The following steps are applied in this process.

Step 1. Identify the objective as maximization or minimization.

Step 2. Mention the objective quantitatively and express it as linear function of decision variables, known as objective function.

Step 3. Identify the constraints in the problem and express them as linear inequalities or equations of the decision variables. The non-negative conditions such as $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$ are also included where $x_1, x_2, \ldots, x_n$ are decision variables.

Standard form of a mathematical model of LPP

A general LPP can be stated as

Find $x_1, x_2, \ldots, x_n$ which optimize

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$ (or $\geq b_1$) **. . . (1)**

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$ (or $\geq b_2$) **. . . (2)**

. . . .

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$ (or $\geq bm_1$) **. . . (m)**

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$ **. . . . (n)** where all c_j's, a_{ij}'s, b_j's are constants and x_j's are decision variables.

The conditions (1) to (m) are called structural constraints and the condition (n) is called non-negative constraint. c_j's are called cost or profit coefficients. a_{ij}'s are called structural coeffiients and b_j's are called requirements.

Example. A company has three operational departments weaving, processing and packing with capacity to produce three different types of clothes namely suiting, shirting and woolens yielding a profit of Rs.2, Rs.4, Rs.3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woolen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively. Formulate the LPP to find the product mix to maximize the profit.

Answer:

Types of clothes	Departments (time taken in minutes)			Profit (Rs. per metre)
	Weaving	Processing	Packing	
Suiting (x_1 metre)	3	2	1	2
Shirting (x_2 metre)	4	1	3	4
Woolens (x_3 metre)	3	3	3	3
Available time (in minutes)	3600	2400	4800	

Let x_1 = quantity of suiting in metre

x_2 = quantity of shirting in metre

x_3 = quantity of woolen in metre

Objective function is profit function, Z = $2x_1 + 4x_2 + 3x_3$

Constraints are $3x_1 + 4x_2 + 3x_3 \leq 3600$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The mathematical formulation of LPP is

Maximize Z = $2x_1 + 4x_2 + 3x_3$

Subject to the constraints $3x_1 + 4x_2 + 3x_3 \leq 3600$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Eg. 2. An animal feed company produce 200lbs of a mixture containing the ingredients X_1 and X_2 X_1 costs Rs.3 per lbs and X_2 costs Rs. 8 per lbs. Not more than 80 lbs of X_1 can be used and minimum quantity to be used for X_2 is 60 lbs. Find how much of each ingredient should be used if the company wants to minimize the cost. Formulate LPP.

Answer:

Ingredients	Cost per lbs (Rs.)	Quantity allowed
X_1	3	≤ 80
X_2	8	≥ 60
Total		200

Let x_1 = quantity of X_1 used (in lbs)

x_2 = quantity of X_2 used

Objective function is cost function, Z = $3x_1 + 8x_2$

Constraints are $x_1 + x_2 = 200$

$x_1 \leq 80$, $x_2 \geq 60$ and $x_1 \geq 0$

The mathematical formulation of LPP is

Minimize Z = $3x_1 + 8x_2$

Subject to the constraints $x_1 + x_2 = 200$

$x_1 \leq 80$, $x_2 \geq 60$ and $x_1 \geq 0$

Eg. 3. A factory engaged in the manufacturing of pistons, rings and valves, for which the profit per unit is 10, 6, 4 respectively, wants to decide the most profitable mix. It takes 1 hour preparatory work, 10 hours of machining and 2 hours of packing for a piston. Corresponding time requirements for rings and valves are 1, 4, 2 and 1, 5, 6 hours respectively. The total number of hours available for preparatory work, machining and packing are 100, 600, 300 respectively. Determine the most profitable mix, assuming that what all produced can be sold. Formulate the LPP

Answer:

Items	Departments (time taken in hours)			Profit (Rs.)
	Preparation	Machining	Packing	
Piston (x_1)	1	10	2	10
Rtings (x_2)	1	4	2	6
Valves (x_3)	1	5	6	4
Available time (in hours)	100	600	300	

Let x_1 = No. of pistons

x_2 = No. of rings

x_3 = No. of valves

Objective function is profit function, Z = $10x_1 + 6x_2 + 4x_3$

Constraints are $x_1 + x_2 + x_3 \leq 100$

$10x_1 + 4x_2 + 5x_3 \leq 600$

$2x_1 + 2x_2 + 6x_3 \leq 300$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The mathematical formulation of LPP is

Maximize Z = $10x_1$ + $6x_2$ + 4 x_3

Subject to the constraints $x_1 + x_2 + x_3 \leq 100$

$10x_1 + 4x_2 + 5x_3 \leq 600$

$2x_1 + 2x_2 + 6x_3 \leq 300$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

(L.R Potti – Operations Research Page C.11)

Qn.8 A dealer wishes to purchase a number of fans and sewing machines. He has only rupees 5760 to invest and has space at most for 20 items. A fan costs him Rs.360 and a sewing machine Rs.240. His expectation is that he can sell a fan at a profit of Rs.22 and a sewing machine at a profit of Rs.18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

Answer:

Items	Cost per unit (Rs.)	Profit per unit
Fans (x_1)	360	22
Sewing machines (x_2)	240	18
Storage capacity – 20 items	Available amount - Rs.5760	

x_1 = No. of fans purchased

x_2 = No. of sewing machines

Objective function is profit function, Z = $22x_1 + 18x_2$

Constraints are $360x_1 + 240x_2 \leq 5760$

$x_1 + x_2 \leq 20$

$x_1 \geq 0, x_2 \geq 0$

The mathematical formulation of LPP is

Maximize Z = $22x_1 + 18x_2$

Subject to $360x_1 + 240x_2 \leq 5760$

$x_1 + x_2 \leq 20$

$x_1 \geq 0, x_2 \geq 0$

Page C.11 Qn. 9

A furniture manufacturer produces three products, ordinary chairs, executive chairs and office tables. All these products use raw material, machine processing time and manual labour as shown below.

Items	Raw material (in cubic feet)	Machine processing time (in hours)	Manual labour (in hours)
Ordinary chairs	10	3	2
Executive chairs	12	5	4
Office tables	20	6	5

The estimated profit from ordinary chair, executive chair, and office table are Rs.20, Rs.35 and Rs.50 respectively. If 300 cubic feet of raw material, 120 hours of machine time and 90 hours of manual labour are available, the problem is to determine as to how many units of each product should be produced to maximize the total profit. Formulate the above problem as LPP.

Answer:

Items	Raw material (in cubic feet)	Machine time (in hours)	Manual labour (in hours)	Profit per unit (in Rs.)
Ordinary chairs	10	3	2	20
Executive chairs	12	5	4	35
Office tables	20	6	5	50
Availability	300	120	90	

Let x_1 = No. of ordinary chairs

x_2 = No. of executive chairs

x_3 = No. of office tables

Objective function is profit function, Z = $20x_1 + 35x_2 + 50x_3$

Constraints are $10x_1 + 12x_2 + 20x_3 \leq 300$

$3x_1 + 5x_2 + 6x_3 \leq 120$

$2x_1 + 4x_2 + 5x_3 \leq 90$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The mathematical formulation of LPP is

Maximize, Z = $20x_1 + 35x_2 + 50x_3$

Subject to $10x_1 + 12x_2 + 20x_3 \leq 300$

$3x_1 + 5x_2 + 6x_3 \leq 120$

$2x_1 + 4x_2 + 5x_3 \leq 90$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Page C.12 Qn. 10.

A company has two types of pens say A and B. Pen A is a superior quality and pen B is a lower quality. Profits on pen A and pen B are Rs.5 and Rs.3 per pen respectively. Raw materials required for each pen A is twice as that of pen B. The supply of raw material is sufficient only for 1000 pens of B per day. Pen A requires a special clip and only 400 such clips are available per day. For pen B, only 700 clips are available per day. Formulate the problem into a LPP.

Answer:

Items	Cost per unit (Rs.)	Profit per unit	Availability of clips
Pen A (x_1)	360	5	400
Pen B (x_2)	240	3	700
Raw material available – 1000 pens of type B = 1000k units			

x_1 = No. of pens of type A

x_2 = No. of pens of type B

Objective function is profit function, Z = $5x_1 + 3x_2$

Raw material required for pen B = k units

Raw material required for pen A = 2k units

Raw material available = 1000k units

So 2k x_1 + k $x_2 \leq$ 1000k $\Longrightarrow$ $2x_1 + x_2 \leq 1000$

$$x_1 \leq 400$$

$$x_2 \leq 700$$

The mathematical formulation of LPP is

Maximize Z = $5x_1 + 3x_2$

Subject to the constraints

$$2x_1 + x_2 \leq 1000$$

$$x_1 \leq 400$$

$$x_2 \leq 700$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution to LPP

A linear programming problem can be solved by Graphic method and by Simplex method.

Graphic method

LPP involving two variables can be solved by graphical method.

Merits of graphic method

1. Graphical method is simple and easy to understand
2. A layman can easily apply this method

Demerits of graphical method

1. LPP involving more than two variables cannot be solved by graphical method
2. Each constraint is represented by a line. So if there are many constraints, many lines are to be drawn. This is so difficult to understand.

Steps for solving LPP by graphical method

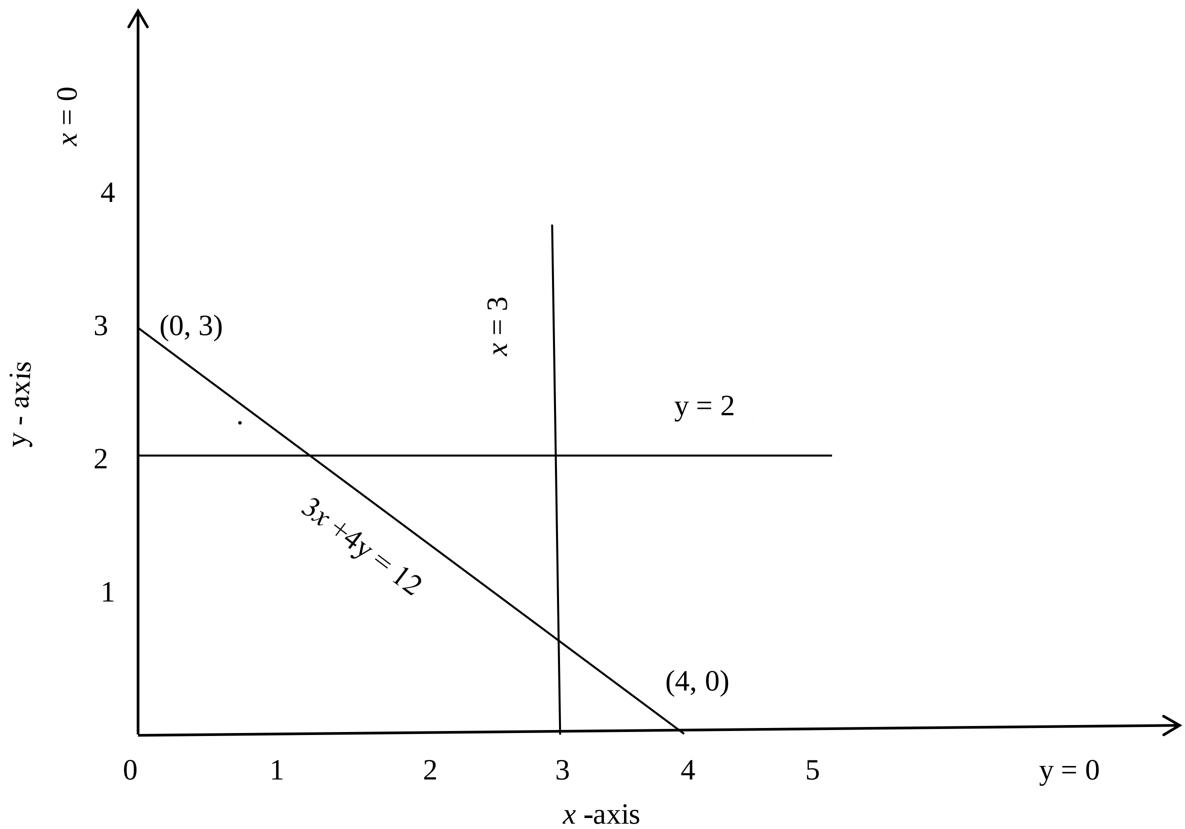
1. Formulate the problem into a LPP
2. Each inequality in the constraint may be written as equality
3. Draw straight lines corresponding to the equations obtained in step 2. So there will be as many straight lines as there are equations
4. Identify the feasible region. Feasible region is the area which satisfies all the constraints simultaneously.
5. The vertices of the feasible region are to be located and their co-ordinates are to be measured
6. Calculate the value of the objective function Z at each vertex.
7. The solution is the co-ordinates of the vertex which optimizes the objective function.

Graphs of equations

Draw lines corresponding to the equations (i) $x = 0$ (ii) $y = 0$ (iii) $x = 3$ (iv) $y = 2$ (v) $3x + 4y = 12$

1. $x = 0$ is the y-axis
2. $y = 0$ is the x-axis
3. $x = 3$ is the line parallel to the y-axis, 3 units away
4. $y = 2$ is the line parallel to the x-axis, 2 units away

5. To draw lines of the type $3x + 4y = 12$, find at least two points on the line. For this put $x = 0$ in the equation, we get $0 + 4y = 12 \implies 4y = 12 \implies y = \frac{12}{4} = 3$. So one point is (0, 3) Then put y = 0, then $3x + 0 = 12 \implies 3x = 12 \implies x = \frac{12}{3} = 4$. So another point is (4, 0). Plot the points and join them by a straight line.



Note: We can find the point of intersection of two lines by solving the equations of the lines

Eg. Find the point of intersection of the lines $x_1 + x_2 = 20$ and $3x_1 + 2x_2 = 48$

Answer. The lines are $x_1 + x_2 = 20$ **. . . (1)** and $3x_1 + 2x_2 = 48$ **. . . (2).** Solving (1) and (2) we get the point of intersection.

$x_1 + x_2 = 20$ . . . (1)

$3x_1 + 2x_2 = 48$ . . . (2).

$2 \times (1)$ gives $2x_1 + 2x_2 = 40$ . . . (3)

(2) – (3) gives $x_1 = 8$

Substituting in (1) we get $8 + x_2 = 20 \implies x_2 = 20 - 8 = 12$. The point of intersection is (8, 12)

Page C.30 Qn. 6

Solve g raphically the following problem;

Maximize $Z = 22x_1 + 18x_2$

Subject to $3x_1 + 2x_2 \leq 48$

$x_1 + x_2 \leq 20, x_1 \geq 0, x_2 \geq 0$

Answer: Writing the constraints as equations,

$3x_1 + 2x_2 = 48 \ldots$ (1)

$x_1 + x_2 = 20 \ \ldots$ (2)

$x_1 = 0, \quad x_2 = 0$

$x_1 = 0$ is the x_2 –axis and $x_2 = 0$ is the x_1 -axis

$3x_1 + 2x_2 = 48$

x_1	x_2
0	24
16	0

Put $x_1 = 0$, then $2x_2 = 48 \quad \Longrightarrow x_2 = \frac{48}{2} = 24$. One point is (0, 24)

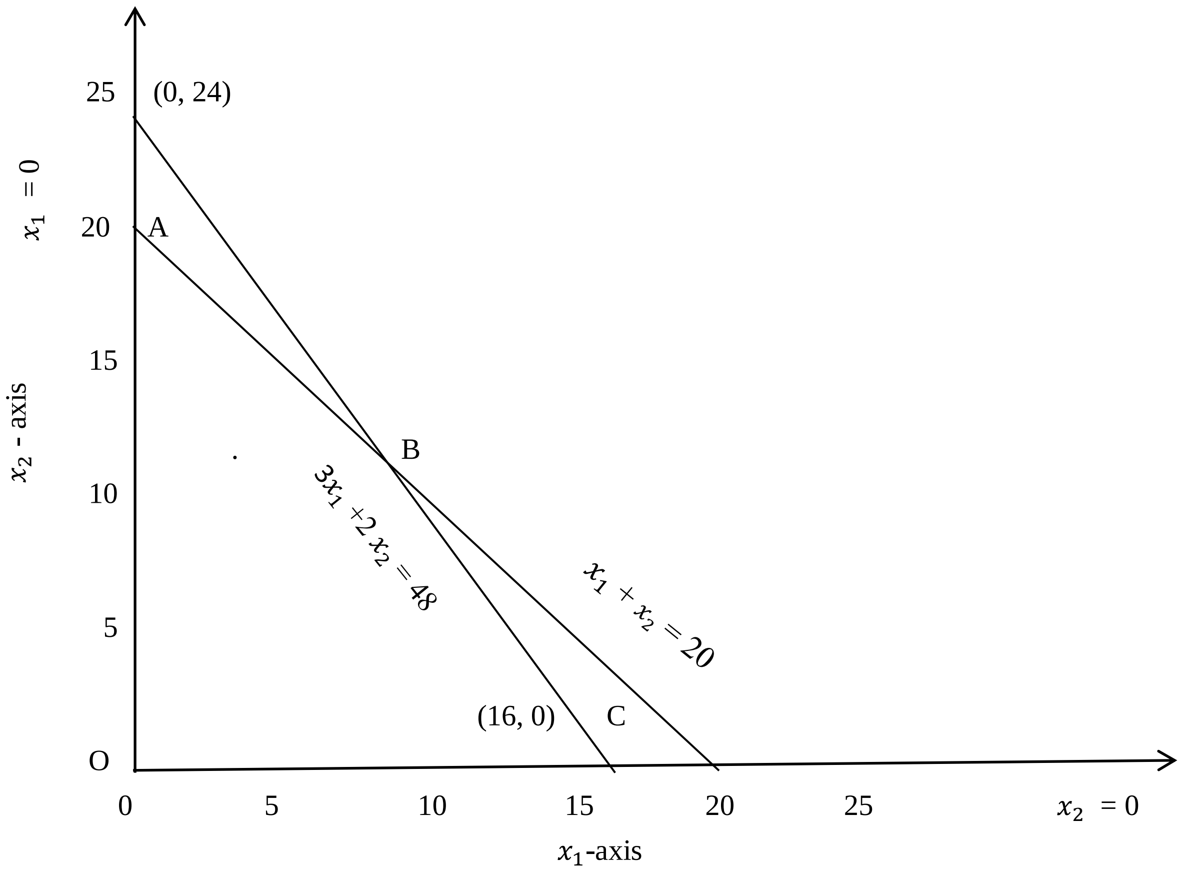
Put $x_2 = 0$, then $3x_1 = 48 \quad \Longrightarrow x_1 = \frac{48}{3} = 16$. Another point is (16, 0)

$x_1 + x_2 = 20$

x_1	x_2
0	20
20	0

Put $x_1 = 0$, then $x_2 = 20$ One point is (0, 20)

Put $x_2 = 0$, then $x_1 = 20$ Another point is (20, 0)



The feasible region is OABC. Coordinates of O, A, B, C are (0, 0), (0, 20), (8, 12), (16, 0).

To find the coordinates of B, solve equations $3x_1 + 2x_2 = 48$ and $x_1 + x_2 = 20$

$x_1 + x_2 = 20 \ \ldots$ (1)

$3x_1 + 2x_2 = 48 \ \ldots$ (2).

$2 \times$ (1) gives $2x_1 + 2x_2 = 40 \ \ldots$ (3)

(2) – (3) gives $x_1 = 8$

Substituting in (1) we get 8 + x_2 = 20 $\Longrightarrow$ x_2 = 20 – 8 = 12. The point of intersection is (8, 12)

Points	x_1	x_2	Z = 22 x_1 +18 x_2
O	0	0	0
A	0	20	0 + 18x20 = 360
B	8	12	22x8 + 18x12 =392
C	16	0	22x16 + 18x0 = 352

Z is highest for the point B. So the solution is x_1 = 8, x_2 = 12 and Z = 392

Page C. 30 Qn. 7

Solve g raphically the following problem;

Maximize Z = $3x_1 + 4x_2$

Subject to $x_1 + x_2 \leq 450$

$2x_1 + x_2 \leq 600$

$x_1 \geq 0$, $x_2 \geq 0$

Answer: Writing the constraints as equations,

$x_1 + x_2$ = 450 . . . (1)

2 $x_1 + x_2$ = 600 . . . (2)

x_1 = 0, x_2 = 0

x_1 = 0 is the x_2 –axis and x_2 = 0 is the x_1 -axis

$\boldsymbol{x_1 + x_2 = 450}$

x_1	x_2
0	450
450	0

Put x_1 = 0, then x_2 = 450. One point is (0, 450)

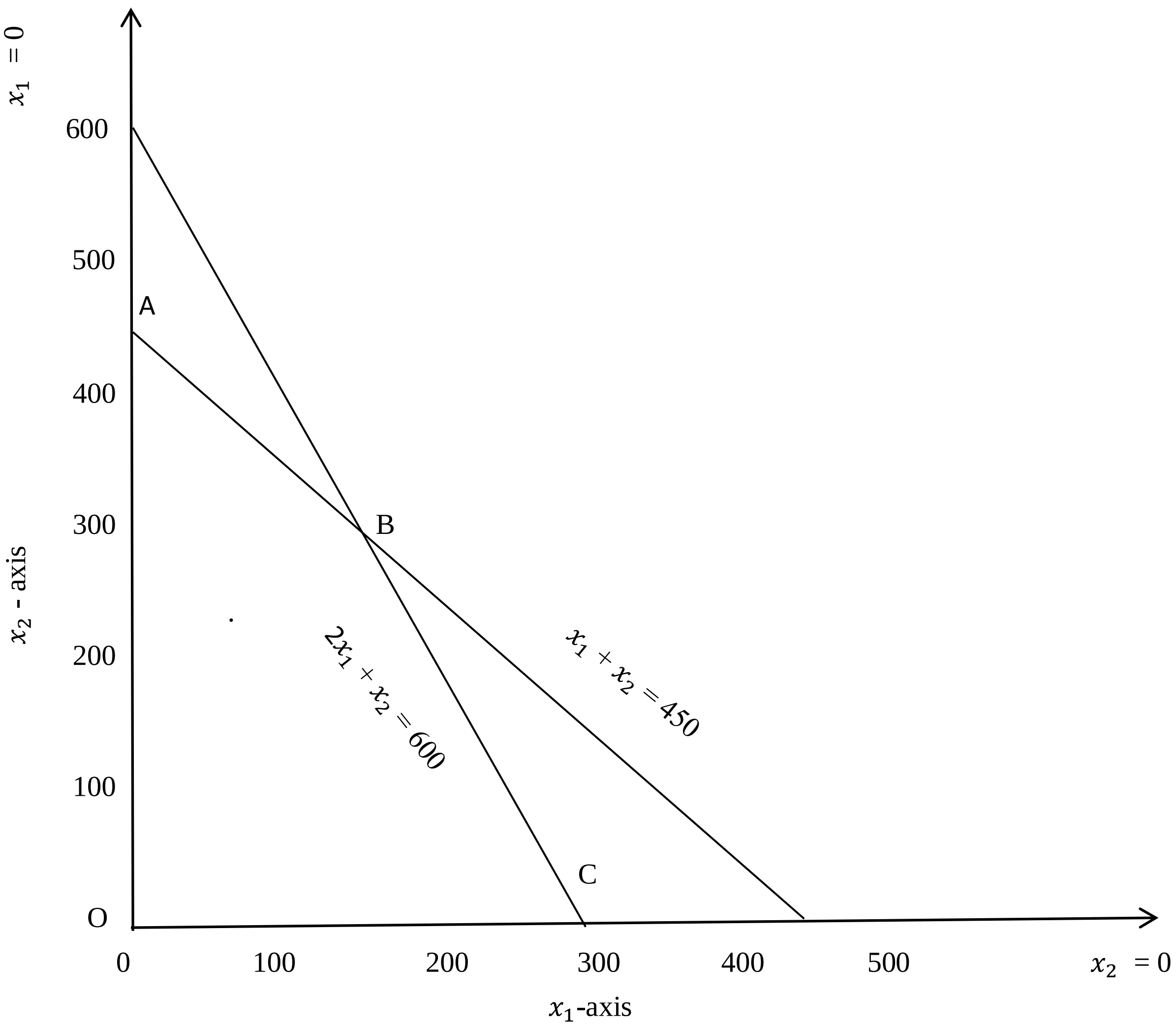
Put x_2 = 0, then x_1 = 450 Another point is (450, 0)

$\boldsymbol{2x_1 + x_2 = 600}$

x_1	x_2
0	600
300	0

Put x_1 = 0, then x_2 = 600 One point is (0, 600)

Put x_2 = 0, then $2x_1$ = 600, so x_1 = 300 Another point is (300, 0)



The feasible region is OABC. Coordinates of O, A, B, C are (0, 0), (0, 450), (150, 300), (300, 0).

To find the coordinates of B, solve equations $x_1 + x_2 = 450$ and $2x_1 + x_2 = 600$

$x_1 + x_2 = 450$ . . . (1)

$2x_1 + x_2 = 600$ . . . (2)

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(2) – (1) gives $x_1 = 150$

Substituting in (1) we get $150 + x_2 = 450 \implies x_2 = 450 – 150 = 300$. The point of intersection is B (150, 300)

Points	x_1	x_2	$Z = 3x_1 + 4x_2$
O	0	0	0
A	0	450	0 + 4x450 = 1800
B	150	300	3x150 + 4x300 = 1650
C	300	0	3x300+ 4x0 = 900

Z is highest for the point A. So the solution is x_1 = 0, x_2 = 450 and Z = 1800

Linear Programming Problem – Solution by Simplex method

Simplex method is a linear programming technique in which we start with a certain solution which is feasible. We improve this solution in a number of consecutive steps until we arrive at an optimal solution. For arriving at the solution of LPP by this method, the constraints and the objective function are presented in a table known as simplex table.

The simplex method is an iterative (step by step) procedure in which we proceed in systematic steps from an initial basic feasible solution to another basic feasible solution and finally arrive at an optimal solution. The simplex algorithm consists of the following steps.

1. Find a trial basic feasible solution of the LPP
2. Test whether it is an optimal solution or not
3. If not optimal, improve the first basic feasible solution by a set of rules
4. Repeat the steps 2 and 3 till an optimal solution is obtained

Feasible solution

A feasible solution to a LPP is the set of values of the variables which satisfy all the constraints and non-negative restrictions of the problem.

Optimal (Otimum) solution

A feasible solution to a LPP is said to be optimum if it optimizes the objective function Z of the problem.

Basic feasible solution

A feasible solution to a LPP in which the vectors associated to non-zero variables are linearly independent is called a basic feasible solution.

Slack variables

If a constraint has a sign ≤ then in order to make it an equality we have to add some variable to the L.H.S. The variables which are added to the L.H.S of the constraints to convert them into equalities are called the slack variables.

Eg. Consider the constraint $2x_1 + x_2 \leq 600$. To convert the constraint into equation we add s_1 to L.H.S then we have $2x_1 + x_2 + s_1 = 600$. Then s_1 is the slack variable.

Surplus variable

If a constraint has a sign ≥ then in order to make it an equality we have to subtract some variable from the L.H.S. The variables which are subtracted from the L.H.S of the constraints to convert them into equalities are called surplus variables.

Eg. Consider the constraint $2x_1 + x_2 \geq 600$. To convert the constraint into equation we subtract s_2 to L.H.S then we have $2x_1 + x_2 - s_2 = 600$. Then s_2 is the surplus variable.

To construct a simplex table

Simplex table consists of rows and columns. If there are m original variables and n introduced variables (slack, surplus or artificial variables) then there will be 3 + m+ n columns in the simplex table. First column (B) contains the basic variables. Second column (C) shows the coefficients of the basic variables in the objective function. Third column (x_B) gives the values of basic variables. Each of next m+n columns contains coefficients of variables in the constraints, when they are onverted into equations.

Basis (B). The variables whose values are not restricted to zero in the current basic solution, are listed in the first column of the simple table known as basis.

Basic variables. The variables which are listed in the basis are called basic variables and others are known as non- basic variables.

Vector. Any column or row of a simplex table is called a vector. In simplex table there is a vetor associated with every variable. The vectors associated with the basic variables are unit vectors.

Unit vector. A vector with one element 1 and all other elements zero is a unit vector.

Eg. $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix}$ are unit vectors.

Net evaluation (Δ_j)

Δ_j is the net profit or loss if one unit of x_j is introduced.the row containing Δ_j values is called net evaluation row or index row. $\Delta_j = Z_j - C_j$, where C_j is the coefficient of x_j variables in the objective function and Z_j is the sum of the product of coefficients of basic variables in the objective function and the vector x_j .

Minimum ratio. Minimum ratio is the lowest non negative ratio in the replacing ratio column. The replacing ratio column (θ) contains values obtained by dividing each element in (x_B) by the corresponding elements in the incoming vector.

Key column (incoming vector)

The column which has highest negative Δ_j in a maximization problem or the highest positive Δ_j in the minimization problem, is called incoming vector.

Key row (outgoing vector)

The row which relates to the minimum ratio, is the outgoing vector.

Key element. Key element is the element of the simplex table which lies both in key row and key column.

Iteration. Iteration means step by step process followed in simplex method to move from one basic feasible solution to another.

Simplex algorithm.

Step 1. Formulate the problem into a LPP

Step 2. Convert the constraints into equations by introducing the slack variables or surplus variables wherever necessary.

Step 3. Construct starting simplex table

Step 4. Conduct the test of optimality. This is done by computing the net evaluation $\Delta_j = Z_j - C_j$. The solution is not optimal if at least one Δ_j is negative for maximization case. If at least one Δ_j is positive the solution is not optimal for minimization case. Otherwise the solution is optimal. If the solution is not optimal we proceed to the next step

Step 5. Find incoming and outgoing vectors. The incoming vector corresponds to highest negative Δ_j for maximization cases and highest positive Δ_j for minimizationcases. Outgoing vector corresponds to minimum ratio. Minimum ratio is the lowest non negative ratio in the replacing ratio column.

Step 6. The element which is at the intersection of incoming vector and outgoing vector is called the key element. We mark this element in

There should be 1 at the position of key element. If it is not 1 then divide all the elements of the row , containing key element , by the key element. Then add appropriate multiples of the corresponding elements of this changed row to the elements of all other rows. Now obtain the next simplex table with the changes. Improve basic feasible solution can be read out from the simplex table. The solution is obtained by reading B column and (x_B) column together.

Step 7. Now test the above improved basic feasible solution for optimality as in step 4. If the solution is not optimal then repeat steps 5 and 6 until an optimal solution is finally obtained.

Note. A minimization problem can be converted into maximization problem by changing the sign of coefficients in the objective function. **Eg.** To minimize Z = $4x_1$ -2 x_2 + x_3 , we can maximize Z' = - $4x_1$ + $2x_2$ - x_3 subject to the same constraints.

Page C.52 Question 5.

Max. Z = $3x_1$ + $2x_2$

s.t $x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1 \geq 0, x_2 \geq 0$

Answer: Converting the constraint inequalities as equalities, by introducing the slack variables,

$x_1 + x_2 + s_1 = 4$, $x_1 - x_2 + s_2 = 2$

i.e $x_1 + x_2 + 1s_1 + 0\, s_2 = 4$ **. . . (1)** $x_1 - x_2 + 0s_1 + 1\, s_2 = 2$ **. . . (2)**

Then the objective function becomes, Z = $3x_1$ +2 x_2 +0 s_1 + 0 s_2

C_j values are 3, 2, 0, 0

Starting simplex table

B (1)	C_B (2)	x_B (3)	x_1 (4)	x_2 (5)	s_1 (6)	s_2 (7)	Repl. Ratio θ = $x_B \div x_1$
s_1 s_2	0 0	4 2	1 $\boxed{\mathbf{1}}$	1 -1	1 0	0 1	$4 \div 1 = 4$ $2 \div 1 = 2$ ⟵
		Z_j C_j	0 3	0 2	0 0	0 0	
		$\Delta_j = Z_j - C_j$	-3 ↑	-2	0	0	

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -3 which relates to x_1

Incoming vector is x_1. Minimum ratio (θ) = 2. Outgoing vector is s_2 . key elment = 1

II Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	Repl. Ratio $\theta = x_B \div x_2$
s_1	0	2	0	$\boxed{2}$	1	-1	$2 \div 2 = 1 \leftarrow$
x_1	3	2	1	-1	0	1	$2 \div -1 = -2$
		Z_j	3	-3	0	3	
		C_j	3	2	0	0	
		$\Delta_j = Z_j - C_j$	0	-5 ↑	0	3	

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -5 which relates to x_2

Incoming vector is x_2. Minimum ratio (θ) = 1. Outgoing vector is s_1. Key element = 2

III Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2
x_2	2	**1**	0	1	$\frac{1}{2}$	$\frac{-1}{2}$
x_1	3	**3**	1	0	$\frac{1}{2}$	$\frac{1}{2}$
		Z_j	3	2	$\frac{5}{2}$	$\frac{1}{2}$
		C_j	3	2	0	0
		$\Delta_j = Z_j - C_j$	0	0	$\frac{5}{2}$	$\frac{1}{2}$

No negative Δ_j. So the solution is optimal. The solution is $x_1 = 3$, $x_2 = 1$

$Z = 3x_1 + 2x_2 = 3 \times 3 + 2 \times 1 = 9 + 2 = 11$

Page C.52 Question 6.

Max. $Z = 5x_1 + 3x_2$

s.t $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1 \geq 0,\ x_2 \geq 0$

Answer: converting the constraint inequalities as equalities, by introducing the slack variables,

$3x_1 + 5x_2 + s_1 = 15 \qquad 5x_1 + 2x_2 + s_2 = 10$

i. e $3x_1 + 5x_2 + 1s_1 + 0s_2 = 15$ ... **(1)** $\quad 5x_1 + 2x_2 + 0s_1 + 1s_2 = 10$ ... **(2)**

Then the objective function becomes, $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

C_j values are 5, 3, 0, 0

Starting simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	Repl. Ratio $\theta = x_B \div x_1$
s_1	0	15	3	5	1	0	$15 \div 3 = 5$
s_2	0	10	$\boxed{5}$	2	0	1	$10 \div 5 = 2 \leftarrow$
		Z_j	0	0	0	0	
		C_j	5	3	0	0	
		$\Delta_j = Z_j$ - C_j	-5	-3	0	0	
			↑				

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -5 which relates to x_1

Incoming vector is x_1. Minimum ratio (θ) = 2. Outgoing vector is s_2. Key element = 5

II Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	Repl. Ratio $\theta = x_B \div x_2$
s_1	0	9	0	$\boxed{\frac{19}{5}}$	1	$\frac{-3}{5}$	$9 \div \frac{19}{5} = \frac{45}{19} \leftarrow$
x_1	5	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	$2 \div \frac{2}{5} = 5$
		Z_j	5	2	0	1	
		C_j	5	3	0	0	
		$\Delta_j = Z_j$ - C_j	0	-1	0	1	
				↑			

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -1 which relates to x_2

Incoming vector is x_2. Minimum ratio (θ) = $\frac{45}{19}$. Outgoing vector is s_1. Key elment = $\frac{19}{5}$

III Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2
x_2	3	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$\frac{-3}{19}$
x_1	5	$\frac{20}{19}$	1	0	$\frac{-2}{19}$	$\frac{5}{19}$
		Z_j	5	3	$\frac{5}{19}$	$\frac{16}{19}$
		C_j	5	3	0	0
		$\Delta_j = Z_j$ - C_j	0	0	$\frac{5}{19}$	$\frac{16}{19}$

No negative Δ_j. So the solution is optimal. The solution is $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

Z = $5x_1 + 3x_2 = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{100+135}{19} = \frac{235}{19}$ = 12.36

Eg. Min. Z = $x_1 - 3x_2 + 2x_3$

s.t $3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Answer. This is a minimization problem. Convert it into a maximization problem.

Maximize Z' = $-x_1 + 3x_2 - 2x_3$ (changing the sign of the coefficients)

Converting the constraint inequalities as equalities, by introducing the slack variables,

$3x_1 - x_2 + 3x_3 + s_1 = 7$, $-2x_1 + 4x_2 + s_2 = 12$, $-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

Then the objective function becomes, Z' = $-x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

C_j values are -1, 3, -2, 0, 0, 0

Starting simplex table

B	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Repl. Ratio $\theta = x_B \div x_2$
s_1	0	7	3	-1	3	1	0	0	$7 \div -1 = -7$
s_2	0	12	-2	$\boxed{4}$	0	0	1	0	$12 \div 4 = 3$ ⟵
s_3	0	10	-4	3	8	0	0	1	$10 \div \quad = \frac{10}{3}$
		Z_j	0	0	0	0	0	0	
		C_j	-1	3	-2	0	0	0	
		$\Delta_j = Z_j - C_j$	1	-3	2	0	0		

↑

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -3 which relates to x_2

Incoming vector is x_2. Minimum ratio (θ) = 3. Outgoing vector is s_2. Key elment = 4

II Simplex table

B	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Repl. Ratio $\theta = x_B \div x_1$
s_1	0	10	$\boxed{\frac{5}{2}}$	0	3	1	$\frac{1}{4}$	0	$10 \div \frac{5}{2} = 4$ ←
x_2	3	3	$\frac{-1}{2}$	1	0	0	$\frac{1}{4}$	0	$3 \div \frac{-1}{2} = -6$
s_3	0	1	$\frac{-5}{2}$	0	8	0	$\frac{-3}{4}$	1	$1 \div \frac{-5}{2} = \frac{-2}{5}$
		Z_j	$\frac{-3}{2}$	3	0	0	$\frac{3}{4}$	0	
		C_j	-1	3	-2	0	0	0	
		$\Delta_j = Z_j - C_j$	$\frac{-1}{2}$	0	2	0	$\frac{3}{4}$	0	
			↑						

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is $\frac{-1}{2}$- which relates to x_1

Incoming vector is x_1. Minimum ratio (θ) = 4. Outgoing vector is s_1. Key elment = $\frac{5}{2}$

III Simplex table

B	C_B	$\boldsymbol{x_B}$	x_1	x_2	x_3	s_1	s_2	s_3
$\boldsymbol{x_1}$	-1	**4**	1	0	$\frac{6}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0
$\boldsymbol{x_2}$	3	**5**	0	1	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0
$\boldsymbol{s_3}$	0	**11**	0	0	11	1	$\frac{-1}{2}$	1
		Z_j	-1	3	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{8}{10}$	0
		C_j	-1	3	-2	0	0	0
		$\Delta_j = Z_j - C_j$	0	0	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{8}{10}$	0

No negative Δ_j. So the solution is optimal. The solution is $x_1 = 4$, $x_2 = 5$, $x_3 = 0$.

Minimum of $Z = x_1 - 3x_2 + 2x_3 = 4 - (3 \times 5) + (2 \times 0) = -11$.

Simplex method – Artificial Variable Technique

Artificial variable

Artificial variables are introduced when the constraints are of the form $\geq$ or = .

Eg. $2x_1 + 3x_2 + x_3 \geq 10$ can be written as $2x_1 + 3x_2 + x_3 - s_1 + A_1 = 10$. Here s_1 is the surplus variable, A_1 is the artificial variable.

$3x_1 + x_2 + 2x_3 = 6$ can be written as $3x_1 + x_2 + 2x_3 + A_1 = 6$, A_1 is the artificial variable

Artificial variable technique

Constraints of some LPP may have ≥ or = signs. In such problems even after introducing surplus variable, the simplex table may not contain identity matrix or unit vectors. So to get identity matrix in the simplex table, we introduce artificial variables to the constraints as per requirement. By introducing artificial variables, we are able to get initial basic feasible solution. (the basic variables are those which have unit vectors in the simplex table). This technique of LPP in which artificial variables are used for solving is known as artificial variable method. Such problems are solved by Big - M Method.

Big -M Method

Big -M Method is a modified simplex method for solving a LPP when a high penalty cost (or profit) M has been assigned to the artificial variable in the objective function. When artificial variables are introduced, we include these artificial variables in the first column of the starting simplex table. We assign a very large value M to each of the artificial variables as coefficient in the objective function. The quantity M is called penalty. In maximization case –M and in minimization case +M are assigned to the artificial variables as their coefficients in the objective function. This method can be applied in minimization problems and maximization problems.

Eg. Min $Z = 5x_1 + 6x_2$

s.t $2x_1 + 5x_2 \geq 1500$

$3x_1 + x_2 \geq 1200$

$x_1 \geq 0, \; x_2 \geq 0$

Answer: This is a minimization problem. Convert it into maximization problem by changing the sign of the coefficients in Z. that is Maximize $Z' = -5x_1 - 6x_2$. Converting the constraint inequalities as equalities, by introducing surplus and artificial variables.

$2x_1 + 5x_2 - s_1 + A_1 = 1500$

$3x_1 + x_2 - s_1 + A_2 = 1200$

Then the objective function becomes, $Z' = -5x_1 - 6x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

C_j values are -5,-6, 0, 0, -M, -M

Starting simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	A_1	A_2	Repl. Ratio $\theta = x_B \div x_2$
A_1	-M	1500	2	$\boxed{5}$	-1	0	1	0	1500 ÷ 5 = 300 ←
A_2	-M	1200	3	1	0	-1	0	1	1200 ÷ 1 = 1200
		Z_j	-5M	-6M	M	M	-M	-M	
		C_j	-5	-6	0	0	-M	-M	
		$\Delta_j = Z_j - C_j$	-5M+5	-6M+6 ↑	M	M	0	0	

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -6M + 6 which relates to x_2

Incoming vector is x_2. Minimum ratio (θ) = 300. Outgoing vector is A_1. Key elment = 5

II Simplex table (dropping A_1)

B	C_B	x_B	x_1	x_2	s_1	s_2	A_2	Repl. Ratio $\theta = x_B \div x_1$
x_2	-6	300	$\frac{2}{5}$	1	$\frac{-1}{5}$	0	0	$300 \div \frac{2}{5} = 750$
A_2	-M	900	$\boxed{\frac{13}{5}}$	0	$\frac{1}{5}$	-1	1	$900 \div \frac{13}{5} = \frac{4500}{13}$ ←
		Z_j	$\frac{-12}{5} - \frac{13}{5}M$	-6	$\frac{6}{5} - \frac{1}{5}M$	M	-M	
		C_j	-5	-6	0	0	-M	
		$\Delta_j = Z_j - C_j$	$\frac{-13}{5}M + \frac{13}{5}$	0	$\frac{6}{5} - \frac{1}{5}M$	M	0	

↑

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is $\frac{-13}{5}M + \frac{13}{5}$, which relates to x_1

Incoming vector is x_1. Minimum ratio (θ) = $\frac{4500}{13}$. Outgoing vector is A_2. Key element = $\frac{13}{5}$

III Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2
x_2	-6	$\frac{2100}{13}$	0	1	$\frac{-3}{13}$	$\frac{2}{13}$
x_1	-5	$\frac{4500}{13}$	1	0	$\frac{1}{13}$	$\frac{-5}{13}$
		Z_j	-5	-6	1	1
		C_j	-5	-6	0	0
		$\Delta_j = Z_j - C_j$	0	0	1	1

No negative Δ_j. So the solution is optimal. The solution is $x_1 = \frac{4500}{13}$, $x_2 = \frac{2100}{13}$.

Minimum of $Z = 5x_1 + 6x_2 = (5 \times \frac{4500}{13}) + (6 \times \frac{2100}{13}) = 2700$

Eg. Min $Z = 3x_1 + 8x_2$

s.t $x_1 + x_2 = 200$

$x_1 \leq 80$

$x_2 \geq 60$

$x_1 \geq 0, \; x_2 \geq 0$

Answer: This is a minimization problem. Convert it into maximization problem by changing the sign of the coefficients in Z. That is, Maximize Z' = $-3x_1 - 8x_2$. Converting the constraint inequalities as equalities, by introducing, slack, surplus and artificial variables.

$x_1 + x_2 + A_1 = 200$

$x_1 + s_1 = 80$

$x_2 - s_2 + A_2 = 60$

Then the objective function becomes, $Z' = -3x_1 - 8x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

C_j values are -3, -8, 0, 0, -M, -M

Starting simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	A_1	A_2	Repl. Ratio $\theta = x_B \div x_2$
A_1	-M	200	1	1	0	0	1	0	$200 \div 1 = 200$
s_1	0	80	1	0	1	0	0	0	$80 \div 0 = \infty$
A_2	-M	60	0	$\boxed{1}$	0	-1	0	1	$60 \div 1 = 60 \leftarrow$
		Z_j	-M	-2M	0	M	-M	-M	
		C_j	-3	-8	0	0	-M	-M	
		$\Delta_j = Z_j - C_j$	-M+3	-2M+8	0	M	0	0	
				$\uparrow$					

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -2M + 8 which relates to x_2

Incoming vector is x_2. Minimum ratio (θ) = 60. Outgoing vector is A_2. Key element = 1

II Simplex table (dropping A_2)

B	C_B	x_B	x_1	x_2	s_1	s_2	A_1	Repl. Ratio $\theta = x_B \div x_1$
A_1	-M	140	1	0	0	1	1	$140 \div 1 = 140$
s_1	0	80	$\boxed{1}$	0	1	0	0	$80 \div 1 = 80 \leftarrow$
x_2	-8	60	0	1	0	-1	0	$60 \div 0 = \infty$
		Z_j	-M	-8	0	-M + 8	-M	
		C_j	-3	-8	0	0	-M	
		$\Delta_j = Z_j - C_j$	- M + 3	0	0	-M + 8	0	
			$\uparrow$					

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is - M + 3, which relates to x_1

Incoming vector is x_1. Minimum ratio (θ) = 80. Outgoing vector is s_1. Key element = 1

III Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	A_1	Repl. Ratio $\theta = x_B \div s_2$
A_1	-M	60	0	0	-1	$\boxed{1}$	1	$60 \div 1 = 60$ ←
x_1	-3	80	1	0	1	0	0	$80 \div 0 = \infty$
x_2	-8	60	0	1	0	-1	0	$60 \div -1 = -60$
		Z_j	-3	-8	M - 3	-M + 8	-M	
		C_j	-3	-8	0	0	-M	
$\Delta_j = Z_j - C_j$			0	0	M -3	-M + 8	0	
						↑		

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is $-M+8$, which relates to s_2

Incoming vector is s_2. Minimum ratio (θ) = 80. Outgoing vector is A_1. Key element = 1

IV Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2
s_2	0	**60**	0	0	-1	1
x_1	-3	**80**	1	0	1	0
x_2	-8	**120**	0	1	-1	0
		Z_j	-3	-8	5	0
		C_j	-3	-8	0	0
$\Delta_j = Z_j - C_j$			0	0	5	0

No negative Δ_j. So the solution is optimal. The solution is $x_1 = 80$, $x_2 = 120$

.Minimum of $Z = 3x_1 + 8x_2 = (3 \times 80) + (8 \times 120) = 1200$

Eg. Max. $Z = x_1 + 2x_2 + 3x_3 - x_4$

s.t $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Answer: Since all the constraints are equations only artificial variables are introduced. Introducing the artificial variables, the constraints become,

$x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_3 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

(the third equation does not require an artificial variable since the vector x_4 has the unit vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, as its coefficients)

Then the objective function becomes, $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

C_j values are 1, 2, 3, -1, -M, -M

Starting simplex table

B	C_B	x_B	x_1	x_2	x_3	x_4	A_1	A_2	Repl. Ratio $\theta = x_B \div x_3$
A_1	-M	15	1	2	3	0	1	0	$15 \div 3 = 5$
A_2	-M	20	2	1	$\boxed{5}$	0	0	1	$20 \div 5 = 4 \leftarrow$
x_4	-1	10	1	2	1	1	0	0	$10 \div 1 = 10$
		Z_j	-3M -1	-3M-2	-8M-1	-1	-M	-M	
		C_j	1	2	3	-1	-M	-M	
		$\Delta_j = Z_j - C_j$	-3M-2	-3M-4	-8M-4	0	0	0	
					↑				

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -8M + 4 which relates to x_3

Incoming vector is x_3. Minimum ratio (θ) = 4. Outgoing vector is A_2. Key elment = 5

II Simplex table (dropping A_2)

B	C_B	x_B	x_1	x_2	x_3	x_4	A_1	Repl. Ratio $\theta = x_B \div x_2$
A_1	-M	3	$\frac{-1}{5}$	$\boxed{\frac{7}{5}}$	0	0	1	$3 \div \frac{7}{5} = \frac{15}{7} \leftarrow$
x_3	3	4	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	$4 \div \frac{1}{5} = 20$
x_4	-1	6	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	$6 \div \frac{9}{5} = \frac{10}{3}$
		Z_j	$\frac{-M}{5} + \frac{3}{5}$	$\frac{-7M}{5} - \frac{6}{5}$	3	-1	-M	
		C_j	1	2	3	-1	-M	
		$\Delta_j = Z_j - C_j$	$\frac{-M}{5} - \frac{2}{5}$	$\frac{-7M}{5} - \frac{16}{5}$	0	0	0	
				↑				

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is $\frac{-7M}{5} - \frac{16}{5}$ which relates to x_2.

Incoming vector is x_2. Minimum ratio (θ) = $\frac{15}{7}$. Outgoing vector is A_1. Key element = $\frac{7}{5}$

III Simplex table

B	C_B	x_B	x_1	x_2	x_3	x_4	Repl. Ratio $\theta = x_B \div x_1$
x_2	2	$\frac{15}{7}$	$\frac{-1}{7}$	1	0	0	$\frac{15}{7} \div \frac{-1}{7}$ = -15
x_3	3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	$\frac{25}{7} \div \frac{3}{7} = \frac{25}{3}$
x_4	-1	$\frac{15}{7}$	$\boxed{\frac{6}{7}}$	0	0	1	$\frac{15}{7} \div \frac{6}{7} = \frac{5}{2}$ ←
		Z_j	$\frac{1}{7}$	2	3	-1	
		C_j	1	2	3	-1	
		$\Delta_j = Z_j - C_j$	$\frac{-6}{7}$	0	0	0	

↑

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is $\frac{-6}{7}$, which relates to x_1

Incoming vector is x_1. Minimum ratio (θ) = $\frac{5}{2}$. Outgoing vector is x_4. Key element = $\frac{6}{7}$

IV Simplex table

B	C_B	x_B	x_1	x_2	x_3	x_4
x_2	2	$\frac{5}{2}$	0	1	0	$\frac{1}{6}$
x_3	3	$\frac{5}{2}$	0	0	1	$\frac{-1}{2}$
x_1	1	$\frac{5}{2}$	1	0	0	$\frac{7}{6}$
		Z_j	1	2	3	0
		C_j	1	2	3	-1
		$\Delta_j = Z_j - C_j$	0	0	0	1

No negative Δ_j. So the solution is optimal. The solution is $x_1 = \frac{5}{2}$, $x_2 = \frac{5}{2}$, $x_3 = \frac{5}{2}$, $x_4 = 0$

Maximum of $Z = x_1 + 2x_2 + 3x_3 - x_4 = (1 \times \frac{5}{2}) + (2 \times \frac{5}{2}) + (3 \times \frac{5}{2}) - 0 = 15.$

Unbounded solution

When a LPP does not have finite valued solutions, the solution is said to be unbounded. In graphic method the feasible region will be unbounded. In simplex method, minimum ratio cannot be obtained.

Eg. Solve, Max. $Z = 2x_1 + x_2$

s.t $x_1 - x_2 \leq 2$

$2x_1 - x_2 \leq 3$

$x_1 , x_2 \geq 0$

Answer. Applying graphic method. Writing the constraints as equations,

$x_1 - x_2 = 2$

$2x_1 - x_2 = 3$

$x_1 = 0, x_2 = 0$

$x_1 = 0$ is the x_2 –axis and $x_2 = 0$ is the x_1 -axis

x_1	x_2
0	-2
2	0

$\boldsymbol{x_1 - x_2 = 2}$

Put $x_1 = 0$, then $- x_2 = 2 \implies x_2 = -2$. One point is (0, -2)

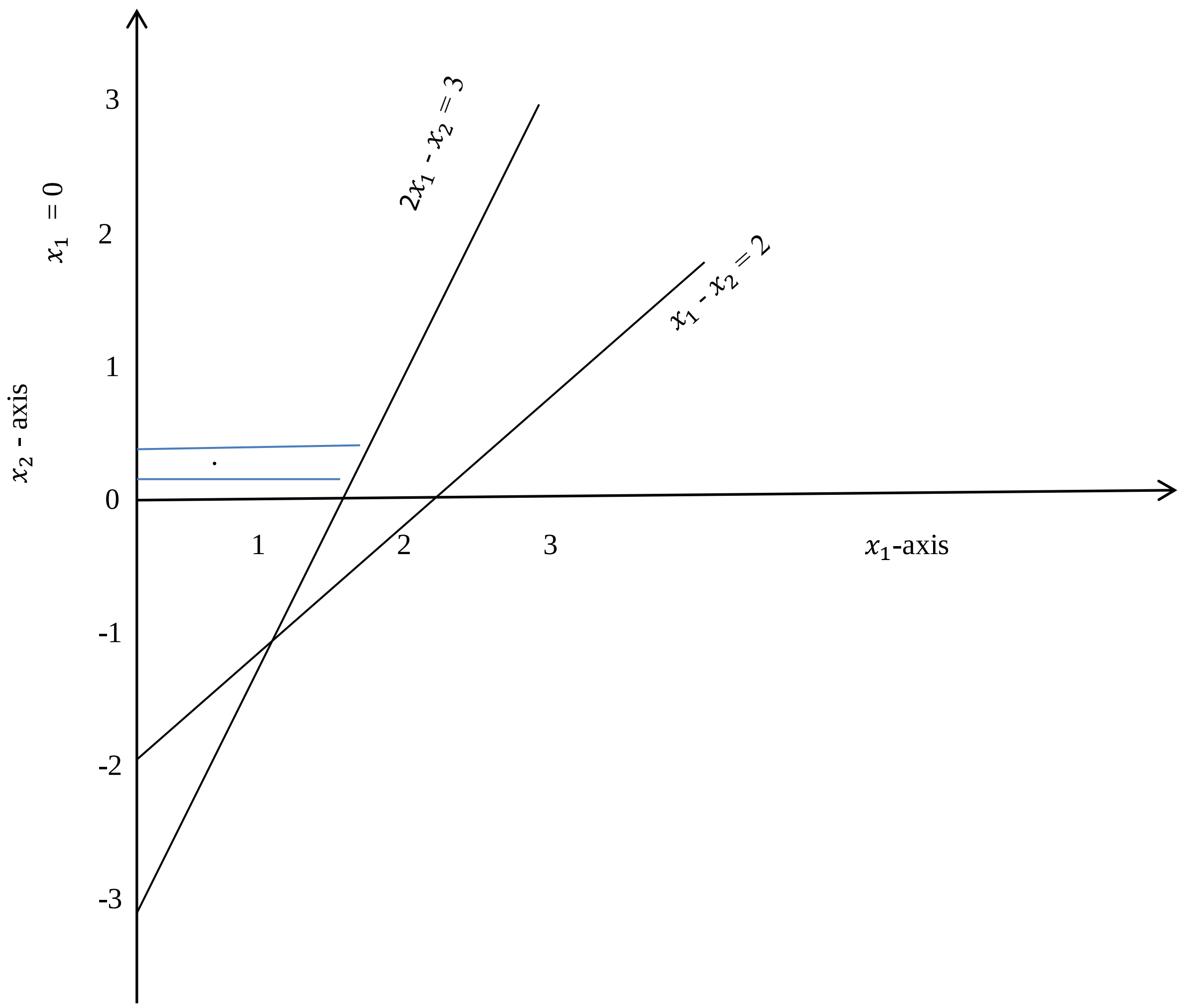
Put $x_2 = 0$, then $x_1 = 2$. Another point is (2, 0)

$\boldsymbol{2x_1 - x_2 = 3}$

Put $x_1 = 0$, then $x_2 = -3$ One point is (0,-3)

Put $x_2 = 0$, then $x_1 = \frac{3}{2}$ Another point is ($\frac{3}{2}$, 0)

x_1	x_2
0	-3
$\frac{3}{2}$	0



The feasible region is unbounded.

Applying simplex method

Max. Z = $2x_1 + x_2$

s.t $\quad x_1 - x_2 \leq 2$

$2x_1 - x_2 \leq 3$

$x_1 \geq 0,\ x_2 \geq 0$

Answer: Converting the constraint inequalities as equalities, by introducing the slack variables,

$x_1 - x_2 + s_1 = 2,\ 2x_1 - x_2 + s_2 = 3$

Then the objective function becomes, $Z = 2x_1 + x_2 + 0s_1 + 0s_2$

C_j values are 2, 1, 0, 0

Starting simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	Repl. Ratio $\theta = x_B \div x_1$
s_1	0	2	1	-1	1	0	$2 \div 1 = 2$
s_2	0	3	$\boxed{2}$	-1	0	1	$3 \div 2 = 1.5$ ⟵
		Z_j	0	0	0	0	
		C_j	2	1	0	0	
		$\Delta_j = Z_j - C_j$	-2	-1	0	0	
			↑				

Solution is not optimal as some Δ_j are negative. Highest negative Δ_j is -2 which relates to x_1

Incoming vector is x_1. Minimum ratio $(\theta) = 1.5$. Outgoing vector is s_2. Key element = 2

II Simplex table

B	C_B	x_B	x_1	x_2	s_1	s_2	Repl. Ratio $\theta = x_B \div x_2$
s_1	0	$\frac{1}{2}$	0	$\frac{-1}{2}$	1	$\frac{-1}{2}$	$\frac{1}{2} \div \frac{-1}{2} = -1$
x_1	2	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2} \div \frac{-1}{2} = -3$
		Z_j	2	-1	0	1	
		C_j	2	1	0	0	
		$\Delta_j = Z_j - C_j$	0	-2	0	1	
				↑			

Now both the values of θ are negative. There is no outgoing vector. The solution is unbounded.