Module 4

Game theory

Many practical problems require decision making in a competitive situation. A competitive situation is a situation where there are two or more opposing parties with conflicting interests and where the action of one depends upon the action taken by the opponent. Competitive situations occur frequently on Economic and Business activities. Management and Labour relations, Political battles and elections, wars are examples.

Game theory is a theory of conflict and it is a mathematical theory which deals with competitive situations.

Game

Game is defined as an activity between two or more persons according to a set of rules at the end of which each person receives some benefit or satisfaction or suffers or loss. In a game there are two or more opposite parties with conflicting interests. They know the objectives and the rules of the game. An experienced player usually predicts with accuracy how his opponent will react if a particular strategy is adopted. When one player wins, his opponent loses.

Characteristics (features) of a competitive game

A competitive situation is a game if it has the following properties

1. There are finite number of competitors called players
2. Each player has a list of finite number of possible courses of action.
3. A play is said to be played when each of the players chooses a single course of action from the list of courses of action available to him.
4. Every play is associated with an outcome known as pay off.
5. The possible gain or loss of each player depends not only on the choice made by him but also the choice made by his opponent.

Assumptions of a game

1. The players act rationally and intelligently
2. Each player has a finite set of possible courses of action
3. The players attempt to maximize gains or minimize losses
4. All relevant information are known to each player
5. The players make individual decisions
6. The players simultaneously select their respective courses of action
7. The pay offs is fixed and determined in advance

Strategy

The strategy of a player is the predetermined rule by which a player decides his course of action during the game. that is, a strategy for a given player is a set of rules or programmes that specify which of the available courses of action, he should select at each play.

Pure strategy

A pure strategy is a decision (in advance of all plays) always to choose a particular course of action. It is a predetermined course of action. The player knows it in advance.

Mixed strategy

A player is said to adopt mixed strategy when he does not adopt a single strategy all the time but would play different strategies each at a certain time. A mixed strategy is a decision to choose a course of action for each play in accordance with some particular probability distribution. In a mixed strategy we cannot definitely say which course of action the player will choose. We can only guess on the basis of probability.

Player

Each participant of the game is called a player.

Pay offs

The outcome of a game in the form of gains and losses to the competitive players for choosing different course of action is known as pay offs.

Pay off matrix

In a game, the gains and the losses, resulting from different moves and counter moves, when represented in the form of a matrix, is known as a pay off matrix. Each element of the pay off matrix is the gain of the maximizing player when a particular course of action is chosen by him as against the course of action chosen by the opponent.

Given below is a pay off matrix

$$\begin{array}{ccc} & & \mathrm{B} \\ & & \begin{array}{cc} \mathrm{B}_1 & \mathrm{B}_2 \end{array} \\ \mathrm{A} & \begin{array}{c} \mathrm{A}_1 \\ \mathrm{A}_2 \end{array} & \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \end{array}$$

Here A is the maximizing player and B is the minimizing player. Each element in the matrix is the gain for A when he chooses a particular course of action against which B chooses another course of action. When A chooses A_2 and B chooses B_1, the gain for A is shown in the second row, first column, which is 0. When A chooses A_2 and B chooses B_2, the gain for A is shown in the second row, second column, which is 1.

Value of a game

The value of the game is the maximum guaranteed gain to the maximizing player (A) if both the players use their best strategies. It is the expected pay off of a play when all the players of the game follow their optimal strategies.

Maximizing and minimizing players

If there are two players A and B, generally the pay offs given in a pay off matrix indicate gains to A for each possible outcomes of the game. That is, each outcome of a game results into a gain for A. all such gains are shown in the pay off matrix. Usually each row of the pay off matrix indicates gain to A for his particular strategy. A is called the maximizing player and B is called the minimizing player. The pay off values given in each column of pay off matrix indicates the losses for B for his particular course of action. So if the element in the position $(\mathrm{A}_2, \mathrm{B}_1)$ is 'a', then A's gain is 'a' and B's gain is – a or B's loss is 'a', when A chooses strategy A_2 and B chooses the strategy B_1.

Maximin and Minimax

Each row in a pay off matrix represents pay offs in respect of every strategy of the maximizing player A. Similarly each column represents pay offs in respect of every strategy of the minimizing player B. Maximin is the maximum of minimum payoffs in each row. Minimax is the minmum of maximum pay offs in each column.

Eg.

$$\begin{array}{c c} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 5 & 3 & 2 \\ 1 & -2 & 0 \\ 8 & -1 & 1 \end{bmatrix} \end{array}$$

Minimum in row A_1 = 2, Minimum in row A_2 = - 2, Minimum in row A_3= -1.

Maximum of these minima = maximum of {2, -2, -1} = 2. So maximin = 2.

Maximum of column B_1 = 8, Maximum of column B_2 = 3, Maximum of column B_3 = 2.

Minimum of these maxima = minimum of {8, 3, 2} = 2. So minimax = 2.

Maximin principle

The maximizing player A, lists worst possible pay offs of all potential strategies and chooses that strategy which corresponds to the best. This is maximin principle.

Minimax principle

The minimizing player B, lists his maximum losses from each strategy and chooses that strategy which corresponds to the least. This is minimax principle.

Saddle point

A saddle point of a pay of matrix is that position in the pay off matrix where the maximin coincides with the minimax. Pay off at the saddle point is the value of the game. In a game having a saddle point, optimum strategy of maximizing player is always to choose the row containing saddle point and for minimizing player to choose the column containing saddle point. If there are more than one saddle point there will be more than one solution.

A game for which maximin for A = minimax for B, is called a game with saddle point. The element at the saddle point position is the value of the game and it is denoted by v.

Eg.

$$\begin{array}{c c c} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{row minimum} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 5 & 3 & 2 \\ 1 & -2 & 0 \\ 8 & -1 & 1 \end{bmatrix} & \begin{array}{c} 2 \leftarrow \\ -2 \\ -1 \end{array} \\ \text{column maximum} & \begin{array}{ccc} 8 & 3 & 2 \\ & & \uparrow \end{array} & \end{array}$$

Maximin = maximum of row minimum = maximum of {2, -2, -1} = 2.

Minimax = minimum of column maximum = minimum of {8, 3, 2} = 2.

Maximin = minimax = 2, which refers to (A_1, B_3)

Saddle point is (A_1, B_3), value of the gme v = 2

Different kinds of games

Games are categorized on the basis of (1) number of players (2) number of moves (3) nature of the pay offs (4) nature of rules.

Zero sum game

In a game, if the algebraic sum of the outcomes (or gains) of all the players together is zero, the fame is called zero sum game, otherwise it is called non-zero sum game. That is, in a zero sum game, the amount won by all winners together is equal to the amount lost by all together.

n - person game

A game involving 'n' players is called n-person game and a game with two players is called two person game.

Two person zero sum game (rectangular games)

It is the simplest of game models. There will be two persons in the conflict and the sum of the pay offs of both together is zero. That is, the gain of one is at the expense of the other.

Two person zero sum game may be pure strategy game or mixed strategy game.

Basic assumptions in two person zero sum game

1. There are two players
2. They have opposite interests
3. The number strategies available to each player is finite
4. For each specific strategy, selected by a player, there results a pay off
5. The amount won by one player is exactly equal to the amount lost by the other.

Limitations of game theory

1. In fact, a player may have infinite number of strategies. But we assume that there are only a finite number of strategies.
2. It is assumed that each player has the knowledge of opponent's strategies. But it is not necessary in all cases.
3. The assumption that gain of one person is the loss of his opponent, need not be true in all situations.
4. Game theory usually ignores the presence of risk and uncertainty
5. It is assumed that pay off is always known in advance. But sometimes it is impossible to know the pay off accurately
6. It is assumed that the two persons involved in the game have equal intelligence. But it need not be true.

Fair game

A game is said to be fair if the value of the game is zero.

Solution of pure strategy games

- **By Maximin-minimax principle**

The maximizing player arrives at his optimal strategy on the basis of maximin criterion, while minimizing player's strategy is based on the minimax criterion. The game is solved by equating maximin value with minimax value. In this type of problems saddle point exists.

Eg. Fpr the following pay off matrix determine the optimal strategies for both the firms and the vaue of the game.

$$\begin{array}{cc} & \text{Firm B} \\ \text{Firm A} & \begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix} \end{array}$$

Answer.

$$\begin{array}{c|ccccc|l} & B_1 & B_2 & B_3 & B_4 & B_5 & \text{row min.} \\ A_1 & 3 & -1 & 4 & 6 & 7 & -1 \\ A_2 & -1 & 8 & 2 & 4 & 12 & -1 \\ A_3 & 16 & 8 & \mathbf{6} & 14 & 12 & 6 \leftarrow \\ A_4 & 1 & 11 & -4 & 2 & 1 & -4 \\ \text{column max.} & 16 & 8 & 6 & 14 & 12 & \\ & & & \uparrow & & & \end{array}$$

Maximin = maximum of row minimum = maximum of {-1, -1, 6, -4} = 6.

Minimax = minimum of column maximum = minimum of {16, 8, 6, 14, 12} = 6.

Maximin = minimax = 6, which refers to (A_3, B_3)

Saddle point is (A_3, B_3). The optimal strategy for A is A_3, The optimal strategy for B is B_3 value of the game v = 6

Eg. The following is a pay off matrix. Find the value of the game. Who will be the winner off the game? Why?

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \end{array}$$

Answer.

$$\begin{array}{c|cc|l} & B_1 & B_2 & \text{row min.} \\ A_1 & 1 & -2 & -2 \\ A_2 & 2 & \mathbf{-1} & -1 \leftarrow \\ \text{column max.} & 2 & -1 & \\ & & \uparrow & \end{array}$$

Maximin = maximum of row minimum = maximum of {-2, -1} = -1.

Minimax = minimum of column maximum = minimum of {2 , -1} = -1.

Maximin = minimax = -1, which refers to (A_2, B_2)

Saddle point is (A_2, B_2), value of the gme v = -1. So gain of X = -1. Since the value of the game is negative, Y wins the game.

Solution of mixed strategy problems

When there is no saddle point for a game, the minimax-maximin principle cannot be applied to solve the game. In those cases the concept of chance move is introduced. Here the choice among a number of strategies is not the decision of the player but by some chance mechanism. That is, predetermined probabilities are used for deciding the course of action. The strategies thus used are called mixed strategies.

Solution to mixed strategy problem can be solved by any one of the following methods.

1. Probability method (equal gain method)
2. Graphic method
3. Linear programming method.

Probability method

This method is used when there is no saddle point and he pay off matrix has two rows and two columns only. The players may adopt mixed strategies with certain probabilities. Here the problem is to determine the probabilities of different strategies of both players and the expected value of the game.

Consider the following pay of matrix.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

Let p be the probability for A using strategy A_1 and 1 – p be the probability for A using A_2 .

Then expected gain of A if B chooses B_1 is ap + c(1 – p).

Expected gain of A if B chooses B_2 is bp + d(1 – p). So ap + c(1 – p) = bp + d(1 – p).

Solving the equation we get, $p = \dfrac{d-c}{(a+d)-(b+c)}$

Similarly let q be the probability for B using strategy B_1 and 1 – q be the probability for B using B_2 .

Then expected gain of B if A chooses A_1 is aq + b(1 – q).

Expected gain of B if A chooses A_2 is cq + d(1 – q). So aq + b(1 – q) = cq + d(1 – q).

Solving the equation we get, $q = \dfrac{d-b}{(a+d)-(b+c)}$

Expected value of the game, $v = \dfrac{ad-bc}{(a+d)-(b+c)}$

So the solution of mixed strategy problem using probability method is,

strategies for A having probabilities (p, 1 – p), strategies for B having probabilities (q, 1 – q) value of the game = v, where $p = \frac{d-c}{(a+d)-(b+c)}$, $q = \frac{d-b}{(a+d)-(b+c)}$, $v = \frac{ad-bc}{(a+d)-(b+c)}$

Eg. Find p, q and the value of the game by probability method.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -2 & -1 \\ 2 & -3 \end{bmatrix} \end{array}$$

Answer. Value of the game, $v = \frac{ad-bc}{(a+d)-(b+c)}$ where $p = \frac{d-c}{(a+d)-(b+c)}$, $q = \frac{d-b}{(a+d)-(b+c)}$

Here a = -2, b = -1, c = 2, d = -3.

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{-3-2}{(-2-3)-(-1+2)} = \frac{-5}{-5-1} = \frac{5}{6}$$

$$q = \frac{d-b}{(a+d)-(b+c)} = \frac{-3--1}{-5-1} = \frac{-3+1}{-6} = \frac{-2}{-6} = \frac{2}{6} = \frac{1}{3}$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(-2\times-3)-(-1\times2)}{-6} = \frac{6+1}{-6} = \frac{8}{-6} = \frac{-4}{3}$$

Eg. Solve the following game

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Answer.

$$\begin{array}{cccc} & B_1 & B_2 & \text{row min.} \\ A_1 & 3 & 5 & 3 \leftarrow \\ A_2 & 4 & 1 & 1 \\ \text{column max.} & 4 & 5 & \\ & \uparrow & & \end{array}$$

Maximin = maximum of row minimum = maximum of {3, 1} = 3.

Minimax = minimum of column maximum = minimum of {4 , 5} = 4.

Here maximin ≠ minimax. So there is no saddle point. So we use probability method

Let probability for player A using strategy A_1 = p

Probability for player A using strategy A_2= 1 - p

Probability for player B using strategy B_1 = q

Probability for player B using strategy B_2 = 1 – q

Value of the game, $v = \frac{ad-bc}{(a+d)-(b+c)}$ where $p = \frac{d-c}{(a+d)-(b+c)}$, $q = \frac{d-b}{(a+d)-(b+c)}$

Here a = 3, b = 5, c = 4, d = 1.

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{1-4}{(3+1)-(4+5)} = \frac{-3}{4-9} = \frac{-3}{-5} = \frac{3}{5}$$

$q = \frac{d-b}{(a+d)-(b+c)} = \frac{1-5}{-5} = \frac{-4}{-5} = \frac{4}{5}$

$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(3\times 1)-(5\times 4)}{-5} = \frac{3-20}{-5} = \frac{-17}{-5} = \frac{17}{5}$

The solution is, A's strategy (A_1, A_2), with probabilities (p, 1 – p) = $(\frac{3}{5}, 1 - \frac{3}{5}) = (\frac{3}{5}, \frac{2}{5})$

B's strategy (B_1, B_2), with probabilities (q, 1 – q) = $(\frac{4}{5}, 1 - \frac{4}{5}) = (\frac{4}{5}, \frac{1}{5})$

Expected value of the game, $v = \frac{17}{5}$

Principle of dominance

The principle of dominance states that if the strategy of a player dominates over another strategy in all conditions, then the latter strategy can be ignored, because it will not affect the solution in any way. A strategy dominates over the other only if it is preferable in all conditions.

1. If all the elements in a row of a pay off matrix are less than or equal to the corresponding elements of another row , then the latter dominates and so the former is ignored.

Eg. Consider the matrix,

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ -2 & 1 & 4 & 0 \end{bmatrix}$$

Here every element of second row is less than or equal to the corresponding elements of the first row. So first row dominates and so the second row can be ignored.

2. If all the elements in a column of a pay off matrix are greater than or equal to the corresponding elements of another column, then the latter dominates and so the former is ignored.

Eg. Consider the matrix,

$$\begin{bmatrix} 2 & 2 \\ -3 & 1 \\ -1 & 1 \\ 4 & 5 \end{bmatrix}$$

Here all the elements of second column are greater than or equal to the corresponding elements of the first column. So the first column dominates and second column can be ignored.

3. If the linear combination of two or more rows (or columns) dominates a row (or column), then the latter is ignored.
   - If all the elements of a row are less than or equal to the average of the corresponding elements of two other rows, then the former is ignored.

Eg. Consider the matrix,

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 1 & 4 \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

Here every element of first row is less than or equal to the average of the corresponding elements of second and third rows. So first row can be ignored.

- If all the elements of a column are greater than or equal to the average of the corresponding elements of two other columns, then the former is ignored.

Eg. Consider the matrix,

$$\begin{bmatrix} 2 & 0 & 2 \\ 3 & -1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

Here every element of third column is greater than or equal to the average of the corresponding elements of first and second columns. So third column can be ignored.

Principle of dominance can be applied to pure strategy and mixed strategy problems.

Eg. Using principle of dominance, obtain the optimum strategies for both the players and determine the value of the game.

Player B

$$\text{Player A} \quad \begin{array}{c c} & \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 2 & 4 & 3 & 3 & 4 \\ 5 & 6 & 3 & 7 & 8 \\ 6 & 7 & 9 & 8 & 7 \\ 4 & 2 & 8 & 4 & 3 \end{bmatrix} \end{array}$$

Answer. All the elements of column 4 (B_4) are greater than or equal to the corresponding elements of column 1 (B_1). So B_1 dominates B_4. We eliminate column 4. The resulting matrix is

$$\begin{array}{c c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 2 & 4 & 3 & 4 \\ 5 & 6 & 3 & 8 \\ 6 & 7 & 9 & 7 \\ 4 & 2 & 8 & 3 \end{bmatrix} \end{array}$$

All the elements in row 4 are less than the corresponding elements of roe 3. So row 3 dominates row 4 and so we eliminate row 4. The resulting matrix is

$$\begin{array}{c c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 2 & 4 & 3 & 4 \\ 5 & 6 & 3 & 8 \\ 6 & 7 & 9 & 7 \end{bmatrix} \end{array}$$

All the elements of column B_5 are greater than the corresponding elements of column B_1. we delete column B_5. The resulting matrix is,

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 2 & 4 & 3 \\ 5 & 6 & 3 \\ 6 & 7 & 9 \end{bmatrix} \end{array}$$

All the elements of row 1 and 2 are less than the corresponding elements of row 3.
Delete row 1 and 2. The resulting matrix is,

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ A_3 & \begin{bmatrix} 6 & 7 & 9 \end{bmatrix} \end{array}$$

Elements in column 2 and 3 ar greater than the element in column 1.
Eliminate column 2 and 3. The resulting matrix is,

$$\begin{array}{cc} & B_1 \\ A_3 & \begin{bmatrix} 6 \end{bmatrix} \end{array}$$

The solution is, optimum strategy for A is A_3 and optimum strategy for B is B_1
Value of the game = 6

Eg. Solve the following game whose pay off matrix is given by,

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix} \end{array}$$

Answer.

Maximin = maximum of row minimum = maximum of {1, 2, 1} = 1. Minimax = minimum of column maximum = minimum of {6, 7, 7} = 6. So minimax $\neq$ maximin.
So minimax-maximin principle canot be used to solve this problem. So we apply principle of dominance.

B_1 dominates B_3. So eliminate B_3. The reduced matrix is

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 7 \\ 6 & 2 \\ 5 & 1 \end{bmatrix} \end{array}$$

A_2 dominates A_3. So eliminate A_3. The reduced matrix is

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \end{array}$$

Now neither a row nor a column is dominating. So we cannot proceed further. So we apply probability method.

Let probability for player A using strategy A_1 = p

Probability for player A using strategy A_2 = 1 - p

Probability for player B using strategy B_1 = q

Probability for player B using strategy B_2 = 1 – q

Value of the game , $v = \frac{ad-bc}{(a+d)-(b+c)}$ where $p = \frac{d-c}{(a+d)-(b+c)}$, $q = \frac{d-b}{(a+d)-(b+c)}$

Here a = 1, b = 7, c = 6, d = 2.

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{2-6}{(1+2)-(6+7)} = \frac{-4}{3-13} = \frac{-4}{-10} = \frac{2}{5}, \; 1-p = 1-\frac{2}{5} = \frac{3}{5}$$

$$q = \frac{d-b}{(a+d)-(b+c)} = \frac{2-7}{-10} = \frac{-5}{-10} = \frac{1}{2}, \; 1-q = 1-\frac{1}{2} = \frac{1}{2}$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(1\times 2)-(7\times 6)}{-10} = \frac{2-42}{-10} = \frac{-40}{-10} = 4$$

The solution is, A’s strategy (A_1, A_2, A_3), with probabilities (p, 1 – p, 0) = $(\frac{2}{5}, \frac{3}{5}, 0)$

B’s strategy (B_1, B_2, B_3), with probabilities (q, 1 – q, 0) = $(\frac{1}{2}, \frac{1}{2}, 0)$

Expected value of the game, v = 4.