Module III
Transportation and assignment problems

Transportation problems

Transportation problems are particular class of allocation problems. The objective is to transport various amounts of a single homogeneous commodity that are stored at several origins, to a number of destinations. The transportation is effected in such a way that the destination's demands are satisfied within the capacity of origins and the total transportation cost is a minimum.

Eg. A manufacturing concern has 'm' plants located in 'm' different cities in India. There are 'n' retail shops in 'n' different cities, which can absorb all the products stored. Then the transportation problem is to determine the transportation schedule that minimizes the total cost of transporting the manufactured products, from various plants to various retail shops.

Transportation technique can be applied to other problems also. Machine allocation, product mix etc. Transportation technique can be applied not only to the cost minimization problems, but also to time minimizing problems, distance minimizing problems, profit maximizing problems etc.

Transportation table

Denote the origins as $O_1, O_2, \ldots, O_m$ and destinations as $D_1, D_2, \ldots, D_n$. Let the quantity produced at the origins be respectively $a_1, a_2, \ldots, a_m$. Let the requirements in various destinations be respectively $b_1, b_2, \ldots, b_n$. The total quantity produced and the total quantity required must be equal. That is, $a_1 + a_2 + \ldots + a_m = b_1 + b_2 + \ldots + b_n$. Or $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Let c_{ij} be the cost of transportation of one unit from the i^{th} origin to the j^{th} destination.

	D_1	D_2	D_j	D_n	Available
O_1	c_{11}	c_{12}	c_{1j}	c_{1n}	a_1
O_2	c_{21}	c_{22}	c_{2j}	c_{2n}	a_2
O_i	c_{i1}	c_{i2}	c_{ij}	c_{in}	a_i
O_m	c_{m1}	c_{m2}	c_{mj}	c_{mn}	a_m
Required	b_1	b_2	b_j	b_n	

This matrix is known as transportation table or cost effectiveness matrix.
Here total availability = total requirement. That is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

The problem is to determine the quantity x_{ij} to be transported fom the i^{th} origin to the j^{th} destination such that the total cost, $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$ is minimum.

Transportation problem in the form of a LPP (Mathematical formulation)

Let x_{ij} be the number of units transported from the i^{th} origin to the j^{th} destination.

Let c_{ij} be the cost of transportation of one unit from the i^{th} origin to the j^{th} destination.

Let a_i be the units available in the i^{th} origin and b_j be he units required in the j^{th} destination.

Then the problem is

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = a_i$, for $i = 1, 2, \ldots, m$

$\sum_{i=1}^{m} x_{ij} = b_j$, for $j = 1, 2, \ldots, n$

$x_{ij} \geq 0$ for i, j

Basic assumptions in transportation problem

1. Total quantity available for distribution is equal to the total requirement in different destinations together. That is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$
2. The unit transportation cost from one origin to a destination is certain.
3. The unit cost is independent of the quantity transported.
4. Objective is to minimize the total transportation cost.

Uses of transportation technique

1. To minimize transportation cost from factories to warehouses or from warehouses to markets.
2. To determine lowest cost location for new factory
3. To determine minimum cost production schedule.

Feasible solution

A feasible solution to a transportation problem is a set of non-negative individual allocations which satisfy the row and column sum restrictions. So the sum of the allocations in the rows must be equal to the availability in that row. Similarly sum of the allocations in the columns must be equal to the demand in that column.

Basic feasible solution

A feasible solution to a m×n transpotation problem is said to be a basic feasible solution if the total number of allocations is exactly equal to m + n+ 1.

Optimal solution

A feasible solution is said to be optimal if it minimizes the total transportation cost.

Non degenerate basic feasible solution

A feasible solution of a m×n transpotation problem is said to be non degenerate basic feasible solution if

(1) The number of allocations is equal to m + n+ 1
(2) The allocations are in independent positions.

Loops in transportation table (non-independent position)

Allocations are said to be in independent positions, if it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. Therefore when the allocations are in independent positions, it is impossible to travel from any allocation back to itself through a series of horizontal or vertical jumps.

For example;

Table I

5	3	**2** 2
4	**3** 7	2
5	**7** 6	**3** 2

In table I, the allocations are in independent positions.

Table II

5	**8** 3	**1** 2
4	7	2
5	**7** 6	**3** 2

In table II, the allocations are not in independent positions.

Table III

8 5	3	**2** 2	3
5 4	7	2	**2** 4
5	6	**5** 2	**2** 1

In table III, the allocations are not in independent positions.

Steps for solving a transportation problem

1. Set up a transportation table with ‘m’ rows representing the origins and ‘n’ columns representing the destinations.
2. Develop an initial feasible solution to the problem.
3. Test whether the solution is optimal or not.
4. If the solution is not optimal modify the allocations.
5. Repeat steps 4 and 5 until an optimal solution is obtained.

Note. If there are ‘m’ rows and ‘n’ columns, there will be ‘mn’ cells. Each cell is known by its row number and column number. For example, cell (2, 3) means the cell in the second row and third column.

Initial (basic) feasible solution

Initial feasible solutions are those which satisfy the rim requirement. That is, the allocations made in every row taken together is equal to the availability shown in that row. Similarly for each column, the total allocation should be equal to the requirement in that column.

The initial feasible solution can be obtained either by inspection or by some rules. The commonly used methods for finding initial feasible solution are (1) North west corner rule (2) Lowest cost entry method (matrix minima method) (3) Vogel’s approximation method (unit cost penalty method)

North-West corner rule

This method is used to find initial feasible solution.

Procedure

Step 1

Allocate to cell (1, 1) maximum possible amount, which is minimum of row total and column total. So either a row total or column total gets exhausted. Cross off that row or column as the case may be.

Step 2

Consider the reduced matrix. In that matrix, allocate to the cell (1, 1) maximum possible amount, which is minimum of present row total and column total. So either a row total or column total gets exhausted. So cross off that row or column as the case may be.

Step 3

Repeat the above steps until all the available quantities are exhausted.

Eg. Find the initial feasible solution to the transportation problem given below, by north west corner rule.

Destinations

Origins	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	

Answer. Allocate to cell (1, 1), minimum of 5 and 7 (row total and column total), which is 5. Then O_1 row total is exhausted, since the supply of O_1 is completely met. So cross off row O_1

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	D_1	D_2	D_3	Supply
O_1	$\boxed{5}$ 2	7	4	5 X
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7 (2)	9	18	

Consider the reduced matrix after deleting O_1 row.
Allocate to cell (1, 1), minimum of 8 and 2,
which is 2. Then column D_1 is exhausted,
and it is crossed off.

	D_1	D_2	D_3	
O_2	$\boxed{2}$ 3	3	1	8 (6)
O_3	5	4	7	7
O_4	1	6	2	14
	2 X	9	18	

Consider the reduced matrix after deleting D_1 column.
Allocate to cell (1, 1), minimum of 6 and 9,
which is 6. Then row Q_2 is exhausted,
and it is crossed off.

	D_2	D_3	
O_2	$\boxed{6}$ 3	1	6 X
O_3	4	7	7
O_4	6	2	14
	9 (3)	18	

Consider the reduced matrix after deleting Q_2 row.
Allocate to cell (1, 1), minimum of 7 and 3,
which is 3. Then column D_2 is exhausted,
and it is crossed off.

	D_2	D_3	
O_3	$\boxed{3}$ 4	7	7 (4)
O_4	6	2	14
	3 X	18	

Consider the reduced matrix after deleting D_2 column.
Allocate 4 to cell (1, 1) and 14 to cell (2, 1).

	D_3	
O_3	$\boxed{4}$ 7	4
O_4	$\boxed{14}$ 2	14
	18	

Thus the various allocations made to the cells are shown below and the solution is

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	D_1	D_2	D_3	Supply
O_1	$\boxed{5}$ 2	7	4	5
O_2	$\boxed{2}$ 3	$\boxed{6}$ 3	1	8
O_3	5	$\boxed{3}$ 4	$\boxed{4}$ 7	7
O_4	1	6	$\boxed{14}$ 2	14
Demand	2	9	18	

Total transportation cost = $(5 \times 2) + (2 \times 3) + (6 \times 3) + (3 \times 4) + (4 \times 7) + (14 \times 2)$
$= 10 + 6 + 18 + 12 + 28 + 28 = 102.$

Lowest cost entry method

Choose the cell, having the lowest cost in the matrix. Allocate there, the minimum of row total and column total. Thus either a row total or column total is exhausted. Cross of the corresponding row or column. From the reduced matrix, locate the cell having the lowest cost. Allocate to that cell, the minimum of row total and column total. Thus either a row total or column total is exhausted. Cross of the corresponding row or column. This leading to a reduced matrix. Continue this process until all the available quantities are exhausted.

Eg. Find the initial feasible solution to the following transportation problem by lowest cost entry method.

	W_1	W_2	W_3	Supply
F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
Demand	7	9	18	

Answer. The lowest cost is 1 in cells (2, 3) and (4, 1). Select one of these, say (2, 3). Allocate to cell (2, 3), minimum of 8 and 18 (row total and column total), which is 8. Then F_2 row total is exhausted. So cross off the row F_2.

	W_1	W_2	W_3	Supply
F_1	2	7	4	5
F_2	3	3	$\boxed{8}$ 1	8 X
F_3	5	4	7	7
F_4	1	6	2	14
Demand	7	9	18 (10)	

The lowest cost in the reduced matrix is 1 in the cell (3, 1). Allocate to this cell, minimum of 7 and 14, which is 7. Then W_1 column total is exhausted. So cross off the column W_1.

	W_1	W_2	W_3	
F_1	2	7	4	5
F_3	5	4	7	7
F_4	$\boxed{7}$ 1	6	2	14 (7)
	7 X	9	10	

The lowest cost in the reduced matrix is 2 in the cell (3, 2). Allocate to this cell, minimum of 7 and 10, which is 7. Then F_4 row total is exhausted. So cross off the row F_4.

	W_2	W_3	
F_1	7	4	5
F_3	4	7	7
F_4	6	$\boxed{7}$ 2	7 X
	9	10 (3)	

Allocate to the cell (2, 1), minimum of 7 and 9, which is 7. Then F_3 row total is exhausted. So cross off the row F_3.

	W_2	W_3	
F_1	7	4	5
F_3	$\boxed{7}$ 4	7	7 X
	9 (2)	3	

Finally allocate 2 to the cell (1, 1) and 3 to the cell (1, 2).

	W_2	W_3	
F_1	$\boxed{2}$ 7	$\boxed{3}$ 4	5
	2	3	

The solution is given in the following matrix.

Total transportation cost =
= (2 × 7) + (3 × 4) + (8 × 1) + (7 × 4) + (7 × 1) + (7 × 2) = 14 + 12 + 8 + 28 + 7 + 14 = 83.

	W_1	W_2	W_3	Supply
F_1	2	$\boxed{2}$ 7	$\boxed{3}$ 4	5
F_2	3	3	$\boxed{8}$ 1	8
F_3	5	$\boxed{7}$ 4	7	7
F_4	$\boxed{7}$ 1	6	$\boxed{7}$ 2	14
Demand	7	9	18	

Vogel's approximation method

Step 1

Under this method we write the difference between smallest and second smallest costs in each column below the corresponding column, within brackets. Similarly write similar differences in each row to the right of corresponding row. These differences are known as penalty.

Step 2

Select the row or column having the largest penalty and allocate the maximum possible amount to the cell with lowest cost in that row or column as the case may be. Thus either the row total or column total is completely exhausted. Cross off that row or column. Construct the reduced matrix with the remaining rows and columns.

Step 3

For the reduced matrix obtained, apply steps 1 and 2 until all rows and columns totals are exhausted.

Note. Vogel's approximation method is better than the other two methods.

Eg. Find the initial feasible solution for the transportation problem by Vogel's method.

	W_1	W_2	W_3	Supply
F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
Demand	7	9	18	

Answer. Write penalties in brackets for all rows and columns.
Penalties (difference between smallest and second smallest costs)
Row 1; 4 – 2 = 2, row 2; 3 – 1 = 2 , row 3; 5 – 4 = 1, row 4; 2 – 1 = 1
Column 1; 2 – 1 =1, column 2; 4 – 3 = 1, column 3; 2 – 1 = 1 .

Maximum penalty 2 is associated with row F_1 and row F_2. Select one of these, say row F_1. The lowest cost in the row F_1 is in the cell (1, 1). Allocate to cell (1, 1), minimum of 5 and 7 (row total and column total), which is 5. Then F_1 row total is exhausted. So cross off the row F_1.

	W_1	W_2	W_3	Supply
F_1	**5** 2	7	4	5 (2) X
F_2	3	3	1	8 (2)
F_3	5	4	7	7 (1)
F_4	1	6	2	14 (1)
Demand	7 (1)	9 (1)	18 (1)	

Maximum penalty 2 is associated with column W_1 and row F_2. Select one of these, say row F_2. The lowest cost in the row F_2 is in the cell (1, 3). Allocate to cell (1, 3), minimum of 8 and 18, which is 8. Then F_2 row total is exhausted. So cross off the row F_2.

	W_1	W_2	W_3	
F_2	3	3	**8** 1	8 (2) X
F_3	5	4	7	7 (1)
F_4	1	6	2	14 (1)
	2 (2)	9 (1)	18 (1)	

Maximum penalty 5 is column W_3. The lowest cost in the column W_3 is in the cell (1, 3). Allocate to cell (2, 3), minimum of 14 and 10, which is 10. So cross off column W_3.

	W_1	W_2	W_3	
F_3	5	4	7	7 (1)
F_4	1	6	**10** 2	14 (1)
	2 (4)	9 (2)	10 (5) X	

Maximum penalty 5 is row F_4 The lowest is in the cell (2, 1). Allocate 2 to cell (2, 1) and cross off column W_1.

	W_1	W_2	
F_3	5	4	7 (1)
F_4	**2** 1	6	4 (5)
	2 (4) X	9 (2)	

Allocate 2 to cell (2, 1) and 7 to cell (1, 1)

	W_2	
F_3	**7** / 4	7
F_4	**2** / 6	2
	9	

The solution is given below.

	W_1	W_2	W_3	Supply
F_1	**5** / 2	7	4	5
F_2	3	3	**8** / 1	8
F_3	5	**7** / 4	7	7
F_4	**2** / 1	**2** / 6	**10** / 2	14
Demand	7	9	18	

Total transportation cost = $(5 \times 2) + (8 \times 1) + (7 \times 4) + (2 \times 1) + (2 \times 6) + (10 \times 2)$
$= 10 + 8 + 28 + 2 + 12 + 20 = 80$.

Optimal solution for transportation problem

By applying Vogel's method or lowest cost entry method or north west corner rule, an initial feasible solution is obtained. The next step is to examine whether the solution is optimal or not. For this we conduct test of optimality.

For this we use the modified distribution method (MODI method)

MODI method

Step 1

When the initial feasible solution is obtained some cells are occupied (allocation made) and others unoccupied. Number of occupied cells is m + n – 1 (m – number of rows, n – number columns in the matrix). Let c_{ij} be the cost of the cell (i, j).

Then we form m + n -1 equations of the form $U_i + V_j = C_{ij}$ corresponding to each occupied cell. Determine m + n values U_i and V_j, satisfying the above equation. For example if one of the occupied cells is (2, 3), then the equation is $U_2 + V_3 = C_{23}$, where = C_{23} is the cost in the cell (2, 3). For solving the equations, we take one of U_i or V_j values as zero. (since the number of unknowns is more than the number of equations)

Step 2

Then we calculate cell evaluations known as d_{ij} values for unoccupied cells, by the formula $d_{ij} = C_{ij} - (U_i + V_j)$.

Step 3

If all d_{ij} values are positive, the solution is optimal and unique. If at least one of them is zero and others positive the solution is optimal but alternative solution exists. If at least one d_{ij} is negative, the solution is not optimal.

Step 4

If the solution is not optimal, make reallocations. Give maximum allocation to the cell for which d_{ij} is negative, making one of the occupied ells empty.

Then we repeat the steps 1 to 4 until solution becomes optimal.

Eg. Solve the following transportation problem

	W_1	W_2	W_3	Supply
F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
Demand	7	9	18	

Answer. To find the initial basic feasible solution, we apply Vogel's method

Table I

Allocate to cell (1, 1), minimum of 5 and 7, which is 5. Then cross off the row F_1.

	W_1	W_2	W_3	Supply
F_1	$\boxed{5}$ 2	7	4	5 (2) X
F_2	3	3	1	8 (2)
F_3	5	4	7	7 (1)
F_4	1	6	2	14 (1)
Demand	7 (2)	9 (1)	18 (1)	

Table II

Allocate to cell (1, 3), minimum of 8 and 18, which is 8. Then cross off the row F_2.

	W_1	W_2	W_3	
F_2	3	3	$\boxed{8}$ 1	8 (2) X
F_3	5	4	7	7 (1)
F_4	1	6	2	14 (1)
	3 (2)	9 (1)	18 (1)	

Table III

Allocate to cell (2, 3), minimum of 10 and 14, which is 10. Then cross off the column W_3.

	W_1	W_2	W_3	
F_3	5	4	7	7 (1)
F_4	1	6	**10** 2	14 (1)
	2 (4)	9 (2)	10 (5) X	

Allocate 2 to cell (2, 1) and cross off column W_1.

	W_1	W_2	
F_3	5	4	7 (1)
F_4	**2** 1	6	4 (5)
	2 (4) X	9 (2)	

Allocate 2 to cell (2, 1) and 7 to cell (1, 1)

	W_2	
F_3	**7** 4	7
F_4	**2** 6	2
	9	

The initial basic feasible solution is

	W_1	W_2	W_3	Supply
F_1	**5** 2	7	4	5
F_2	3	3	**8** 1	8
F_3	5	**7** 4	7	7
F_4	**2** 1	**2** 6	**10** 2	14
Demand	7	9	18	

To test the optimality; form the equations $U_i + V_j = C_{ij}$ for occupied cells.

Occupied cells are (1, 1), (2, 3), (3, 2), (4, 1), (4, 2), (4, 3). The costs (C_{ij}) in these occupied cells are 2, 1, 4, 1, 6, 2. The equations are,

$U_1 + V_1 = 2$ $\qquad$ $U_4 + V_1 = 1$

$U_2 + V_3 = 1$ $\qquad$ $U_4 + V_2 = 6$

$U_3 + V_2 = 4$ $\qquad$ $U_4 + V_3 = 2$

To solve these equations, take $U_4 = 0$ (which occurs more number of times)

$U_4 + V_1 = 1 \implies V_1 = 1$

$U_4 + V_2 = 6 \implies V_2 = 6$

$U_4 + V_3 = 2 \implies V_3 = 2$

$U_1 + V_1 = 2 \implies U_1 + 1 = 2 \implies U_1 = 2 - 1 = 1$

$U_2 + V_3 = 1 \implies U_2 + 2 = 1 \implies U_2 = 1 - 2 = -1$

$U_3 + V_2 = 4 \implies U_3 + 6 = 4 \implies U_3 = 4 - 6 = -2$

Calculate the cell evaluation for unoccupied cells; $d_{ij} = C_{ij} - (U_i + V_j)$.

C_{ij}

×	7	4
3	3	×
5	×	7
×	×	×

$C_{ij} - (U_i + V_j) = d_{ij}$

$U_i + V_j$

				U_i
	×	1 + 6 = 7	1 + 2 = 3	1
	-1 + 1 = 0	-1 + 6 = 5	×	-1
	-2 + 1 = -1	×	-2 + 2 = 0	-2
	×	×	×	0
V_j	1	6	2	

d_{ij}

×	7 – 7 = 0	4 – 3 = 1
3 – 0 = 3	3 – 5 = - 2	×
5 + 1 = 6	×	7 – 0 = 7
×	×	×

Since one of the d_{ij} values is negative, the solution is not optimal.

Make reallocations. Give maximum allocation to the cell having negative d_{ij}, that is the cell (2, 2).

Consider the four cells below

	W_2	W_3	
F_2	3	**8** 1	8
F_4	**2** 6	**10** 2	12
	2	9	

Transfer 2 from (F_4, W_2) to (F_2, W_2). Correspondingly subtract 2 from (F_2, W_3) and add 2 to (F_4, W_3). The resulting table is given below.

	W_2	W_3	
F_2	**2** 3	**8-2** 1	8
F_4	**2-2** 6	**10+2** 2	12
	2	18	

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	W_2	W_3	
F_2	**2** 3	**6** 1	8
F_4	6	**12** 2	12
	2	18	

The reallocation gives the following matrix.

	W_1	W_2	W_3	
F_1	**5** 2	7	4	5
F_2	3	**2** 3	**6** 1	8
F_3	5	**7** 4	7	7
F_4	**2** 1	6	**12** 2	14
	7	9	18	

Test the optimality again; the equations $U_i + V_j = C_{ij}$ for occupied cells are

$U_1 + V_1 = 2$ $\quad$ $U_3 + V_2 = 4$

$U_2 + V_2 = 3$ $\quad$ $U_4 + V_1 = 1$

$U_2 + V_3 = 1$ $\quad$ $U_4 + V_3 = 2$

To solve these equations, take $U_4 = 0$

Then $V_1 = 1$, $V_2 = 4$, $V_3 = 2$, $U_1 = 1$, $U_2 = -1$, $U_3 = 0$

Calculate the cell evaluation for unoccupied cells; $d_{ij} = C_{ij} - (U_i + V_j)$.

$$C_{ij} - (U_i + V_j) = d_{ij}$$

C_{ij}

×	7	4
3	×	×
5	×	7
×	6	×

$U_i + V_j$

			U_i
×	1 + 4 = 5	1 + 2 = 3	1
-1 + 1 = 0	×	×	-1
0 + 1 = 1	×	0 + 2 = 2	0
×	0 + 4 = 4	×	0
V_j: 1	4	2	

d_{ij}

×	7 – 5 = 2	4 – 3 = 1
3 – 0 = 3	×	×
5 - 1 = 4	×	7 – 2 = 5
×	6 – 4 = 2	×

No d_{ij} value is negative, the solution is optimal. The optimal solution is;

F_1 to W_1 : 5 $\quad$ F_3 to W_2 : 7

F_2 to W_2 : 2 $\quad$ F_4 to W_1 : 2

F_2 to W_3 : 6 $\quad$ F_4 to W_3 : 12

Total transportation cost = $(5 \times 2) + (2 \times 3) + (6 \times 1) + (7 \times 4) + (2 \times 1) + (12 \times 2) = 10 + 6 + 6 + 28 + 2 + 24 = 76$.

Degeneracy in transportation problems

In transportation problem, degeneracy occurs whenever the number of individual allocations is less than m + n – 1, where m and n are respectively the number of rows and columns of the transportation matrix. Degeneracy in transportation problem can develop in two ways

(1) The basic feasible solution may degenerate from the initial stage onwards.

(2) They may degenerate at any intermediate stage.

In such cases we allocate Δ to one o more empty cells so that the total number of allocations is m + n – 1. Δ is a very small number almost equal to zero.

Eg. Solve the following transportation problem.

	A	B	C	Available
X	50	30	220	1
Y	90	45	170	3
Z	250	200	50	4
Required	4	2	2	

Answer. To find the initial basic feasible solution, we apply Vogel's method

Table I

Allocate to cell (3, 3), minimum of 4 and 2, which is 2. Then cross off column C.

	A	B	C	
X	50	30	220	1 (20)
Y	90	45	170	3 (45)
Z	250	200	$\boxed{2}$ 50	4 (150) ←
	4 (40)	2 (15)	2 (120) X	

Allocate 2 to cell (3, 2 and cross off row Z and column B

	A	B	
X	50	30	1 (20)
Y	90	45	3 (45)
Z	250	$\boxed{2}$ 200	2 (50) ← X
	4 (40)	2 (15) X	

Allocate 1 to cell (1, 1) and 3 to cell (2, 1)

	A	
X	$\boxed{1}$ 50	1
Y	$\boxed{3}$ 90	3
	4	

The initial basic feasible solution is given below.

	A	B	C	
X	[1] 50	30	220	1
Y	[3] 90	45	170	3
Z	250	[2] 200	[2] 50	4
	4	2	2	

Here m + n -1 = 3 + 3 -1 = 5. Total number of allocations is 4, which is less than m + n -1. So the solution is degenerate. Now to resolve this degeneracy, we allocate a very small amount Δ to the cell (2, 2), getting 5 allocations at independent positions.

The new basic feasible solution is;

	A	B	C	
X	[1] 50	30	220	1
Y	[3] 90	[Δ] 45	170	3
Z	250	[2] 200	[2] 50	4
	4	2	2	

To test the optimality; form the equations $U_i + V_j = C_{ij}$ for occupied cells.

Occupied cells are (1, 1), (2, 1), (2, 2), (3, 2), (3, 3). The costs (C_{ij}) in these occupied cells are 50, 90, 45, 200, 50. The equations are,

$U_1 + V_1 = 50 \qquad U_3 + V_2 = 200$

$U_2 + V_1 = 90 \qquad U_3 + V_3 = 50$

$U_2 + V_2 = 45$

To solve these equations, take $U_3 = 0$, then $V_2 = 200$, $V_3 = 50$,

$U_2 + V_2 = 45 \implies U_2 + 200 = 45 \implies U_2 = 45 - 200 = -155$

$U_2 + V_1 = 90 \implies -155 + V_1 = 90 \implies V_1 = 90 + 155 = 245$

$U_1 + V_1 = 50 \implies U_1 + 245 = 50 \implies U_1 = 50 - 245 = -195.$

Calculate the cell evaluation for unoccupied cells; $d_{ij} = C_{ij} - (U_i + V_j)$.

C_{ij}

×	30	220
×	×	170
250	×	×

$U_i + V_j$

				U_i
	×	5	-145	-195
	×	×	-105	-155
	245	×	×	0
V_j	245	200	50	

d_{ij}

×	25	365
×	×	275
5	×	×

No d_{ij} is negative, so the solution is optimal. The optimal solution is
X to A; 1, Y to A: 3, Z to B: 2, Z to C: 2.
Total transportation cost = $(1 \times 50) + (3 \times 90) + (\Delta \times 45) + (2 \times 200) + (2 \times 50)$
$= 50 + 270 + 0 + 400 + 100 = 8200$.

Unbalanced transportation problems

A transportation problem is said to be unbalanced if the sum of all available amounts is not equal to the sum of all requirements in all destinations together. That is, $\sum a_i \neq \sum b_j$. An unbalanced transportation problem is converted into a balanced transportation problem, by introducing a fictitious source or destination. The cost of transportation corresponding to it is taken to be zero.

Eg. Find the initial feasible solution for the transportation problem.

	W_1	W_2	W_3	W_4	Availability
F_1	11	20	7	8	50
F_2	21	16	10	12	40
F_3	8	12	18	9	70
Requirement	30	25	35	40	

Answer. Here total requirement = 30 + 25 + 35 + 40 = 130. Total availability = 50 + 40 + 70 = 160. They are not equal, so this is an unbalanced transportation problem. Here total availability is 30 more than the total requirement. We convert it into a balanced problem by introducing a fictitious destination, W_5 with requirement 30 and cost or transportation 0. The balanced transportation problem is given below.

	W_1	W_2	W_3	W_4	W_5	
F_1	11	20	7	8	0	50
F_2	21	16	10	12	0	40
F_3	8	12	18	9	0	70
	30	25	35	40	30	

To find initial basic feasible solution, we use Vogel's method.

Allocate to cell (2, 5), minimum of 40 and 30, which is 30. Then cross off column W_5.

	W_1	W_2	W_3	W_4	W_5	
F_1	11	20	7	8	0	50 (7)
F_2	21	16	10	12	**30** 0	40 (10) ←
F_3	8	12	18	9	0	70 (8)
	30 (3)	25 (4)	35 (3)	40 (1)	30 (0) X	

Allocate to cell (3, 2), minimum of 70 and 25, which is 25. Then cross off column W_2.

	W_1	W_2	W_3	W_4	
F_1	11	20	7	8	50 (1)
F_2	21	16	10	12	10 (2)
F_3	8	**25** 12	18	9	70 (1)
	30 (3)	25 (4) ↑ X	35 (3)	40 (1)	

Allocate to cell (3, 1), minimum of 30 and 45, which is 30. Then cross off column W_1.

	W_1	W_3	W_4	
F_1	11	7	8	50 (1)
F_2	21	10	12	10 (2)
F_3	**30** 8	18	9	45 (1)
	30 (3) ↑ X	35 (3)	40 (1)	

Allocate to cell (3, 2), minimum of 40 and 15, which is 15. Then cross off row F_3.

	W_3	W_4	
F_1	7	8	50 (1)
F_2	10	12	10 (2)
F_3	18	**15** 9	15 (9) ← X
	35 (3)	40 (1)	

Allocate to cell (1, 2), minimum of 50 and 25, which is 25. Then cross off column W_4.

	W_3	W_4	
F_1	7	**25** 8	50 (1)
F_2	10	12	10 (2) ←
	35 (3)	25 (4) ↑ X	

Allocate 25 to cell (1, 2) and 10 to cell (2, 1) .

	W_3	
F_1	**25** 7	25
F_2	**10** 10	10
	35	

The initial basic feasible solution is given below.

	W_1	W_2	W_3	W_4	W_5	
F_1	11	20	**25** 7	**25** 8	0	50
F_2	21	16	**10** 10	12	**30** 0	40
F_3	**30** 8	**25** 12	18	**15** 9	0	70
	30	25	35	40	30	

Maximization in transportation problems

A transportation problem in which the objective is to maximize, can be solved by converting it into a minimization problem. For this select the highest value and subtract all other values from this highest value. Then the given problem becomes minimization problem.

Eg. Solve the following transportation problem to maximize profit.

Profit in Rs./unit distribution

	A	B	C	D	Supply
I	15	51	42	33	23
II	80	42	26	81	44
III	90	40	66	60	33
Demand	23	31	16	30	

Answer. This is a maximization problem. Convert it into a minimization problem. For this select the highest profit, which is 90. Subtract each profit from 90. The modified matrix is,

	A	B	C	D	Supply
I	75	39	48	57	23
II	10	48	64	9	44
III	0	50	24	30	33
Demand	23	31	16	30	

Then by using Vogel's method find initial basic feasible solution and by using MODI method, find optimal solution as before, in the minimization problem.

Assignment problem

Assignment problem is a special case of transportation problem, in which the objective is to assign a number of origins (persons) to the equal number of destinations (tasks) at a minimum cost. For example, a department has four persons available for assignment and four jobs to fill. Then the interest is to find the best assignment which will be in the best interest of the department.

The assignment problem can be stated in the form of n x n matrix called cost of effectiveness matrix.

Destination

$$
\begin{array}{c|cccc|}
 & 1 & 2 & \cdots & n \\
\hline
1 & c_{11} & c_{12} & \cdots & c_{1n} \\
2 & c_{21} & c_{22} & \cdots & c_{2n} \\
 & . & . & & \\
n & c_{n1} & c_{n2} & \cdots & c_{nn} \\
\hline
\end{array}
$$

Mathematical formulation of assignment problems

Let c_{ij} be the cost of assigning i^{th} source (person) to the j^{th} destination (job) .

Mathematically an assignment problem can be stated as follows.

The problem is

Minimize the total cost Z = $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to the conditions

$\sum_{i=1}^{n} x_{ij}$ = 1, for j = 1, 2, . . . , n, which means that only one job is done by the i^{th} person

where i = 1, 2, 3, . . . , n

$\sum_{j=1}^{n} x_{ij}$ = 1, for i = 1, 2, . . . , n, which means that one person should be assigned to the j^{th} job , where j = 1, 2, 3, . . . , n

Difference between transportation problem and assignment problem

1. Transportation problem is a LPP in which the objective is to transport various quantum of commodity that are stored at various origins, to different destinations with minimum transportation cost.

   The assignment problem is a special case of transportation problem in which the objective is to assign a number of origins to the equal number of destinations at a minimum cost.

2. In transportation problems number of origins and number of destinations need not be equal, so that the number of rows and the number of columns of the matrix need not be equal.
   In assignment problems the number of persons and the number of jobs are equal, so that the number of rows and the number of columns of the matrix are equal.
3. Transportation problems are said to be unbalanced if the total demand and the total supply are not equal.
   Assignment problems are said to be unbalanced if the number of rows are not equal to the number of columns.
4. In transportation problems a positive quantity is allocated from a source to a destination.
   In assignment problems a source (job) is assigned to a destination (person).

Method of solving assignment problem

Assignment algorithm (Hungarian method)

Step 1

Subtract the smallest element of each row, in the cost matrix, from every element of that row.

Step 2

Subtract the smallest element of each column, in the reduced matrix, from every element of that column.

Step 3

(a) Starting with row 1 of the matrix obtained, examine all rows having exactly one zero (0) element. Enclose this zero within $\square$ showing that assignment is made there. Cross out all other zeros in the column in which zero is enclosed. Proceed in this way until the last row is examined.
(b) Examine all columns with one unmarked zero. Mark $\square$ at this zero. Cross out all other zeros in the row in which zero is marked. Proceed in this way until the last column is examined.
(c) Continue these operations (a) and (b) successively until we reach any of the following two situations.

(i) All the zeros are enclosed by $\square$ or crossed off.

(ii) The remaining unmarked zeros lie in at least two rows or columns.

In case (i) we have a maximal assignment.

In case (ii) still we have some zeros to be treated. For this we use trial and error method.

After the above operations, there arise two situations.

(1) It has an assignment in every row and column so that we get the solution.

(2) It does not contain assignment in all rows and columns. In this case the following method is used.

Step 4

- Mark ✓ all rows for which assignments have not been made.
- Mark ✓ all columns which have zeros in marked rows.
- Mark ✓ all rows (not already marked) which have assignment in marked columns.
- Draw lines through unmarked rows and through marked columns to cover all the zeros.

Step 5

Select the smallest of the elements that is not covered by lines. Subtract it from all the elements that do not have a line through them. Add it to every element that lies at the intersection of two lines. Leave the remaining elements of the matrix unchanged.

Step 6

Reapply the steps 3 to 5 to the modified matrix.

Eg. Find the optimum solution to the following assignment problem showing the cost for assigning workers to jobs.

Job

		X	Y	Z
Workers	A	18	17	16
	B	15	13	14
	C	19	20	21

Answer.

Subtracting the smallest element of every row from all the elements of that row

	X	Y	Z
A	2	1	0
B	2	0	1
C	0	1	2

Subtracting the smallest element of every column from all the elements of that column

	X	Y	Z
A	2	1	0
B	2	0	1
C	0	1	2

Mark □ to zero in every row, starting from the first row. Now all the rows and all the columns have assignment. So the solution is optimum. The solution is A to Z, B to Y, and C to X. Total cost of assignment = 16 + 13 + 19 = 48

	X	Y	Z
A	2	1	$\boxed{0}$
B	2	$\boxed{0}$	1
C	$\boxed{0}$	1	2

Eg. Solve the following assignment problem.

$$
\begin{array}{cc|cccc|}
 & & & \text{Man} & & \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & \text{I} & 12 & 30 & 21 & 15 \\
\text{Job} & \text{II} & 18 & 33 & 9 & 31 \\
 & \text{III} & 44 & 25 & 21 & 21 \\
 & \text{IV} & 14 & 30 & 28 & 14 \\
\hline
\end{array}
$$

Answer.

Subtracting the smallest element of each row from every element of that row

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\text{I} & 0 & 18 & 9 & 3 \\
\text{II} & 9 & 24 & 0 & 22 \\
\text{III} & 23 & 4 & 0 & 0 \\
\text{IV} & 0 & 16 & 14 & 0 \\
\hline
\end{array}
$$

Subtracting the smallest element of every column from all the elements of that column

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\text{I} & 0 & 14 & 9 & 3 \\
\text{II} & 9 & 20 & 0 & 22 \\
\text{III} & 23 & 0 & 3 & 0 \\
\text{IV} & 0 & 12 & 14 & 0 \\
\hline
\end{array}
$$

Starting from the first row, mark $\square$ to zero in the row containing only one zero and cross out the zeros in the column in which it lies,

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\text{I} & \boxed{0} & 14 & 9 & 3 \\
\text{II} & 9 & 20 & \boxed{0} & 22 \\
\text{III} & 23 & 0 & 3 & \cancel{0} \\
\text{IV} & \cancel{0} & 12 & 14 & \boxed{0} \\
\hline
\end{array}
$$

Starting from the first column, mark $\square$ to zero in the column containing only one unmarked or uncrossed zero and cross out the zeros in the row in which it lies.

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\text{I} & \boxed{0} & 14 & 9 & 3 \\
\text{II} & 9 & 20 & \boxed{0} & 22 \\
\text{III} & 23 & \boxed{0} & 3 & \cancel{0} \\
\text{IV} & \cancel{0} & 12 & 14 & \boxed{0} \\
\hline
\end{array}
$$

Now all the rows and all the columns have assignment.
So the solution is optimum. Th e solution is

Job:	I	II	III	IV.
Man:	1	3	2	4

Total cost of assignment = 12 + 9 + 25 + 14 = 60

Eg. Solve the following assignment problem.

Man

$$\begin{array}{cc|cccc|}
 & & 1 & 2 & 3 & 4 \\
\hline
 & \mathrm{I} & 12 & 10 & 8 & 9 \\
\text{Job} & \mathrm{II} & 8 & 9 & 11 & 7 \\
 & \mathrm{III} & 11 & 14 & 12 & 10 \\
 & \mathrm{IV} & 9 & 9 & 8 & 9 \\
\hline
\end{array}$$

Answer.

Subtracting the smallest element of each row from every element of that row.

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\mathrm{I} & 4 & 2 & 0 & 1 \\
\mathrm{II} & 1 & 2 & 4 & 0 \\
\mathrm{III} & 1 & 4 & 2 & 0 \\
\mathrm{IV} & 1 & 1 & 0 & 1 \\
\hline
\end{array}$$

.

Subtracting the smallest element of each column from every element of that column.

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\mathrm{I} & 3 & 1 & 0 & 1 \\
\mathrm{II} & 0 & 1 & 4 & 0 \\
\mathrm{III} & 0 & 3 & 2 & 0 \\
\mathrm{IV} & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Starting from the first row, mark $\boxed{}$ to zero in the row containing only one zero and cross out the zeros in the column in which it lies,

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
\mathrm{I} & 3 & 1 & \boxed{0} & 1 \\
\mathrm{II} & 0 & 1 & 4 & 0 \\
\mathrm{III} & 0 & 3 & 2 & 0 \\
\mathrm{IV} & 0 & 0 & \cancel{0} & 1 \\
\hline
\end{array}$$

Starting from the first column, mark $\square$ to zero in the column containing only one unmarked or uncrossed zero and cross out the zeros in the row in which it lies,

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline \text{I} & 3 & 1 & \boxed{0} & 1 \\ \text{II} & 0 & 1 & 4 & 0 \\ \text{III} & 0 & 3 & 2 & 0 \\ \text{IV} & \cancel{0} & \boxed{0} & \cancel{0} & 1 \\ \hline \end{array}$$

Again starting row wise, assignment is done already in the first row. Consider the second row. There are two zeros. Select any one zero in that row, say first one and complete the assignment.

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline \text{I} & 3 & 1 & \boxed{0} & 1 \\ \text{II} & \boxed{0} & 1 & 4 & \cancel{0} \\ \text{III} & \cancel{0} & 3 & 2 & \boxed{0} \\ \text{IV} & \cancel{0} & \boxed{0} & \cancel{0} & 1 \\ \hline \end{array}$$

We can have an alternative solution if we had selected last zero in the second row.

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline \text{I} & 3 & 1 & \boxed{0} & 1 \\ \text{II} & \cancel{0} & 1 & 4 & \boxed{0} \\ \text{III} & \boxed{0} & 3 & 2 & \cancel{0} \\ \text{IV} & \cancel{0} & \boxed{0} & \cancel{0} & 1 \\ \hline \end{array}$$

The two solutions are

(a) I to 3, II to 1, III to 4, IV to 2. Total cost = 8 + 8 + 10 + 9 = 35
(b) I to 3, II to 4, III to 1, IV to 2. Total cost = 8 + 7 + 11 + 9 = 35

Eg. Solve the following assignment problem showing the cost for assigning workers to jobs.

		Job		
		X	Y	Z
Workers	A	17	25	31
	B	10	25	16
	C	12	14	11

Answer.

Subtracting the smallest element of every row from all the elements of that row

	X	Y	Z
A	0	8	14
B	0	15	6
C	1	3	0

Subtracting the smallest element of every column from all the elements of that column

$$\begin{array}{c|ccc|} & X & Y & Z \\ \hline A & 0 & 5 & 14 \\ B & 0 & 12 & 6 \\ C & 1 & 0 & 0 \\ \hline \end{array}$$

Mark $\square$ to zero in every row, starting from the first row and then from each column.

$$\begin{array}{c|ccc|} & X & Y & Z \\ \hline A & \boxed{0} & 5 & 1 \\ B & \cancel{0} & 12 & 6 \\ C & 1 & \boxed{0} & \cancel{0} \\ \hline \end{array}$$

Now all the rows and columns do not have assignment. So mark ∨ in the second row as it has no assignment. Then mark ∨ in the first column as it has zeros in the ticked row. Then mark ∨ in the first row as it has assigned zero of ticked column. Draw lines in the unticked rows and ticked columns.

$$\begin{array}{c|ccc|c} & X & Y & Z & \\ \hline A & \boxed{0} & 5 & 14 & \vee \\ B & 0 & 12 & 6 & \vee \\ C & 1 & \boxed{0} & 0 & \\ \hline & \vee & & & \end{array}$$

The smallest element not covered by the lines is 5

(i) Subtract 5 from all elements not covered by the lines

(ii) Add 5 to the elements at the intersection of two lines. Then make assignment.

Now every row and column has assignment. the solution is optimal. The solution is
A to Y, B to X, C to Z
Total cost = 25 + 10 + 11 = 46

$$\begin{array}{c|ccc|} & X & Y & Z \\ \hline A & \cancel{0} & \boxed{0} & 9 \\ B & \boxed{0} & 7 & 1 \\ C & 6 & \cancel{0} & \boxed{0} \\ \hline \end{array}$$

Maximization in assignment problems

The objective of some assignment problem is to maximize the effectiveness like maximizing profit. Such problems can be converted into minimization problems. For this subtract each element of the matrix from the highest element. Then minimization of the resulting matrix gives the solution.

Eg. Given below is the profit for different jobs done through different machines. Find the assignment programme which maximizes the total profit.

		Machines			
		A	B	C	D
Job	I	51	53	54	50
	II	47	50	48	50
	III	49	50	60	61
	IV	63	64	60	61

Answer.

.

Since the problem is a maximization problem, convert it into a minimization problem. The highest element is 64. Subtract all elements from 64.

	A	B	C	D
I	13	11	10	14
II	17	14	16	14
III	15	14	4	3
IV	1	0	4	3

.

Subtracting the smallest element of each row from every element of that row.

	A	B	C	D
I	3	1	0	4
II	3	0	2	0
III	12	11	1	0
IV	1	0	4	3

.

.

Subtracting the smallest element of each column from every element of that column.

	A	B	C	D
I	2	1	0	4
II	2	0	2	0
III	11	11	1	0
IV	0	0	4	3

.

.

Making zero assignment starting from the first row. Then making assignment starting from the first column.

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline I & 2 & 1 & \boxed{0} & 4 \\ II & 2 & \boxed{0} & 2 & \cancel{0} \\ III & 11 & 11 & 1 & \boxed{0} \\ IV & \boxed{0} & \cancel{0} & 4 & 3 \\ \hline \end{array}$$

.

Now all rows and colums have zero assignment. So the solution is optimal.

The solution is I to C, II to B, III to D, IV to A. Total profit = 54 + 50 + 61 + 63 = 228.

Unbalanced assignment problems

An assignment problem is sad to be unbalanced, whenever the number of jobs is not equal to the number of persons. That is, in the cost matrix the number of rows is not equal to the number of columns. It is not a square matrix. To solve such a problem, we add dummy rows or dummy columns to the given matrix to make it a square matrix. The cost in the dummy rows or columns are taken to be zero. Now it is a balanced problem and we can solve it.

Eg. Solve the assignment problem given below.

Machines

$$\begin{array}{cc|cccc|} & & A & B & C & D \\ \hline & I & 18 & 24 & 28 & 32 \\ \text{Job} & II & 8 & 13 & 17 & 19 \\ & III & 10 & 15 & 19 & 22 \\ \hline \end{array}$$

Answer.

Given problem is unbalanced since the number of rows (3) is less than the number of columns (4). So introducing a dummy row with cost 0.

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline I & 18 & 24 & 28 & 32 \\ II & 8 & 13 & 17 & 19 \\ III & 10 & 15 & 19 & 22 \\ IV & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

.

Subtracting the smallest element of each row from every element of that row and subtracting the smallest element of eachcolumn from every element of that column.

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline I & 0 & 6 & 10 & 14 \\ II & 0 & 5 & 9 & 11 \\ III & 0 & 5 & 9 & 12 \\ IV & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

.

Making zero assignment.

$$\begin{array}{c|cccc|}
 & A & B & C & D \\
\hline
I & \boxed{0} & 6 & 10 & 14 \\
II & \cancel{0} & 5 & 9 & 11 \\
III & \cancel{0} & 5 & 9 & 12 \\
IV & \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \\
\hline
\end{array}$$

All the rows and columns do not have assignment. We proceed further.

So mark ✓ in the second and third rows as they have no assignment. Then mark ✓ in the first column as it has zeros in the ticked row. Then mark ✓ in the first row as it has assigned zero of ticked column. Draw lines in the unticked rows and ticked columns.

$$\begin{array}{c|cccc|c}
 & A & B & C & D & \\
\hline
I & \boxed{0} & 6 & 10 & 14 & ✓ \\
II & 0 & 5 & 9 & 11 & ✓ \\
III & 0 & 5 & 9 & 12 & ✓ \\
IV & 0 & \boxed{0} & 0 & 0 & \\
\hline
 & ✓ & & & & \\
\end{array}$$

The smallest element not covered by the lines is 5. Subtract 5 from all elements not covered by the lines. Add 5 to the elements at the intersection of two lines. Then make assignment.

$$\begin{array}{c|cccc|}
 & A & B & C & D \\
\hline
I & \boxed{0} & 1 & 5 & 9 \\
II & \cancel{0} & \boxed{0} & 4 & 6 \\
III & \cancel{0} & \cancel{0} & 4 & 7 \\
IV & 5 & \cancel{0} & \boxed{0} & \cancel{0} \\
\hline
\end{array}$$

Again the assignment is not complete. We proceed further. Making ✓ marks in row 1, 2 and 3 and columns 1, 2. Then draw lines in unticked rows and ticked columns.

$$\begin{array}{c|cccc|c}
 & A & B & C & D & \\
\hline
I & \boxed{0} & 1 & 5 & 9 & ✓ \\
II & \cancel{0} & \boxed{0} & 4 & 6 & ✓ \\
III & \cancel{0} & \cancel{0} & 4 & 7 & ✓ \\
IV & 5 & \cancel{0} & \boxed{0} & \cancel{0} & \\
\hline
 & ✓ & ✓ & & & \\
\end{array}$$

The smallest element not covered by the lines is 4. Subtract 4 from all elements not covered by the lines. Add 4 to the elements at the intersection of two lines. Then make assignment.

$$\begin{array}{c|cccc|}
 & A & B & C & D \\
\hline
I & \boxed{0} & 1 & 1 & 5 \\
II & \cancel{0} & \boxed{0} & \cancel{0} & 2 \\
III & \cancel{0} & \cancel{0} & \boxed{0} & 3 \\
IV & 9 & 4 & \cancel{0} & \boxed{0} \\
\hline
\end{array}$$

One solution is I to A, II to B, III to C, IV to D. Another solution is I to A, II to C, III to B, IV to D.

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline I & \boxed{0} & 1 & 1 & 5 \\ II & \cancel{0} & \cancel{0} & \boxed{0} & 2 \\ III & \cancel{0} & \boxed{0} & \cancel{0} & 3 \\ IV & 9 & 4 & \cancel{0} & \boxed{0} \end{array}$$