Module - 4

Boolean Algebra

The operation of a circuit is defined by a Boolean function that specifies the value of an output for each set of inputs. The first step in constructing a circuit is to represent its Boolean function by an expression built up using the basic operations of Boolean algebra.

Boolean Functions

Boolean algebra provides the operations and the rules for working with the set $\{0, 1\}$.

The three operations in Boolean algebra that we will use most are complementation, the Boolean sum, and the Boolean product

The complement of an element, denoted with a bar, is defined by $\bar{0} = 1$ and $\bar{1} = 0$

The Boolean sum, denoted by + or by OR, has the following values:-

$1 + 1 = 1$
$1 + 0 = 1$
$0 + 1 = 1$
$0 + 0 = 0$

The Boolean product, denoted by · or by AND, has the following values

$1 \cdot 1 = 1$
$1 \cdot 0 = 0$
$0 \cdot 1 = 0$
$0 \cdot 0 = 0$

Precedence

unless parentheses are used the rules of precedence for Boolean operations are :

1. Complements
2. Boolean products
3. Boolean sums

Ques) Find the value of $1 \cdot 0 + (\overline{0+1})$

$= 1 \cdot 0 + 0$

$= 0 + 0$

$= \underline{\underline{0}}$

The complement, Boolean sum and Boolean product correspond to the logical operators $\neg$, $\vee$ and $\wedge$ respectively, where 0 corresponds to F (False) and

1 corresponds to T (true).
equalities in Boolean algebra can be directly translated into equivalences of compound propositions. and conversely

Logical Equivalence

We obtain logical equivalence when we translate each 1 into a T each 0 into a F, each Boolean sum (+) into a disjunction (V), each Boolean product (·) into a conjunction (Λ) and each ~~comp~~ complementation (¯) into a negation (¬).

$1 \longrightarrow T$

$0 \longrightarrow F$

OR $+ \longrightarrow \vee$ disjunction

AND $\cdot \longrightarrow \wedge$ conjunction

$\bar{} \longrightarrow \neg$

$= \longrightarrow \equiv$

Ques Translate $1 \cdot 0 + (\overline{0 + 1}) = 0$ into a logical equivalence

$(T \wedge F) \vee \neg(F \vee T) = F$

Ques Translate the logical equivalence $(T \wedge T) \vee \neg F \equiv T$ into an identity in Boolean algebra.

$(1 \cdot 1) + \overline{0} = 1$

Boolean Expressions & Boolean Functions

Ques Find the values of the Boolean function represented by $F(x,y) = xy$

x	y	$F(x,y)$ OR xy
1	1	1
1	0	0
0	1	0
0	0	0

Ques Find the values of the Boolean function represented by $F(x,y,z) = xy + z$

x	y	z	xy	$\bar{z}$	$F(x,y,z) = xy + \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Equivalent - Two different Boolean fx expressions that represent the same function are called equivalent.

eg:- The Boolean expressions xy, $xy+0$ and $xy \cdot 1$ are equivalent.

Ques How many different Boolean functions of degree n are there?

There are 2^n possible different Boolean functions of degree n.

No. of Boolean functions of degree n

Degree	Number
1	4
2	16
3	256
4	65,536
5	4,294,967,296
6	18,446,744,073,709,551,616

Identities of Boolean Algebra

Identity	Name
$x = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent Laws
$x + 0 = x$ $x \cdot 1 = x$	Identity Laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination Laws
$x + y = y + x$ $xy = yx$	Commutative Laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative Laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive Laws

$\overline{(xy)} = \bar{x} + \bar{y}$ $\overline{(x+y)} = \bar{x}\,\bar{y}$	De Morgan's Laws
$x + xy = x$ $x(x+y) = x$	Absorption Laws
$x + \bar{x} = 1$	Unit property
$x \cdot \bar{x} = 0$ OR $x\bar{x} = 0$	Zero property

Ques Verify one of the distrubutive Laws
$[x(y+z) = xy + xz]$

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Ques Translate the destributive law $x + yz = (x+y)\cdot(x+z)$ into a logical equivalence

$\Rightarrow$ $p \vee qr \equiv (p \vee q) \wedge (p \vee r)$

Ques Prove the absorption law $x(x+y) = x$ using the other identities of Boolean algebra

$x(x+y) = \underline{(x+0)(x+y)}$ Identity law for the Boolean sum

$= \underline{x + 0 \cdot y}$ Distrubutive law of the Boolean sum over the Boolean product.

$= \underline{x + y \cdot 0}$ Commutative law for the Boolean product

$= \underline{x + 0}$ Domination law for the Boolean product

$= \underline{x}$ Identity law for the Boolean sum

Duality

The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

ie $0 \longleftrightarrow 1$

$\cdot \longleftrightarrow +$

Ques Find the duals of (i) $x(y+0)$

(ii) $\bar{x}\cdot 1+(\bar{y}+z)$

(i) $x + y\cdot 1$

(ii) $(\bar{x}+0)\cdot(\bar{y}z)$

Duality Principle

The Abstract Definition of Boolean Algebra

A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1 and a unary operation $-$ such that these properties hold for all x, y and z in B

$$\left.\begin{aligned} x \vee 0 &= x \\ x \wedge 1 &= x \end{aligned}\right\} \text{ Identity laws}$$

$$\left.\begin{aligned} x \vee \bar{x} &= 1 \\ x \wedge \bar{x} &= 0 \end{aligned}\right\} \text{ Complement laws}$$

$$\left.\begin{aligned} (x \vee y) \vee z &= x \vee (y \vee z) \\ (x \wedge y) \wedge z &= x \wedge (y \wedge z) \end{aligned}\right\} \text{ Associative laws}$$

$$\left.\begin{aligned} x \vee y &= y \vee x \\ x \wedge y &= y \wedge x \end{aligned}\right\} \text{ Commutative laws}$$

$$\left.\begin{array}{l} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{array}\right\}$$ Distributive laws

Representing Boolean Functions

A literal is a boolean variable or its compliment

Minterm (standard product)

The minterm of the boolean variable $x_1, x_2 \ldots x_n$ is a boolean product $y_1, y_2 \cdots y_n$ where $y_i = x_i$ or $y_i = \overline{x_i}$

Hence a minterm is a product of n literals with one literal for each variable.

The sum of minterms that represents the function is called the sum-of-products expansion or the disjunctive normal form of the Boolean function.

Sum of Products (SOP)

A sum of products expression is a single product term / minterm or several product terms / minterms logically added (ORed) together.

OR Group of product terms that is summed together

Also called DNF- ~~Dist~~ Disjunctive Normal Form

eg:- $(A \cdot B) + (C \cdot D)$

Maxterm (standard sum)

Maxterm is a sum of n literals with one literal for each variable

The product of minterms that represents the function is called product-of-sums expansion or the conjunctive normal form.

Product of Sums (POS)

A product of sums expression is a single sum term/ maxterm or several sum terms/ maxterms logically multiplied (ANDed) together.

OR Group of sum terms that is producted together.

Also called CNF - Conjunctive Normal Form

eg:- $(A+B)\cdot(C+D)$

Minterm & Maxterm

A	B	Minterm
0	0	$\bar{A} \cdot \bar{B}$
0	1	$\bar{A} \cdot B$
1	0	$A \cdot \bar{B}$
1	1	$A \cdot B$

0 - compliment
1 - variable

A	B	Maxterm
0	0	$A + B$
0	1	$A + \bar{B}$
1	0	$\bar{A} + B$
1	1	$\bar{A} + \bar{B}$

0 - variable
1 - compliment

Ques Find a minterm that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$ and equals to 0 otherwise maxterm

Minterm

$x_1 = x_3 = 0$

$x_2 = x_4 = x_5 = 1$

~~To prove $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 1$~~

LHS $\overline{x_1} \cdot x_2 \cdot \overline{x_3} \cdot x_4 \cdot x_5$

$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

$= \underline{\underline{1}}$

Maxterm

$x_1 = x_3 = 0$

$x_2 = x_4 = x_5 = 1$

LHS $x_1 + \overline{x_2} + x_3 + \overline{x_4} + \overline{x_5}$

$0 + 0 + 0 + 0 + 0$

$= \underline{\underline{0}}$

Ques 1 Find the sum of products of expansion for the function

$F(x,y,z) = (x+y)\cdot\bar{z}$

Two methods

1st $F(x,y,z) = (x+y)\bar{z}$

$= x\bar{z} + y\bar{z}$ Distributive law

$= x1\bar{z} + 1y\bar{z}$ Identity law

$= x(y+\bar{y})\bar{z} + (x+\bar{x})y\bar{z}$ Unit property

$= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z}$ Distributive law

$= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$ Idempotent law

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Example 9 Translate the distributive law $x + yz = (x + y)(x + z)$ in Table ...

2nd

x	y	z	$\bar{z}$	$x+y$	$(x+y)\cdot\bar{z}$
1	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	1	0
1	0	0	1	1	1
0	1	1	0	1	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	1	0	0

The corresponding minterms of the above combinations are

110, 100, 010. (convert to $x\,y\,z$ form)

ie $xy\bar{z}$, $x\bar{y}\bar{z}$, $\bar{x}y\bar{z}$

Taking the sum (OR) of all these minterms

$$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$$

Ques Find the product of sum expansion for the function $F(x, y, z) = (x+y)\cdot\bar{z}$

x	y	z	$\bar{z}$	$x+y$	$(x+y)\cdot\bar{z}$
1	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	1	0
1	0	0	1	1	1
0	1	1	0	1	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	1	0	0

The corresponding maxterms of the above combination are

111, 101, 011, 001, 000

$\bar{x}+\bar{y}+\bar{z}$, $\bar{x}+y+\bar{z}$, $x+\bar{y}+\bar{z}$, $x+y+\bar{z}$, $x+y+z$

Taking the product (AND) of all these maxterms

$(\bar{x}+\bar{y}+\bar{z})\cdot(\bar{x}+y+\bar{z})\cdot(x+\bar{y}+\bar{z})\cdot$
$(x+y+\bar{z})\cdot(x+y+z)$

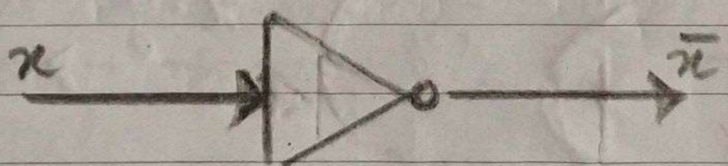
Logic Gates

The basic elements of circuits are called gates.

Each type of gate implements a Boolean operation.

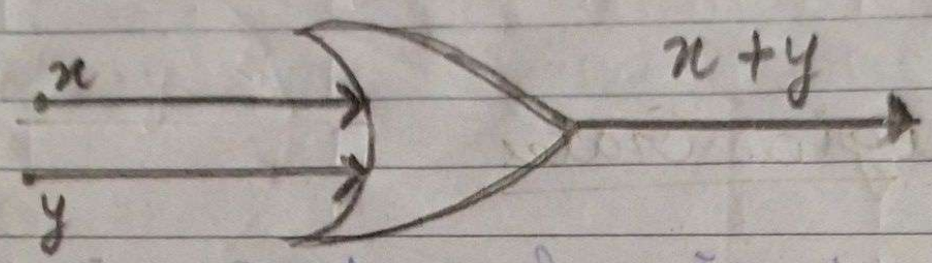
Basic Types of gates

① Inverter (NOT gate)



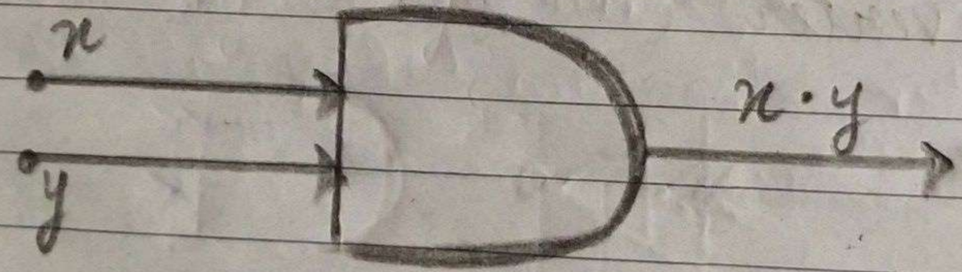
Inverter accepts the value of one Boolean variable as input and, produces the complement of this value as its output.

② OR gate



In OR gate, the inputs to this gate are the values of two or more Boolean variables. The output is the Boolean sum of their values.

③ AND gate



In AND gate, the inputs to this gate are the values of two or more Boolean variables. The output is the Boolean product of their values.

Combination of Gates

Combinational circuits can be constructed using a combination of inverters, OR gates and AND gates.

when combinations of circuits are formed, some gates may share inputs.

One method is to use branchings that indicate all the gates that use a given method.

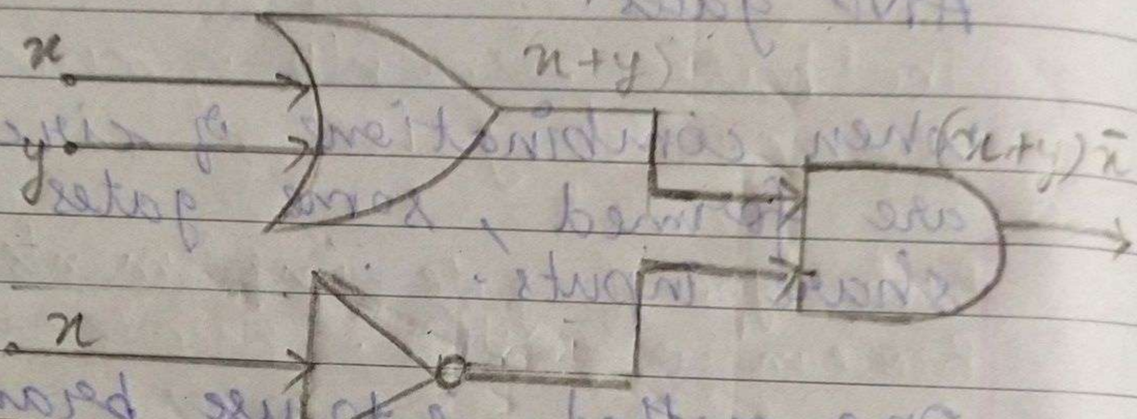
The other method is to indicate this input separately for each gate.

Also the output from a gate may be used as input by one or more other elements.

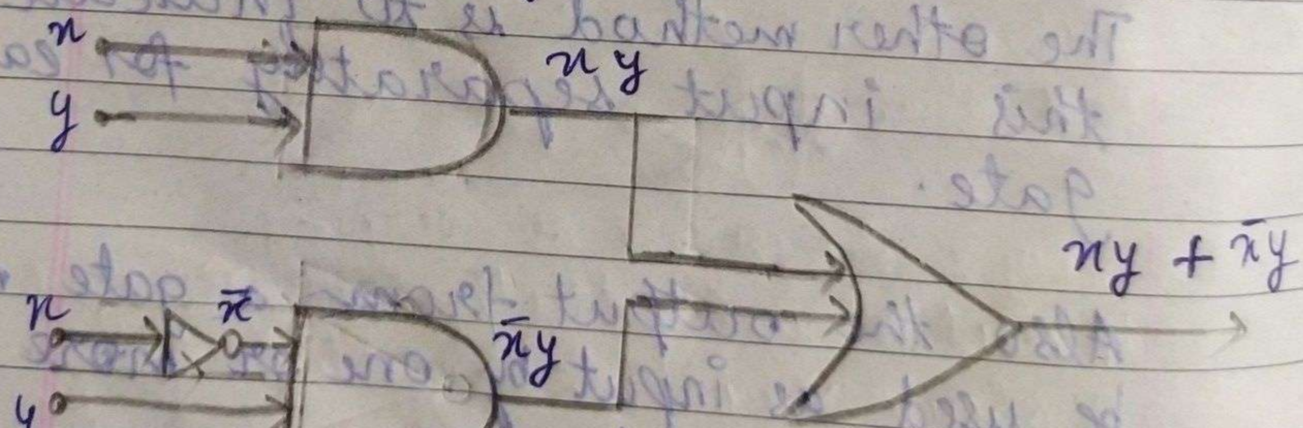
Ques Construct circuits that produce the following outputs

(a) $(x+y)\bar{x}$

Ans:-



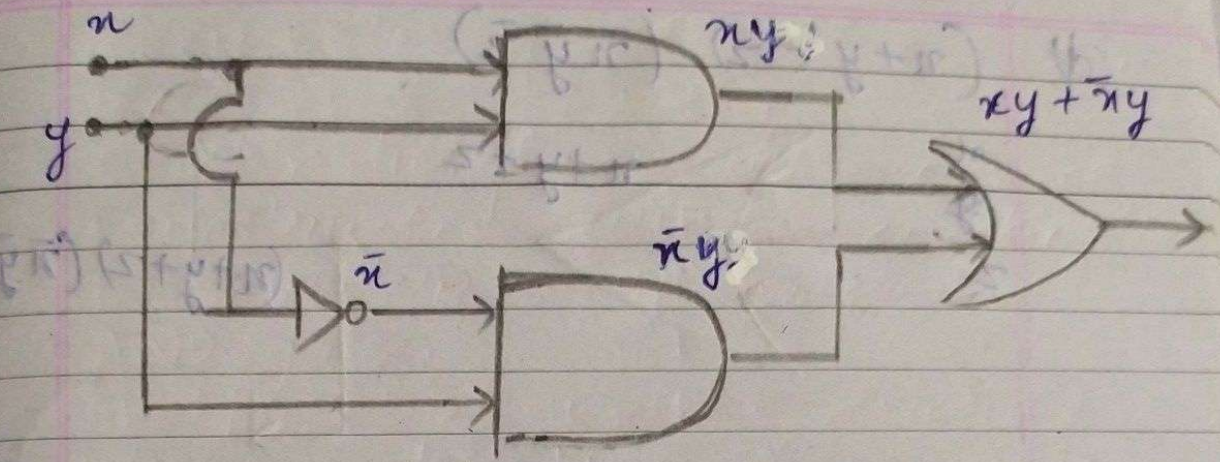
(b) ~~xy~~ $xy + \bar{x}y$.



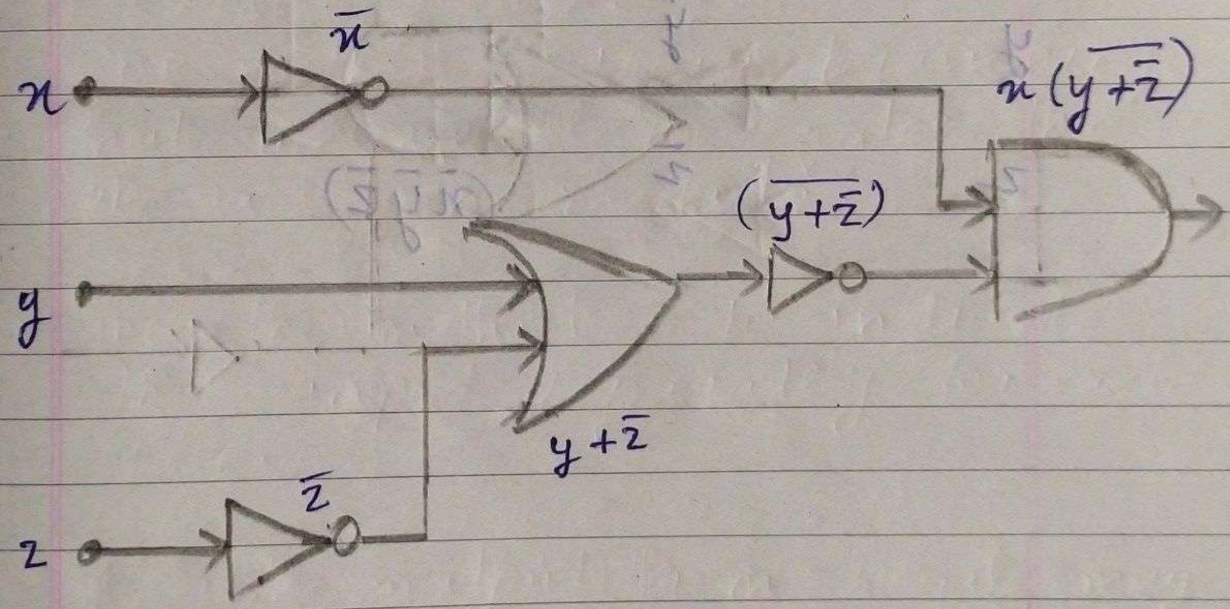
OR

Example 9 *Translate the distributive law $x + yz = (x + y)(x + z)$ in Table 5 ...*

(6)



(c) $\bar{x}\,(\overline{y + \bar{z}})$



(d) $(x+y+z)(\bar{x}\bar{y}\bar{z})$